# CHAPTER IV

# WEATHER FORECASTING MODEL

From the point of view of Theoretical Physics the problem of weather prediction may be regarded as an initial-value problem. Thus, if the initial state of the atmosphere and the laws that govern its motion were known, the future behavior of the atmosphere could be determined by mathematical deduction.

Since, in standard practice meteorological data is reported on constant pressure surface rather than constant height surface, so it is advantageous to compute the closed set of prediction equation relative to surface of constant pressure rather than surfaces of constant height. Thus, in this chapter we will compute the closed set of prediction equation relative to surface of constant pressure and then, present some basic weather prediction models.

## Isobaric Coordinates

Isobaric coordinates are the coordinates in which pressure is the independent vertical coordinate. Since in chapter II, the closed set of prediction equation was expressed used height as a vertical coordinate. Therefore, this involves a transformation from z to p as the independent vertical coordinate. The expression of the horizontal pressure gradient in terms of the height gradient at constant pressure may be carried out with the aid of Fig.4.1

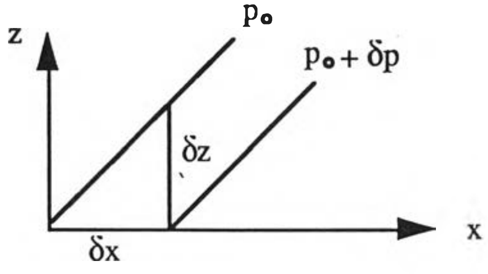


Figure 4.1 Slope of pressure surfaces in the x, z plane

From fig.4.1 we see that

$$\left[\frac{(p_0 + \delta p) - p_0}{\delta x}\right]_z = \left[\frac{(p_0 + \delta p) - p_0}{\delta z}\right]_x \left(\frac{\delta z}{\delta x}\right)_p$$

taking the limit as $\delta x \rightarrow 0$ and $\delta y \rightarrow 0$, we obtain

$$\left(\frac{\partial p}{\partial x}\right)_z = -\left(\frac{\partial p}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_p$$

which after substitution from the hydrostatic approximation may be written

$$\frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right)_z = g\left(\frac{\partial z}{\partial x}\right)_p = \left(\frac{\partial \Phi}{\partial x}\right)_p \qquad (4.1)$$

Similarly, it is easy to show that

$$-\frac{1}{\rho}\left(\frac{\partial p}{\partial y}\right)_z = -\left(\frac{\partial \Phi}{\partial y}\right)_p \qquad (4.2)$$

Thus, in the isobaric coordinate system the horizontal pressure gradient force is measured by the gradient of geopotential at constant pressure.

## The Horizontal Momentum and Hydrostatic Equation

The approximate horizontal momentum eq.(2.10) and eq.(2.11) may be written in vectorial form as

$$\frac{d\vec{V}}{dt} + f\hat{k} \times \vec{V} = \frac{1}{\rho}\nabla p \tag{4.3}$$

where $\vec{V} = \hat{i}\,u + \hat{j}\,v$ is the horizontal velocity vector. In order to express eq.(4.3) in isobaric coordinate form, we transform the pressure gradient force using eq.(4.1) and eq.(4.2) to obtain

$$\frac{d\vec{V}}{dt} + f\hat{k} \times \vec{V} = -\nabla_p \Phi \tag{4.4}$$

where $\nabla_p$ is the horizontal gradient operator applied with pressure held constant. And the hydrostatic relation in isobaric coordinate can be obtained as follows

$$\frac{\partial \Phi}{\partial p} = g\frac{\partial z}{\partial p} = g\left(\frac{-\alpha}{g}\right)$$

$$\frac{\partial \Phi}{\partial p} = -\alpha = -\frac{RT}{p} \tag{4.5}$$

Since p is the independent vertical coordinate, we must expand the total derivative as follows:

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \frac{dx}{dt}\frac{\partial}{\partial x} + \frac{dy}{dt}\frac{\partial}{\partial y} + \frac{dp}{dt}\frac{\partial}{\partial p}$$

$$\equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + \omega\frac{\partial}{\partial p} \tag{4.6}$$

Here $\omega = \frac{dp}{dt}$ ( usually called the " *omega* " vertical motion ) is the pressure change following the motion, which plays the same role in the isobaric coordinate system as $w = \frac{dz}{dt}$ plays in height coordinate.

