CHAPTER III

WAVE MOTION IN THE ATMOSPHERE

There are a variety of phenomena occurring which are extremely complex motions of the atmosphere. If we wish to gain physical insight into the fundamental nature of atmospheric motion, it is necessary to isolate and analyze some simple type of motion. In this chapter we will use a simple technique, the perturbation method, which is ideally suited for qualitative analysis of the nature of atmospheric motions to examine several types of pure waves in the atmosphere.

Linearized Equations

In order to simplify the governing equation, we consider motion only in the x-z plane and assume uniformity in the lateral direction (y) and also neglect the rotation of the earth, friction, and diabatic heating. The Newtonian momentum equations, thermodynamic equation, and the continuity equation are then expressible in the form

$$\begin{aligned} \frac{du}{dt} + \alpha\frac{\partial P}{\partial x} &= 0 \\ \frac{dw}{dt} + \alpha\frac{\partial P}{\partial z} + g &= 0 \\ \alpha\frac{dP}{dt} + P\gamma\frac{d\alpha}{dt} &= 0 \\ \alpha\nabla.\vec{V} - \frac{d\alpha}{dt} &= 0 \end{aligned} \tag{3.1}$$

where

$$\gamma = \frac{C_p}{C_v}\ ,\ \alpha = \frac{1}{\rho}$$

Now these equations will be linearized by the so-called *perturbation method.* For simplicity assume a constant basic current $\bar{u}$ and basic state thermodynamic variables $\bar{p}(z)$ and $\bar{\alpha}(z)$ in hydrostatic balance $\left(\bar{\alpha}\,\partial\bar{p}/\partial z = -g\right)$. Next, express the dependent variables as the sum of the basic or undisturbed value plus a perturbation.

$$\begin{aligned} u &= \bar{u} + u' \\ w &= 0 + w' \\ p &= \bar{p} + p' \\ \alpha &= \bar{\alpha} + \alpha' \end{aligned} \tag{3.2}$$

Substitute eq.(3.2) into eq.(3.1) and neglect the products of perturbation quantities, the resulting linear equations for the perturbation quantities are:

$$\begin{aligned} &\frac{\partial u'}{\partial t} + \bar{u}\frac{\partial u'}{\partial x} + \bar{\alpha}\frac{\partial p'}{\partial x} = 0 \\ &\delta 1\left(\frac{\partial w'}{\partial t} + \bar{u}\frac{\partial w'}{\partial x}\right) + \bar{\alpha}\frac{\partial p'}{\partial z} - g\frac{\alpha'}{\bar{\alpha}} = 0 \\ &\bar{\alpha}\left(\frac{\partial p'}{\partial t} + \bar{u}\frac{\partial p'}{\partial x}\right) - gw' + \bar{p}\,\gamma\left(\frac{\partial \alpha'}{\partial t} + \bar{u}\frac{\partial \alpha'}{\partial x} + w'\frac{\partial \bar{\alpha}}{\partial z}\right) = 0 \\ &\bar{\alpha}\left(\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z}\right) - \delta 2\left(\frac{\partial \alpha'}{\partial t} + \bar{u}\frac{\partial \alpha'}{\partial x}\right) - w'\frac{\partial \bar{\alpha}}{\partial z} = 0 \end{aligned} \tag{3.3}$$

The symbols $\delta 1$ and $\delta 2$ will take on values of either unity or zero according to whether the terms are omitted or included.

Sound Waves

Sound waves are compression waves that can be isolated by setting $g = 0$, $\delta 1 = \delta 2 = 1$, and by letting $\bar{p}$ and $\bar{\alpha}$ be constants. Now assume the perturbation quantities are harmonic in x , z, and t with constant coefficients as follows:

$$
\begin{aligned}
u' &= S\, e^{i(\mu x + kz - \nu t)} , \quad w' = W\, e^{i(\mu x + kz - \nu t)} \\
p' &= P\, e^{i(\mu x + kz - \nu t)} , \quad \alpha' = A\, e^{i(\mu x + kz - \nu t)}
\end{aligned}
\tag{3.4}
$$

