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GRAPH GRABBING GAMES AND TOUCHER-ISOLATOR GAMES

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ในงานวิจัยนี้ เราศึกษาเกมการคว้ากราฟและเกมทัชเชอร์-ไอโซเลเทอร์ ในเกมการคว้ากราฟ เราตอบปัญหาบางส่วนของข้อคาดการณ์ของ Seacrest และ Seacrest ซึ่งกล่าวว่า อลิซชนะเกมบนกราฟคู่สองส่วนเชื่อมโยงถ่วงน้ำหนักทุกกราฟ ในเกมทัชเชอร์-ไอโซเลเทอร์ เราให้บทพิสูจน์ใหม่อย่างง่ายของผลลัพธ์ของ Räty ซึ่งหากราฟต้นไม้ n จุดยอดที่เหมาะสมที่สุดสำหรับทัชเชอร์ ซึ่งตอบคำถามของ Dowden, Kang, Mikalački และ Stojaković

ภาควิชา คณิตศาสตร์และวิทยาการคอมพิวเตอร์ ลายมือชื่อนิสิต ....

สาขาวิชา คณิตศาสตร์ ลายมือชื่อ อ.ที่ปรึกษาหลัก ....

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In this research, we study the graph grabbing game and the Toucher-Isolator game. In the graph grabbing game, we partially confirm a conjecture of Seacrest and Seacrest which states that Alice wins the game on every weighted connected bipartite even graph. In the Toucher-Isolator game, we give a simple alternative proof of a result of Räty that determines the most suitable tree on n vertices for Toucher which answers a question of Dowden, Kang, Mikalački and Stojaković.

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CHAPTER I
INTRODUCTION

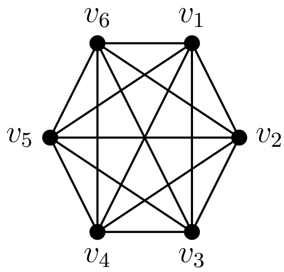
In this dissertation, we study two combinatorial games on simple graphs, namely, the graph grabbing game and the Toucher-Isolator game. We first recall some basic definitions in graph theory which will be used for this dissertation and we then talk about combinatorial game theory.

1.1 Graph Theory

This dissertation follows most of basic graph theory terminology from a textbook of West [33] and a textbook of Bondy and Murty [2].

A *graph* G is a pair of a *vertex set* $V(G)$ of G and an *edge set* $E(G)$, a collection of 2-subsets of $V(G)$, of G. A *subgraph* H of a graph G is a graph such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. An element in $V(G)$ (resp. $E(G)$) is called a *vertex* (resp. an *edge*) of G. The vertices u and v are *adjacent* in G if and only if $\{u, v\} \in E(G)$. For convenience, we write uv for $\{u, v\}$. For a vertex $v \in V(G)$, a vertex $u \in V(G)$ is a *neighbor* of v if and only if $uv \in E(G)$. For a graph G and a set $S \subseteq V(G)$, let $N_G(S)$ denote the *neighborhood* of S, i.e., the set of vertices having a neighbor in S and we write $N_G(v)$ for $N_G(\{v\})$. For a vertex $v \in V(G)$, the *degree* of v is $|N_G(v)|$, denoted by $\deg(v)$. A graph G is *even* (resp. *odd*) if $|V(G)|$ is even (resp. odd).

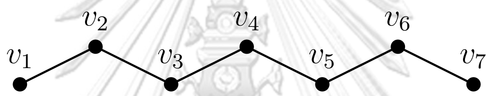
The *complete graph* K_n on n vertices is a graph on n vertices in which any two vertices are adjacent. That is, $V(K_n) = \{v_1, v_2, v_3, \ldots, v_n\}$ and $E(K_n) = \{v_i v_j : 1 \leq i < j \leq n\}$.



K_6

Figure 1.1: The complete graph K_6.

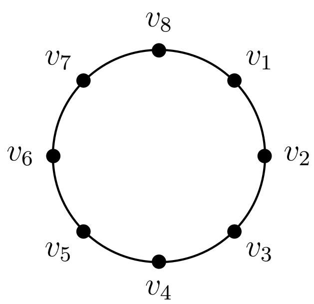
The *path* P_n on n vertices is a graph on n vertices whose vertices can be arranged in a line such that two vertices are adjacent if and only if they are consecutive in the line. That is, $V(P_n) = \{v_1, v_2, v_3, \ldots, v_n\}$ and $E(P_n) = \{v_i v_{i+1} : 1 \leq i \leq n-1\}$.



P_7

Figure 1.2: The path P_7.

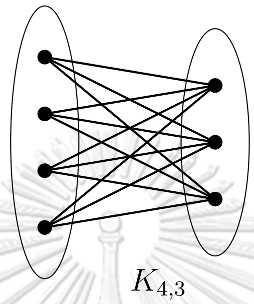
The *cycle* C_n on n vertices is a graph on n vertices whose vertices can be arranged in a circle such that two vertices are adjacent if and only if they are consecutive in the circle. That is, $V(C_n) = \{v_1, v_2, v_3, \ldots, v_n\}$ and $E(C_n) = \{v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{v_n v_1\}$.



C_8

Figure 1.3: The cycle C_8.

