CHAPTER III

STRUCTURE THEOREMS

3.1 A Structure Theorem for Elementary Functions

In 1979 R.H. Risch [8] gave a so-called structure theorem for elementary functions which shows all possible algebraic relationships among a set of elementary functions, which will be described in this section.

Let F be a differential field. Here the expression "$K_1 = F(t_1, t_2,\ldots,t_n)$ is an elementary extension of F" means that the tower of fields $F = F_0 \subset F_1 \subset \cdots \subset F_n = K_1$ with for each i, $1 \le i \le n$, $F_i = F_{i-1}(t_i)$ and t_i satisfies one of the conditions for elementary extension of F.

Let $E = \{i \mid t_i = \exp(a_i),\ a_i \in F_{i-1},\ 1 \le i \le n\}$ and

$L = \{i \mid t_i = \log(a_i),\ a_i \in F_{i-1},\ 1 \le i \le n\}$.

Theorem 3.1.1([8]). Let F be a differential field of characteristic zero, K an elementary extension of F with the same subfield of constants C. Let $K_1 = F(t_1,\ldots,t_n)$ be an elementary extension of F and $K = K_1(t)$ for some t in K.

(1) If $t = \log(v)$ is algebraic over K_1 where v is in K_1, then there are $\{c_i \in C \mid i \in L\}$, $\{d_i \in C \mid i \in E\}$ and f in the algebraic closure of F in K such that

$$t + \sum_{i\in L} c_i t_i + \sum_{i\in E} d_i a_i = f,$$

where $t_i = \exp(a_i)$ for $i \in E$.

(2) If $t = \exp(v)$ is algebraic over K_1 where v is in K_1,then there are $\{n_i \in \mathbf{Z} \mid i \in L\}$, $\{m_i \in \mathbf{Z} \mid i \in E\}$, $n \in \mathbf{Z}\setminus\{0\}$ and g in the algebraic closure of F in K such that

$$t^n\left(\prod_{i\in L} a_i^{n_i}\right)\left(\prod_{i\in E} t_i^{m_i}\right) = g,$$

where $t_i = \log(a_i)$ for $i \in L$.

Proof. The proof is analogous to the proof of Lemma 5.1.4 in Chapter V.

3.2 A Structure Theorem for Exponential and Primitive Functions

This section describes M. Rothstein and B.D. Caviness's generalization of a structure theorem of Risch [8] in the case that the ground field is the field of constants.

Let F be a differential field and K a differential extension field of F. An element x in K is called a primitive over F if $D(x) \in F$ for each given derivation D of K.

Let $t \in K$ be primitive over F. t is a simple logarithm over F if there exist $u_1,\ldots,u_m$ in F such that for some constant c in K, $t + c \in F(\log(u_1),\ldots,\log(u_m))$. We say that t is nonsimple if it is not a simple logarithm over F.

We call K a generalized log-explicit extension of F if there exists a finite tower of fields $F = F_0 \subset F_1 \subset \cdots \subset F_n = K$ such that for each i, $1 \le i \le n$, $F_i = F_{i-1}(t_i)$ and one of the following holds:

(i) t_i is algebraic over F_{i-1},

(ii) $t_i = \exp(u)$ for some u in F_{i-1},

(iii) $t_i = \log(u)$ for some nonzero u in F_{i-1},

(iv) t_i is primitive and nonsimple over F_{i-1}.

Remark. The differential extension field K of F equipped with cases (i) - (iii) is an elementary extension of F.

Here the expression "$K = C(t_1,...,t_n)$ is a generalized log - explicit extension field of C, where C is the field of constants of K" means that K is a differential field and the tower of fields $C = K_0 \subset K_1 \subset \cdots \subset K_n = K$ with for each i, $1 \le i \le n$, $K_i = K_{i-1}(t_i)$ and t_i satisfying one of the conditions of generalized log-explicit extension field.

Let $L = \{ i \mid t_i = \log(a_i), a_i \in K_{i-1}, 1 \le i \le n \}$,

$E = \{ i \mid t_i = \exp(a_i), a_i \in K_{i-1}, 1 \le i \le n \}$.