These are plane waves with μ and k as wave numbers in the x and z directions and with ν as the frequency. The actual physical quantities are obtained by taking the real parts of the solutions. Substituting eq.(3.4) into eq.(3.3) leads to a system of homogeneous algebraic equations for amplitudes S, W, P and A as follows:

$$
\begin{aligned}
(\mu\bar{u} - \nu)\, S + \bar{\alpha}\mu P &= 0 \\
(\mu\bar{u} - \nu)\, W + \bar{\alpha} k P &= 0 \\
\bar{\alpha}\,(\mu\bar{u} - \nu)\, P + \bar{P}\gamma(\mu\bar{u} - \nu)\, A &= 0 \\
\mu\bar{\alpha} S + \bar{\alpha} k W - (\mu\bar{u} - \nu)\, A &= 0
\end{aligned}
\tag{3.5}
$$

Which can be rewritten in matrix form as

$$
\begin{bmatrix}
(\mu\bar{u} - \nu) & 0 & \bar{\alpha}\mu & 0 \\
0 & (\mu\bar{u} - \nu) & \bar{\alpha} k & 0 \\
0 & 0 & \bar{\alpha}(\mu\bar{u} - \nu) & \bar{P}\gamma(\mu\bar{u} - \nu) \\
\bar{\alpha}\mu & \bar{\alpha} k & 0 & -(\mu\bar{u} - \nu)
\end{bmatrix}
\begin{bmatrix} S \\ W \\ P \\ A \end{bmatrix} = 0
\tag{3.6}
$$

Nonzero values for the amplitudes S, W, P, and A, are possible only if the determinant of the set of homogeneous equations vanishes. When the determinant is expanded the following frequency equation is obtained.

$$\bar{\alpha}\,(\mu\bar{u} - \nu)^2\left[-(\mu\bar{u} - \nu)^2 + \gamma\bar{p}\bar{\alpha}\left(k^2 + \mu^2\right)\right] = 0 \tag{3.7}$$

Then, we get

$$\nu = \mu\bar{u} \tag{3.8a}$$

or

$$\nu = \mu\bar{u} \pm \left(k^2 + \mu^2\right)^{\frac{1}{2}} \sqrt{\gamma R T} \tag{3.8b}$$

The phase speed is the velocity of the phase lines $(\mu x + kz = \text{constant})$ in the normal direction, and it is related to the frequency as follows:

$$c = \nu\left(k^2 + \mu^2\right)^{-\frac{1}{2}} \tag{3.9}$$

setting $\bar{u} = 0$ and using (3.8b) in (3.9) gives

$$c = \pm\sqrt{\gamma R T} \tag{3.10}$$

Which is the well-known formula for the speed of sound.

Gravity Waves

Gravity waves are transverse oscillations which arise from the differential effect of gravity on air parcels of different density at the same level. Gravity waves can only exit in a medium which is stably stratified. In this section we will discuss some properties of atmospheric gravity waves.

A. <u>Internal Gravity Waves</u>

To introduce the internal gravity waves, we consider a dry atmosphere which is governed by the hydrostatic and state equations:

$$0 = -\overline{\alpha}\frac{\partial \overline{p}}{\partial z} - g$$

$$\overline{p}\,\overline{\alpha} = R\,T$$

หอสมุดกลาง สถาบันวิทยบริการ จุฬาลงกรณ์มหาวิทยาลัย

For an isothermal atmosphere the solutions to these equations give

$$\overline{p} = \overline{p}(0)\,e^{-z/H} \quad , \quad \overline{\alpha} = \overline{\alpha}(0)\,e^{z/H} \qquad (3.11)$$

where $H = R\overline{T}/g$ is called the *scale height.* Under these conditions simple wave solutions of the form:

$$\begin{aligned} u' &= S\,\overline{\alpha}^{1/2}e^{i(\mu x + kz - \nu t)} \quad , \quad & w' &= W\,\overline{\alpha}^{1/2}e^{i(\mu x + kz - \nu t)} \\ p' &= P\,\overline{\alpha}^{-1/2}e^{i(\mu x + kz - \nu t)} \quad , \quad & \alpha' &= A\,\overline{\alpha}^{3/2}e^{i(\mu x + kz - \nu t)} \end{aligned} \qquad (3.12)$$

lead to the matrix equation

$$\begin{bmatrix} -\nu & 0 & \mu & 0 \\ 0 & -\delta_1\nu & k + \dfrac{i}{2\overline{\alpha}}\dfrac{\partial\overline{\alpha}}{\partial z} & i\,g \\ 0 & \left(-g + \dfrac{\gamma R \overline{T}}{\overline{\alpha}}\dfrac{\partial\overline{\alpha}}{\partial z}\right)i & \nu & \gamma R \overline{T}\nu \\ \mu & k + \dfrac{i}{2\overline{\alpha}}\dfrac{\partial\overline{\alpha}}{\partial z} & 0 & \delta_2\nu \end{bmatrix} \begin{bmatrix} S \\ W \\ P \\ A \end{bmatrix} = 0 \qquad (3.13)$$