A *bipartite graph* G with partite classes X and Y is a graph whose vertex set $V(G)$ can be partitioned into two subsets X and Y and there is no edge having both endpoints in the same class, i.e., $E(G) \subseteq \{xy : x \in X, y \in Y\}$. The *complete bipartite graph* $K_{m,n}$ is a bipartite graph with $|X| = m$, $|Y| = n$ and two vertices are adjacent if and only if they are in different classes, i.e., $E(G) = \{xy : x \in X, y \in Y\}$.



$K_{4,3}$

Figure 1.4: The complete bipartite graph $K_{4,3}$.

For a vertex $v \in V(G)$ and a subset $S \subseteq V(G)$, we write $G - v$ (resp. $G - S$) for the subgraph obtained by deleting the vertex v (resp. the set S). A graph G is *connected* if for any $x, y \in V(G)$ there is a path from x to y; otherwise G is *disconnected*. A vertex v of a connected graph G is a *cut vertex* if $G - v$ is disconnected.

A *forest* is a graph with no cycle. A *tree* is a connected graph with no cycle.

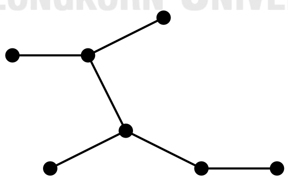


Figure 1.5: A tree.

A *weighted graph* G is a graph G with a weighted function $w : V(G) \rightarrow \mathbb{R}^{+} \cup \{0\}$.

Unless stated otherwise, $[k]$ means the set of the natural numbers from one to k.

1.2 Combinatorial Game Theory

As defined in a textbook of Seigel [31], a *combinatorial game* is a two-player game with perfect information and no chance elements, such as a dice, shuffled cards, or a roulette. This includes well-known games such as the Tic-Tac-Toe game, the Dots and Boxes game, the chess game, the checkers game and the Go game.

Recently, there are many research studies about combinatorial game on graphs For example, the Maker-Breaker games (see [13, 14, 17, 18]), the cop and robber games (see [4, 22, 32]) and the graph coloring games (see [3, 5, 7]).

In this dissertation, we study two combinatorial games, i.e., the graph grabbing game and the Toucher-Isolator game. The graph grabbing game is played on a non-negatively weighted connected graph by Alice and Bob who alternately claim a non-cut vertex from the remaining graph, where Alice plays first, to maximize the weights on their respective claimed vertices. Seacrest and Seacrest [30] conjectured that Alice can secure at least half of the total weight of every weighted connected bipartite even graph. Later, Egawa, Enomoto and Matsumoto [10] partially confirmed this conjecture by showing that Alice wins the game on a class of weighted connected bipartite even graphs called $K_{m,n}$-trees. We extend the result on this class to include a number of graphs, e.g. even blow-ups of trees and cycles.

In the Toucher-Isolator game, introduced recently by Dowden, Kang, Mikalački and Stojaković [9], Toucher and Isolator alternately claim an edge from a graph such that Toucher aims to touch as many vertices as possible, while Isolator aims to isolate as many vertices as possible, where Toucher plays first. Among trees with n vertices, they showed that the star is the best choice for Isolator and they asked for the most suitable tree for Toucher. Later, Räty [28] showed that the answer is the path with n vertices. We give a simple alternative proof of this result. The method to determine where Isolator should play is by breaking down the gains and losses in each move of both players.

CHAPTER II
GRAPH GRABBING GAMES

2.1 Introduction

The *graph grabbing game* is played on a non-negatively weighted connected graph by two players: Alice and Bob alternately claim a non-cut vertex from the remaining graph and collect the weight on the vertex, where Alice plays first. The aim of each player is to maximize the weights on their respective claimed vertices at the end of the game when all vertices have been claimed. Alice *wins* the game if she gains at least half of the total weight of the graph.

The first version of the graph grabbing game appeared in the first problem in Winkler's puzzle book (2003) [34], where he gave a winning strategy for Alice on every weighted even path and he observed that there is a weighted odd path on which Alice cannot win. In 2009, Rosenfeld [29] proposed the game for trees and call it the *gold grabbing game*. In 2011, Micek and Walczak [24] generalized the game to general graphs and call it the graph grabbing game. They showed that Alice can secure at least a quarter of the total weight of every weighted even tree and they conjectured that Alice can in fact secure at least half of the total weight of every weighted even tree. Later in 2012, Seacrest and Seacrest [30] solved this conjecture by considering a vertex-rooted version of the game and they posed the following conjecture.

Conjecture 2.1 ([30]). *Alice wins the game on every weighted connected bipartite even graph.*

In 2018, Egawa, Enomoto and Matsumoto [10] gave a supporting evidence for this conjecture. They generalized the proof of Seacrest and Seacrest by considering a set-rooted version of the game to prove that Alice wins the game on every weighted even $K_{m,n}$-*tree*, namely a bipartite graph obtained from a complete bipartite graph $K_{m,n}$ on $[m+n]$ and trees $T_1, T_2, T_3, \ldots, T_{m+n}$ by identifying vertex i of $K_{m,n}$ with exactly one vertex of T_i for each $i \in [m+n]$.

For a graph G with vertices $v_1, v_2, v_3, \ldots, v_k$ and non-empty sets $V_1, V_2, V_3, \ldots, V_k$, a *blow-up* $\mathrm{B}(G)$ of G is a graph obtained from G by replacing $v_1, v_2, v_3, \ldots, v_k$ with $V_1, V_2, V_3, \ldots, V_k$, respectively where, for each $i, j \in [k]$, vertices $x \in V_i$ and $y \in V_j$ are adjacent in $\mathrm{B}(G)$ if and only if v_i and v_j are adjacent in G. For a graph G on $[k]$ and trees $T_1, T_2, T_3, \ldots, T_k$, a G-*tree* is a graph obtained from G by identifying vertex i of G with exactly one vertex of T_i for each $i \in [k]$. For a tree T, we note that a $\mathrm{B}(T)$-tree and $\mathrm{B}(C_{2n})$ are connected bipartite graphs, and a $\mathrm{B}(T)$-tree is a $K_{m,n}$-tree when T is the path on two vertices, (see Figure 2.1).