The Continuity Equation

From the conservation of mass of the fluid element, we can write

$$\frac{1}{\delta M}\frac{d\,\delta M}{dt} = \frac{g}{\delta x\,\delta y\,\delta p}\frac{d\left(\frac{\delta x\,\delta y\,\delta p}{g}\right)}{dt} = 0$$

After differentiating, using the chain rule, and changing the order of the differential operators we obtain

$$\frac{1}{\delta x}\delta\left(\frac{dx}{dt}\right) + \frac{1}{\delta y}\delta\left(\frac{dy}{dt}\right) + \frac{1}{\delta p}\delta\left(\frac{dp}{dt}\right) = 0$$

or

$$\frac{\delta u}{\delta x} + \frac{\delta v}{\delta y} + \frac{\delta \omega}{\delta p} = 0$$

Taking the limit $\delta x, \delta y, \delta p \to 0$ we obtain the continuity equation in the isobaric system

$$\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)_p + \frac{\partial \omega}{\partial p} = 0 \qquad (4.7)$$

## The Thermodynamic Energy Equation

The first law of Thermodynamics eq.(2.21) can be expressed in the isobaric system by letting $\frac{dp}{dt} = \omega$ and expanding $\frac{dT}{dt}$ by using eq.(4.5)

$$c_p\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + \omega\frac{\partial T}{\partial p}\right) - \alpha\,\omega = \dot{q}$$

This may be rewritten as

$$\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) - S_p\,\omega = \dot{q}\,/\,c_p \qquad (4.8)$$

Where $S_p = -T\frac{\partial \ln\theta}{\partial p}$, with the aid of the definition of entropy eq.(2.22) and the hydrostatic eq.(4.5), then eq.(4.8) can be rewritten as follow:

$$\frac{\partial\left(-\frac{\partial \Phi}{\partial p}\right)}{\partial t} + u\frac{\partial\left(-\frac{\partial \Phi}{\partial p}\right)}{\partial x} + v\frac{\partial\left(-\frac{\partial \Phi}{\partial p}\right)}{\partial y} - \sigma\,\omega = \frac{\alpha}{c_p}\frac{dS}{dt} \qquad (4.9)$$

where $\sigma$, the static stability parameter, is defined by

$$\sigma \equiv -\frac{\alpha}{\theta}\frac{\partial \theta}{\partial p} = R\frac{S_p}{p}$$

## The Vorticity and the Divergence Equation

Starting with the horizontal momentum eq.(4.4) we can derive the vorticity equation by operating on eq.(4.4) with the operator $\hat{k}\,.\,\nabla\times$ , where $\nabla$ now indicates the horizontal gradient on a surface of constant pressure. However to facilitate this process it is desirable to first use the vector identity

$$\left(\vec{V}.\nabla\right)\vec{V} = \nabla\left(\frac{\vec{V}.\vec{V}}{2}\right) + \hat{k}\times\vec{V}\,\zeta \tag{4.10}$$

where $\zeta = \hat{k}.\left(\nabla\times\vec{V}\right)$, to rewrite eq.(4.4) as

$$\frac{\partial\vec{V}}{\partial t} = -\nabla\left(\frac{\vec{V}.\vec{V}}{2} + \Phi\right) - \hat{k}\times\vec{V}\,\zeta - \omega\frac{\partial\vec{V}}{\partial p} + f\,\vec{V}\times\hat{k} \tag{4.11}$$

We now apply the operator $\hat{k}\,.\,\nabla\times$ to eq.(4.11). Using the facts that for any scalar A, $\nabla\times\nabla A = 0$ and for any vector $\vec{a}$, $\vec{b}$

$$\nabla\times\left(\vec{a}\times\vec{b}\right) = \left(\nabla.\vec{b}\right)\vec{a} - \left(\vec{a}.\nabla\right)b - \left(\nabla.\vec{a}\right)b + \left(\vec{b}.\nabla\right)a \tag{4.12}$$

We can eliminate the first term on the right and simplify the second term so that the resulting vorticity equation becomes

$$\frac{\partial\zeta}{\partial t} = -\vec{v}.\nabla(\zeta + f) - \omega\frac{\partial\zeta}{\partial p} - (\zeta + f)\,\nabla.\vec{v} + \hat{k}.\left(\frac{\partial\vec{v}}{\partial p}\times\nabla\omega\right) \tag{4.13}$$