Rothstein and Caviness's structure theorem for exponential and primitive functions is as follows:

Theorem 3.2.1 ([9]). Let $K = C(t_1,...,t_n)$ be a generalized log-explicit extension field of C, where C is the subfield of constants of K having characteristic zero. If $u \neq 0$ and v are members of K such that $D(u)/u = D(v)$ for each given derivation D of K, then there exist rational numbers r_i, s_j for all $i \in L$, $j \in E$ and a constant $c \in C$ such that

$$v = c + \sum_{i \in L} r_i t_i + \sum_{j \in E} s_j a_j,$$

where $t_j = \exp(a_j)$ for $j \in E$.

Proof. The proof is by induction on m, the number of primitive and nonsimple among $t_1,...,t_n$. For $m = 0$, then $t_1,...,t_n \in C$ and hence $K = C$. So $v \in C$. Let $m \ge 1$. Assume that the theorem is true for all smaller values of m. Among the m t_i's that are primitive and nonsimple with respect to $C(t_1,...,t_{i-1})$ and let t_M be the one with the largest subscript. For notational simplicity, let $F = C(t_1,...,t_{M-1})$ if $M > 1$ or $F = C$ if $M = 1$ and $t = t_M$. There are $u_1,...,u_p$, $v_1,...,v_p \in K$, with $u_1,...,u_p$ nonzero such that

(i) $D(u_i)/u_i = D(v_i)$ for $i = 1,\dots,p$ and all derivations D of K,

(ii) precisely one member of each pair (u_i, v_i) is algebraic over $F(t, u_1,\dots,u_{i-1},v_1,\dots,v_{i-1})$ if $i > 1$ or algebraic over $F(t)$ if $i = 1$,

(iii) one member of each pair (u_i, v_i) will be t_j for some $M+1 \le j \le n$,

(iv) K is algebraic over $F(t, u_1,\dots,u_p, v_1,\dots,v_p)$.

Note that $D(t) \in F$,

$$D(u_i)/u_i - D(v_i) = 0 \in F, \text{ for } i = 1,\dots,p,$$

$$D(u)/u - D(v) = 0 \in F,$$

and that tr.deg. $F(t,u_1,\dots,u_p, u, v_1,\dots,v_p, v)/F < p + 2$. Then by Theorem 1.8, the elements dt, $du_1/u_1-dv_1,\dots,du_p/u_p-dv_p$, and $du/u-dv$ of $\Omega_{K/F}$ are linearly dependent over C.

In fact $du/u-dv$ depends linearly on dt and the $du_i/u_i - dv_i$. We can therefore find constants $\gamma_1,\dots,\gamma_p, \gamma$ such that

$$du/u - dv + \sum_{i=1}^{p} \gamma_i(du_i/u_i - dv_i) + \gamma dt = 0. \tag{3.1}$$

Let $c_o = 1, c_1,\dots,c_q$ be a basis for the vector space over **Q** spanned by $\{\gamma_o = 1, \gamma_1, \gamma_2,\dots,\gamma_p\}$ and write $\gamma_i = \sum_{j=0}^{q} n_{ij}c_j$ with each $n_{ij} \in \mathbf{Q}$.

Replacing each c_j by c_j/least common denominator of $\{n_{ij} \mid i = 0,1,\dots,p, j = 0,1,\dots,q\}$, if necessary, we can assume $n_{ij} \in \mathbf{Z}$. This means, in particular,

$$1 = \gamma_o = \sum_{j=0}^{q} n_{oj}c_j = n_{oo}c_o; \text{ that is } n_{o1} = n_{02} = \dots = n_{oq} = 0.$$

We can write (3.1) as

$$\sum_{j=0}^{q} c_j \left\{ \frac{d\left(u^{n_{0j}} u_1^{n_{1j}} \cdots u_p^{n_{pj}}\right)}{u^{n_{0j}} u_1^{n_{1j}} \cdots u_p^{n_{pj}}} - d\left(n_{0j}v + n_{1j}v_1 + \cdots + n_{pj}v_p\right) \right\} + \gamma dt = 0.$$

