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GREYBODY FACTORS FOR PERFECT FLUID BLACK HOLES

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สมการไอน์สไตน์เป็นสมการสนามที่อธิบายถึงแรงโน้มถ่วงที่เกิดจากความโค้งสมการไอน์สไตน์ ซึ่งเป็นสมการสนามที่อธิบายถึงแรงโน้มถ่วงที่เกิดจากความโค้งของกาลอวกาศ สสารทำให้เกิดความโค้งกาลอวกาศอย่างไร ผลเฉลยของสมการสนามไอน์สไตน์คือเมตริกของกาลอวกาศซึ่งสามารถหาได้จากความโค้งของกาลอวกาศ เนื่องจากสมการสนามไม่เป็นเชิงเส้นมีความซับซ้อนในการแก้ปัญหา ดังนั้นจึงมีการตั้งสมมติฐานบางอย่างเพื่อลดความซับซ้อนของสมการไอน์สไตน์ หนึ่งในสมมติฐานเหล่านั้นคือทรงกลมของไหลสมบูรณ์ ซึ่งเป็นหนึ่งในสมมติฐานที่ลดความซับซ้อนของสมการสนามไอน์สไตน์ ทรงกลมของไหลสมบูรณ์มีคุณสมบัติดังต่อไปนี้ คือ ไม่มีความหนืด ไม่มีการนำความร้อน และมีสมบัติไอโซโทรปี หลุมดำของไหลสมบูรณ์เป็นผลเฉลยหลุมดำของสมการสนามไอน์สไตน์ ซึ่งจำแนกจากรัศมีชวาร์ซชิลด์ ทรงกลมของไหลสมบูรณ์ที่มีรัศมีน้อยกว่ารัศมีชวาร์ซชิลด์จะกลายเป็นหลุมดำ ในงานวิจัยนี้เราศึกษาผลเฉลยหลุมดำของสมการสนามไอน์สไตน์ ตัวประกอบวัตถุเทาคือการส่งผ่านและการสะท้อนของความน่าจะเป็นของรังสีฮอว์คิงที่แผ่ออกมาจากหลุมดำ ตัวประกอบวัตถุเทาสามารถคำนวณหาได้จากพลังงานศักย์ และคำนวณอุณหภูมิและเอนโทรปีของรังสีฮอว์คิงซึ่งเป็นคุณสมบัติของอุณหพลศาสตร์ อุณหภูมิแสดงอยู่ในรูปแรงโน้มถ่วงพื้นผิวของหลุมดำในขณะที่เอนโทรปีแสดงอยู่ในรูปของพื้นที่ของขอบฟ้าเหตุการณ์ และสุดท้ายนี้เราได้ศึกษาเอนโทรปีของระบบหลุมดำและคำนวณแต่ละระบบของหลุมดำ

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KUNLAPAT SANSUK : GREYBODY FACTORS FOR PERFECT FLUID BLACK HOLES. ADVISOR : ASSOC. PROF. PETARPA BOONSERM, Ph.D., CO-ADVISOR : ASST. PROF. TRITOS NGAMPITIPAN, Ph.D., 72 pp.

The Einstein equation is a field equation which describes gravity in terms of the curvature of spacetime. It states how matter curves spacetime. The solutions of the Einstein field equation are the metrics of spacetime from which the curvature of spacetime can be found. The field equations are non-linear, which are complicated to solve. Therefore, some assumptions are needed to reduce the complexity of the Einstein equation. One of these assumptions is a perfect fluid sphere. Perfect fluid sphere satisfies the following: no viscosity, no heat conduction, and isotropy. Perfect fluid black holes are black hole solutions of the Einstein field equation, which are classified according to the Schwarzschild radius. Perfect fluid spheres that have a radius smaller than the Schwarzschild radius will be transformed into black holes. In this work, we are interested in studying a black hole solution of the Einstein field equation. Greybody factors are the transmission and reflection probabilities of the Hawking radiation which are emitted from a black hole. Greybody factors can be obtained from their structure of potential. We then calculated the Hawking radiation temperature and entropy, which are the properties of thermodynamics. Temperature is expressed in terms of the surface gravity of a black hole, while entropy is expressed in terms of the area of the event horizon. Finally, we are interested in the entropy composition of the black hole systems, and then we calculated it for each system of black holes.

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CHAPTER I

INTRODUCTION

In this chapter, we will study the relativity of Albert Einstein that is divided into two theories, namely, special relativity and general relativity, and describe the difference between these two theories. Since the objective of this thesis is to calculate the greybody factors of perfect fluid black holes, the theory of Albert Einstein becomes important in studying the perfect fluid black holes. He developed two theories of relativity, which are special relativity and general relativity. In this thesis, we are interested in general relativity because this theory is a gravitational theory that explains how matter is responsible for the spacetime curvature. The concept of general relativity has resulted in the formulation of the Einstein field equation, which is an important equation to study the modeling of stars in the structure of perfect fluid black holes. First, we will introduce special relativity, which forms the basic knowledge of the theory of relativity in special cases. Before studying about general relativity, we should start with special relativity.

1.1 Special Relativity

The theory of relativity is the theory of space and time. The theory of relativity is divided into special and general relativities [1]. In 1905, Einstein introduced special relativity that explains the motion of non-accelerating bodies, which also describes the speed of light being constant and is independent of the motion of all observers [2]. The postulates of special relativity are in four-dimensional forms, the existence of globally inertial frames, the speed of light being constant, and the principle of special relativity [3]. Moreover, a counterintuitive concept of special relativity involves time dilation, which is that the time interval linking two events does not change from one spectator to another, but depends on the relative velocity; the simultaneity of events, which is when two events occur simultaneously in two different places for one observer, but may not be at the same time for another observer, the length contraction, which is the dimension of objects when

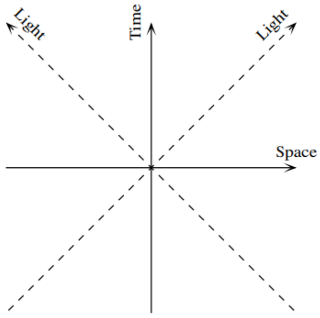

Figure 1.1: Spacetime diagram showing events in space and time [3].

measured by one spectator, which may be tinier than when measured by another observer; the combination of velocities, where the velocities are not simply combined; inertia and momentum, when the velocity of an object approaches the speed of light, objects are accelerating faster and the equation in mathematics that comes from Einstein's special theory of relativity is the Mass-Energy equivalence that is the relationship between mass and the energy [4]. The mass-energy equivalence is given by

$$E = mc^2. \tag{1.1}$$

In Figure 1.1, the three space dimensions are horizontal and the time dimension is vertical, and the speed of light is equal to one. So, the direction of light is at 45° from vertical [3].

Einstein's special theory of relativity is a theory in Minkowski spacetime, where the line element ds between two events is given by [3]

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2, \tag{1.2}$$

where dt is the time interval, and $dr = \sqrt{dx^2 + dy^2 + dz^2}$ is the spatial interval.

In Figure 1.2, spacetime interval is called timelike if $ds^2 < 0$, that is, the area has a velocity less than the velocity of light, lightlike if $ds^2 = 0$, that is, the path has the velocity equal to the velocity of light, and called spacelike if $ds^2 > 0$, that is, the area has the velocity more than the light.

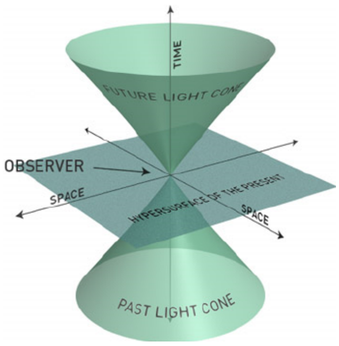

Figure 1.2: Spacetime diagram showing timelike, lightlike, and spacelike spacetime intervals [5].

In special relativity, the light speed has the identical value in any inertial that is non-accelerating, which does not deal with gravity, while general relativity does [6].

1.2 General Relativity

In 1915, Albert Einstein developed the theory of gravitation that is general relativity. General relativity describes every motion such as the motion of acceleration, curving or spinning around. Gravitation is the curvature of spacetime. Light and objects move according to the spacetime. Therefore, spacetime will bend around every object with mass [7]. The Einstein field equation (EFE) is the centerpiece of general relativity.

In special relativity, the metric of spacetime is flat, which is given by

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu, \tag{1.3}$$

where

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{1.4}$$

and in Einstein's general theory of relativity, the metric of spacetime is curved, which is given by

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \tag{1.5}$$

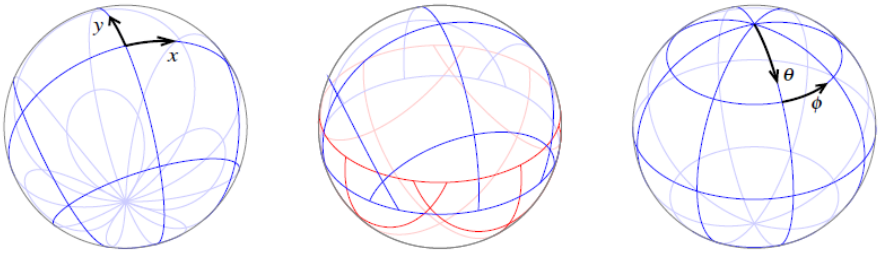

Figure 1.3: The 2-sphere is a 2-manifold [3].

where

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{pmatrix}. \tag{1.6}$$

Special relativity considers the motion of non-acceleration; however, general relativity considers the motion of acceleration, that is, gravity results in spacetime curvature.

In chapter I, we study about special and general relativities, and analyze the difference between special and general relativities. Since the objective of this thesis is to calculate the greybody factors of black holes, we are, therefore, interested in general relativity and have continued with the equation in general relativity.

We have divided this thesis into six chapters. In chapter II, we study about the Einstein field equation (EFE), which is an important equation in studying the modeling of stars in the universe. In chapter III, we study the modeling of stars that is perfect fluid spheres, and study the generating theorem of perfect fluid spheres, as well as classify black holes in perfect fluid spheres in two coordinates. In chapter IV, we study the method to obtain the greybody factors (transmission probability), that is the bogoliubov coefficients, and calculate the greybody factors of black holes. In chapter V, we calculate the Hawking temperature, Hawking radiation, and the entropy of black holes, and also calculate the entropy composition of the black hole systems. Finally in chapter IV, we present the conclusion to this thesis and the future work.

CHAPTER II

EINSTEIN'S FIELD EQUATION

In this chapter, we are interested in the most important equation in general relativity that is the Einstein field equation. The Einstein field equation was published in 1915 by Albert Einstein as a tensor equation, which describes gravity as the result of mass and energy causing spacetime curvature [8]. The Einstein tensor is used for describing the spacetime curvature. The stress-energy tensor is used for representing the energy and the momentum density of the gravitational field [9]. Einstein added a constant (8πG) to the right hand side of the Einstein field equation that is calculated from the Poisson's equation [10]. The Einstein field equation can be written in this form

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{2.1}$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ is the Einstein tensor, $R_{\mu\nu}$ is the Ricci curvature tensor, $g_{\mu\nu}$ is the metric tensor, R is the scalar curvature, G is the Newtonian constant of gravitation, $T_{\mu\nu}$ is the stress-energy-momentum tensor, and $\mu\nu$ are labels that take on the values 0, 1, 2, 3.

Next, we introduce the quantities in the Einstein's general field equation, namely, the metric tensor, the Christoffel symbols, covariant derivative, the Riemannian curvature tensor, the Ricci curvature tensor, the Ricci scalar, the Einstein tensor, and the stress energy tensor.

2.1 Metric tensor

The tensor of metric is the distance between two events. It is a 4×4 matrix, which also gives ten independent coefficients.

First, we will introduce the metric in two dimensional coordinates that is the cartesian coordinates, we then obtain the metric [11]

$$ds^2 = dx^2 + dy^2, \tag{2.2}$$

and after we expand ds^2 in three dimensions, we obtain in this form

$$ds^2 = \sum_{i=0}^{2}\sum_{j=0}^{2}[dx^i dy^j], \tag{2.3}$$

where

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.4}$$

Next, we will introduce the metric in two dimensional coordinates that is the polar coordinates, we then obtain the metric

$$ds^2 = dr^2 + r^2 d\theta^2, \tag{2.5}$$

where

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}. \tag{2.6}$$

In Einstein's general theory of relativity, the metric of spacetime is given by

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \tag{2.7}$$

where

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{pmatrix}, \tag{2.8}$$

which is the metric tensor. The properties of metric tensor are symmetric [12]

$$g_{\mu\nu} = g_{\nu\mu}, \tag{2.9}$$

and the inverse matrix $g^{\mu\nu}$ is defined by

$$g_{\mu\nu} g^{\nu\alpha} = \delta^{\alpha}_{\mu}. \tag{2.10}$$

The metric tensor is a very important variable because it is one of the factors in other variables such as the Christoffel symbols, the Riemann curvature tensor, the Ricci curvature tensor and the Ricci scalar. Next, we will introduce the Christoffel symbol that is associated with the tensor of metric.

2.2 Christoffel symbols

The Christoffel symbols are the tensors that can be extracted dealing with the partial derivatives of metric $g_{\mu\nu}$. They are used to inquiry the geometry of the metric. We start with [13]

$$ds^2 = \eta_{\alpha\beta} d\zeta^{\alpha} d\zeta^{\beta} = \eta_{\alpha\beta} \frac{\partial \zeta^{\alpha}}{\mu} \frac{\partial \zeta^{\beta}}{\mu} = g_{\mu\nu} dx^{\mu} dx^{\nu}. \tag{2.11}$$

$$\frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} + \frac{\partial g_{\lambda\nu}}{\partial x^{\mu}} + \frac{\partial g_{\mu\lambda}}{\partial x^{\nu}} = \eta \left[\frac{\partial^2 \zeta^{\alpha}}{\partial x^{\mu} \partial x^{\lambda}} \frac{\partial \zeta^{\beta}}{\partial x^{\nu}} + \frac{\partial \zeta^{\alpha}}{\partial x^{\mu}} \frac{\partial^2 \zeta^{\beta}}{\partial x^{\nu} \partial x^{\lambda}} \right]$$

$$+\eta \left[\frac{\partial^2 \zeta^{\alpha}}{\partial x^{\lambda} \partial x^{\mu}} \frac{\partial \zeta^{\beta}}{\partial x^{\nu}} + \frac{\partial \zeta^{\alpha}}{\partial x^{\lambda}} \frac{\partial^2 \zeta^{\beta}}{\partial x^{\nu} \partial x^{\mu}} \right] - \eta \left[\frac{\partial^2 \zeta^{\alpha}}{\partial x^{\mu} \partial x^{\nu}} \frac{\partial \zeta^{\beta}}{\partial x^{\lambda}} + \frac{\partial \zeta^{\alpha}}{\partial x^{\mu}} \frac{\partial^2 \zeta^{\beta}}{\partial x^{\nu} \partial x^{\lambda}} \right]. \tag{2.12}$$

Using $\eta_{\alpha\beta} = \eta_{\beta\alpha}$, this gives

$$\frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} + \frac{\partial g_{\lambda\nu}}{\partial x^{\mu}} + \frac{\partial g_{\mu\lambda}}{\partial x^{\nu}} = 2\eta_{\alpha\beta} \frac{\partial^2 \zeta^{\alpha}}{\partial x^{\mu} \partial x^{\lambda}} \frac{\partial \zeta^{\beta}}{\partial x^{\nu}}. \tag{2.13}$$

Also,

$$g_{\nu\sigma}\Gamma^{\sigma}_{\mu\lambda} = \eta_{\alpha\beta}\frac{\partial\zeta^{\alpha}}{\partial x^{\nu}}\frac{\partial\zeta^{\beta}}{\partial x^{\sigma}}\frac{\partial x^{\sigma}}{\partial\zeta^{\rho}}\frac{\partial^2\zeta^{\rho}}{\partial x^{\mu}\partial x^{\lambda}} \tag{2.14}$$

$$= \eta_{\alpha\beta}\frac{\partial\zeta^{\alpha}}{\partial x^{\nu}}\frac{\partial^2\zeta^{\beta}}{\partial x^{\mu}\partial x^{\lambda}} \tag{2.15}$$

$$= \frac{1}{2}\left[\frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} + \frac{\partial g_{\lambda\nu}}{\partial x^{\mu}} + \frac{\partial g_{\mu\lambda}}{\partial x^{\nu}}\right]. \tag{2.16}$$

Therefore, the Christoffel symbols are given by

$$\Gamma^{k}_{\mu\lambda} = \frac{1}{2}g_{k\nu}\left[\frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} + \frac{\partial g_{\lambda\nu}}{\partial x^{\mu}} + \frac{\partial g_{\mu\lambda}}{\partial x^{\nu}}\right], \tag{2.17}$$

which is a term of the first order derivative of the metric tensor. Then, we will introduce the Riemann curvature tensor that is related with the Christoffel symbols.

2.3 Riemannian curvature tensor

The Riemannian curvature tensor $R_{\alpha\beta\gamma\delta}$ is a four-index tensor. It expresses the curvature of the Riemann manifolds and can be simplified in the form of the second derivative of the metric tensor [3].

$$R_{\rho\lambda\mu\nu} = \frac{\partial\Gamma_{\mu\nu\lambda}}{\partial x^{\rho}} - \frac{\partial\Gamma_{\mu\nu\rho}}{\partial x^{\lambda}}, \tag{2.18}$$

which is a term of the first order derivative of the Christoffel symbols.

$$R_{\rho\lambda\mu\nu} = \frac{1}{2}\left(\frac{\partial^2 g_{\mu\nu}}{\partial x^{\rho}\partial x^{\lambda}} + \frac{\partial^2 g_{\mu\lambda}}{\partial x^{\rho}\partial x^{\nu}} - \frac{\partial^2 g_{\nu\lambda}}{\partial x^{\rho}\partial x^{\mu}} - \frac{\partial^2 g_{\mu\nu}}{\partial x^{\rho}\partial x^{\lambda}} - \frac{\partial^2 g_{\mu\rho}}{\partial x^{\lambda}\partial x^{\nu}} + \frac{\partial^2 g_{\nu\rho}}{\partial x^{\lambda}\partial x^{\mu}}\right) \tag{2.19}$$

$$= \frac{1}{2}\left(\frac{\partial^2 g_{\mu\lambda}}{\partial x^{\rho}\partial x^{\nu}} - \frac{\partial^2 g_{\nu\lambda}}{\partial x^{\rho}\partial x^{\mu}} - \frac{\partial^2 g_{\mu\rho}}{\partial x^{\lambda}\partial x^{\nu}} + \frac{\partial^2 g_{\nu\rho}}{\partial x^{\lambda}\partial x^{\mu}}\right), \tag{2.20}$$

which is a term of the second order derivative of the metric tensor. Then, we will introduce the Ricci curvature tensor that has a relation with the Riemann curvature tensor.