The divergence equation can be obtain by operating on eq.(4.11) with the operator $\nabla.$ as follow:

$$\frac{\partial(\nabla.\vec{v})}{\partial t} = -\nabla^2\left(\Phi + \frac{\vec{v}.\vec{v}}{2}\right) - \nabla.\left[\hat{k}\times\vec{v}\,(\zeta + f)\right]$$

$$-\,\omega\frac{\partial(\nabla.\vec{v})}{\partial p} - \frac{\partial\vec{v}}{\partial p}.\nabla\omega \tag{4.14}$$

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Eq.(4.13) and eq.(4.14) are independent scalar equation which can be used in place of the horizontal equations of motion.

A. Quasi-Geostrophic Vorticity Equation

The quasi-geostrophic vorticity equation can be obtained by referring back to the vorticity equation (4.13). The terms in eq.(4.13) in order reading from left to right are as follows:

1. The local rate of change of relative vorticity
2. The horizontal advection of absolute vorticity
3. The vertical advection of relative vorticity
4. The divergence term
5. The twisting or tilting term.

By evaluating the order of magnitude or scaling consideration, we may simplify the vorticity equation (4.13) for synoptic scale motions by

1. neglecting the vertical advection and twisting terms,
2. neglecting $\zeta$ compared to f in the divergence term,
3. approximating the horizontal velocity by the geostrophic wind in the advection term, and
4. replacing the relative vorticity by its geostrophic value.

As a further simplification, we may expand the Coriolis parameter in a Taylor series about the latitude $\phi_0$ as

$$f = f_0 + \beta y + (\text{higher - order terms})$$

where $\beta = (df / dy)_{\phi_0}$ , and $y = 0$ at $\phi_0$ . If we let L designate the latitudinal scale of the motion, then the ratio of the first two terms in the expansion of f has order of magnitude

$$\frac{\beta L}{f_0} \sim \frac{\cos \phi_0}{\sin \phi_0} \frac{L}{a}$$

Thus, when the latitude scale of the motions is small compared to the radius of the earth ($L / a << 1$) we can let the Coriolis parameter have a constant value $f_0$ except where it appears differentiated in the advection term.

Applying all the above approximations, we obtain *the quasi-geostrophic vorticity equation,*

$$\frac{\partial \zeta_g}{\partial t} = - \vec{v}_g . \nabla(\zeta + f) - f_0 \nabla . \vec{v} \tag{4.15}$$

where $\zeta_g = \nabla^2 \Phi / f_0$ and $\vec{v}_g = \hat{k} \times \nabla\Phi / f_0$ are both evaluated using constant Coriolis parameter $f_0$ .

B. Quasi-Geostrophic Potential Vorticity Equation

According to Helmholtz's theorem in (Appendix B), any velocity field can be divided into nondivergent part $\vec{v}_\psi$ plus a divergent part $\vec{v}_e$ such that

$$\vec{v} = \vec{v}_\psi + \vec{v}_e$$

where $\nabla . \vec{v}_\psi = 0$ and $\nabla \times \vec{v}_e = 0$. If the velocity field is two dimensional, the nondivergent part can be expressed in terms of the streamfunction defined by letting

$$\vec{v}_\psi = \hat{k} \times \nabla\psi \tag{4.16}$$

or in Cartesian components,

$$u_\psi = -\frac{\partial\psi}{\partial y} \quad , \quad v_\psi = \frac{\partial\psi}{\partial x}$$

and

$$\nabla.\vec{v}_\psi = 0$$

$$\zeta = \hat{k}.\nabla \times \vec{v}_\psi = \nabla^2\psi$$

In case of quasi-nondivergent, that is

$$|\vec{v}_\psi| >> |\vec{v}_e|$$

Thus to a first approximation we can replace $\vec{v}$ by $\vec{v}_\psi$ everywhere in eq.(4.13) and eq.(4.14) except in terms involving the horizontal divergence and use filtering of waves conditions then the vorticity and the divergence equation become

$$\frac{\partial\zeta}{\partial t} = -\vec{v}_\psi.\nabla(\zeta + f) - f_0\,\nabla.\vec{v}_e \tag{4.17}$$

and

$$\nabla^2\Phi = -f_0\,\nabla.(\hat{k} \times \vec{v}_\psi) = f_0\,\nabla^2\psi \tag{4.18}$$