For j = 0, ...,q let $z_j = u^{n_{0j}} u_1^{n_{1j}} \cdots u_p^{n_{pj}}$,

$$y_j = n_{0j}v + n_{1j}v_1 + \cdots + n_{pj}v_p,$$

and we have that $D(z_j)/z_j = D(y_j)$ for all derivations D of K and

$$\sum_{j=0}^{q} c_j dz_j/z_j - d\left(\sum_{j=0}^{q} c_j y_j - \gamma t\right) = 0.$$

Since $c_0, c_1, \ldots, c_q$ are linearly independent over **Q**, we have, by Theorem 1.7, that each z_j and $w = \sum_{j=0}^{q} c_j y_j - \gamma t$ are algebraic over F.

First of all γ must be zero. To see this, note that $\sum_{j=0}^{q} c_j(D(z_j)/z_j - D(y_j)) = 0$ and hence $\gamma D(t) = \sum_{j=0}^{q} c_j D(z_j)/z_j - D(w)$ for all derivations D of K. $z_0, \ldots, z_q$ and w are algebraic over F, so taking σ an element of the Galois group of $F' = F(z_0, z_1, \ldots, z_q, w)$ over F and summing over all σ and dividing by [F'. F] gives

$$\gamma D(t) = \sum_{j=0}^{q} \frac{c_j}{[F':F]} \frac{D(N(z_j))}{N(z_j)} + D\left(\frac{T(w)}{[F':F]}\right)$$

where T and N denote trace and norm respectively.

If $\gamma \neq 0$, then t would be simple logarithm over F, contrary to the hypotheses.

In particular, we can conclude that $\sum_{j=0}^{q} c_j y_j = w$ is algebraic over F.

Now, let $K' = F(z_o,\dots,z_q, w, y_1,\dots,y_q) \subset K$. $F(z_o,\dots,z_q, w)$ is an algebraic extension of F and K' is an extension of $F(z_o,\dots,z_q,w)$.

Furthermore $y_o = w - \sum_{j=0}^{q} c_j y_j \in K'$ and $D(z_o)/z_o = D(y_o)$ for all derivations D of K. K' has one less primitive and nonsimple than K so by the induction hypothesis, we have

$$(3.2) \qquad y_o = c + \sum_{i\in L'} r_i t_i + \sum_{i\in E'} s_i a_i + \sum_{i=1}^{q} \bar{r}_i y_i$$

where $E' = \{i \mid t_i = \exp(a_i),\ a_i \in F_{i-1} \text{ and } t_i \in F\}$,

$L' = \{i \mid t_i = \log(a_i),\ a_i \in F_{i-1} \text{ and } t_i \in F\}$,

and r_i, s_i, $\bar{r}_i$ are rational numbers.

Now recall

$$(3.3) \quad \begin{cases} y_o = n_{oo}v + n_{1o}v_1 + \dots + n_{po}v_p = v + n_{1o}v_1 + \dots + n_{po}v_p & \text{since } n_{oo} = 1, \\ y_1 = n_{o1}v + n_{11}v_1 + \dots + n_{p1}v_p = n_{11}v_1 + \dots + n_{p1}v_p & \text{since } n_{o1} = 0, \\ \vdots & \\ y_q = n_{oq}v + n_{1q}v_1 + \dots + n_{pq}v_p = n_{1q}v_1 + \dots + n_{pq}v_p & \text{since } n_{oq} = 0. \end{cases}$$

Substitute the expressions (3.3) in (3.2) and note that each v_i either equals some t_i with $i \in L$ or equals some a_i where $t_i = \exp(a_i)$, $i \in E$, to obtain

$$v = c + \sum_{i\in L} \hat{r}_i t_i + \sum_{i\in E} \hat{s}_i a_i,$$

where $\hat{r}_i$ and $\hat{s}_i$ are rational numbers.

This completes the proof. #