2.4 Ricci curvature tensor and Ricci scalar

The Ricci curvature tensor is the retraction of the Riemann curvature tensor that is a second order tensor and represents the curvature [14]. The Ricci curvature tensor is given by

$$R_{\mu\nu} = R^{\lambda}_{\mu\nu\lambda} = \frac{\partial \Gamma^{\lambda}_{\mu\nu}}{\partial x^{\lambda}} - \frac{\partial \Gamma^{\lambda}_{\lambda\nu}}{\partial x^{\mu}} + \Gamma^{\rho}_{\mu\nu}\Gamma^{\lambda}_{\lambda\rho} - \Gamma^{\rho}_{\lambda\nu}\Gamma^{\lambda}_{\mu\rho}. \tag{2.21}$$

The Ricci scalar is the curvature scalar that can be extracted in terms of the Ricci tensor and the metric tensor, which is defined by [3]

$$R = g^{\mu\nu}R_{\mu\nu}. \tag{2.22}$$

Next, we will introduce the Einstein tensor that is important in the Einstein field equation.

2.5 Einstein tensor

The Einstein tensor can be expressed in this form

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}. \tag{2.23}$$

On the left hand side of the Einstein filed equation, $G_{\mu\nu}$ is the Einstein tensor, which represents the curvature of spacetime. Einstein created the Einstein tensor, which combines the Ricci curvature tensor $R_{\mu\nu}$, the metric tensor $g_{\mu\nu}$ and the scalar curvature R [13]. Next, we introduce the stress-energy-momentum tensor that is on the right hand side of the Einstein field equation.

2.6 Stress-energy-momentum tensor or stress energy tensor

The stress energy tensor is a property of the object that curves spacetime, which is represented by $T_{\mu\nu}$ on the right hand side of the Einstein field equation. The stress energy tensor describes the forces on the surface of this elementary volume [9]. The stress-energy-momentum tensor can be expressed in this form

$$T_{\mu\nu} = \begin{pmatrix} T_{00} & T_{01} & T_{02} & T_{03} \\ T_{10} & T_{11} & T_{12} & T_{13} \\ T_{20} & T_{21} & T_{22} & T_{23} \\ T_{30} & T_{31} & T_{32} & T_{33} \end{pmatrix}, \tag{2.24}$$

where T_{00} is the energy density, T_{01}, T_{02}, T_{03} are the energy flux, T_{10}, T_{20}, T_{30} are the momentum density, T_{12}, T_{13}, T_{23}, T_{21}, T_{31}, T_{32} are the shear stress, which is the force that causes deformation of objects by slipping along a plane, or from pressure, and T_{11}, T_{22}, T_{33} are the normal stress or the isostatic pressure.

In chapter II, we studied about the Einstein field equation and the quantities in this equation such as the Einstein tensor, the stress energy tensor, the metric tensor, the Christoffel symbols, the Ricci curvature tensor, the Riemann curvature tensor, and the Ricci scalar. Afterwards, we studied the Einstein field equation and used all the quantities in the Einstein field equation to obtain the important constraint to solve the Einstein field equation. Next, we will study about perfect fluid spheres, which is an assumption to reduce the complexity of the Einstein field equation.

CHAPTER III

PERFECT FLUID SPHERE

In this chapter, we will study the perfect fluid spheres because we use it in modeling of stars like black holes. We also study about generating theorems of perfect fluid spheres developed by Boonserm, et al. [15]. We are also interested in perfect fluid spheres in two coordinates, namely, Schwarzschild and isotropic coordinates developed by Boonserm and Thairatana [16, 17]. In this thesis, we will classify black holes in perfect fluid spheres and use these black holes to calculate the greybody factors.

3.1 Perfect fluid sphere

The perfect fluid sphere is one of the assumptions to reduce the complexity of the Einstein field equation. Through the three properties of the perfect fluid sphere (no viscosity, no heat conductivity, and isotropy ($p_r = p_t$)), the stress energy tensor takes the form [15]

$$T_{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}, \tag{3.1}$$

where the general form of $T_{\mu\nu}$ is given by

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + p g_{\mu\nu}. \tag{3.2}$$

By substituting equation (3.1) into the Einstein field equation, we obtain the constraint of the perfect fluid sphere especially given by

$$G_{\hat{r}\hat{r}} = G_{\hat{\theta}\hat{\theta}} = G_{\hat{\phi}\hat{\phi}}, \tag{3.3}$$

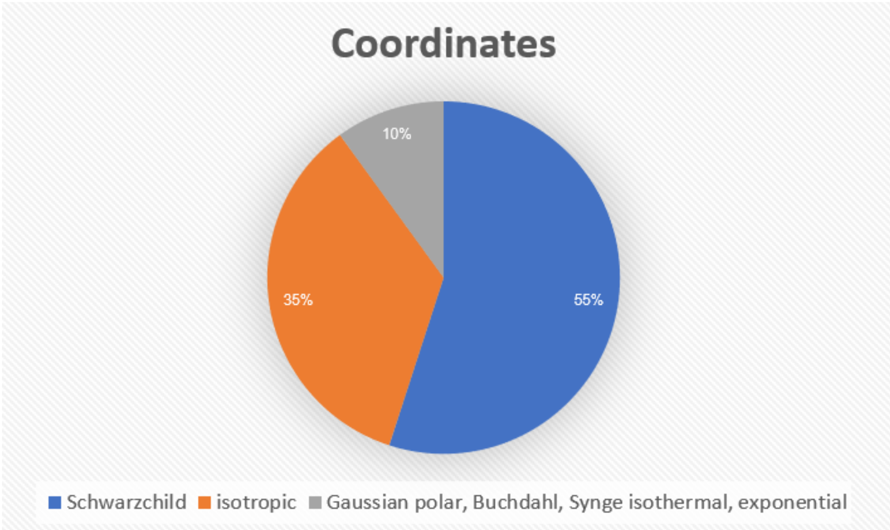

Figure 3.1: The proportion of coordinates.

and use this constraint to solve the Einstein field equation.

In this thesis, we are interested in perfect fluid spheres in two coordinates, namely, Schwarzschild and isotropic coordinates. The Schwarzschild coordinates is estimated to be about 55% and the isotropic coordinates is estimated to be about 35%. Both coordinates constitute the major proportion of the overall coordinates.

3.2 Schwarzschild Coordinates

The specific geometry in the Schwarzschild metric must first be defined by

$$ds^2 = -\zeta(r)^2 dt^2 + \frac{1}{\beta(r)} dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta) d\phi^2, \tag{3.4}$$

and then, we study about the generating theorems that can be used to apply with the Schwarzschild metric to analyze the perfect fluid spheres in Schwarzschild coordinate. From the perfect fluid constraint, we derive [16]

$$G_{\hat{r}\hat{r}} = -\frac{2\beta(r) r \zeta'(r) - \zeta(r) + \zeta(r)\beta(r)}{r^2 \zeta(r)}, \tag{3.5}$$

and

$$G_{\hat{\theta}\hat{\theta}} = -\frac{1}{2}\frac{\beta'(r)\zeta(r) + 2\beta(r)\zeta'(r) + 2\beta(r)\zeta''(r) + r\zeta'(r)\beta'(r)}{r\zeta(r)}. \tag{3.6}$$

Set $G_{\hat{r}\hat{r}} = G_{\hat{\theta}\hat{\theta}}$, we obtain [16]

$$[r(r\zeta(r))']\beta'(r) + [2r^2\zeta''(r) - 2(r\zeta(r))']\beta(r) + 2\zeta(r) = 0, \tag{3.7}$$

or by rearranging the above equation, we derive

$$2r^2\beta(r)\zeta''(r) + (r^2\beta'(r) - 2r\beta(r))\zeta'(r) + (r\beta'(r) - 2\beta(r) + 2)\zeta = 0. \tag{3.8}$$

Next, we will study the generating theorem in Schwarzschild coordinates, which is presented by Boonserm, et al. [15], and is divided into four theorems.

3.2.1 Generating theorems

Generating theorems developed by Boonserm, et al. [15] that explains the four theorems, when applied with perfect fluid spheres, still result in the generation of the same perfect fluid spheres. In addition, the generating theorems can also be used to obtain new perfect fluid spheres in Schwarzschild coordinates.

Theorem 1 (1^{st} BVW theorem [16, 15]). Suppose $\{\zeta(r), \beta(r)\}$ represents a perfect fluid sphere, and is described by

$$\Delta(r) = [\frac{\zeta(r)}{\zeta(r) + r\zeta'(r)}]^2 r^2 \exp\{2\int \frac{\zeta'(r)}{\zeta(r)}\frac{\zeta(r) - r\zeta'(r)}{\zeta(r) + r\zeta'(r)}dr\}. \tag{3.9}$$

Then, $\{\zeta(r), \beta(r) + \lambda\Delta(r)\}$ is also a perfect fluid sphere. That is, the mapping of

$$T_1 : \{\zeta(r), \beta r\} \rightarrow \{\zeta(r), \beta(r) + \lambda\Delta(r)\}\} \tag{3.10}$$

takes perfect fluid spheres into perfect fluid spheres.

Theorem 2 (2^{nd} BVW theorem [16, 15]). Suppose $\{\zeta(r), \beta(r)\}$ represents a perfect fluid sphere, and is defined by

$$Z(r) = \sigma + \varepsilon \int \frac{rdr}{\zeta(r)^2\sqrt{\beta(r)}}. \tag{3.11}$$

Then, $\{\zeta(r)Z(\zeta, \beta), \beta(r)\}$ is also a perfect fluid sphere. That is, the mapping of

$$T_2 : \{\zeta(r), \beta r\} \to \{\zeta(r)Z(\zeta, \beta), \beta(r)\} \tag{3.12}$$

takes perfect fluid spheres into perfect fluid spheres.

Theorem 3 (3^{rd} BVW theorem [16, 18]). Suppose $\{\zeta(r), \beta(r)\}$ represents a perfect fluid sphere, and is defined by

$$\Delta(r) = [\frac{\zeta(r)}{\zeta(r) + r\zeta'(r)}]^2 r^2 \exp\{2 \int \frac{\zeta'(r)}{\zeta(r)} \frac{\zeta(r) - r\zeta'(r)}{\zeta(r) + r\zeta'(r)} dr\}. \tag{3.13}$$

Then, $\{\zeta(r)Z(\zeta, \beta + \lambda\Delta(r)), \beta(r) + \lambda\Delta(r)\}$ is also a perfect fluid sphere. That is, the mapping of

$$T_3 : \{\zeta(r), \beta(r)\} \to \{\zeta(r)Z(\zeta, \beta + \lambda\Delta(r)), \beta(r) + \lambda\Delta(r)\} \tag{3.14}$$

takes perfect fluid spheres into perfect fluid spheres.

Theorem 4 (4^{th} BVW theorem [16, 18]). Suppose $\{\zeta(r), \beta(r)\}$ represents a perfect fluid sphere, and is defined by

$$\Delta_0(r) = [\frac{\zeta_0(r)}{\zeta_0(r) + r\zeta_0'(r)}]^2 r^2 \exp\{2 \int \frac{\zeta_0'(r)}{\zeta_0(r)} \frac{\zeta_0(r) - r\zeta_0'(r)}{\zeta_0(r) + r\zeta_0'(r)} dr\}, \tag{3.15}$$

$$Z(r) = \sigma + \varepsilon \int \frac{rdr}{\zeta(r)^2\sqrt{\beta(r)}}. \tag{3.16}$$

Then, $\{\zeta_0(r), \beta(r) + \lambda\Delta(\zeta_0)\}$ is also a perfect fluid sphere. That is, the mapping of

$$T_4 : \{\zeta(r), \beta r\} \to \{\zeta_0(r), \beta(r) + \lambda\Delta(\zeta_0)\} \tag{3.17}$$

takes perfect fluid spheres into perfect fluid spheres.

We study about the generating theorem of perfect fluid spheres because these theorems classify seed and non-seed metrics that generate new solutions of perfect fluid spheres, and we will also categorize the black holes in these perfect fluid spheres. Next, we will study about the generating theorem in isotropic coordinates and apply it on the isotropic metric to analyze the perfect fluid spheres in isotropic coordinates.

3.3 Isotropic Coordinates

The specific geometry in the isotropic metric must first be defined by

$$ds^2 = -\zeta(r)^2 dt^2 + \frac{1}{\zeta(r)^2\beta(r)^2}\{dr^2 + r^2 d\theta^2 + r^2\sin^2(\theta)d\phi^2\}. \tag{3.18}$$

Next, we will study the generating theorem in isotropic coordinates, which is presented by Boonserm and Visser [18], and is divided into two theorems

3.3.1 Generating theorems in isotropic coordinates

Generating theorems in isotropic coordinates developed by Boonserm and Visser [18] that explains the four theorems, when applied with perfect fluid spheres still result in the generation of the same perfect fluid spheres. In addition, a generating theorem developed by [17] can be used to obtain new perfect fluid spheres in isotropic coordinates.

Theorem 5 (7^{th} BVW theorem [18]). Suppose $\{\zeta, \beta\}$ represents a perfect fluid sphere, and the transformation of Buchdahl in disguise is given by

$$ds^2 = -\frac{1}{\zeta(r)^2}dt^2 + \frac{\zeta(r)^2}{\beta(r)^2}\{dr^2 + r^2 d\theta^2 + r^2\sin^2(\theta)d\phi^2\}. \tag{3.19}$$

Then, $\{\zeta(r)^{-1}, \beta(r)\}$ is also a perfect fluid sphere. After applying theorem 5 n times, $\zeta(r) = \zeta(r)^{-1}$ if n is an odd number, and $\zeta(r) = \zeta(r)$ if n is an even number.

Theorem 6 (8^{th} BVW theorem [18]). Suppose $\{\zeta, \beta\}$ represents a perfect fluid sphere, and is defined by

$$Z(r) = \sigma + \varepsilon \int \frac{rdr}{\sqrt{\beta(r)^2}}. \tag{3.20}$$

Then, $\{\zeta(r), \beta(r)Z(\beta)\}$ is also a perfect fluid sphere, and the geometry in isotropic coordinates is defined by

$$ds^2 = -\zeta(r)^2 dt^2 + \frac{1}{\zeta(r)^2\beta(r)^2 Z(r)^2}\{dr^2 + r^2 d\theta^2 + r^2\sin^2(\theta)d\phi^2\}. \tag{3.21}$$

In generating theorem, these six theorems can be used to obtain the new solution of perfect fluid spheres in two coordinates. After this, we study about the perfect fluid spheres and the generating theorem of perfect fluid spheres that is developed by [16] and then, we will classify black holes in perfect fluid spheres.

3.4 Black holes in perfect fluid spheres

Black holes are the large theoretical objects. In 1971, the first physical black hole was discovered. Black holes are regions in space that can absorb everything that comes close to its surface, including light. Scientists have proven that black holes exist, with one present at the center of our galaxy [19]. There exist only four types of black holes, where these black holes are described as whether they are with or without rotation and charge; namely, Schwarzschild black hole, Kerr black hole, Reissner–Nordström black hole, and Kerr-Newman black hole.

3.4.1 Types of black holes

Types of black holes	**Non-rotating (J = 0)**	**Rotating (J > 0)**
Uncharged (Q = 0)	Schwarzschild black hole	Kerr black hole
Charged (Q ≠ 0)	Reissner-Nordstrom black hole	Kerr-Newman black hole

Table 3.1: Type of black holes

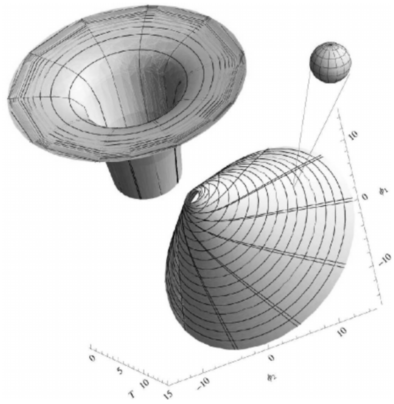

Figure 3.2: Embedded Schwarzschild black hole [20].

In this thesis, we are focused on the theory of black holes that are classified by charge and rotation. There are four types of black holes in the theory of black holes, namely, the Schwarzschild black hole, Kerr black hole, Reissner–Nordström black hole, and Kerr-Newman black hole.

3.4.2 Schwarzschild black hole

The Schwarzschild black hole is a static and spherical symmetrical black hole with mass [19]. The Schwarzschild black hole in Schwarzschild coordinates is given by [15]

$$ds^2 = -\left(1 - \frac{2m}{r}\right)^2 dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta) d\phi^2. \tag{3.22}$$

3.4.3 Kerr black hole

The Kerr black hole is stationary, axisymmetric and depends on angular momentum [19]. The Kerr metric is given by

$$ds^2 = -dt^2 + (r^2 + a^2)\sin^2(\theta)d\phi^2 + \frac{2Mr(dt - a\sin^2(\theta)d\phi^2)^2}{r^2 + a^2\cos^2(\theta)}$$

$$+(r^2 + a^2\cos^2(\theta)) + (d\theta^2 + \frac{dr^2}{r^2 - 2Mr + a^2}). \tag{3.23}$$

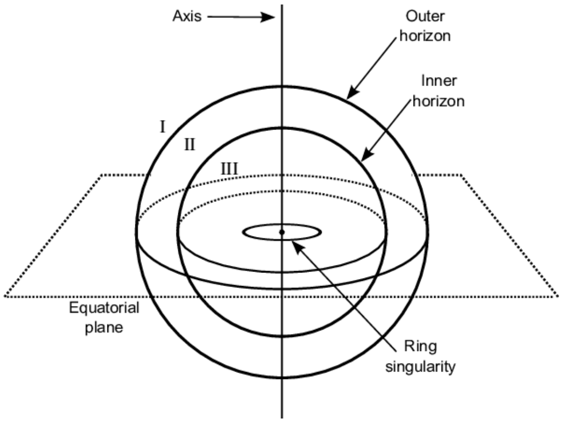

Figure 3.3: Sketch of Kerr black hole [21].