Where $f_0$ is the average Coriolis parameter. From eq.(4.18) the streamfunction can be given approximately by the relation

$$\psi = \frac{\Phi}{f_0} \tag{4.19}$$

and the nondivergent part of velocity is

$$\vec{v}_{\psi} = \frac{\hat{k} \times \nabla\Phi}{f_0} \tag{4.20}$$

The geostrophic vorticity equation and hydrostatic thermodynamic energy equation can now be written in terms of $\psi$ and $\omega$ as

$$\frac{\partial \nabla^2 \psi}{\partial t} = - \vec{v}_{\psi}.\nabla\left(\nabla^2\psi + f\right) + f_0 \frac{\partial \omega}{\partial p} \tag{4.21}$$

$$\frac{\partial\left(\frac{\partial \psi}{\partial p}\right)}{\partial t} = - \vec{v}_{\psi}.\nabla\left(\frac{\partial \psi}{\partial p}\right) - \frac{\sigma\,\omega}{f_0} \tag{4.22}$$

Differentiation of eq.(4.13) with respect to p after multiplying through by $f_0^2 / \sigma$ and adding the result to eq.(4.21) gives *the quasi-geostrophic potential vorticity equation.*

$$\left(\frac{\partial}{\partial t} + \vec{v}_{\psi}.\nabla\right) q = 0 \tag{4.23}$$

Where

$$q = \nabla^2\psi + f + f_0^2 \frac{\partial\left(\frac{1}{\sigma}\frac{\partial \psi}{\partial p}\right)}{\partial p}$$

and we have assumed that $\sigma$ is a function of pressure only. This equation states that the geostrophic potential vorticity q is conserved following the nondivergent wind in pressure coordinates. If the time derivatives are eliminated between eq.(4.21) and eq.(4.22), we obtain the diagnostic omega equation

$$\left(\nabla^2 + \frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega = \frac{f_0}{\sigma}\frac{\partial\left[\vec{v}_{\psi}.\nabla\left(\nabla^2\psi + f\right)\right]}{\partial p} - \frac{f_0}{\sigma}\nabla^2\left[\vec{v}_{\psi}.\nabla\left(\frac{\partial \psi}{\partial p}\right)\right] \tag{4.24}$$

Eq.(4.24) can be used to diagnose the $\omega$ field at any instant provided that the $\psi$ field is known.

## One-Level Barotropic Model

Krishnamurti and Pearce designed the one-level barotropic model by using *the principle of conservation of absolute vorticity* ( Krishnamurti and Pearce,1977 ), which can be obtained by referring back to the vorticity equation (4.13) and retaining only the first term on the right-hand side, then we get

$$\frac{\partial \zeta}{\partial t} = - \vec{v}.\nabla(\zeta + f) \tag{4.25}$$

because of $\frac{\partial f}{\partial t} = 0$ and $\zeta_a = \nabla^2\psi + f$ is the absolute vorticity then we can rewritten eq.(4.25) as

$$\frac{d\zeta_a}{dt} = 0 \tag{4.26}$$

where $\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{v}.\nabla.$

Eq.(4.26) is called *the principle of conservation of absolute vorticity* which can be written as:

$$\frac{\partial \zeta_a}{\partial t} = - u\frac{\partial \zeta}{\partial x} - v\frac{\partial \zeta}{\partial y} - v\frac{\partial f}{\partial y}$$

And can be simplified to the form:

$$\frac{\partial \nabla^2\psi}{\partial t} = - J\left(\psi,\nabla^2\psi\right) - \beta\frac{\partial \psi}{\partial x} \tag{4.27}$$

where $\beta = \frac{\partial f}{\partial y}$ is the beta parameter and J is the Jacobian operator. Eq.(4.27) is the basic framework of the one-level barotropic model. Adejokun and Krishnamurtiused used this model to forecast the weather ( Adejokun and Krishnamurti ,1983).