With elements that are not functions of position or time. Here $\overline{u}$ is omitted since it merely adds to the propagation in the x direction. Next substitute eq.(3.11) into eq.(3.13) and set the determinant equal to zero, which gives the following frequency equation:

$$\delta_1\delta_2\nu^4 - \left\{\gamma R\overline{T}\left(k^2 + \mu^2\delta_1\right) + \frac{g}{4H}\left[\left(2\delta_2 - 1\right)\gamma + 2\left(1 - \delta_2\right)\right] + g\left(\gamma - 1\right)\left(\delta_2 - 1\right) ik\right\}\nu^2 + \mu^2 g^2\left(\gamma - 1\right) = 0$$

The g terms in the ν^2 coefficient may be dropped in comparison to $\gamma R\overline{T} k^2$ when the vertical wavelength $2\pi / k$ is smaller than $4\pi H$, which is always the case. Also, the last term can be rewritten with the potential temperature, which gives

$$\delta_1\delta_2\nu^4 - \gamma R\overline{T}\left(k^2 + \mu^2\delta_1\right)\nu^2 + \mu^2\gamma\frac{R\overline{T}}{\overline{\theta}}\frac{g\partial\overline{\theta}}{\partial z} = 0 \tag{3.14}$$

The four roots of this equation correspond to a pair of sound waves and a pair of internal gravity waves. The gravity waves can be excluded by setting g to zero, which give the same result as in the previous section. To isolate the gravity waves, take $\delta_2 = 0$ (incompressibility) and $\delta_1 = 1$, which gives

$$\nu^2 = \frac{\dfrac{\mu^2 g\,\partial\overline{\theta}}{\overline{\theta}\,\partial z}}{k^2 + \mu^2} \tag{3.15}$$

Using eq.(3.9), the phase speed becomes

$$c = \pm\frac{\mu}{k^2 + \mu^2}\left(\frac{g\partial\overline{\theta}}{\overline{\theta}\partial z}\right)^{1/2} \tag{3.16}$$

If the depth of the disturbance is large compared to the horizontal scale, $\mu^2 >> k^2$, then, from eq.(3.15)

$$\nu = \pm\left(\frac{g\partial\overline{\theta}}{\overline{\theta}\partial z}\right)^{1/2} \tag{3.17}$$

This is the Brunt-Vaisala frequency for essential vertical oscillations. On the other hand when $k^2 >> \mu^2$

$$\nu = \pm\frac{\mu}{k}\left(\frac{g}{\bar{\theta}}\frac{\partial\bar{\theta}}{\partial z}\right)^{1/2} \tag{3.18}$$

and, the phase speed becomes

$$c = \frac{\nu}{k} = \pm\frac{\mu}{k^2}\left(\frac{g}{\bar{\theta}}\frac{\partial\bar{\theta}}{\partial z}\right)^{1/2} \tag{3.19}$$

There are no meteorological phenomena in which sound waves play a significant dynamical role, and it is often desirable to eliminate them from the equations. It can be seen from eq.(3.14) that the sound waves are excluded if the hydrostatic approximation ($\delta_1 = 0$) or the incompressibility condition ($\delta_2 = 0$) is used.

B. Surface Gravity Waves

The surface gravity waves arise at the interface between two fluids of differing density. In order to examine the basic nature of gravity waves, we consider the atmosphere which is incompressible and homogeneous, $\alpha' = 0$, and an upper surface will exist that will be permitted to be free, the linearized equations of motion eq.(3.3) are as follows:

$$\frac{\partial u'}{\partial t} + \bar{u}\frac{\partial u'}{\partial x} + \frac{1}{\rho}\frac{\partial p'}{\partial x} = 0 \tag{3.20a}$$

$$\delta\left(\frac{\partial w'}{\partial t} + \bar{u}\frac{\partial w'}{\partial x}\right) + \frac{1}{\rho}\frac{\partial p'}{\partial z} = 0 \tag{3.20b}$$