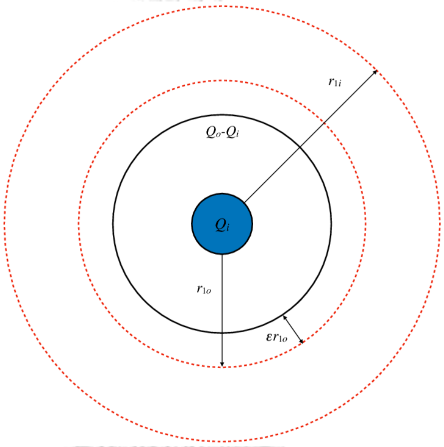

Figure 3.4: The space-time geometry of a Reissner-Nordstrom black hole of charge Q [22].

3.4.4 Reissner-Nordstrom black hole

The Reissner-Nordstrom black hole is static and spherically symmetric and depends on electric charge [19]. The Reissner-Nordstrom metric is expressed as

$$ds^2 = -Adt^2 + Bdr^2 + r^2(d\theta^2 + \sin^2(\theta)d\phi^2), \tag{3.24}$$

where $A = \frac{2M}{r} + \frac{Q^2}{4\pi r^2}$, and $B = 1/A$.

3.4.5 Kerr-Newman black hole

The Kerr-Newman black hole is the most overall asymptotically flat black hole. The Kerr-Newman black hole spins and be dependent on electric charge and rotational

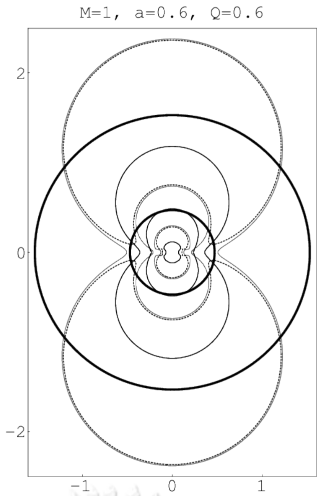

Figure 3.5: Sketch of Kerr-Newman black hole [23].

momentum (angular momentum). The Kerr-Newman metric is given by [19]

$$ds^2 = \left(\frac{dr^2}{\triangle(r)} + d\theta^2\right)\rho^2 - (dt - a\sin^2\theta d\phi)^2\frac{(r)}{\rho^2} + ((r^2+a^2)d\phi - adt)^2\frac{\sin^2}{\rho^2}. \tag{3.25}$$

In this thesis, we are focused on the Schwarzschild black hole. Thus, we will compare the radius of perfect fluid spheres with the Schwarzschild radius.

3.4.6 Schwarzchild Radius

The event horizon is the borderline of a black hole and the Schwarzschild radius is the distance from the singularity or the center of the black hole. Stars with sizes smaller than the Schwarzschild radius will be transformed into black holes [24]. The Schwarzschild radius is given by [19]

$$r_s = \frac{2GM}{c^2}, \tag{3.26}$$

where r_s is Schwarzschild radius, G is the gravitational constant, c is the speed of light, and M is the solar mass. Several spherical body of mass M restricted within the critical radius r_s should be a black hole. Next, we calculate the radius of perfect fluid sphere in Schwarzschild coordinates.

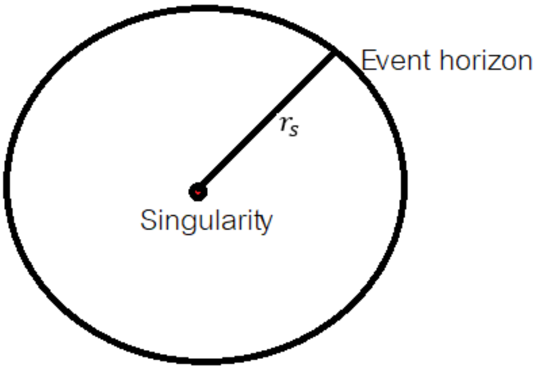

Figure 3.6: Schwarzschild radius.

3.4.7 Perfect fluid black hole in Schwarzschild coordinates

We calculate the radius of perfect fluid spheres in the Schwarzschild coordinates by matching with Schwarzschild exterior black hole. We start with spherical symmetrical geometry in Schwarzschild (curvature) coordinates

$$ds^2 = -\zeta(r)^2 dt^2 + \frac{1}{B(r)} dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta) d\phi^2. \tag{3.27}$$

Using a perfect fluid constraint

$$G_{\hat{r}\hat{r}} = G_{\hat{\theta}\hat{\theta}} = G_{\hat{\phi}\hat{\phi}}. \tag{3.28}$$

We consider [16]

$$G_{\hat{r}\hat{r}} = \frac{-2rB(r)\zeta(r) + \zeta(r) - \zeta(r)B(r)}{r^2\zeta(r)}. \tag{3.29}$$

Considering the Einstein field equation

$$G_{\hat{r}\hat{r}} = 8\pi G T_{\hat{r}\hat{r}}, \tag{3.30}$$

the pressure inside the perfect fluid sphere is expressed by

$$p = \frac{G_{\hat{r}\hat{r}}}{8\pi G} = \frac{1}{8\pi G}\frac{-2rB(r)\zeta(r) + \zeta(r) - \zeta(r)B(r)}{r^2\zeta(r)}. \tag{3.31}$$

So, the radius r of a perfect fluid sphere, must satisfy [25]

$$p(r) = 0. \tag{3.32}$$

We obtian black holes in the structure of a perfect fluid sphere in Schwarzschild coordinates as shown in Table 3.2.

Black holes	**Metrics**
Schwarzschild Exterior	$-(1-\frac{2m}{r})dt^2 + (1-\frac{2m}{r})^{-1}dr^2 + r^2d\Omega^2$
Tolman VI	$-(Ar^{1-n} + Br^{1+n})^2dt^2 + (2-n^2)dr^2 + r^2d\Omega^2$
Kuch 68 II	$-(1-\frac{2m}{r})dt^2 + [(1-\frac{2m}{r})(1+C(2r-2m)^2)]^{-1}dr^2 + r^2d\Omega^2$
M-W III	$-Ar(r-a)dt^2 + \frac{7/4}{1-r^2/a^2}dr^2 + r^2d\Omega^2$

Table 3.2: A black hole in the form of a perfect fluid sphere in Schwarzschild coordinates

Matese and Whitman considered M-W III and Kuchowicz considered Kuch 68 II. Next, we calculate the radius of perfect fluid sphere in isotropic coordinates.

3.4.8 Perfect fluid black hole in isotropic coordinates

We calculate the radius of perfect fluid sphere in isotropic coordinates by matching with Schwarzschild exterior black hole. We start with the spherically symmetric geometry in isotropic coordinates

$$ds^2 = -\zeta(r)^2dt^2 + \frac{1}{\zeta(r)^2B(r)^2}\{dr^2 + r^2d\theta^2 + r^2\sin^2(\theta)d\phi^2\}. \tag{3.33}$$

Using a perfect fluid constraint

$$G_{\hat{r}\hat{r}} = G_{\hat{\theta}\hat{\theta}} = G_{\hat{\phi}\hat{\phi}}, \tag{3.34}$$

and we use $G_{\hat{r}\hat{r}}$ to calculate the radius of perfect fluid sphere.

We consider [16]

$$G_{\hat{r}\hat{r}} = (\zeta^{'})^2 B^2 - (B^{'})^2 \zeta^2 + 2B^{'} B \frac{\zeta^2}{r}. \tag{3.35}$$

Considering the Einstein field equation

$$G_{\hat{r}\hat{r}} = 8\pi G T_{\hat{r}\hat{r}}, \tag{3.36}$$

the pressure inside the perfect fluid sphere is given by

$$p = \frac{G_{\hat{r}\hat{r}}}{8\pi G} = \frac{1}{8\pi G}(\zeta^{'})^2 B^2 - (B^{'})^2 \zeta^2 + 2B^{'} B \frac{\zeta^2}{r}. \tag{3.37}$$

So, the radius r of a perfect fluid sphere must satisfy [25]

$$p(r) = 0. \tag{3.38}$$

We obtain black holes in the structure of a perfect fluid sphere in isotropic coordinates as shown in Table 3.3.

Black holes	Metrics
Schwarzschild Exterior	$-\frac{(1-\frac{M}{2r})^2}{(1+\frac{M}{2r})^2}dt^2 + (1+\frac{M}{2r})^4\{dr^2 + r^2 d\Omega^2\}$
N-P-V Ia	$-(ar^{1+\frac{x}{2}} + br^{1-\frac{x}{2}})^2(Ar^{1+\frac{n}{2}} + Br^{1-\frac{n}{2}})^{-2}dt^2 +$ $(\mathrm{Ar}^{1+\frac{n}{2}} + Br^{1-\frac{n}{2}})^{-2}\{dr^2 + r^2 d\Omega^2\}$
Burl I	$-A(1+r^2)^{\frac{4(a+1)}{2a^2+4a+1}}dt^2 + (1+r^2)^{\frac{4a}{2a^2+4a+1}}\{dr^2 + r^2 d\Omega^2\}$

Table 3.3: A black hole in the structure of a perfect fluid sphere in isotropic coordinates.

Then, we will find the potential of black holes by starting with the Klein-Gordon equation because this equation is one of the wave equations. After we derive this equation, we can obtain an equation that is similar to the Schrödinger equation, which is a combination of kinetic energy and potential energy.

3.4.9 Klein-Gordon Equation

The Klein-Gordon equation is the relativistic wave equation that describes the behavior of spinless particles [26]. The Klein-Gordon equation can be expressed in the form

$$\frac{1}{\sqrt{-g}}\partial_\mu \sqrt{-g} g^{\mu\nu} \partial_\nu \psi = 0, \tag{3.39}$$

where $g_{\mu\nu}$ is the metric tensor, $g^{\mu\nu}$ is the inverse of the metric tensor, and g is the determinant of the metric tensor. We can use this equation and transform it into the Regge-Wheeler equation that is equivalent to the Schrödinger equation.

3.4.10 Regge-Wheeler equation

The Regge-Wheeler equation explains the perturbations of the Schwarzschild metric and also plays a significant role in Schwarzschild black hole [27]. The Regge-Wheeler equation can be written in this form [28]

$$\frac{d^2\Psi}{dr_*^2} + [\omega^2 - V(r)]\Psi(r) = 0, \tag{3.40}$$

where $V(r) = \frac{l(l+1)\zeta(r)^2}{r^2} + r^{-1}\sqrt{\zeta(r)^2 B(r)}\frac{d}{dr}\sqrt{\zeta(r)^2 B(r)}$ is the potential of black holes. We can then use this potential to obtain the greybody factors. For the Schwarzschild black holes that have a coefficient in front of dr^2 equal to dt^2, we can obtain the potential in the form [29]

$$V(r) = \frac{l(l+1)f(r)}{r^2} + \frac{f(r)f'(r)}{r^2}. \tag{3.41}$$

Using the general potential of the Schwarzschild black hole developed by Ngampitipan [28], we can obtain the potential of the perfect fluid black hole in isotropic coordinates using this concept [28]. We begin with static spherical symmetrical geometry in isotropic coordinates, which then gives

$$ds^2 = -\zeta(r)^2 dt^2 + \frac{1}{\zeta(r)^2 B(r)^2} dr^2 + r^2 d\Omega^2, \tag{3.42}$$

where $d\Omega^2 = d\theta^2 + \sin^2(\theta)d\phi^2$. Let $A(r) = \zeta(r)^2$ and $B(r) = \zeta(r)^2 B(r)^2$.

We start with the Klein-Gordon equation

$$\frac{1}{\sqrt{-g}}\partial_\mu \sqrt{-g} g^{\mu\nu}\partial_\nu \psi = 0, \tag{3.43}$$

where $g_{\mu\nu}$ is the metric tensor, $g^{\mu\nu}$ is the inverse of the metric tensor, and g is the determinant of the metric tensor.

$$g_{\mu\nu} = \begin{bmatrix} -A(r) & 0 & 0 & 0 \\ 0 & B(r)^{-1} & 0 & 0 \\ 0 & 0 & r^2 B(r)^{-1} & 0 \\ 0 & 0 & 0 & r^2 \sin^2(\theta) B(r)^{-1} \end{bmatrix}, \tag{3.44}$$

$$g^{\mu\nu} = \begin{bmatrix} -A(r)^{-1} & 0 & 0 & 0 \\ 0 & B(r) & 0 & 0 \\ 0 & 0 & r^{-2} B(r) & 0 \\ 0 & 0 & 0 & r^{-2} \sin^{-2}(\theta) B(r) \end{bmatrix}, \tag{3.45}$$

$$g = -\frac{A(r)}{B(r)^3} r^4 \sin^2(\theta), \tag{3.46}$$

$$\sqrt{-g} = \sqrt{\frac{A(r)}{B(r)^3}} r^2 \sin(\theta). \tag{3.47}$$

Then, the Klein-Gordon equation becomes

$$\frac{1}{\sqrt{-g}} \partial_\mu \sqrt{-g} g^{\mu\nu} \partial_\nu \psi = 0, \tag{3.48}$$

where μ, ν are labels that take on the values $0, 1, 2, 3$ or t, r, θ, ϕ.

$$\frac{1}{\sqrt{-g}} \partial_t (\sqrt{-g} g^{tt} \partial_t \psi) + \frac{1}{\sqrt{-g}} \partial_r (\sqrt{-g} g^{rr} \partial_r \psi)$$

$$+ \frac{1}{\sqrt{-g}} \partial_\theta (\sqrt{-g} g^{\theta\theta} \partial_\theta \psi) + \frac{1}{\sqrt{-g}} \partial_\phi (\sqrt{-g} g^{\phi\phi} \partial_\phi \psi) = 0. \tag{3.49}$$

$$\partial_t (g^{tt} \partial_t \psi) + \frac{1}{\sqrt{-g}} \partial_r (\sqrt{-g} g^{rr} \partial_r \psi) + \frac{1}{\sqrt{-g}} \partial_\theta (\sqrt{-g} g^{\theta\theta} \partial_\theta \psi) + \partial_\phi (g^{\phi\phi} \partial_\phi \psi) = 0. \tag{3.50}$$

$$-\frac{1}{A(r)} \frac{\partial^2 \psi}{\partial t^2} + r^{-2} B(r) \sqrt{\frac{B(r)}{A(r)}} \partial_r \left(\sqrt{\frac{A(r)}{B(r)^3}} r^2 B(r) \partial_r \psi \right)$$

$$+ \frac{1}{r^2 \sin(\theta)} \partial_\theta (\sin(\theta) B(r) \partial_\theta \psi) + r^{-2} B(r) \sin^{-2}(\theta) \frac{\partial^2 \psi}{\partial \phi^2} = 0. \tag{3.51}$$

$$\psi(t, r, \Omega) = e^{i\omega t} \varphi(r) X(\Omega). \tag{3.52}$$

Substituting $\psi(t, r, \Omega)$ into the above equation, we derive

$$A(r)^{-1} \omega^2 e^{i\omega t} \varphi(r) X(\Omega) + r^{-2} e^{i\omega t} X(\Omega) B(r) \sqrt{\frac{B(r)}{A(r)}} \frac{d}{dr} \left(\sqrt{\frac{A(r)}{B(r)^3}} r^2 B(r) \frac{d\varphi(r)}{dr} \right)$$

$$+ r^{-2} \sin^{-1}(\theta) e^{i\omega t} \varphi(r) \partial_\theta (B(r) \sin(\theta) \partial_\theta X(\Omega)) + r^{-2} B(r) \sin^{-2}(\theta) e^{i\omega t} \varphi(r) \frac{\partial^2 X(\Omega)}{\partial \phi^2} = 0. \tag{3.53}$$

Multiplying by $r^2 / e^{i\omega t} \varphi(r) X(\Omega)$ into equation (3.53),

$$A(r)^{-1} r^2 \omega^2 + \varphi^{-1} B(r) \sqrt{\frac{B(r)}{A(r)}} \frac{d}{dr} \left(\sqrt{\frac{A(r)}{B(r)^3}} r^2 B(r) \frac{d\varphi(r)}{dr} \right)$$

$$+B(r)\sin^{-1}(\theta)X(\Omega)^{-1}\partial_\theta(\sin(\theta)\partial_\theta X(\Omega))+\frac{B(r)}{\sin^2(\theta)}X(\Omega)^{-1}\frac{\partial^2 X(\Omega)}{\partial\phi^2}=0. \quad (3.54)$$

Since

$$B(r)(\sin^{-1}(\theta)\partial_\theta(\sin(\theta)\partial_\theta X(\Omega))+\frac{1}{\sin^2(\theta)}\frac{\partial^2 X(\Omega)}{\partial\phi^2}=B(r)[-l(l+1)X(\Omega)], \quad (3.55)$$

$$A(r)^{-1}r^2\omega^2+\varphi^{-1}B(r)\sqrt{\frac{B(r)}{A(r)}}\frac{d}{dr}\left(\sqrt{\frac{A(r)}{B(r)^3}}r^2B(r)\frac{d\varphi(r)}{dr}\right)-B(r)[l(l+1)]=0. \quad (3.56)$$