Equivalent Barotropic Model

In the previous model the atmosphere was assumed to be essentially barotropic and a number of factors were neglected. It will now be seen that similar results may be obtained with somewhat less restrictive assumptions by assuming that the wind speed changes with height but the direction remain constant. Thus the horizontal wind field is assumed to be of the form

$$V_\psi(x,y,p) = A(p)\langle v_\psi(x,y)\rangle \qquad (4.28)$$

The angle brackets here denote a vertical average

$$\langle(\ )\rangle = \frac{1}{p_0}\int_0^{p_0} (\ )\, dp \qquad (4.29)$$

where $p_0 = 1000$ hPa. From the definition of $\zeta$ and $\psi$ we also can write

$$\zeta = A(p)\langle\zeta\rangle \quad , \quad \psi = A(p)\ \langle\psi\rangle$$

Using the above notation, the vorticity equation (4.13) can be rewritten as

$$\frac{\partial\left[A(p)\,\nabla^2\langle\psi\rangle\right]}{\partial t} = -A(p)^2\ \langle V_\psi\rangle.\nabla\left(\nabla^2\langle\psi\rangle\right) - A(p)\frac{\partial\langle\psi\rangle}{\partial x}\beta + f_0\frac{\partial\omega}{\partial p} \qquad (4.30)$$

We next average eq.(4.30) in the vertical by applying eq.(4.29), noting that

$$\langle A(p) \rangle = \frac{1}{p_0} \int_0^{p_0} A(p)\, dp = 1$$

and $\omega(0) = 0$. The result is

$$\frac{\partial \nabla^2 \langle \psi \rangle}{\partial t} = - \langle A(p)^2 \rangle \langle V_\psi \rangle . \nabla \left( \nabla^2 \langle \psi \rangle \right) - \beta \frac{\partial \langle \psi \rangle}{\partial x} + f_0 \frac{\omega(p_0)}{p_0} \qquad (4.31)$$

We now define a level $p^*$, called *the starred level,* according to

$$V_\psi(x,y,p^*) = V^* = \langle A(p)^2 \rangle \langle V_\psi \rangle \qquad (4.32)$$

so that

$$\psi(x,y,p^*) = \psi^* = \langle A(p)^2 \rangle \langle \psi \rangle$$

Multiplying through by $\langle A(p)^2 \rangle$ in eq.(4.31) and using the definitions in eq.(4.32) we find that at the starred level the vorticity equation is

$$\frac{\partial \nabla^2 \psi^*}{\partial t} = - V^* . \nabla \left( \nabla^2 \psi^* + f \right) + \frac{\langle A(p)^2 \rangle f_0 \, \omega(p_0)}{p_0} \qquad (4.33)$$

Therefore, at the starred level the prediction equation reduces to the barotropic vorticity equation with one addition term due to vertical motion at the lower boundary.

One-Level Primitive Equation Model

This model is base on *the principle of conservation of potential vorticity*. It is described by the following three equations for the three unknowns u, v, and z. The equations of motion are:

$$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} + f v - g\frac{\partial(z+h)}{\partial x} \tag{4.34}$$

$$\frac{\partial v}{\partial t} = -u\frac{\partial v}{\partial x} - v\frac{\partial v}{\partial y} - f u - g\frac{\partial(z+h)}{\partial y} \tag{4.35}$$

and the mass continuity equation:

$$\frac{\partial z}{\partial t} = -u\frac{\partial z}{\partial x} - v\frac{\partial z}{\partial y} - z\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \tag{4.36}$$

Where z is the height of a free surface and h is a smoothed mountain height. The pacel invariant of this system are potential vorticity $\frac{(\zeta + f)}{z} = \zeta_p$ and all its powers. In order to show this, we start with the vorticity eq.(4.13) by retaining the first two terms on the right side, then we obtain

$$\frac{\partial \zeta_a}{\partial t} = -\vec{v}.\nabla\zeta_a + \zeta_a \nabla.\vec{v} \tag{4.37}$$

Upon elimination of $\nabla.\vec{v}$ from the above equation and the mass continuity equation, we obtain

$$\left(\frac{\partial}{\partial t} + \vec{v}.\nabla\right)\left(\frac{\zeta + f}{z}\right) = 0 \tag{4.38}$$

or

$$\frac{d\left(\frac{\zeta + f}{z}\right)}{dt} = 0 \tag{4.39}$$

Eq.(4.39) is *the principle of conservation of potential vorticity.* Adejokun and Krishnamurti used this model to forecast the weather ( Adejokun and Krishnamurti ,1983).