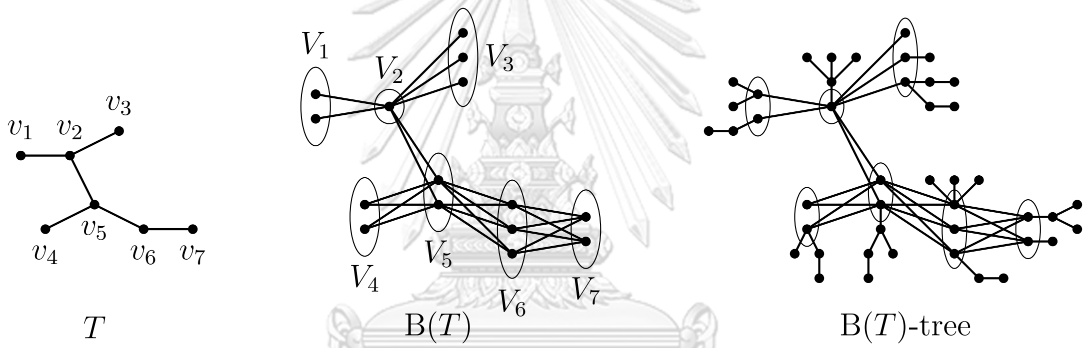


Figure 2.1: Examples of a tree T, a blow-up $\mathrm{B}(T)$ and a $\mathrm{B}(T)$-tree.

In this chapter, we partially confirm Conjecture 2.1 as follows.

Theorem 2.2. *Alice wins the game on every weighted even B(T)-tree, where T is a tree.*

Corollary 2.3. *Alice wins the game on every weighted even B(C_n).*

The proof is based on the method of Egawa, Enomoto and Matsumoto [10], where their main lemmas dealt with the score of the game on a $K_{m,n}$-tree rooted at a partite class. We generalize their method by considering instead the scores of the game on an H-tree rooted at V_i and the game on the H-tree rooted at $N_H(V_i)$, where H is a blow-up of a tree.

The rest of this chapter is organized as follows. In Section 2.2, we recall some observations and a lemma on $K_{m,n}$-trees given by Egawa, Enomoto and Matsumoto [10]. Section 2.3 is devoted to proving Theorem 2.2 and then applying it to prove Corollary 2.3. In Section 2.4, we give some concluding remarks.

2.2 Preliminaries

In this section, we prepare some observations and a lemma on $K_{m,n}$-trees which will be useful for the proof of Theorem 2.2.

We first give definitions of a rooted version of the graph grabbing game and some related terms introduced by Egawa, Enomoto and Matsumoto. For a weighted graph G, a *root set* S of G is a set of vertices intersecting every component of G and the *game on* G *rooted at* S is a graph grabbing game, where each player does not have to claim a non-cut vertex, but instead they claim a vertex v such that every component of $G - v$ contains at least one vertex in S. Therefore, a move v in the game on G is *feasible* if $G - v$ is connected, and a move v in the game on G rooted at S is *feasible* if every component of $G - v$ contains at least one vertex in S. A move v in the game on G (rooted at S) is *optimal* if there is an optimal strategy in the game on G (rooted at S) having v as the first move. The first (resp. second) player is called *Player* 1 (resp. *Player* 2). The last (resp. second from last player) is called *Player* -1 (resp. *Player* -2). For $k \in \{1, 2, -1, -2\}$, assuming that both players play optimally, let $N(G, k)$ denote the score of Player k in the game on G and let $R(G, S, k)$ denote the score of Player k in the game on G rooted at S and we write $R(G, v, k)$ for $R(G, \{v\}, k)$. For a set S and an element x, we write $S - x$ for $S \setminus \{x\}$.

Egawa, Enomoto and Matsumoto [10] observed some relationships between the scores of both players in the normal version and the rooted version of the game. Note that the equation/inequality in the brackets in each observation is an equivalent form of the first one because of the fact that, assuming that both players play optimally, the sum of their scores equals the total weight of the graph.

Observation 2.4 ([10]). *If x is a feasible move in the game on G, then*

$$N(G,2) \leq N(G-x,1) \quad (\Leftrightarrow N(G,1) \geq N(G-x,2)+w(x)).$$

If x is an optimal move in the game on G, then

$$N(G,2) = N(G-x,1) \quad (\Leftrightarrow N(G,1) = N(G-x,2)+w(x)).$$

Observation 2.5 ([10]). *Let S be a root set of G. If x is a feasible move in the game on G rooted at S, then*

$$R(G,S,2) \leq R(G-x,S-x,1) \quad (\Leftrightarrow R(G,S,1) \geq R(G-x,S-x,2)+w(x)).$$

If x is an optimal move in the game on G rooted at S, then

$$R(G,S,2) = R(G-x,S-x,1) \quad (\Leftrightarrow R(G,S,1) = R(G-x,S-x,2)+w(x)).$$

Observation 2.6 ([10]). *If v is a root of G, then*

$$R(G,v,-2) = R(G-v,N_G(v),-1)(\Leftrightarrow R(G,v,-1) = R(G-v,N_G(v),-2)+w(v)).$$

The next lemma is a part of their main results which will help us in the proof.