Multiplying by $A(r)\varphi(r)/r^2B(r)^2$ into equation (3.56),

$$B(r)^{-2}\omega^2\varphi(r)+r^{-2}\sqrt{A(r)B(r)}^{-1}\frac{d}{dr}\left(\sqrt{A(r)B(r)^{-1}}r^2\frac{d\varphi(r)}{dr}\right)$$
$$-B(r)^{-1}[l(l+1)]\frac{A(r)\varphi(r)}{r^2}=0. \quad (3.57)$$

Multiplying by $B(r)^2$ into equation (3.57),

$$\omega^2\varphi(r)+r^{-2}B(r)\sqrt{A(r)B(r)}\frac{d}{dr}\left(B(r)^{-1}\sqrt{A(r)B(r)}r^2\frac{d\varphi(r)}{dr}\right)$$
$$-B(r)[l(l+1)]\frac{A(r)\varphi(r)}{r^2}=0. \quad (3.58)$$

Let $r_*=c(r)$ in order to simplify the above equation and to use the chain rule; then, we get

$$\frac{d}{dr}=\frac{dr_*}{dr}\frac{d}{dr_*}=c'(r)\frac{d}{dr_*}. \quad (3.59)$$

Substituting the above equation into equation (3.58)

$$\omega^2\varphi(r)+r^{-2}B(r)\sqrt{A(r)B(r)}c'(r)\frac{d}{dr_*}\left(B(r)^{-1}\sqrt{A(r)B(r)}c'(r)r^2\frac{d\varphi(r)}{dr_*}\right)$$
$$-B(r)[l(l+1)]\frac{A(r)\varphi(r)}{r^2}=0. \quad (3.60)$$

Choose $c'(r)=1/\sqrt{A(r)B(r)}$ in equation (3.60),

$$\omega^2\varphi(r)+r^{-2}B(r)\frac{d}{dr_*}\left(B(r)^{-1}r^2\frac{d\varphi(r)}{dr_*}\right)-B(r)[l(l+1)]\frac{A(r)\varphi(r)}{r^2}=0. \quad (3.61)$$

Let $\varphi(r) = f(r)\Psi(r)$,

$$\frac{d\varphi(r)}{dr_*} = f(r)\frac{d\Psi(r)}{dr_*} + f'(r)\sqrt{A(r)B(r)}\Psi(r). \tag{3.62}$$

$$B(r)^{-1}r^2\frac{d\varphi(r)}{dr_*} = B(r)^{-1}r^2 f(r)\frac{d\Psi(r)}{dr_*} + B(r)^{-1}r^2 f'(r)\sqrt{A(r)B(r)}\Psi(r). \tag{3.63}$$

$$\begin{aligned}\frac{d}{dr_*}\left(B(r)^{-1}r^2\frac{d\varphi(r)}{dr_*}\right) &= B(r)^{-1}r^2 f(r)\frac{d^2\Psi(r)}{dr_*^2}\\ &+(r^2B(r)^{-1}f'(r) - r^2f(r)B(r)^{-2}B'(r) + 2rB(r)^{-1}f(r))\sqrt{A(r)B(r)}\frac{d\Psi(r)}{dr_*}\\ &+B(r)^{-1}r^2f'(r)\sqrt{A(r)B(r)}\frac{d\Psi(r)}{dr_*} + \frac{d}{dr_*}B(r)^{-1}r^2f'(r)\sqrt{A(r)B(r)}\Psi(r).\end{aligned} \tag{3.64}$$

$$\frac{d}{dr_*}\left(B(r)^{-1}r^2\frac{d\varphi(r)}{dr_*}\right) = B(r)^{-1}r^2 f(r)\frac{d^2\Psi(r)}{dr_*^2} + \frac{d}{dr_*}B(r)^{-1}r^2f'(r)\sqrt{A(r)B(r)}\Psi(r). \tag{3.65}$$

From the above equation, when the second and the third terms are cancelled, the equation becomes

$$(r^2B(r)^{-1}f'(r) - r^2f(r)B(r)^{-2}B'(r) + 2rB(r)^{-1}f(r)) + B(r)^{-1}r^2f'(r) = 0. \tag{3.66}$$

$$2r^2B(r)^{-1}f'(r) - r^2f(r)B(r)^{-2}B'(r) + 2rB(r)^{-1}f(r) = 0. \tag{3.67}$$

Multiplying by $B(r)/r$ into equation (3.67),

$$2rf'(r) - rf(r)B(r)^{-1}B'(r) + 2f(r) = 0. \tag{3.68}$$

We rearrange equation (3.68)

$$2rf'(r) + (-rB(r)^{-1}B'(r) + 2)f(r) = 0. \tag{3.69}$$

Then, we derive $f(r)$

$$\frac{f'(r)}{f(r)} = \frac{rB(r)^{-1}B'(r) - 2}{2r}. \tag{3.70}$$

$$\frac{df(r)}{f(r)} = \frac{B(r)^{-1}B'(r)}{2}dr - \frac{1}{r}dr. \tag{3.71}$$

$$\ln f(r) = \frac{1}{2}\ln B(r) - \ln r. \tag{3.72}$$

We obtain

$$f(r) = \frac{\sqrt{B(r)}}{r} = B(r)^{\frac{1}{2}}r^{-1}. \tag{3.73}$$

Thus, the first derivative is given by

$$f'(r) = \frac{\left(\frac{rB'(r)}{2\sqrt{B(r)}} - \sqrt{B(r)}\right)}{r^2} = \frac{1}{2}r^{-1}B(r)^{-\frac{1}{2}}B'(r) - r^{-2}B(r)^{\frac{1}{2}}. \tag{3.74}$$

Substituting $f(r)$ and $f'(r)$ into equation (3.65), we derive

$$\frac{d}{dr_*}\left(B(r)^{-1}r^2\frac{d\varphi(r)}{dr_*}\right) = B(r)^{-1}r^2B(r)^{\frac{1}{2}}r^{-1}\frac{d^2\Psi(r)}{dr_*^2} + \frac{d}{dr_*}\left(B(r)^{-1}r^2\left(\frac{1}{2}r^{-1}B(r)^{-\frac{1}{2}}B'(r) - r^{-2}B(r)^{\frac{1}{2}}\right)\right)\sqrt{A(r)B(r)}\Psi(r). \tag{3.75}$$

$$\frac{d}{dr_*}\left(B(r)^{-1}r^2\frac{d\varphi(r)}{dr_*}\right) = rB(r)^{-\frac{1}{2}}\frac{d^2\Psi(r)}{dr_*^2} + \frac{d}{dr_*}\left(\frac{1}{2}rB(r)^{-\frac{3}{2}}B'(r) - B(r)^{-\frac{1}{2}}\right)\sqrt{A(r)B(r)}\Psi(r). \tag{3.76}$$

Substituting $\varphi(r)$ and equation (3.76) into equation (3.61), we derive

$$\omega^2(\sqrt{B(r)}/r)\Psi(r) + r^{-2}B(r)\left[rB(r)^{-\frac{1}{2}}\frac{d^2\Psi(r)}{dr_*^2}\right] + r^{-2}B(r)\left[\frac{d}{dr_*}\left(\frac{1}{2}rB(r)^{-\frac{3}{2}}B'(r) - B(r)^{-\frac{1}{2}}\right)\sqrt{A(r)B(r)}\Psi(r)\right] - B(r)[l(l+1)]\frac{A(r)(\sqrt{B(r)}/r)\Psi(r)}{r^2} = 0. \tag{3.77}$$

Multiplying by $\frac{r}{\sqrt{B(r)}}$ into equation (3.77),

$$\omega^2\Psi(r) + \frac{d^2\Psi(r)}{dr_*^2} + r^{-1}\sqrt{B(r)}\left[\frac{d}{dr_*}\left(\frac{1}{2}rB(r)^{-\frac{3}{2}}B'(r) - B(r)^{-\frac{1}{2}}\right)\sqrt{A(r)B(r)}\Psi(r)\right] - l(l+1)\frac{A(r)B(r)\Psi(r)}{r^2} = 0. \tag{3.78}$$

From the Regge-Wheeler equation

$$\frac{d^2\Psi(r)}{dr_*^2} + [\omega^2 - V(r)]\Psi(r) = 0, \tag{3.79}$$

and then,

$$V(r) = \frac{l(l+1)\zeta(r)^4 B(r)^2}{r^2} - r^{-1}\zeta(r)^3 B^2(r)\frac{d}{dr}\left[\frac{r(\zeta(r)^2 B(r)^2)'}{2\zeta(r)B(r^2)} - \zeta(r)\right] \tag{3.80}$$

is the potential of black holes in the form of perfect fluid spheres in isotropic coordinates. We use these potential formula in two coordinates to obtain the potentials of black holes.

3.4.11 Potentials of black holes

We obtain the potentials of black holes in Schwarzschild and isotropic coordinates as shown in Tables 3.4 and 3.5.

Black holes	**Potentials**
Schwarzschild Exterior	$V(r) = \frac{l(l+1)(1-\frac{2m}{r})}{r^2} + \frac{\frac{2m}{r^2}(1-\frac{2m}{r})}{r}$
Tolman VI	$V(r) = \frac{(A+Br^8)(3A+14Al+14Al^2)+(-5Br^8+14Br^8l+14Bl^2r^8)}{14r^8}$
Kuch 68 II	$V(r) = \frac{l(l+1)(1-\frac{2m}{r})}{r^2} + \frac{2C(2r-2m)}{r}$
M-W III	$V(r) = \frac{A(a-r)[2a^2+7la^2+7l^2a^2-2ar-8r^2]}{7ra^2}$

Table 3.4: Potentials of black holes in the form of the perfect fluid spheres in Schwarzschild coordinates.

Black holes	**Potentials**
Schwarzschild Exterior	$V(r) = \frac{16(M-2r)^2r^2(-8Mr+l(M+2r)^2+l^2(M+2r)^2)}{(M+2r)^8}$
N-P-V Ia	$V(r) = a^2l(l+1) + \frac{b^2l(l+1)}{r^2} + \frac{ab(1+2l+2l^2)}{r}$
Burl I	$V(r) = \frac{A(1+r^2)^{-2+\frac{4}{k}}[-8a(2+2a-k)r^4-4(3ak+kr^2)r^2(1+r^2)+k^2l(l+1)(1+r^2)^2]}{k^2r^2}$

Table 3.5: Potentials of black holes in the form of perfect fluid spheres in isotropic coordinates [30].

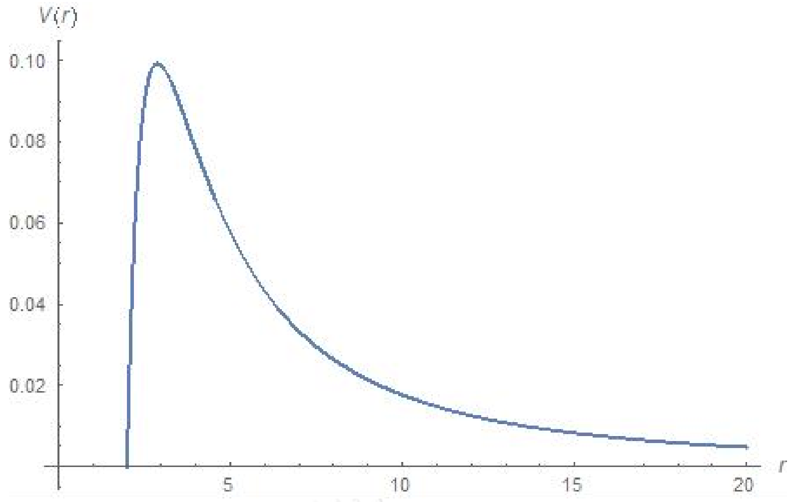

Figure 3.7: Potential of Schwarzschild Exterior black hole.

Potentials as shown in Tables 3.4 and 3.5 are potentials of black holes in the form of perfect fluid spheres in Schwarzschild coordinates and isotropic coordinates, respectively. Potentials are functions that depend on r (radius of perfect fluid black hole). We use these potentials to calculate the greybody factors.

Potentials of perfect fluid black holes are plotted as shown in Figures 3.7 - 3.10 represent the potentials of black holes in the form of perfect fluid spheres in Schwarzschild coordinates and Figures 3.11 - 3.13 represent the potentials of black holes in the form of perfect fluid spheres in isotropic coordinates. The figures of the potentials show the gravity of black holes and we derive greybody factors for perfect fluid black holes and depend on the potentials of the black holes.

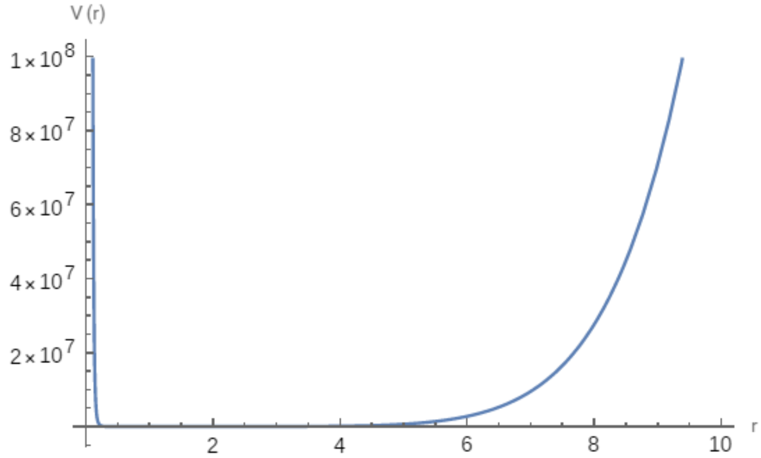

Figure 3.8: Potential of Tolman VI black hole.

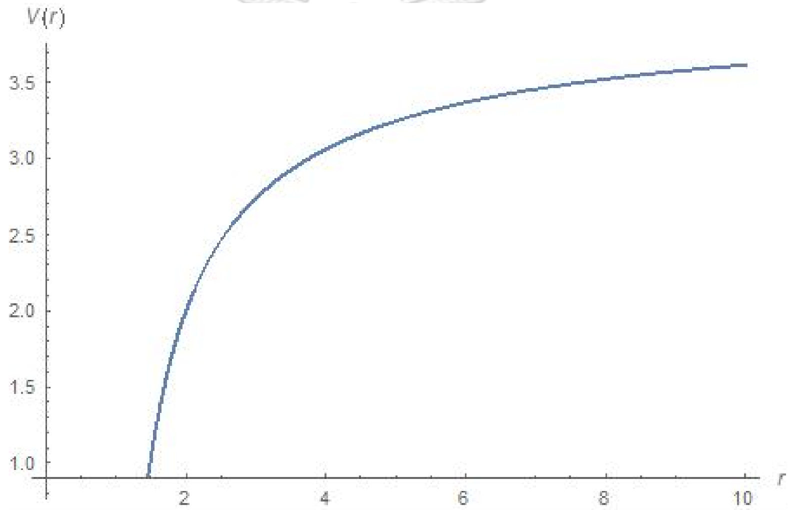

Figure 3.9: Potential of Kuch 68 II black hole.

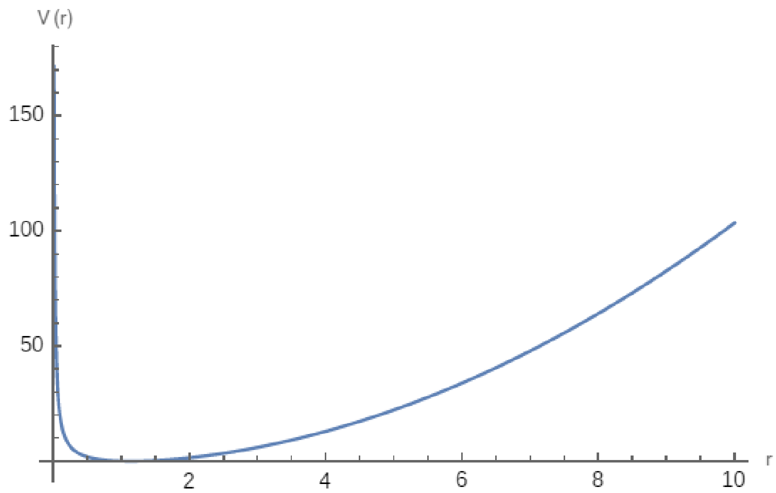

Figure 3.10: Potential of M-W III black hole.

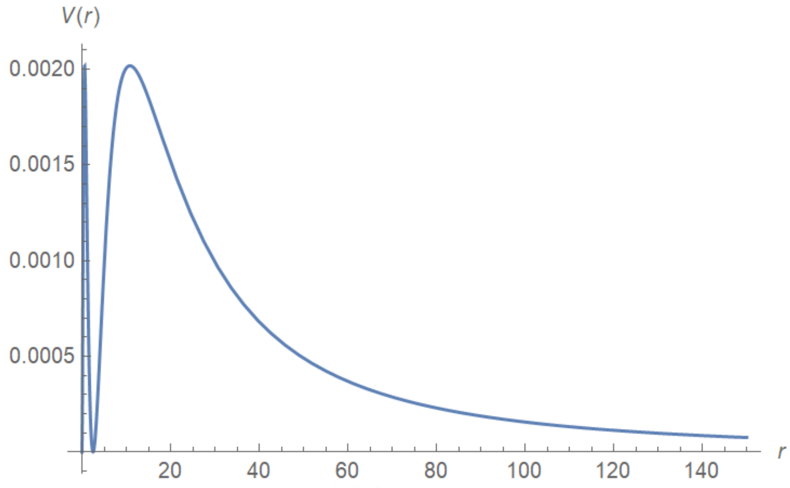

Figure 3.11: Potential of Schwarzschild Exterior (isotropic) black hole.