$$\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0 \tag{3.20 c}$$

The hydrostatic equation applies to the undisturbed flow

$$\frac{\partial \bar{p}}{\partial z} = - \rho g$$

Integrating this equation from z = 0 to the top of the undisturbed fluid H gives

$$g \bar{\rho} H = p_0 \tag{3.21}$$

Next assume the perturbation quantities to be of the harmonic form

$$\left.\begin{aligned} u' &= \psi(z)\, e^{i\mu(x - ct)} \\ w' &= \Phi(z)\, e^{i\mu(x - ct)} \\ \frac{p'}{\rho} &= p(z)\, e^{i\mu(x - ct)} \end{aligned}\right\} \tag{3.22}$$

Substituting eq.(3.22) into eq.(3.20) and simplifying leads to

$$\left.\begin{aligned} (\bar{u} - c)\, \psi(z) + p(z) &= 0 \\ i\mu\delta\, (\bar{u} - c)\, \Phi(z) + p'(z) &= 0 \\ i\mu\psi(z) + \Phi'(z) &= 0 \end{aligned}\right\} \tag{3.23}$$

Eliminating p(z) from the first two equations of eq.(3.23) and then further elimination of between the resulting equation and the last equation of eq.(3.23) gives

$$\Phi''(z) - \mu^2 \delta \Phi(z) = 0 \tag{3.24}$$

We consider only in the hydrostatic case $(\delta = 0)$ and, at the lower boundary, which is assumed to be horizontal, the vertical velocity vanishes, then we get

$$\Phi(z) = a\, z \tag{3.25}$$

The second boundary condition is that the total pressure of a surface particle remains unchanged. Hence

$$\frac{d(\bar{p} + p')}{dt} = 0 \qquad (3.26)$$

at the free surface. These may be approximated by linearizing and applying this condition at z = H . Thus,

$$\frac{\partial p'}{\partial t} + \bar{u}\frac{\partial p'}{\partial x} + w'\frac{\partial \bar{p}}{\partial z} = 0 \qquad ,(z = H) \qquad (3.27)$$

Utilizing the solutions eq.(3.25) and eq.(3.23) gives the following results.

$$\left.\begin{aligned} \psi(z) &= a\,i/\mu \\ p(z) &= \frac{-i\,a(\bar{u} - c)}{\mu} \end{aligned}\right\} \qquad (3.28)$$

Substituting eq.(3.25) and eq.(3.28) into eq.(3.27) and simplifying gives

$$c = \bar{u} \pm \sqrt{g\,H} \qquad (3.29)$$

Waves traveling with the phase velocity given by eq(3.29) are generally referred to as " *Shallow-Water* ". The quantity is called the shallow water wave speed. It is a valid approximation only for wave whose wavelengths are much greater than the depth of the fluid.

Rossby Waves

The wave type which is of importance for large-scale atmospheric flow processes is the *Rossby wave*. Rossby waves owe their existence to the variation of the Coriolis force with latitude, the so-called β effect. In this subsection the effects of the earth's rotation will be added to the equations of motion as follows:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - f\,v + g\frac{\partial h}{\partial x} = 0 \tag{3.30a}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + f\,u + g\frac{\partial h}{\partial y} = 0 \tag{3.30b}$$

with the incompressibility assumption, the continuity equation is expressible in the form

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{3.31}$$

Integrating eq.(3.31) with respect to z gives

$$\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) h + w_h - w_0 = 0 \tag{3.32}$$

In accordance with the kinematic boundary condition, w must vanish at the lower boundary (i.e. , $w_0 = 0$) . On the other hand, the vertical velocity $w = dz / dt$ at the upper boundary represents the rate at which the free surface is rising. Thus $w_h = \frac{dh}{dt}$, and eq.(3.32) becomes

$$h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = -\frac{dh}{dt} = -\left(\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y}\right) \tag{3.33}$$