Lemma 2.7 ([10]). *Let G be a $K_{m,n}$-tree with partite classes X, Y of size $m, n \geq 1$, respectively. Then*

$$R(G,Y,-2) \leq N(G,-2) \quad (\Leftrightarrow R(G,Y,-1) \geq N(G,-1)).$$

2.3 Proofs of Theorem 2.2 and Corollary 2.3

In this section, we start by proving Lemma 2.8 which will be used repeatedly in the proof of our main lemmas, namely, Lemmas 2.9 and 2.10. We then prove Theorem 2.2 by applying the main lemmas and deduce Corollary 2.3 from Theorem 2.2.

The following lemma shows the relationship between the scores of both players in the game on an even graph rooted at two different sets of some structure.

Lemma 2.8. *Let G_1 and G_2 be subgraphs of an even graph G such that $V(G_1)$ and $V(G_2)$ partition $V(G)$. If $U_1 = V(G_1) \cap N_G(V(G_2))$ and $U_2 = V(G_2) \cap N_G(V(G_1))$ are root sets of G_1 and G_2, respectively, and every vertex in U_1 is joined to every vertex in U_2, (see Figure 2.2), then*

2.8.1 $R(G, U_1, 1) \geq R(G_1, U_1, -2) + R(G_2, U_2, -1)$.

2.8.2 $R(G, U_1, 1) \geq R(G, U_2, 2)$.

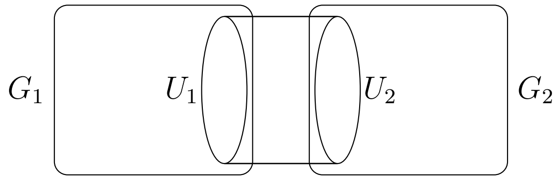


Figure 2.2: The graph G in Lemma 2.8.

Proof. First, we shall prove Lemma 2.8.1 by considering a strategy for Alice who plays first in the game on G rooted at U_1. She plays optimally as Player -2 in the game on G_1 rooted at U_1 and plays optimally as Player -1 in the game on G_2 rooted at U_2. Since $|V(G_1)| + |V(G_2)|$ is even, she plays as Player 1 in one game and as Player 2 in the other. Now, we check that Alice's moves are feasible in the game on G rooted at U_1, and Bob's moves are feasible in the game on G_1 rooted at U_1 and the game on G_2 rooted at U_2. Indeed, after each move of Alice, every remaining component of G_1 and G_2 contains a vertex in U_1 and U_2, respectively. Together with the fact that every vertex in U_2 is joined to the remaining subset of U_1, we can conclude that every remaining component of G contains a vertex in U_1. That is, her moves are feasible in the game on G rooted at U_1. On the other hand, after each move of Bob, every remaining component of G contains a vertex of U_1. Since the edges between G_1 and G_2 have endpoints only in U_1 and U_2, every remaining component of G_1 or G_2 contains a vertex in U_1 or U_2, respectively. That is, his moves are feasible in the game on G_1 rooted at U_1 and the game on G_2 rooted at U_2. Hence

$$R(G, U_1, 1) \geq R(G_1, U_1, -2) + R(G_2, U_2, -1),$$

which completes the proof of Lemma 2.8.1. By symmetry, we have

$$R(G, U_2, 1) \geq R(G_1, U_1, -1) + R(G_2, U_2, -2),$$

which is equivalent to

$$R(G, U_2, 2) \le R(G_1, U_1, -2) + R(G_2, U_2, -1),$$

by considering the total weight of G, G_1 and G_2. Together with Lemma 2.8.1, we have

$$R(G, U_2, 2) \le R(G_1, U_1, -2) + R(G_2, U_2, -1) \le R(G, U_1, 1),$$

which completes the proof of Lemma 2.8.2. □

We are now ready to prove the main lemmas which generalize the results on $K_{m,n}$-trees to B(T)-trees relating the scores of both players in the normal version and the rooted version of the game.

Lemma 2.9. *Let H be a blow-up graph of a tree with sets of vertices $V_1, V_2, V_3, \ldots, V_k$ and let G be an H-tree.*

2.9.1 For a vertex $v \in V(G)$, $R(G, v, -2) \le N(G, -2)$

$$(\Leftrightarrow R(G, v, -1) \ge N(G, -1)).$$

2.9.2 For each $i \in [k]$, $R(G, V_i, -2) \le N(G, -2)$

$$(\Leftrightarrow R(G, V_i, -1) \ge N(G, -1)).$$

2.9.3 For each $i \in [k]$, $R(G, N_H(V_i), -2) \le N(G, -2)$

$$(\Leftrightarrow R(G, N_H(V_i), -1) \ge N(G, -1)).$$

Lemma 2.10. *Let H be a blow-up graph of a tree with sets of vertices $V_1, V_2, V_3, \ldots, V_k$ and let G be an even H-tree.*

2.10.1 For a vertex $v \in V(G)$, $R(G, v, 1) \ge N(G, 2)$

$$(\Leftrightarrow R(G, v, 2) \le N(G, 1)).$$

2.10.2 For each $i \in [k]$, $R(G, V_i, 1) \ge N(G, 2)$

$$(\Leftrightarrow R(G, V_i, 2) \le N(G, 1)).$$

2.10.3 For each $i \in [k]$, $R(G, N_H(V_i), 1) \geq N(G, 2)$

$$(\Leftrightarrow R(G, N_H(V_i), 2) \leq N(G, 1)).$$

We prove Lemmas 2.9 and 2.10 simultaneously by induction on $n = |V(G)|$. It is easy to check that Lemmas 2.9 and 2.10 hold for $n \leq 2$. Now, we let $n \geq 3$ and suppose that Lemmas 2.9 and 2.10 hold for $|V(G)| < n$. We remark that the following fact will be used throughout the proofs: Let G be an H-tree, where H is a blow-up of a tree and let v be a vertex in G. Then $G - v$ is an H'-tree, where H' is a blow-up of some tree if and only if $G - v$ is connected.