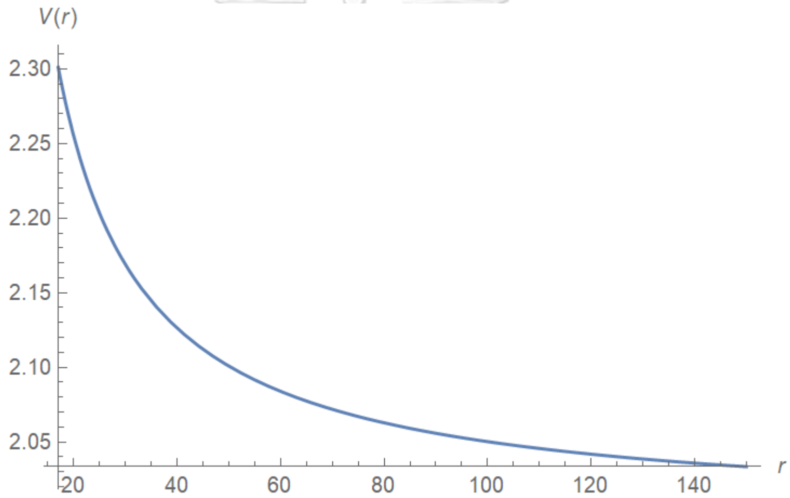

Figure 3.12: Potential of N-P-V Ia black hole.

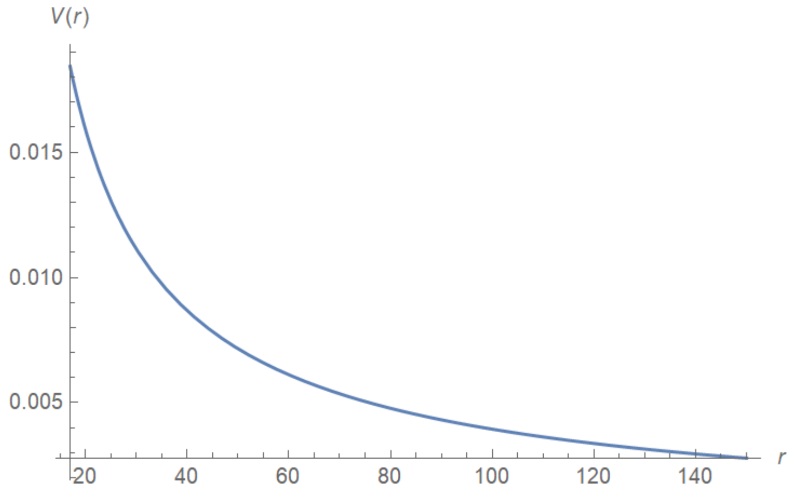

Figure 3.13: Potential of Burl I black hole.

In chapter III, we studied about perfect fluid spheres, the properties of perfect fluid spheres, and considered perfect fluid spheres in two different coordinates, namely, the Schwarzschild coordinate and the isotropic coordinate. Next, we classified black holes in perfect fluid spheres in Schwarzschild coordinate and isotropic coordinate by finding the radius of the perfect fluid spheres compared with the Schwarzschild radius. We obtained four categories of black holes in Schwarzschild coordinate and three categories of black holes in isotropic coordinate. Then, we calculated the potential of black holes by deriving the Regge-Wheeler equation from the Klein-Gordon equation. Finally, we plotted the figures of the potentials of black holes to show the spacetime curvature outside the event horizon.

CHAPTER IV

GREYBODY FACTORS FOR PERFECT FLUID BLACK HOLE

In this chapter, we will learn the method to obtain the greybody factors or the transmission probabilities of black holes that is the Bogoliubov coefficient. There are many techniques to obtain the transmission and reflection probabilities such as the WKB approximation and the 2×2 transfer matrix [31, 32]. In this thesis, we will study the Bogoliubov coefficients developed by Boonserm [33], which is the highly accurate method to obtain the rigorous bound on the reflection and transmission probabilities. The rigorous bound can be used to derive the greybody factors of perfect fluid black holes.

4.1 Bogoliubov coefficients

The Bogoliubov coefficients developed by Boonserm [33] is the method that can be used to obtain the rigorous bound on the reflection and transmission probabilities, which involves α and β. The concept of this method is used to obtain the exact solution of the second order linear ordinary differential equation in the form of a matrix time-ordered exponential and the Bogoliubov coefficient that relate with the constant of this matrix. We then use these coefficients to obtain the rigorous bound on the reflection and transmission probabilities.

We start with the second order differential equation

$$\frac{d^2\phi}{dt^2} + \omega^2(x)\phi(t) = 0. \tag{4.1}$$

The solutions are given by [33]

$$\phi(t \leq t_i) = e^{+i\omega_0 t}, \tag{4.2}$$

$$\phi(t \geq t_f) = \alpha e^{+i\omega_0 t} + \beta e^{-i\omega_0 t}, \tag{4.3}$$

From equation (4.2) and equation (4.3), we get

$$\begin{bmatrix} \phi \\ \frac{\pi}{\omega_0} \end{bmatrix}_{t_i} = \begin{bmatrix} e^{+i\omega_0 t_i} \\ ie^{+i\omega_0 t_i} \end{bmatrix} \tag{4.4}$$

and

$$\begin{bmatrix} \phi \\ \frac{\pi}{\omega_0} \end{bmatrix}_{t_f} = \begin{bmatrix} \alpha e^{+i\omega_0 t_f} + \beta e^{-i\omega_0 t_f} \\ i(\alpha e^{+i\omega_0 t_f} - \beta e^{-i\omega_0 t_f}) \end{bmatrix}. \tag{4.5}$$

We also have

$$\begin{bmatrix} \phi \\ \frac{\pi}{\omega_0} \end{bmatrix}_{t_f} = T \begin{bmatrix} \phi \\ \frac{\pi}{\omega_0} \end{bmatrix}_{t_i}, \tag{4.6}$$

$$\begin{bmatrix} \alpha e^{+i\omega_0 t_f} + \beta e^{-i\omega_0 t_f} \\ i(\alpha e^{+i\omega_0 t_f} - \beta e^{-i\omega_0 t_f}) \end{bmatrix} = \begin{bmatrix} \alpha a e^{+i\omega_0 t_i} + bie^{+i\omega_0 t_i} \\ ce^{+i\omega_0 t_i} + die^{+i\omega_0 t + i}) \end{bmatrix}. \tag{4.7}$$

From equation (4.7), we get [33]

$$\alpha e^{+i\omega_0 t_f} + \beta e^{-i\omega_0 t_f} = ae^{+i\omega_0 t_i} + bie^{+i\omega_0 t_i}, \tag{4.8}$$

$$\alpha e^{+i\omega_0 t_f} - \beta e^{-i\omega_0 t_f} = -ice^{+i\omega_0 t_i} + de^{+i\omega_0 t + i}). \tag{4.9}$$

Solving the two above equations, we get [33]

$$2\alpha e^{+i\omega_0 t_f} = [a + d + i(b - c)e^{+\omega_0 t_i}], \tag{4.10}$$

$$2\beta e^{-i\omega_0 t_f} = [a - d + i(b + c)e^{+\omega_0 t_i}]. \tag{4.11}$$

$$\alpha = \frac{1}{2}[a + d + i(b - c)]e^{-i\omega_0 (t_f - t_i)}, \tag{4.12}$$

$$\beta = \frac{1}{2}[a - d + i(b + c)]e^{i\omega_0 (t_f + t_i)}. \tag{4.13}$$

$$|\alpha|^2 = \frac{1}{4}[(a + d)^2 + (b - c)^2], \tag{4.14}$$

$$|\beta|^2 = \frac{1}{4}[(a-d)^2 + (b+c)^2], \tag{4.15}$$

and

$$|\alpha|^2 - |\beta|^2 = \frac{(a+d)^2 + (b-c)^2 - (a-d)^2 - (b+c)^2}{4},$$
$$= \frac{2ad - 2bc + 2ad - 2bc}{4} = ad - bc = 1. \tag{4.16}$$

From equation (4.14) and equation (4.15), we get [33]

$$|\beta|^2 = \frac{1}{4}[a^2 + d^2 + b^2 + c^2 - 2] = \frac{1}{4}tr(TT^T - I). \tag{4.17}$$

Also,

$$|\alpha|^2 = \frac{1}{4}[a^2 + d^2 + b^2 + c^2 + 2] = \frac{1}{4}tr(TT^T + I), \tag{4.18}$$

Let

$$X(t) = T(t)T(t)^T, \tag{4.19}$$

and

$$tr[x] = tr\{TT^T\} = a^2 + b^2 + c^2 + d^2. \tag{4.20}$$

Then, the differential equation is [33]

$$\frac{dX}{dt} = \begin{bmatrix} 0 & \omega_0 \\ -\omega^2(t)/\omega_0 & 0 \end{bmatrix} x(t) + X(t) \begin{bmatrix} 0 & -\omega^2(t)/\omega_0 \\ \omega_0 & 0 \end{bmatrix}, \tag{4.21}$$

$$\frac{dX}{dt} = \begin{bmatrix} 0 & \omega_0 \\ -\omega^2(t)/\omega_0 & 0 \end{bmatrix} \begin{bmatrix} a^2+b^2 & ac+bd \\ ac+bd & c^2+d^2 \end{bmatrix} + \begin{bmatrix} a^2+b^2 & ac+bd \\ ac+bd & c^2+d^2 \end{bmatrix} \begin{bmatrix} 0 & -\omega^2(t)/\omega_0 \\ \omega_0 & 0 \end{bmatrix}, \tag{4.22}$$

$$\frac{dX}{dt} = \begin{bmatrix} 2\omega_0(ac+bd) & \omega_0(c^2+d^2) - (\omega^2/\omega_0)(a^2+b^2) \\ \omega_0(c^2+d^2) - (\omega^2/\omega_0)(a^2+b^2) & (-2\omega^2/\omega_0)(ac+bd) \end{bmatrix}, \tag{4.23}$$

and so

$$\frac{dtr[X]}{dt} = 2(ac+bd)(\omega_0 - \frac{\omega^2}{\omega_0}), \tag{4.24}$$

$$\frac{d\mathrm{tr}[X]}{dt} \leq 2|ac+bd||\omega_0 - \frac{\omega^2}{\omega_0}|, \tag{4.25}$$

$$\frac{dtr[X]}{dt} \leq 2|\sqrt{(a^2+b^2+c^2+d^2)-4}|\omega_0 - \frac{\omega^2}{\omega_0}| = \sqrt{(tr[x])^2-4}|\omega_0 - \frac{\omega^2}{\omega_0}|. \tag{4.26}$$

From equation (4.24), we get

$$\frac{1}{\sqrt{(tr[x])^2-4}}\frac{dtr[X]}{dt} \leq |\omega_0 - \frac{\omega^2}{\omega_0}|, \tag{4.27}$$

$$\frac{d\cosh^{-1} tr[X/2]}{dt} \leq |\omega_0 - \frac{\omega^2}{\omega_0}|, \tag{4.28}$$

$$tr[X/2] \leq \int_{t_i}^{t_f} |\omega_0 - \frac{\omega^2}{\omega_0}|dt. \tag{4.29}$$

$$|\beta|^2 \leq \sinh^2 \int_{t_i}^{t_f} \frac{1}{2}|\omega_0 - \frac{\omega^2}{\omega_0}|dt. \tag{4.30}$$

$$|\alpha|^2 \leq \cosh^2 \int_{t_i}^{t_f} \frac{1}{2}|\omega_0 - \frac{\omega^2}{\omega_0}|dt. \tag{4.31}$$

We obtain the rigorous bound on the Bogoliubov coefficient. We begin with the Schrödinger's time-independent equation

$$\frac{d^2\psi}{dx^2} + k^2(x)\psi(x) = 0, \tag{4.32}$$

where $k^2(x) = \frac{2m[E-V(x)]}{\hbar^2}$.

The solutions in the asymptotic regions are given by [33, 34]

$$\psi(x) \approx \begin{cases} \alpha\frac{e^{ik_{-\infty}x}}{\sqrt{k_{-\infty}}} + \beta\frac{e^{-ik_{-\infty}x}}{\sqrt{k_{-\infty}}}, & x \to -\infty \\ \frac{e^{ik_{\infty}x}}{\sqrt{k_{\infty}}}, & x \to \infty \end{cases}. \tag{4.33}$$

The transmission and reflection probabilities are given by [33, 34]

$$R = \left|\frac{\beta}{\alpha}\right|^2, T = \left|\frac{1}{\alpha}\right|^2. \tag{4.34}$$

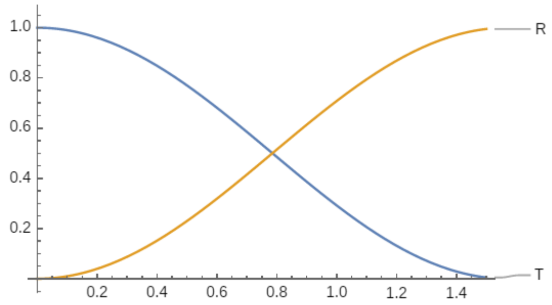

Figure 4.1: The relation between transmission and reflection probabilities.

Through the conservation probability (T+R=1) [33, 34]

$$|\alpha|^2 - |\beta|^2 = 1. \tag{4.35}$$

We obtain [33, 34]

$$|\alpha| \leq \cosh\left[\int_{-\infty}^{\infty} \vartheta(x)dx\right], \tag{4.36}$$

and

$$|\beta| \leq \sinh\left[\int_{-\infty}^{\infty} \vartheta(x)dx\right], \tag{4.37}$$

where

$$\vartheta(x) = \frac{\sqrt{[h'(x)]^2 + [k^2(x) - h^2(x)]^2}}{2|h(x)|}. \tag{4.38}$$

The rigorous bound of the transmission and reflection probabilities are given by [33, 34]

$$T \geq \text{sech}^2\left[\int_{-\infty}^{\infty} \vartheta(x)dx\right], \tag{4.39}$$

and

$$R \leq \tanh^2\left[\int_{-\infty}^{\infty} \vartheta(x)dx\right]. \tag{4.40}$$

4.2 Greybody factors

Hawking radiation, the radiation emitted by black holes, is reflected by potential, while the rest of the radiation is sent out [35]. Greybody factors are the transmission probabilities of the Hawking radiation. The transmission and reflection probabilities of black holes can be calculated using the 2×2 transfer matrix method to obtain the lower bound on the transmission coefficient and the upper bound on the reflection coefficient. The 2×2 transfer matrix method is given by [32]

$$T \geq \operatorname{sech}^2 \frac{1}{2\omega} \int_{-\infty}^{\infty} |V(r)| dr_*, \tag{4.41}$$

where $\frac{dr_*}{dr} = \frac{1}{\sqrt{\zeta(r)^2 B(r)}}$.

For the Schwarzschild black holes that have a coefficient in front of dr^2 equal to dt^2, we can obtain dr_* in this form [29]

$$\frac{dr_*}{dr} = \frac{1}{f(r)}. \tag{4.42}$$

The transmission and reflection probabilities as shown in Tables 4.1 and 4.2, respectively represent the transmission and reflection probabilities of black holes in the form of perfect fluid spheres in Schwarzschild coordinates.

The relation between the transmission and reflection probabilities are plotted as shown in Figures 4.2 - 4.4 represent the relation between the transmission and reflection probabilities of black holes in the form of perfect fluid spheres in Schwarzschild coordinates namely, Schwarzschild exterior, Tolman VI, and Kuch 68 II black holes, respectively. The reflection probability of these two black holes namely, Tolman VI and Kuch 68 II increase if ω increases, and the transmission probability decreases if ω increases.

Black holes	Transmission probabilities
Schwarzschild Exterior	$T \geq \text{sech}^2\left[\frac{1}{2\omega}\frac{m+2l(l+1)m}{4m^2}\right]$
Tolman VI	$T \geq \text{sech}^2\left[\frac{1}{2\omega}\frac{(A+Br^8)(3A+14Al+14Al^2)}{4\sqrt{14}r^7\sqrt{-\frac{(A+Br^8)^2}{r^6}}} + (5Br^8-14Br^8l-14Bl^2r^8)/4\sqrt{14}r^7\sqrt{-\frac{(A+Br^8)^2}{r^6}}\right]$
Kuch 68 II	$T \geq \text{sech}^2\left[\frac{1}{2\omega}\frac{l(l+1)a^*(m+4Cm^3-r+8Cm^2r)}{(r+4Cm^2r)^2} + 2\sqrt{C}\sinh^{-1}[2\sqrt{C}(m-r)] + t\ln[1+a^*b+4Cm(m-r)] - t\ln[r]\right]$
M-W III	$T \geq \text{sech}^2\left[\frac{A(a-r)(\sqrt{a^*}(10+7l+7l^2)\sqrt{\frac{a+r}{a}})}{a^{3/2}\sqrt{-7Ar(a-r)^2(a+r)^2/a^3}} - \frac{A(a-r)(-14\sqrt{r}\sinh^{-1}[\sqrt{r/a}])}{a^{3/2}\sqrt{-7Ar(a-r)^2(a+r)^2/a^3}} - 8A(a-r)\sqrt{a}P/a^{3/2}\sqrt{-7Ar(a-r)^2(a+r)^2/a^3}\right]$

Table 4.1: Transmission probabilities of black holes.

where $a^* = \sqrt{1+4C(m-r)^2}, b = \sqrt{1+4Cm^2}, t = \frac{4Cm(l(-2+4Cm^2)+l^2(-2+4Cm^2)+b^4)}{b^5}$, $P = {}_2F_1[-3/2,-1/2,1/2,-r/a]$ and ${}_2F_1[a,b,c,z] = \sum_{n=0}^{+\infty}[\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n}]$.

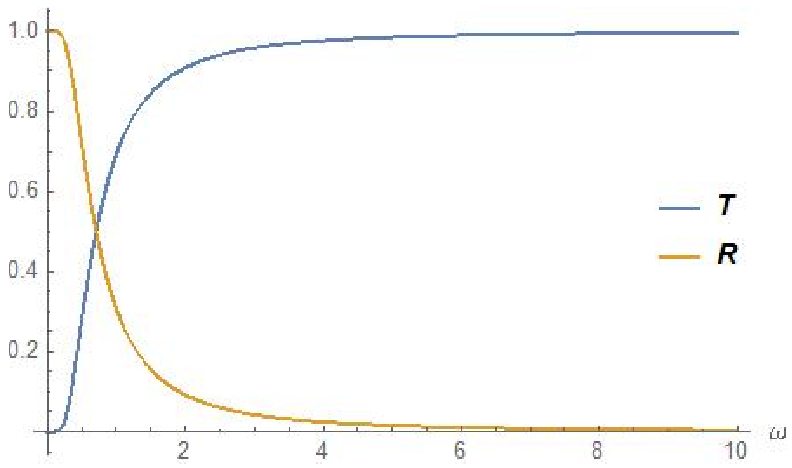

Figure 4.2: The relation between transmission and reflection probabilities of Schwarzschild Exterior black hole.