Eq.(3.30) and eq.(3.33) constitute a system of three equations in three unknowns u, v, and h. These equations are called the *shallow-water equations*. Eq.(3.30) and eq.(3.33) are linearized about H and u, which is constant. These quantities are related geostrophically:

$$\bar{u} = -\frac{g}{f}\frac{\partial H}{\partial y} \tag{3.34}$$

Where H is the depth of a fluid of constant density as before. If perturbations u, v, and h as well as f are taken to be independent of y, the following system of equations results:

$$\begin{aligned} \delta\left(\frac{\partial u}{\partial t} + \bar{u}\frac{\partial u}{\partial x}\right) - fv + g\frac{\partial h}{\partial x} &= 0 \\ \frac{\partial v}{\partial t} + \bar{u}\frac{\partial v}{\partial x} + f u &= 0 \\ \frac{\partial h}{\partial t} + \bar{u}\frac{\partial h}{\partial x} + H\frac{\partial u}{\partial x} + v\frac{\partial H}{\partial y} &= 0 \end{aligned} \tag{3.35}$$

Treating the coefficient H as a constant and assuming harmonic perturbations of the form $u_0\, e^{i(\mu x - ct)}$, $v_0\, e^{i(\mu x - ct)}$, and $h_0\, e^{i(\mu x - ct)}$ transform eq.(3.35) into the following system:

$$\begin{aligned} \delta\,(\bar{u} - c)\, i\mu u_0 - f\, v_0 + g i\mu h_0 &= 0 \\ f u_0 + i\mu\,(\bar{u} - c)\, v_0 &= 0 \\ i\mu H u_0 + \frac{\partial H}{\partial y} v_0 + i\mu\,(\bar{u} - c)\, h_0 &= 0 \end{aligned} \tag{3.36}$$

In order that eq.(3.36) will have nontrivial solutions for u_0, v_0, and h_0 the following condition must be satisfied:

$$\begin{vmatrix} \delta(\bar{u}-c)\,i\mu & -f & gi\mu \\ f & i\mu(\bar{u}-c) & 0 \\ i\mu H & \dfrac{\partial H}{\partial y} & i\mu(\bar{u}-c) \end{vmatrix} = 0$$

Expansion of this determinant leads to a cubic frequency equation

$$\delta(\bar{u}-c)^3 - \left(gH + \frac{f^2}{\mu^2}\right)(\bar{u}-c) - \frac{fg}{\mu^2}\frac{\partial H}{\partial y} = 0 \qquad (3.37)$$

This equation contains a pair of fast gravity wave solutions and one slow meteorological solution. In order to isolate the fast solutions set $\bar{u} = 0$, $\frac{\partial H}{\partial y} = 0$ and $\delta = 1$, which gives

$$c = \pm\sqrt{gH + \frac{f^2}{\mu^2}} \qquad (3.38)$$

When $f = 0$, eq.(3.38) reduces to the formula for shallow-water (gravity) waves. The slow meteorological solution to eq.(3.37) may be obtained by setting $\delta = 0$

$$c = \bar{u} + \frac{(f/H)\dfrac{\partial H}{\partial y}}{\mu^2 + (f^2/gH)} \qquad (3.39)$$

This expression can be written in terms of the basic state potential vorticity $\bar{q} = f/H$, which gives

$$\frac{\partial \bar{q}}{\partial y} = -\frac{f}{H^2}\frac{\partial H}{\partial y} = \frac{f^2\bar{u}}{gH^2} \qquad (3.40)$$

with eq.(3.40) the phase velocity can be written in the following form:

$$c = \bar{u} - \frac{H\dfrac{\partial q}{\partial y}}{\mu^2 + (f^2/gH)} \qquad (3.41)$$

This solution is a type of Rossby wave and it gives phase speeds that are reasonable for observed synoptic disturbances. The formula can be generalized if f is allowed to vary with latitude. In this case the potential vorticity gradient becomes

$$\frac{\partial \bar{q}}{\partial y} = \left(\beta - \frac{f}{H}\frac{\partial H}{\partial y}\right) / H = \left[\beta + \left(f^2 \bar{u} / gH\right)\right] / H \tag{3.42}$$

where $\beta = \partial f / \partial y$. This is an example of the rule that Rossby waves propagate in the direction $k \times \nabla \bar{q}$ relative to the mean flow. The Rossby waves can be isolated in eq.(3.37) by setting $\delta = 0$ in eq.(3.36). This implies that the v component is geostrophic, that is,

$$fv = g\frac{\partial h}{\partial x} \tag{3.43}$$

However, the u component is not evaluated geostrophically when the disturbance fields vary in both x and y, it is necessary to replace eq.(3.30) with the vorticity and divergence equation in order to carry out the analysis of this section. It can be show in this case that the gravity waves will be eliminated when the time derivation of the divergence is neglected in the divergence equation.