Proof of Lemma 2.9.1. Let $v \in V(G)$.

Case 1. G is even.

Let a be an optimal move in the game on G rooted at v. Therefore, $a \neq v$ and a is feasible in the game on G. Thus $G - a$ is connected. Then

$$\begin{aligned} R(G, v, -1 = 2) &= R(G - a, v, 1 = -1) && \text{(Observation 2.5)} \\ &\geq N(G - a, -1 = 1) && \text{(Lemma 2.9.1 by induction)} \\ &\geq N(G, 2 = -1) && \text{(Observation 2.4).} \end{aligned}$$

Case 2. G is odd.

Let b be an optimal move in the game on G. Thus $G - b$ is connected.

Case 2.1. $b \neq v$.

Now, b is a feasible move in the game on G rooted at v. Then

$$\begin{aligned} R(G, v, -2 = 2) &\leq R(G - b, v, 1 = -2) && \text{(Observation 2.5)} \\ &\leq N(G - b, -2 = 1) && \text{(Lemma 2.9.1 by induction)} \\ &= N(G, 2 = -2) && \text{(Observation 2.4).} \end{aligned}$$

Case 2.2. $b = v$ and v is a leaf.

Let u be the unique neighbor of v. Then

$$\begin{aligned} R(G, v, -2) &= R(G - v, u, -1 = 2) && \text{(Observation 2.6)} \\ &\leq N(G - v, 1) && \text{(Lemma 2.10.1 by induction)} \\ &= N(G, 2 = -2) && \text{(Observation 2.4 and } b = v\text{).} \end{aligned}$$

Case 2.3. $b = v$ and v is not a leaf.

Therefore, $v \in V_i$ for some $i \in [k]$ and $N_G(v) = N_H(V_i)$. Then

$$\begin{aligned} R(G, v, -2) &= R(G - v, N_G(v) = N_H(V_i), -1 = 2) && \text{(Observation 2.6)} \\ &\leq N(G - v, 1) && \text{(Lemma 2.10.3 by induction)} \\ &= N(G, 2 = -2) && \text{(Observation 2.4 and } b = v\text{).}\square \end{aligned}$$

Proof of Lemma 2.9.2. Let $i \in [k]$. If $|V_i| = 1$, then we are done by Lemma 2.9.1. Now, suppose that $|V_i| \geq 2$.

Case 1. G is odd.

Let b be an optimal move in the game on G. Thus $G - b$ is connected. Since $|V_i| \geq 2$, we have $V_i - b \neq \varnothing$. Therefore, b is a feasible move in the game on G rooted at V_i. Then

$$\begin{aligned} N(G, -2 = 2) &= N(G - b, 1 = -2) && \text{(Observation 2.4)} \\ &\geq R(G - b, V_i - b, -2 = 1) && \text{(Lemma 2.9.2 by induction)} \\ &\geq R(G, V_i, 2 = -2) && \text{(Observation 2.5).} \end{aligned}$$

Case 2. G is even.

Let a be an optimal move in the game on G rooted at V_i.

Case 2.1. a is a feasible move in the game on G.

Thus $G - a$ is connected. Then

$$R(G, V_i, -1 = 2) = R(G - a, V_i - a, 1 = -1) \qquad \text{(Observation 2.5)}$$

$$\geq N(G - a, -1 = 1) \qquad \text{(Lemma 2.9.2 by induction)}$$
$$\geq N(G, 2 = -1) \qquad \text{(Observation 2.4).}$$

Case 2.2. a is not a feasible move in the game on G.

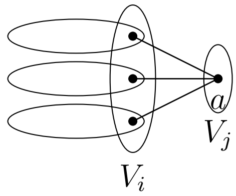


Figure 2.3: The graph G in Case 2.2 of Lemma 2.9.2.

Thus $G - a$ is disconnected. Since a is a feasible move in the game on G rooted at V_i, we have $a \in V_j$ for some $j \in [k]$ and $N_G(V_j) = N_H(V_j)$. Since $G - a$ is disconnected, $V_j = \{a\}$ and a is not a leaf. Suppose that $i = j$. Then every component of $G - a$ does not contain a vertex in V_i, a contradiction. Hence $i \neq j$. Suppose that there is a vertex set V_ℓ, where $\ell \notin \{i, j\}$. Then either $G - a$ is connected or there is a component of $G - a$ which does not contain a vertex in V_i, a contradiction. Hence $V_j = \{a\}$ for some $j \neq i$, $N_H(V_j) = V_i$ and $N_H(V_i) = V_j$, (see Figure 2.3). Therefore, G is a $K_{m,n}$-tree with partite classes V_i and V_j. Then, by Lemma 2.7,

$$N(G, -1) \leq R(G, V_i, -1). \qquad \square$$

Proof of Lemma 2.9.3. We remark that the proofs of Lemmas 2.9.1 and 2.9.2 do not use Lemma 2.9.3. Let $i \in [k]$. If $|N_H(V_i)| = 1$ or $N_H(V_i) = V_j$ for some $j \in [k]$, then we are done by Lemmas 2.9.1 or 2.9.2, respectively. Now, suppose that $|N_H(V_i)| \geq 2$ and V_i is joined to at least two sets in $V_1, V_2, V_3, \ldots, V_k$.