Black holes	**Reflection probabilities**
Schwarzschild Exterior	$R \leq \tanh^2\left[\frac{1}{2\omega}\frac{m+2l(l+1)m}{4m^2}\right]$
Tolman VI	$R \leq \tanh^2\left[\frac{1}{2\omega}\frac{(A+Br^8)(3A+14Al+14Al^2)}{4\sqrt{14}r^7\sqrt{-\frac{(A+Br^8)^2}{r^6}}} + (5Br^8 - 14Br^8 l - 14Bl^2r^8)/4\sqrt{14}r^7\sqrt{-\frac{(A+Br^8)^2}{r^6}}\right]$
Kuch 68 II	$R \leq \tanh^2\left[\frac{1}{2\omega}\frac{l(l+1)a^*(m+4Cm^3-r+8Cm^2r)}{(r+4Cm^2r)^2} + 2\sqrt{C}\sinh^{-1}[2\sqrt{C}(m-r)] + t\ln[1+a^*b+4Cm(m-r)] - t\ln[r]\right]$
M-W III	$R \leq \tanh^2\left[\frac{A(a-r)(\sqrt{a^*}(10+7l+7l^2)\sqrt{\frac{a+r}{a}})}{a^{3/2}\sqrt{-7Ar(a-r)^2(a+r)^2/a^3}} - \frac{A(a-r)(-14\sqrt{r}\sinh^{-1}[\sqrt{r/a}])}{a^{3/2}\sqrt{-7Ar(a-r)^2(a+r)^2/a^3}} - 8A(a-r)\sqrt{a}P/a^{3/2}\sqrt{-7Ar(a-r)^2(a+r)^2/a^3}\right]$

Table 4.2: Reflection probabilities of black holes.

where $a^* = \sqrt{1+4C(m-r)^2}, b = \sqrt{1+4Cm^2}, t = \frac{4Cm(l(-2+4Cm^2)+l^2(-2+4Cm^2)+b^4)}{b^5}$, and $P = {}_2F_1[-3/2, -1/2, 1/2, -r/a]$.

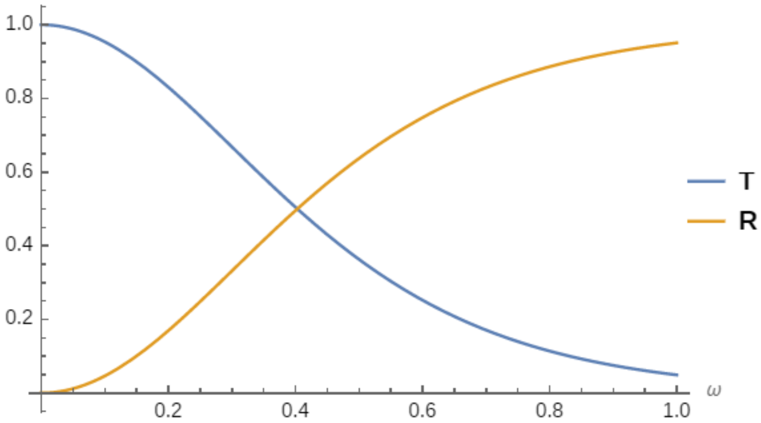

Figure 4.3: The relation between transmission and reflection probabilities of Tolman VI black hole.

Figure 4.4: The relation between transmission and reflection probabilities of Kuch 68 II black hole.

For the isotropic coordinates, we can obtain dr_* in this form

$$\frac{dr_*}{dr} = \frac{1}{\zeta(r)^2 B(r)}. \tag{4.43}$$

The greybody factors of the Schwarzschild Exterior are given by [30]

$$T \geq \text{sech}^2 \left[\frac{2(R-r_1)^3(-4M(R-r_1) + l(M+2(R-r_1))^2 + l^2(M+2(R-r_1))^2)}{(M-2(R-r_1))(M+2(R-r_1))^5 \sqrt{\frac{(R-r_1)^4}{(M^2-4(R-r_1)^2)^2}}\omega} \right], \tag{4.44}$$

and

$$R \leq \tanh^2 \left[\frac{2(R-r_1)^3(-4M(R-r_1) + l(M+2(R-r_1))^2 + l^2(M+2(R-r_1))^2)}{(M-2(R-r_1))(M+2(R-r_1))^5 \sqrt{\frac{(R-r_1)^4}{(M^2-4(R-r_1)^2)^2}}\omega} \right]. \tag{4.45}$$

The greybody factors of N-P-V Ia are given by [30]

$$T \geq \text{sech}^2 \left[\frac{(b+ar)(bl(1+l) - a(1+l+l^2)r\log[r] + ar\log[b+ar])}{2r\sqrt{(b+ar)^2}\omega} \right], \tag{4.46}$$

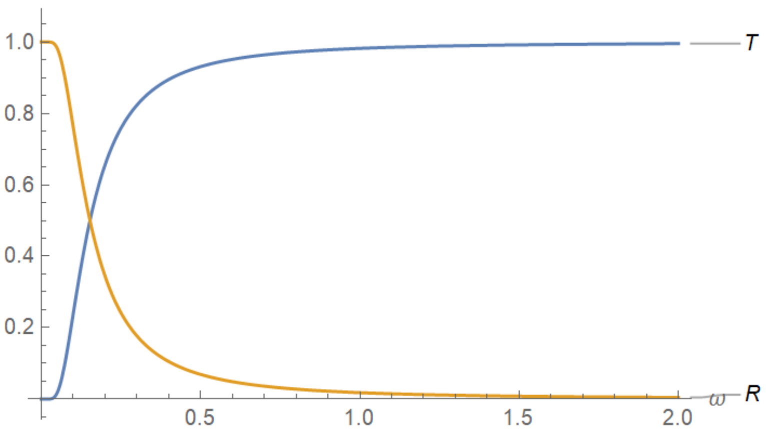

Figure 4.5: The relation between transmission and reflection probabilities of Schwarzschild Exterior (isotropic) black hole.

and

$$R \leq \tanh^2 \left[\frac{(b+ar)(bl(1+l) - a(1+l+l^2)r \log[r] + ar \log[b+ar])}{2r\sqrt{(b+ar)^2}\omega} \right], \tag{4.47}$$

where $r = \frac{10b}{a} + \frac{8\sqrt{\frac{b^3}{a^2}}}{b}$.

The greybody factors of Burl I are given by [30]

$$T \geq \text{sech}^2 \left[\frac{(1+r^2)^6(60r^2 - 28r^4 + l(-3+6r^2+r^4) + l^2(-3+6r^2+r^4))}{6r\sqrt{(1+r^2)^1 2\omega}} \right], \tag{4.48}$$

$$R \leq \tanh^2 \left[\frac{(1+r^2)^6(60r^2 - 28r^4 + l(-3+6r^2+r^4) + l^2(-3+6r^2+r^4))}{6r\sqrt{(1+r^2)^1 2\omega}} \right], \tag{4.49}$$

where $r = \sqrt{c}\left(5 - 4\left(1+c\right)^{\frac{a}{1+4a+2a^2}}\right)$ and $c = \frac{-1-6a-10a^2-4a^3}{1+10a+16a^2+4a^3}$.

In the Figures 4.5 - 4.7 represent the relation between the transmission and reflection probabilities of black holes in the form of perfect fluid spheres in isotropic coordinates namely, Schwarzschild exterior, N-P-V Ia, and Burl I black holes, respectively. The reflection probability of these three black holes in isotropic coordinates decreases if ω increase, and the transmission probability increases if ω increases.

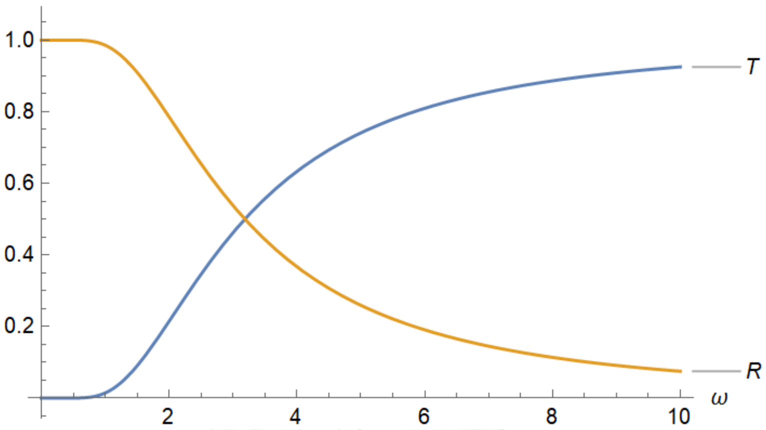

Figure 4.6: The relation between transmission and reflection probabilities of N-P-V Ia black hole.

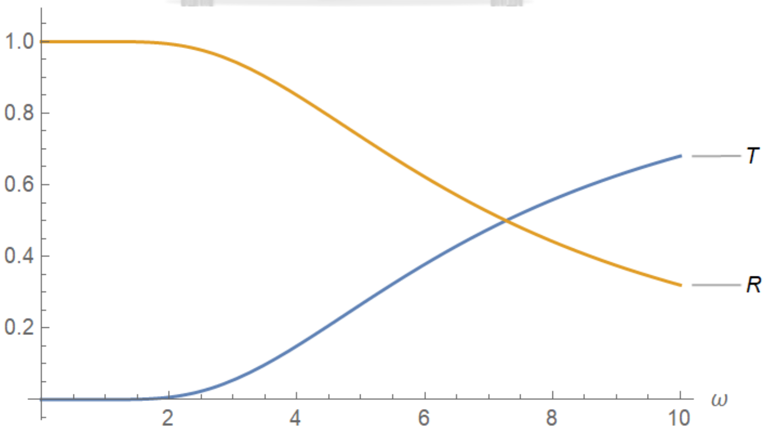

Figure 4.7: The relation between transmission and reflection probabilities of Burl I black hole.

In chapter IV, we studied the method to obtain the greybody factors of black holes. The coefficients we have obtained are α and β that are used to calculate the transmission and reflection probabilities. Then, we calculate the greybody factors or the transmission probability of black holes in two coordinates and plot the graph to show the relation between the transmission and reflection probabilities of these perfect fluid black holes.

CHAPTER V

HAWKING TEMPERATURE AND ENTROPY

In this chapter, we compute the Hawking temperature and entropy from the Hawking radiation and also compute the entropy composition of the black hole systems. After calculating the greybody factors of the Hawking radiation of black holes, we will calculate the Hawking temperature that is the thermal radiation of black holes that has a temperature, and also calculate the entropy that is associated to the Hawking temperature. Moreover, we will also calculate the entropy of the black hole systems in each coordinate.

5.1 Thermodynamics

Thermodynamics explains the relationship between work, energy, heat, and temperature. Thermodynamics can predict the origin and extinction of the universe with the second law of thermodynamics and indicate the direction of time with the increase in entropy [19]. Next, we will introduce the two laws of thermodynamics, namely, the first law and second law of thermodynamics [19].

5.1.1 The First Law of Thermodynamics

The first law of thermodynamics explains that energy cannot be established or disrupted in an isolated system.

In thermodynamics, the quantities of the energy supplied to the system as heat is given by

$$TdS = dE - dW, \tag{5.1}$$

where E is the energy and W is work.

In black hole dynamics, the quantities of the energy supplied to the system as heat is given by

$$TdS = dE - \Omega dJ - \Phi dQ, \tag{5.2}$$

where J is the rotational momentum (the angular momentum) and Q is the electric charge.

5.1.2 The Second Law of Thermodynamics

The second law of thermodynamics describes the the entropy as always increasing in an isolated system.

In thermodynamics, entropy can never decrease:

$$\triangle S \geq 0. \tag{5.3}$$

In black hole dynamics, the area of event horizon can never decrease:

$$\triangle S_0 + \triangle S_{BH} \geq 0. \tag{5.4}$$

5.2 Hawking temperature

The temperature of the Hawking radiation can be calculated in the matter of the surface gravity of the black hole [29];

$$T = \frac{\kappa}{2\pi}, \tag{5.5}$$

where $\kappa = (\sqrt{\zeta^2(r)B(r)})'$.

For Schwarzschild black holes with a coefficient in front of dr^2 equal to dt^2, we can obtain κ in this form [29];

$$\kappa = f'(r). \tag{5.6}$$

For isotropic coordinates, we can obtain κ in this form;

$$\kappa = (\zeta(r)^2 B(r))'. \tag{5.7}$$

5.3 Entropy

Entropy is also one of the fundamental properties in thermodynamics, which can be expressed in the form

$$S = \int \frac{1}{T} dM. \tag{5.8}$$

Black holes	**Temperatures**
Schwarzschild Exterior	$T = \frac{1}{4M\pi}$
Tolman VI	$T = \frac{\sqrt{-\frac{(-A/r^3+Br^5)^2}{14}}}{2\pi}$
Kuch 68 II	$T = \frac{2CM}{d\pi}$
M-W III	$T = \sqrt{-\frac{A(a-r)^2 r(a+r)}{a^2}} / \sqrt{7}\pi$
Schwarzschild Exterior (isotropic)	$T = \frac{24}{125}\pi$
N-P-V Ia	$T = \sqrt{(a+br)^2}/2\pi$
Burl I	$T = \frac{A(1+r^2)^{\frac{4(1+a)}{1+4a+2a^2}}}{2\pi} \sqrt{(1+r^2)^{-\frac{4+8a}{1+4a+2a^2}}} / A$

Table 5.1: Temperatures of black holes

In Table 5.1, the calculation of the temperatures of the Schwarzschild exterior black hole and the Kuch 68 II black hole are obtained by [36]. In this thesis, we compute the temperatures of the Tolman VI black hole, the M-W III black hole, the Schwarzschild exterior (isotropic) black hole, the N-P-V Ia black hole and the Burl I black hole.

Black holes	Entropies
Schwarzschild Exterior	$S = 2M^2\pi$
Tolman VI	$S = \frac{15\pi(A+Br^8)\tan^{-1}[\sqrt{B/A}r^4]}{4r^3\sqrt{-14AB(A+Br^8)^2/r^6}}$
Kuch 68 II	$S = \frac{\pi(d+\ln[M]-\ln[1+d])}{2C}$
M-W III	$S = \int \frac{dM}{dr}\frac{1}{T}dr$
Schwarzschild Exterior (isotropic)	$S = \frac{125M}{24\pi}$
N-P-V Ia	$S = -\frac{4(r(4b+3ar)+3(b+ar)\sqrt{-r^2(b+ar)}\ln[b+ar])}{3a\pi\sqrt{-r^2(b+ar)^3}}$
Burl I	$S = -\frac{4r(1+r^2)^{-q}[(1+4a+2a^2)g-(1+6a+2a^2)h+2aj]}{(1+4a+2a^2)A\pi\sqrt{(1+r^2)^{-2q}/A}}$

Table 5.2: Entropies of black holes

In Table 5.2, the calculation of the entropies of the Schwarzschild exterior black hole and the Kuch 68 II black hole are obtained by [36] where $g = \text{Hypergeometric}_2F_1[1/2, \frac{2}{1+4a+2a^2}, 3/2, -r^2]$, $h = \text{Hypergeometric}_2F_1[1/2, \frac{2-a}{1+4a+2a^2}, 3/2, -r^2]$, $j = \text{Hypergeometric}_2F_1[1/2, \frac{3+3a+2a^2}{1+4a+2a^2}, 3/2, -r^2]$, and $q = \frac{2+4a}{1+4a+2a^2}$. In this thesis, we compute the entropies of the Tolman VI black hole, the M-W III black hole, the Schwarzschild exterior (isotropic) black hole, the N-P-V Ia black hole and the Burl I black hole.

5.3.1 Entropy Composition

The entropy composition is the entropy of the black hole systems. In this thesis, we examine two black holes as a system, and the black holes in the system are in the same coordinates. We then compute the entropy of the black hole systems in two cases, namely, additive entropy composition and nonadditive entropy composition.

5.3.1.1 Additive Entropy Composition

The additive entropy composition is given by [37]

$$S_{12} = S_1 + S_2. \tag{5.9}$$

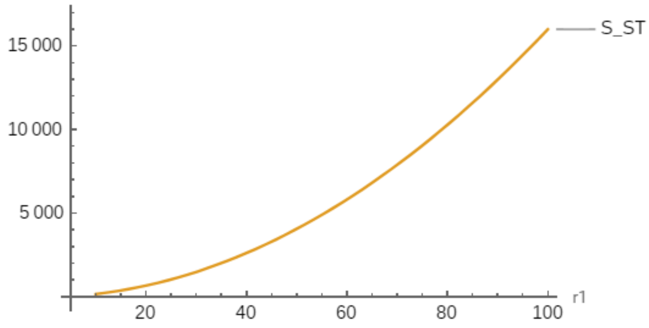

Figure 5.1: The additive entropy composition of Schwarzschild exterior and Tolman VI black holes.

We obtained the additive entropy composition of the black hole systems in Schwarzschild coordinates as shown in Table 5.3.

System of Black holes	Entropy Composition
Schwarzschild Exterior ⊕ Tolman VI	$S_{ST} = 2M^2\pi + \frac{15\pi(A+Br^8)\tan^{-1}[\sqrt{B/Ar^4}]}{4r^3\sqrt{-14AB(A+Br^8)^2/r^6}}$
Schwarzschild Exterior ⊕ Kuch 68 II	$S_{SK} = 2M^2\pi + \frac{\pi(d+\ln[M]-\ln[1+d])}{2C}$
Schwarzschild Exterior ⊕ M-W III	$S_{SM} = 2M^2\pi + \int \frac{dM}{dr}\frac{1}{T}dr$
Tolman VI ⊕ Kuch 68 II	$S_{TK} = \frac{15\pi(A+Br^8)\tan^{-1}[\sqrt{B/Ar^4}]}{4r^3\sqrt{-14AB(A+Br^8)^2/r^6}} + \frac{\pi(d+\ln[M]-\ln[1+d])}{2C}$
Tolman VI ⊕ M-W III	$S_{TM} = \frac{15\pi(A+Br^8)\tan^{-1}[\sqrt{B/Ar^4}]}{4r^3\sqrt{-14AB(A+Br^8)^2/r^6}} + \int \frac{dM}{dr}\frac{1}{T}dr$
Kuch 68 II ⊕ M-W III	$S_{KM} = \frac{\pi(d+\ln[M]-\ln[1+d])}{2C} + \int \frac{dM}{dr}\frac{1}{T}dr$

Table 5.3: The additive entropy composition of the black hole systems in Schwarzschild coordinates

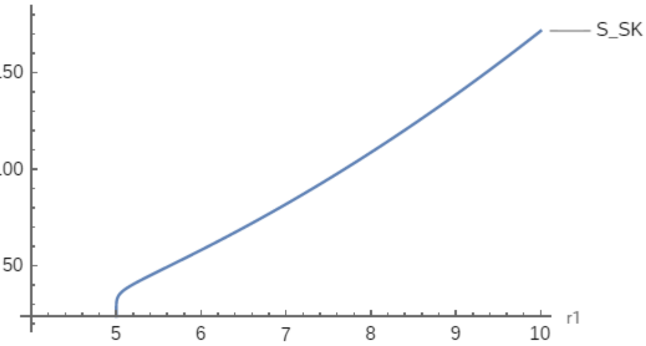

Figure 5.2: The additive entropy composition of Schwarzschild exterior and Kuch 68 II black holes.

System of Black holes	Entropy Composition
Schwarzschild Exterior ⊕ N-P-V Ia	$S_{SN} = \frac{125M(r(4b+3ar)}{18a\pi^2\sqrt{-r^2(b+ar)^3}} + \frac{3(b+ar)\sqrt{-r^2(b+ar)}\ln[b+ar])}{18a\pi^2\sqrt{-r^2(b+ar)^3}}$
Schwarzschild Exterior ⊕ Burl I	$S_{SB} = \frac{125M}{24\pi} - \frac{4r(1+r^2)^{-q}[(1+4a+2a^2)g-(1+6a+2a^2)h+2aj]}{(1+4a+2a^2)A\pi\sqrt{(1+r^2)^{-2q}/A}}$
N-P-V Ia ⊕ Burl I	$S_{NB} = -\frac{4(r(4b+3ar)+3(b+ar)\sqrt{-r^2(b+ar)}\ln[b+ar])}{3a\pi\sqrt{-r^2(b+ar)^3}} - \frac{4r(1+r^2)^{-q}[(1+4a+2a^2)g-(1+6a+2a^2)h+2aj]}{(1+4a+2a^2)A\pi\sqrt{(1+r^2)^{-2q}/A}}$

Table 5.4: The additive entropy composition of the black hole systems in isotropic coordinates

We obtained the additive entropy composition of the black hole systems in isotropic coordinates as shown in Table 5.4 where $g =_2 F_1[1/2, \frac{2}{1+4a+2a^2}, 3/2, -r^2]$, $h =_2 F_1[1/2, \frac{2-a}{1+4a+2a^2}, 3/2, -r^2]$, $j =_2 F_1, \frac{3+3a+2a^2}{1+4a+2a^2}, 3/2, -r^2]$, and $q = \frac{2+4a}{1+4a+2a^2}$.

5.3.1.2 The nonadditive Entropy Composition

The nonadditive entropy composition rule can be obtained by the Abe's equation [37]

$$S_{12} = S_1 + S_2 + \lambda S_1 S_2, \tag{5.10}$$

where $0 < \lambda < 1$.

We obtained the nonadditive entropy composition of the black hole systems in Schwarzschild coordinates as shown in Table 5.5.

System of Black holes	**Entropy Composition**
Schwarzschild Exterior $\oplus$ Tolman VI	$S_{ST} = 2M^2\pi + \frac{15\pi(A+Br^8)\tan^{-1}[\sqrt{B/A}r^4]}{4r^3\sqrt{-14AB(A+Br^8)^2/r^6}}$ $+\lambda(\frac{15M^2\pi^2(A+Br^8)\tan^{-1}[\sqrt{B/A}r^4]}{2r^3\sqrt{-14AB(A+Br^8)^2/r^6}})$
Schwarzschild Exterior $\oplus$ Kuch 68 II	$S_{SK} = 2M^2\pi + \frac{\pi(d+\ln[M]-\ln[1+d])}{2C}$ $+\lambda(\frac{M^2\pi^2(d+\ln[M]-\ln[1+d])}{C})$
Schwarzschild Exterior $\oplus$ M-W III	$S_{SM} = 2M^2\pi + \int \frac{dM}{dr}\frac{1}{T}dr$ $+\lambda(2M^2\pi \int \frac{dM}{dr}\frac{1}{T}dr)$
Tolman VI $\oplus$ Kuch 68 II	$S_{TK} = \frac{15\pi(A+Br^8)\tan^{-1}[\sqrt{B/A}r^4]}{4r^3\sqrt{-14AB(A+Br^8)^2/r^6}}$ $+\frac{\pi(d+\ln[M]-\ln[1+d])}{2C}$ $+\lambda(\frac{15\pi^2(A+Br^8)\tan^{-1}[\sqrt{B/A}r^4](d+\ln[M]-\ln[1+d])}{8Cr^3\sqrt{-14AB(A+Br^8)^2/r^6}})$
Tolman VI $\oplus$ M-W III	$S_{TM} = \frac{15\pi(A+Br^8)\tan^{-1}[\sqrt{B/A}r^4]}{4r^3\sqrt{-14AB(A+Br^8)^2/r^6}} + \int \frac{dM}{dr}\frac{1}{T}dr$ $+\lambda(\frac{15\pi(A+Br^8)\tan^{-1}[\sqrt{B/A}r^4]}{4r^3\sqrt{-14AB(A+Br^8)^2/r^6}})(\int \frac{dM}{dr}\frac{1}{T}dr)$
Kuch 68 II $\oplus$ M-W III	$S_{KM} = \frac{\pi(d+\ln[M]-\ln[1+d])}{2C} + \int \frac{dM}{dr}\frac{1}{T}dr$ $+\lambda(\frac{\pi(d+\ln[M]-\ln[1+d])}{2C})(\int \frac{dM}{dr}\frac{1}{T}dr)$

Table 5.5: The nonadditive entropy composition of the black hole systems in Schwarzschild coordinates

We obtained the nonadditive entropy composition of the black hole systems in isotropic coordinates as shown in Table 5.6.

System of Black holes	Entropy Composition
Schwarzschild Exterior $\oplus$ N-P-V Ia	$S_{SN} = \frac{125M(r(4b+3ar)}{18a\pi^2\sqrt{-r^2(b+ar)^3}}$ $+\frac{3(b+ar)\sqrt{-r^2(b+ar)}\ln[b+ar])}{18a\pi^2\sqrt{-r^2(b+ar)^3}}$ $-\lambda(\frac{125M}{24\pi})(\frac{4(r(4b+3ar)}{3a\pi\sqrt{-r^2(b+ar)^3}})$ $+\lambda(\frac{125M}{24\pi})(\frac{3(b+ar)\sqrt{-r^2(b+ar)}\ln[b+ar])}{3a\pi\sqrt{-r^2(b+ar)^3}})$
Schwarzschild Exterior $\oplus$ Burl I	$S_{SB} = \frac{125M}{24\pi} - \frac{4r(1+r^2)^{-q}[(1+4a+2a^2)g}{(1+4a+2a^2)A\pi\sqrt{(1+r^2)^{-2q}/A}}$ $-\frac{(1+6a+2a^2)h+2aj]}{(1+4a+2a^2)A\pi\sqrt{(1+r^2)^{-2q}/A}}$ $-\lambda(\frac{125M}{24\pi})(\frac{4r(1+r^2)^{-q}[(1+4a+2a^2)g}{(1+4a+2a^2)A\pi\sqrt{(1+r^2)^{-2q}/A}})$ $+\lambda(\frac{125M}{24\pi})\frac{(1+6a+2a^2)h+2aj]}{(1+4a+2a^2)A\pi\sqrt{(1+r^2)^{-2q}/A}})$
N-P-V Ia $\oplus$ Burl I	$S_{NB} = -\frac{4(r(4b+3ar)+3(b+ar)\sqrt{-r^2(b+ar)}\ln[b+ar])}{3a\pi\sqrt{-r^2(b+ar)^3}}$ $-\frac{4r(1+r^2)^{-q}[(1+4a+2a^2)g-(1+6a+2a^2)h+2aj]}{(1+4a+2a^2)A\pi\sqrt{(1+r^2)^{-2q}/A}}$ $+\lambda[(\frac{4(r(4b+3ar)+3(b+ar)\sqrt{-r^2(b+ar)}\ln[b+ar])}{3a\pi\sqrt{-r^2(b+ar)^3}})$ $(\frac{4r(1+r^2)^{-q}[(1+4a+2a^2)g-(1+6a+2a^2)h+2aj]}{(1+4a+2a^2)A\pi\sqrt{(1+r^2)^{-2q}/A}})]$

Table 5.6: The nonadditive entropy composition of the black hole systems in isotropic coordinates

where $g ={}_2 F_1[1/2, \frac{2}{1+4a+2a^2}, 3/2, -r^2], h ={}_2 F_1[1/2, \frac{2-a}{1+4a+2a^2}, 3/2, -r^2]$, $j ={}_2 F_1[1/2, \frac{3+3a+2a^2}{1+4a+2a^2}, 3/2, -r^2]$, and $q = \frac{2+4a}{1+4a+2a^2}$.

In chapter V, we calculated the Hawking temperatures and the entropy of black holes. We calculated the Hawking temperature and the entropy of perfect fluid black holes and also calculated the entropy composition of black hole systems. Finally, we plotted the graph to show the entropy of each black hole system by using the radius of one of the black holes in the system, where the result of all entropy composition increases when the radius of the black hole increases.

CHAPTER VI

CONCLUSIONS AND FUTURE WORK

In this chapter, we discuss the conclusion of this thesis and provide some possibilities about future work.

6.1 Conclusions

In chapter I, we studied special and general relativities, and explained the difference between special and general relativities. In this thesis, we are interested in general relativity. The objective of this thesis is to calculate the greybody factors of black holes using general relativity, which describes how matter causes spacetime curvature. This concept has led to the formulation of the Einstein field equation. In chapter II, we studied about the Einstein field equation and the quantities in the Einstein field equation, namely, the Einstein tensor, the stress energy tensor, the metric tensor, the Christoffel symbols, the Ricci curvature tensor, the Riemann curvature tensor, and the Ricci scalar, and further described these quantities in more detail. The Einstein field equation in general relativity is the most important equation to study about perfect fluid black holes in this thesis. In chapter III, we studied the perfect fluid spheres, the properties of perfect fluid spheres, and made use of these properties of the perfect fluid spheres to obtain the constraint that can reduce the complexity of the Einstein field equation. We also considered the perfect fluid sphere in two different coordinates, namely, the Schwarzschild and the isotropic coordinates. Moreover, we studied the generating theorem of perfect fluid spheres and how these theorems can generate new solutions of perfect fluid spheres. Next, we classified black holes in perfect fluid spheres in Schwarzschild coordinates and isotropic coordinates by deriving the radius of the perfect fluid sphere and compared it with the Schwarzschild radius. If the radius of the perfect fluid spheres is lower than the Schwarzschild radius, the perfect fluid sphere becomes a black hole. In this thesis, we obtained four black holes in Schwarzschild coordinates, namely, the Schwarzschild exterior black hole, the Tolman VI black hole, the Kuch 68 II black hole, and the M-W III black hole, and three black holes in isotropic coordinates, namely, the Schwarzschild exterior (isotropic) black hole, the N-P-V Ia black hole, and the Burl I black hole. After that, we calculated the potential of the black hole by deriving the Regge-Wheeler equation from the Klein-Gordon equation. The Regge-Wheeler equation is similar to the Schrödinger equation, which contains the kinetic and potential energies. We derived the potential of the perfect fluid black holes in isotropic coordinates by using the concept of Schwarzschild coordinates. Finally, we plotted the figures of the potentials of perfect fluid black holes to show the spacetime curvature outside the event horizon. After calculating the potentials of the black holes, we used these potential to calculate the greybody factors. In chapter IV, we studied the Bogoliubov coefficients method to obtain the greybody factors of black holes. The coefficients that we obtained are α and β that are used to calculate the transmission and reflection probabilities. Then, we calculated the greybody factors of black holes in Schwarzschild coordinates and isotropic coordinates, and plotted the graphs showing the relation between the transmission and reflection probabilities. The results show that the reflection probability of these three black holes in isotropic coordinates decreases if the wave's energy increases, and the transmission probability of these three black holes in isotropic coordinate increases if the wave's energy increases. In chapter V, we calculated the Hawking temperatures and the entropy of black holes. We calculated the Hawking temperature and the entropy of perfect fluid black hole and also compute the entropy composition of black hole systems. In this thesis, we considered two black holes in a system and these two black holes are in the same coordinates. We divided the entropy composition into two cases, namely, the additive entropy composition and the nonadditive entropy composition. These two cases are different. Additive have $\lambda = 0$ but nonadditive have $0 < \lambda < 1$. Finally, we plotted the graph to show the entropy of each system of black hole by using the radius of one of the black hole systems. The results of all entropy composition increases when the radius of black holes increases.

6.2 Future work

We have made suggestions about future work that can be developed from this thesis as follows.

- We are interested in the power spectra of greybody factors of black holes that is associated to the temperature and the entropy of black holes. We will calculate the power spectra in different black holes and compare the power spectra in these black holes.

- We will calculate the greybody factors of different black holes by using the WKB approximation and the Bogoliubov coefficients, and compare the results of the greybody factors in these two methods for each black hole. We will then analyze the results to determine which method is best to obtain the highly rigorous of the greybody factors.

- We will study the new method that is adiabatic approximation that refers to the Schrödinger equation and we will calculate the greybody factors by using this method.

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APPENDIX

In appendix, we show mathematica code to calculate the radius of perfect fluid spheres and compare the radius of perfect fluid spheres with Schwarzschild radius. Next, we will calculate the potential of black holes. Finally, we calculate the greybody factors, the Hawking temperature, and the entropy of black holes.

APPENDIX A : The radius of MW-III black hole, the potential, and the greybody factors of M-W III black hole mathematica code

In[1]:= $f = \sqrt{A \times r \times (r - a)}$

$P = D[f, r]$

Out[1]= $\sqrt{A\, r\, (-a + r)}$

Out[2]= $\dfrac{A\, r + A\, (-a + r)}{2 \sqrt{A\, r\, (-a + r)}}$

In[3]:= $f2 = f \wedge 2$

Out[3]= $A\, r\, (-a + r)$

In[4]:= $g = \left(\dfrac{\frac{7}{4}}{1 - \frac{r^2}{a^2}}\right)^{-1}$

Out[4]= $\dfrac{4}{7}\left(1 - \dfrac{r^2}{a^2}\right)$

In[5]:= $\text{Solve}\left[\dfrac{-(2 \times g \times r \times P) + f - (f \times g)}{r^2 \times f} == 0,\ r\right]$

Out[5]= $\left\{\left\{r \to -\dfrac{a}{2}\right\}, \left\{r \to \dfrac{a}{6}\right\}\right\}$

In[6]:= $\text{Solve}\left[\left(1 - \dfrac{2\, M}{R}\right) == A\, R\, (R - a),\ M\right]$

Out[6]= $\left\{\left\{M \to -\dfrac{1}{2} R\, (-1 - a\, A\, R + A\, R^2)\right\}\right\}$

In[7]:= $R = \frac{a}{6}$

Out[7]= $\frac{a}{6}$

In[8]:= `a = 1`

Out[8]= 1

In[9]:= `A = 1`

Out[9]= 1

In[10]:= $M = -\frac{1}{2} R \left(-1 - a\, A\, R + A\, R^2\right)$

Out[10]= $\frac{41}{432}$

In[11]:= `R < 2 M`

Out[11]= True

In[15]:= `j = Simplify[Sqrt[g f2]]`

Out[15]= $$\frac{2 \sqrt{-\frac{A (a-r)^2 r (a+r)}{a^2}}}{\sqrt{7}}$$

In[16]:= `V = Simplify[((L (L + 1) (f2)) / (r ^ 2)) + (j (D[j, r]) r ^ -1)]`

Out[16]= $$-\frac{A (a-r) \left(a^2 \left(2+7 L+7 L^2\right)-2 a r-8 r^2\right)}{7 a^2 r}$$

In[17]:= `Simplify[Integrate[V / j, r]]`

Out[17]= $$\frac{A \left(a^2-r^2\right)\left(\sqrt{a}\left(10+7 L+7 L^2\right)\sqrt{\frac{a+r}{a}}-14 \sqrt{r}\,\mathrm{ArcSinh}\left[\frac{\sqrt{r}}{\sqrt{a}}\right]-8 \sqrt{a}\,\mathrm{Hypergeometric2F1}\left[-\frac{3}{2},-\frac{1}{2},\frac{1}{2},-\frac{r}{a}\right]\right)}{\sqrt{7}\, a^{3/2} \sqrt{\frac{a+r}{a}} \sqrt{-\frac{A (a-r)^2 r (a+r)}{a^2}}}$$