Case 1. G is odd.

Let b be an optimal move in the game on G. Thus $G - b$ is connected. Since $|N_H(V_i)| \geq 2$, we have $N_H(V_i) - b \neq \varnothing$. Then b is a feasible move in the game on

G rooted at $N_H(V_i)$. Then

$$\begin{aligned} N(G, -2 = 2) &= N(G - b, 1 = -2) && \text{(Observation 2.4)} \\ &\geq R(G - b, N_H(V_i) - b, -2 = 1) && \text{(Lemma 2.9.3 by induction)} \\ &\geq R(G, N_H(V_i), 2 = -2) && \text{(Observation 2.5).} \end{aligned}$$

Case 2. G is even.

Let a be an optimal move in the game on G rooted at $N_H(V_i)$.

Case 2.1. a is a feasible move in the game on G.

Thus $G - a$ is connected. Then

$$\begin{aligned} &R(G, N_H(V_i), -1 = 2) \\ &\qquad = R(G - a, N_H(V_i) - a, 1 = -1) && \text{(Observation 2.5)} \\ &\qquad \geq N(G - a, -1 = 1) && \text{(Lemma 2.9.3 by induction)} \\ &\qquad \geq N(G, 2 = -1) && \text{(Observation 2.4).} \end{aligned}$$

Case 2.2. a is not a feasible move in the game on G.

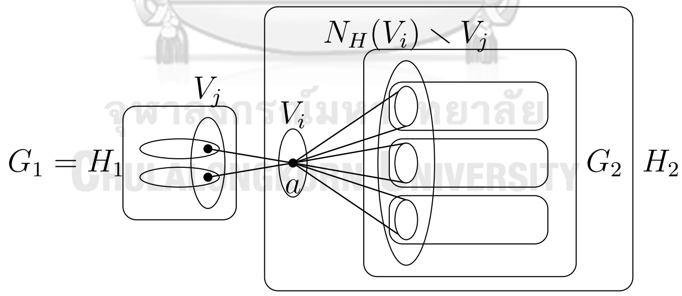


Figure 2.4: The graph G in Case 2.2 of Lemma 2.9.3.

Thus $G - a$ is disconnected. Since a is a feasible move in the game on G rooted at $N_H(V_i)$, we have $a \in V_\ell$ for some $\ell \in [k]$ and $N_G(V_\ell) = N_H(V_\ell)$. Since $G - a$ is disconnected, $V_\ell = \{a\}$ and a is not a leaf. Suppose that $i \neq \ell$. Since V_i is joined to at least two sets, V_i and $N_H(V_i)$ lie in the same component of $G - a$, but other components of $G - a$ do not contain a vertex in $N_H(V_i)$, a contradiction.

Hence $V_i = \{a\}$. Let $V_j \subseteq N_H(V_i)$ and let G_1 be the union of components in $G - a$ containing some vertices of V_j and let $G_2 = G - a - G_1$. By assumption, G_2 is not empty.

First, we shall show that

$$R(G, N_H(V_i), -1) \geq R(G_1, V_j, -1) + R(G_2, N_H(V_i) \setminus V_j, -1),$$

by considering a strategy for Bob who plays second in the game on G rooted at $N_H(V_i)$ after Alice grabs a. He plays optimally as Player -1 in the game on G_1 rooted at V_j and plays optimally as Player -1 in the game on G_2 rooted at $N_H(V_i) \setminus V_j$. Since $|V(G_1)| + |V(G_2)|$ is odd, he plays as Player 1 in one game and as Player 2 in the other. Now, we check that Bob's moves are feasible in the game on G rooted at $N_H(V_i)$ and Alice's moves are feasible in the game on G_1 rooted at V_j and the game on G_2 rooted at $N_H(V_i) \setminus V_j$. Indeed, after each move of Bob, every remaining component in G_1 or G_2 contains a vertex in V_j or $N_H(V_i) \setminus V_j$, respectively. Then every remaining component of G contains a vertex in $N_H(V_i)$. That is, his moves are feasible in the game on G rooted at $N_H(V_i)$. On the other hand, after each move of Alice, every remaining component of G contains a vertex in $N_H(V_i)$. Then every remaining component of G_1 or G_2 contains a vertex in V_j or $N_H(V_i) \setminus V_j$, respectively. That is, her moves are feasible in the game on G_1 rooted at V_j and the game on G_2 rooted at $N_H(V_i) \setminus V_j$. Hence

$$R(G, N_H(V_i), -1) \geq R(G_1, V_j, -1) + R(G_2, N_H(V_i) \setminus V_j, -1). \tag{2.1}$$