APPENDIX B : The radius of N-P-V Ia black hole, the potential, and the grey-body factors of N-P-V Ia black hole mathematica code

```
In[1]:= Z2 = (a r + b)^2 (a r + b)^-2
Out[1]= 1

In[2]:= Z = 1
Out[2]= 1

In[3]:= B2 = (a r + b)^2
Out[3]= (b + a r)^2

In[4]:= B = Sqrt[B2]
Out[4]= Sqrt[(b + a r)^2]

In[5]:= Solve[((D[Z, r])^2 × B2) - ((D[B, r])^2 × Z2) + (2 × B × D[B, r]) × (Z2/r) == 0, r]
Out[5]= {{r → -2 b/a}}

In[6]:= Solve[(a r + b)^-2 == (1 + M/(2 r))^-2, M]
Out[6]= {{M → 2 (-b r - a r^2 - Sqrt[-b r^2 - a r^3])/(b + a r)}, {M → 2 (-b r - a r^2 + Sqrt[-b r^2 - a r^3])/(b + a r)}, {M → 2 (-b r - a r^2 - Sqrt[b r^2 + a r^3])/(b + a r)}, {M →
        2 (-b r - a r^2 + Sqrt[b r^2 + a r^3])/(b + a r)}}
```

In[7]:= $r = -\frac{2\,b}{a}$

Out[7]= $-\frac{2\,b}{a}$

In[8]:= a = 1

Out[8]= 1

In[9]:= b = 1

Out[9]= 1

In[10]:= $M2 = 2 \times \left(\frac{2\left(-b\,r - a\,r^2 - \sqrt{-b\,r^2 - a\,r^3}\right)}{b + a\,r}\right)$

Out[10]= 16

In[11]:= r < M2

Out[11]= True

In[5]:= $\text{Simplify}\left[\left(\frac{l \times (l+1) \times Z2^2 \times B2}{r^2}\right) + \left(r^{-1} \times Z \times Z2 \times B2 \times D\left[Z \times \left(\frac{r \times D[Z2 \times B2, r]}{2 \times Z2 \times B2} - 1\right), r\right]\right)\right]$

Out[5]= $a^2\,l\,(1+l) + \frac{b^2\,l\,(1+l)}{r^2} + \frac{a\,b\,(1+2\,l+2\,l^2)}{r}$

In[6]:= $\text{Simplify}\left[\text{Integrate}\left[\frac{1}{2\,\omega} \times \frac{a^2\,l\,(1+l) + \frac{b^2\,l\,(1+l)}{r^2} + \frac{a\,b\,(1+2\,l+2\,l^2)}{r}}{Z2\;B}, r\right]\right]$

Out[6]= $-\frac{(b+a\,r)\,(b\,l\,(1+l) - a\,(1+l+l^2)\,r\,\text{Log}[r] + a\,r\,\text{Log}[b+a\,r])}{2\,r\,\sqrt{(b+a\,r)^2}\,\omega}$

APPENDIX C : The radius of Tolman VI black hole, the potential, and the greybody factors of Tolman VI black hole mathematica code

In[1]:=
```
f = (A × r^(1-n)) + (B × r^(1+n))
P = D[f, r]
```

Out[1]= $A\, r^{1-n} + B\, r^{1+n}$

Out[2]= $A\,(1-n)\, r^{-n} + B\,(1+n)\, r^{n}$

In[3]:=
```
f2 = f^2
```

Out[3]= $(A\, r^{1-n} + B\, r^{1+n})^2$

In[4]:=
```
g = 1/(2 - n^2)
```

Out[4]= $\frac{1}{2-n^2}$

In[5]:=
```
n = 4
```

Out[5]= 4

In[6]:=
```
Solve[(-(2 × g × r × P) + f - (f × g))/(r^2 × f) == 0, r]
```

Out[6]= $$\left\{\left\{r \to -\frac{(-1)^{1/8} \left(\frac{3}{5}\right)^{1/4} A^{1/8}}{B^{1/8}}\right\}, \left\{r \to \frac{(-1)^{1/8} \left(\frac{3}{5}\right)^{1/4} A^{1/8}}{B^{1/8}}\right\}, \left\{r \to -\frac{(-1)^{3/8} \left(\frac{3}{5}\right)^{1/4} A^{1/8}}{B^{1/8}}\right\}, \left\{r \to \frac{(-1)^{3/8} \left(\frac{3}{5}\right)^{1/4} A^{1/8}}{B^{1/8}}\right\}, \left\{r \to -\frac{(-1)^{5/8} \left(\frac{3}{5}\right)^{1/4} A^{1/8}}{B^{1/8}}\right\}, \left\{r \to \frac{(-1)^{5/8} \left(\frac{3}{5}\right)^{1/4} A^{1/8}}{B^{1/8}}\right\}, \left\{r \to -\frac{(-1)^{7/8} \left(\frac{3}{5}\right)^{1/4} A^{1/8}}{B^{1/8}}\right\}, \left\{r \to \frac{(-1)^{7/8} \left(\frac{3}{5}\right)^{1/4} A^{1/8}}{B^{1/8}}\right\}\right\}$$

In[7]:=
```
Solve[1 - (2 M)/R == 1/(2 - n^2), M]
```

Out[7]= $\left\{\left\{M \to \frac{15\, R}{28}\right\}\right\}$

In[8]:= A = -2

Out[8]= -2

In[9]:= B = -1

Out[9]= -1

In[10]:= $M = \frac{15\,R}{28}$

Out[10]= $\frac{15\,R}{28}$

In[11]:= R < 2 M

Out[11]= $R < \frac{15\,R}{14}$

In[6]:= j = Sqrt[g f2]

Out[6]= $\frac{\sqrt{-\left(\frac{A}{r^3}+B\,r^5\right)^2}}{\sqrt{14}}$

In[7]:= V = Simplify[((L (L+1) (f2)) / (r^2)) + (j (D[j, r]) r^-1)]

Out[7]= $\frac{(A+B\,r^8)\,(A\,(3+14\,L+14\,L^2)+B\,(-5+14\,L+14\,L^2)\,r^8)}{14\,r^8}$

In[8]:= Simplify[Integrate[V / j, r]]

Out[8]= $-\frac{(A+B\,r^8)\,(A\,(3+14\,L+14\,L^2)+B\,(5-14\,L-14\,L^2)\,r^8)}{4\sqrt{14}\,r^7\sqrt{-\frac{(A+B\,r^8)^2}{r^6}}}$

APPENDIX D : The greybody factors, the Hawking temperature, and the entropy of Schwarzschild Exterior (isotropic) black hole mathematica code

In[1]:=

$$Z2 = \text{Simplify}\left[\left(\frac{1-\frac{M}{2r}}{1+\frac{M}{2r}}\right)^2\right]$$

$$Z = \sqrt{Z2}$$

$$B2 = \text{Simplify}\left[\frac{1}{Z2\left(1+\frac{M}{2r}\right)^4}\right]$$

$$B = \text{Sqrt}[B2]$$

$$\text{Simplify}\left[\left(\frac{l \times (l+1) \times Z2^2 \times B2}{r^2}\right)+\left(r^{-1} \times Z \times Z2 \times B2 \times D\left[Z \times \left(\frac{r \times D[Z2 \times B2, r]}{2 \times Z2 \times B2}-1\right), r\right]\right)\right]$$

Out[1]=

$$\frac{1024\,(M-2r)^4\, r^6 \sqrt{\frac{(M-2r)^2}{(M+2r)^2}}\sqrt{\frac{r^4}{(M^2-4r^2)^2}}\left(-8Mr+l\,(M+2r)^2+l^2\,(M+2r)^2\right)}{(M+2r)^{10}\,(M^2-4r^2)^2}$$

In[2]:=

$$\text{Simplify}\left[\frac{16\,(M-2r)^2\, r^2\left(-8Mr+l\,(M+2r)^2+l^2\,(M+2r)^2\right)}{(M+2r)^8}\right]$$

Out[2]=

$$\frac{16\,(M-2r)^2\, r^2\left(-8Mr+l\,(M+2r)^2+l^2\,(M+2r)^2\right)}{(M+2r)^8}$$

In[3]:=

$$\text{Simplify}\left[\text{Integrate}\left[\frac{1}{2\,\omega} \times \frac{\frac{16\,(M-2r)^2\, r^2\left(-8Mr+l\,(M+2r)^2+l^2\,(M+2r)^2\right)}{(M+2r)^8}}{Z2\,B}, r\right]\right]$$

Out[3]=

$$\frac{2\,r^3\left(-4Mr+l\,(M+2r)^2+l^2\,(M+2r)^2\right)}{(M-2r)\,(M+2r)^5\sqrt{\frac{r^4}{(M^2-4r^2)^2}}\;\omega}$$

In[5]:=

$$T = \text{Simplify}[D[Z2\,B, r]/(2\,pi)]$$

Out[5]=

$$\frac{4M\,(M-4r)\,(M-2r)\sqrt{\frac{r^4}{(M^2-4r^2)^2}}}{pi\; r\,(M+2r)^3}$$

In[7]:=

$$S = \text{Integrate}[1/T, M]$$

Out[7]=

$$\frac{pi\sqrt{\frac{r^4}{(M^2-4r^2)^2}}\,(-M^2+4r^2)\left(-M\,(M^2+18Mr+216r^2)+12\,r^3\,\text{Log}[M]-972\,r^3\,\text{Log}[M-4r]\right)}{12\,r^3}$$

APPENDIX E : The greybody factors, the Hawking temperature, and the entropy of Burl I black hole mathematica code

In[1]:= $Z2 = A\,(1+r^2)^{\frac{4\,(a+1)}{2\,a^2+4\,a+1}}$

Out[1]= $A\,(1+r^2)^{\frac{4\,(1+a)}{1+4\,a+2\,a^2}}$

In[2]:= $Z = \sqrt{Z2}$

Out[2]= $\sqrt{A\,(1+r^2)^{\frac{4\,(1+a)}{1+4\,a+2\,a^2}}}$

In[3]:= $\sqrt{A\,(1+r^2)^{\frac{4\,(1+a)}{1+4\,a+2\,a^2}}}$

Out[3]= $\sqrt{A\,(1+r^2)^{\frac{4\,(1+a)}{1+4\,a+2\,a^2}}}$

In[6]:= $B2 = \text{Simplify}\left[\frac{1}{Z2\,(1+r^2)^{\frac{4\,a}{2\,a^2+4\,a+1}}}\right]$

Out[6]= $\frac{(1+r^2)^{-\frac{4+8\,a}{1+4\,a+2\,a^2}}}{A}$

In[7]:= $B = \sqrt{B2}$

Out[7]= $\sqrt{\frac{(1+r^2)^{-\frac{4+8\,a}{1+4\,a+2\,a^2}}}{A}}$

In[8]:= $\text{Solve}\left[\left((D[Z, r])^2 \times B2\right) - \left((D[B, r])^2 \times Z2\right) + (2 \times B \times D[B, r])\left(\frac{Z2}{r}\right) == 0,\ r\right]$

Out[8]= $\left\{\left\{r \to -\frac{\sqrt{-1-6\,a-10\,a^2-4\,a^3}}{\sqrt{1+10\,a+16\,a^2+4\,a^3}}\right\}, \left\{r \to \frac{\sqrt{-1-6\,a-10\,a^2-4\,a^3}}{\sqrt{1+10\,a+16\,a^2+4\,a^3}}\right\}\right\}$

In[ ]:= $\text{Solve}\left[(1+r^2)^{\frac{4\,a}{2\,a^2+4\,a+1}} == \left(1+\frac{M}{2\,r}\right)^4,\ M\right]$

Out[ ]= $\left\{\left\{M \to 2\,r\left(-1+(1+r^2)^{\frac{a}{1+4\,a+2\,a^2}}\right)\right\}, \left\{M \to -2\,i\,r\left(-i+(1+r^2)^{\frac{a}{1+4\,a+2\,a^2}}\right)\right\}, \left\{M \to 2\,i\,r\left(i+(1+r^2)^{\frac{a}{1+4\,a+2\,a^2}}\right)\right\}, \left\{M \to -2\,r\left(1+(1+r^2)^{\frac{a}{1+4\,a+2\,a^2}}\right)\right\}\right\}$

In[9]:= Simplify[$\left(\frac{l \times (l+1) \times Z2^2 \times B2}{r^2}\right)+\left(r^{-1} \times Z \times Z2 \times B2 \times D\left[Z \times \left(\frac{r \times D[Z2 \times B2, r]}{2 \times Z2 \times B2}-1\right), r\right]\right)$]

Out[9]= $\frac{A (1+r^2)^{-2+\frac{4}{1+4a+2a^2}} (-16 a r^4 - 16 a^2 r^4 + 8 a (1+4a+2a^2) r^4 - 4 (1+4a+2a^2) r^2 (1+r^2) - 12 a (1+4a+2a^2) r^2 (1+r^2) +}{(1+4a+2a^2)^2 r^2}$

Simplify[$\frac{1}{k^2 r^2} A (1+r^2)^{-2+\frac{4}{k}} [(-16 a - 16 a^2 + 8 a k) r^4 - (4 k r^2 + 12 a k) r^2 (1+r^2) + (k^2 l + k^2 l^2)(1+r^2)^2]$]

In[•]:= Simplify[$\frac{A (1+r^2)^{-2+\frac{4}{k}} [-8 a (2+2a-k) r^4 + 4 (3 a k - k r^2) r^2 (1+r^2) + k^2 l (1+l) (1+r^2)^2]}{k^2 r^2}$]

Out[•]= $\frac{A (1+r^2)^{-2+\frac{4}{k}} [-8 a (2+2a-k) r^4 + 4 (3 a k - k r^2) r^2 (1+r^2) + k^2 l (1+l) (1+r^2)^2]}{k^2 r^2}$

$\frac{A (1+r^2)^{-2+\frac{4}{k}} [-8 a (2+2a-k) r^4 - 4 (3 a k + k r^2) r^2 (1+r^2) + k^2 l (1+l) (1+r^2)^2]}{k^2 r^2}$

In[•]:= a = -2

Out[•]= -2

In[10]:= Simplify[Integrate[$\frac{1}{2\omega} \times \frac{1}{Z2\,B} \frac{1}{(1+4a+2a^2)^2 r^2} A (1+r^2)^{-2+\frac{4}{1+4a+2a^2}} (-16 a r^4 - 16 a^2 r^4 + 8 a (1+4a+2a^2) r^4$

$a+2a^2) r^2 (1+r^2) - 12 a (1+4a+2a^2) r^2 (1+r^2) + (1+4a+2a^2)^2 l (1+r^2)^2 + (1+4a+2a^2)^2 l^2 (1+r^2$

Out[10]= $\frac{(1+r^2)^{-\frac{2+4a}{1+4a+2a^2}} \left(-(1+4a+2a^2)^2 l (1+l) \text{Hypergeometric2F1}\left[-\frac{1}{2}, -\frac{2}{1+4a+2a^2}, \frac{1}{2}, -r^2\right] - 4 r^2 \left((1+9a+10a^2+2\right.\right.}{2 (1+4a+2a^2)^2}$

In[15]:= T = Simplify[D[Z2 B] / (2 pi)]

Out[15]= $\frac{A (1+r^2)^{\frac{4(1+a)}{1+4a+2a^2}} \sqrt{\frac{(1+r^2)^{-\frac{4+8a}{1+4a+2a^2}}}{A}}}{2\,pi}$

In[16]:= Q = Simplify[$D\left[2 r \left(-1 + (1+r^2)^{\frac{a}{1+4a+2a^2}}\right), r\right]$]

Out[16]= $-2 + 2 (1+r^2)^{\frac{a}{1+4a+2a^2}} + \frac{4 a r^2 (1+r^2)^{-1+\frac{a}{1+4a+2a^2}}}{1+4a+2a^2}$

In[17]:= Integrate[Q / T, r]

Out[17]= $-\frac{4\,pi\,r (1+r^2)^{-\frac{2+4a}{1+4a+2a^2}} \left((1+4a+2a^2) \text{Hypergeometric2F1}\left[\frac{1}{2}, \frac{2}{1+4a+2a^2}, \frac{3}{2}, -r^2\right] - (1+6a+2a^2) \text{Hypergeometri}\right.}{(1+4a+2a^2) A \sqrt{\frac{(1+r^2)^{-\frac{4+8a}{1+4a+2a^2}}}{A}}}$

APPENDIX F : The entropy composition of black hole systems, namely, Schwarzschild Exterior black hole ⊕ Tolman VI black hole, and Schwarzschild Exterior black hole ⊕ Kuch 68 II black hole mathematica code

```
r = r1 - 5
A = B = 1
pi = Pi
S1 = (Pi r1^2) / 2
```

Out[1]= $-5 + r1$

Out[2]= 1

Out[3]= π

Out[4]= $\frac{\pi\, r1^2}{2}$

In[5]:= $S2 = \frac{15\, pi\, (A + B\, r^8)\, \text{ArcTan}\left[\frac{\sqrt{B}\, r^4}{\sqrt{A}}\right]}{4\sqrt{14}\sqrt{A}\sqrt{B}\, r^3 \sqrt{-\frac{(A+B\, r^8)^2}{r^6}}}$

Out[5]= $\frac{15\pi\, (1 + (-5 + r1)^8)\, \text{ArcTan}[(-5 + r1)^4]}{4\sqrt{14}\sqrt{-\frac{(1+(-5+r1)^8)^2}{(-5+r1)^6}}\, (-5 + r1)^3}$

```
In[6]:= Plot[{S1 + S2}, {r1, 5, 100}, PlotLabels → {S_ST}, AxesLabel → {r1}]
```

```
In[7]:= A = 1
d = Sqrt[1 + 4 A r^2]
r = r1 - 5
S1 = (Pi r1^2) / 2
S3 = (Pi / (2 A)) (d + Log[r] - Log[1 + d])
```

Out[7]= 1

Out[8]= $\sqrt{1 + 4\, (-5 + r1)^2}$

Out[9]= $-5 + r1$

Out[10]= $\frac{\pi\, r1^2}{2}$

Out[11]= $\frac{1}{2}\pi \left(\sqrt{1 + 4\, (-5 + r1)^2} - \text{Log}\left[1 + \sqrt{1 + 4\, (-5 + r1)^2}\right] + \text{Log}[-5 + r1]\right)$

```
In[12]:= Plot[{S1 + S3}, {r1, 4, 10}, PlotLabels → {S_SK}, AxesLabel → {r1}]
```

BIOGRAPHY

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