Next, we let $H_1 = G_1$ and $H_2 = G - G_1$. We observe that $V_j = V(H_1) \cap N_G(V(H_2))$ and $\{a\} = V(H_2) \cap N_G(V(H_1))$ are root sets of H_1 and H_2, respectively, and a is adjacent to all vertices in V_j, (see Figure 2.4). Hence

$$\begin{aligned}
&R(G, V_j, -2 = 1) \\
&\quad \geq R(G_1, V_j, -2) + R(G - G_1, a, -1) && \text{(Lemma 2.8.1)} \\
&\quad = R(G_1, V_j, -2) + R(G_2, N_H(V_i) \setminus V_j, -2) + w(a) && \text{(Observation 2.6)},
\end{aligned}$$

which is equivalent to

$$R(G, V_j, -1) \leq R(G_1, V_j, -1) + R(G_2, N_H(V_i) \smallsetminus V_j, -1), \tag{2.2}$$

by considering the total weight of G, G_1 and G_2. Then

$$\begin{aligned} N(G, -1) &\leq R(G, V_j, -1) && \text{(Lemma 2.9.2)} \\ &\leq R(G_1, V_j, -1) + R(G_2, N_H(V_i) \smallsetminus V_j, -1) && \text{(Inequality (2.2))} \\ &\leq R(G, N_H(V_i), -1) && \text{(Inequality (2.1)).} \end{aligned}$$ □

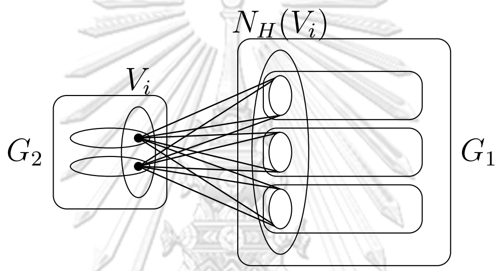


Figure 2.5: The graph G in Lemma 2.10.3.

Proof of Lemma 2.10.3. For $i \in [k]$, let G_1 be the union of components of $G - V_i$ containing some vertices of $N_H(V_i)$ and let $G_2 = G - G_1$. We observe that $N_H(V_i) = V(G_1) \cap N_G(V(G_2))$ and $V_i = V(G_2) \cap N_G(V(G_1))$ are root sets of G_1 and G_2, respectively, and every vertex in $N_H(V_i)$ is joined to every vertex in V_i, (see Figure 2.5). Then

$$\begin{aligned} N(G, 2 = -1) &\leq R(G, V_i, -1 = 2) && \text{(Lemma 2.9.2)} \\ &\leq R(G, N_H(V_i), 1) && \text{(Lemma 2.8.2).} \end{aligned}$$ □

Proof of Lemma 2.10.2. For $i \in [k]$, let G_1 be the union of components of $G - N_H(V_i)$ containing some vertices of V_i and let $G_2 = G - G_1$. We observe that $V_i = V(G_1) \cap N_G(V(G_2))$ and $N_H(V_i) = V(G_2) \cap N_G(V(G_1))$ are root sets of G_1 and G_2, respectively, and every vertex in V_i is joined to every vertex in $N_H(V_i)$.

Then

$$\begin{aligned} N(G, 2 = -1) &\leq R(G, N_H(V_i), -1 = 2) && \text{(Lemma 2.9.3)} \\ &\leq R(G, V_i, 1) && \text{(Lemma 2.8.2).} \qquad \square \end{aligned}$$

Proof of Lemma 2.10.1. Let $v \in V(G)$.

Case 1. There is a cut edge uv incident to v.

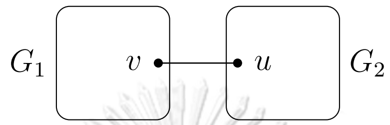


Figure 2.6: The graph G in Case 1 of Lemma 2.10.1.

Let G_1 be the component of $G - uv$ containing v and let $G_2 = G - G_1$. We observe that $\{v\} = V(G_1) \cap N_G(V(G_2))$ and $\{u\} = V(G_2) \cap N_G(V(G_1))$ are root sets of G_1 and G_2, respectively, and v is adjacent to u, (see Figure 2.6). Then

$$\begin{aligned} R(G, v, 1) &\geq R(G, u, 2 = -1) && \text{(Lemma 2.8.2)} \\ &\geq N(G, -1 = 2) && \text{(Lemma 2.9.1).} \end{aligned}$$

Case 2. There is no cut edge incident to v.

Then $v \in V_j$ for some $j \in [k]$ and $N_G(v) = N_H(V_j)$.

Case 2.1. $|V_j| \geq 2$.

Therefore, v is a feasible move in the game on G. Thus $G - v$ is connected. Then

$$\begin{aligned} R(G, v, 1 = -2) &= R(G - v, N_G(v) = N_H(V_j), -1) && \text{(Observation 2.6)} \\ &\geq N(G - v, -1 = 1) && \text{(Lemma 2.9.3 by induction)} \\ &\geq N(G, 2) && \text{(Observation 2.4).} \end{aligned}$$

Case 2.2. $|V_j| = 1$.

Then, by Lemma 2.10.2,

$$R(G, v, 1) = R(G, V_j, 1) \geq N(G, 2). \qquad \square$$

We proceed to prove our main theorem.

Proof of Theorem 2.2. Let G be an even B(T)-tree, where T is a tree and let $v \in V(G)$. Then, by Lemmas 2.9.1 and 2.10.1, it follows that

$$N(G, 2 = -1) \leq R(G, v, -1 = 2) \leq N(G, 1).$$

Therefore, Alice wins the game on G. $\square$

We now deduce Corollary 2.3 from Theorem 2.2.

Proof of Corollary 2.3. We give a proof by induction on the number of vertices. Let G be an even blow-up of a cycle. We note that every vertex of G is a non-cut vertex. Alice claims a maximum weighted vertex of G in her first move, say a vertex a. Let b be the vertex claimed by Bob in his first move. Then $G - \{a, b\}$ is an even blow-up of either a path or a cycle. If $G - \{a, b\}$ is an even blow-up of a path, then Alice wins the game on $G - \{a, b\}$ by Theorem 2.2. Otherwise, Alice wins the game on $G - \{a, b\}$ by the induction hypothesis. In both cases, since $w(a) \geq w(b)$, Alice wins the game on G. $\square$

2.4 Concluding Remarks

We provide two new classes, namely B(T)-trees and B(C_{2n}), of bipartite even graphs which satisfy Conjecture 2.1. However, this conjecture is still open. It was shown in [10] that Lemmas 2.9.1 and 2.10.1 are not true for general bipartite graphs, therefore this method cannot be directly used to solve the full conjecture. There are several variants of the graph grabbing game, for example, the graph sharing game (see [6, 8, 16, 20, 25]), the graph grabbing game on $\{0, 1\}$-weighted graphs (see [11]), and the convex grabbing game (see [23]), where a few problems are left open.

CHAPTER III
TOUCHER-ISOLATOR GAMES

3.1 Introduction

A Maker-Breaker game, introduced by Erdős and Selfridge [12] in 1973, is a positional game played on the complete graph K_n on n vertices, by two players: Maker and Breaker, who alternately claim an edge from the remaining graph, where Maker plays first. Maker wins if she can build a particular structure (e.g., a clique [1, 15], a perfect matching [19, 26] or a Hamiltonian cycle [19, 21]) from her claimed edges, while Breaker wins if he can prevent this. There are several variants of Maker-Breaker games, many of which are studied recently (see [13, 14, 17, 18]).

The *Toucher-Isolator game*, introduced by Dowden, Kang, Mikalački and Stojaković [9] in 2019, is a quantitative version of a Maker-Breaker game played on a finite graph by two players: *Toucher* and *Isolator*, who alternately claim an edge from the remaining graph, where Toucher plays first. A vertex is *touched* if it is incident to at least one edge claimed by Toucher, and a vertex is *untouched* if all edges incident to it are claimed by Isolator. The *score* of the game is the number of untouched vertices at the end of the game when all edges have been claimed. Toucher aims at minimizing the score, while Isolator aims at maximizing the score. For a graph G, let $u(G)$ be the score of the game on G when both players play optimally.

The above mentioned authors gave general upper and lower bounds for $u(G)$, leaving the asymptotic behavior of $u(C_n)$ and $u(P_n)$ as the most interesting unsolved cases. Later in 2019, Räty [27] determined the exact values of $u(C_n)$ and $u(P_n)$, showing that

$$u(C_n) = \left\lfloor \frac{n+1}{5} \right\rfloor \quad \text{and} \quad u(P_n) = \left\lfloor \frac{n+3}{5} \right\rfloor .$$

Moreover, the first set of authors showed that for any tree T on $n \geq 3$ vertices,

$$\frac{n+2}{8} \leq u(T) \leq \frac{n-1}{2},$$

where the upper bound is tight when T is a star, but the only tight example they found for the lower bound is a path on six vertices. Therefore, they asked whether there is an infinite family of tight examples for lower bound, or if it can be improved for large n.

Later in 2020, Räty [28] improved the lower bound for $u(T)$ by showing that the path P_n is the most suitable tree on n vertices for Toucher.

Theorem 3.1. *Let T be a tree on $n \geq 3$ vertices. Then*

$$u(T) \geq \left\lfloor \frac{n+3}{5} \right\rfloor.$$

In this chapter, we give a simple new proof of this theorem. The argument proceeds as follows. The strategy for Isolator is that he claims an edge which immediately creates an untouched vertex in every move for as long as he can (see Figure 3.1: left). When no such an edge exists, we modify the graph before the game continues. The vertices which are incident to only edges claimed by Isolator become untouched vertices. These vertices and the edges claimed by Isolator can be deleted as their disappearance does not change the touched/untouched status of any vertex (see Figure 3.1: middle). Observe that the leaves of the remaining tree are touched otherwise Isolator would have claimed the edge incident to it. Then we delete the edges e claimed by Toucher one by one and, in order to keep the game equivalent to the original game, we replace the edges $u_1v, u_2v, u_3v, \ldots, u_tv$ sharing a vertex v with e by new edges $u_1v_1, u_2v_2, u_3v_3, \ldots, u_tv_t$ keeping their respective Toucher/Isolator status, where the new vertices $v_1, v_2, v_3, \ldots, v_t$ are considered touched. The resulting graph is a forest all of whose leaves are considered touched (see Figure 3.1: right).

Therefore, this motivates us to study the *non-leaf Isolator-Toucher game* on a forest F which is a variant of the Toucher-Isolator game on F where Isolator plays first and the score of the game is the number of untouched vertices which are not leaves of F, at the end of the game. The aim of Toucher is to minimize the score, while the aim of Isolator is to maximize the score. We remark that this game is inspired by the proof of the lower bound for $u(P_n)$ in [27]. Our main lemma gives a lower bound for the minimum score $\alpha(m, k, \ell)$ of the non-leaf Isolator-Toucher game on F when both players play optimally, among all forests F with m edges, k components, and ℓ leaves.

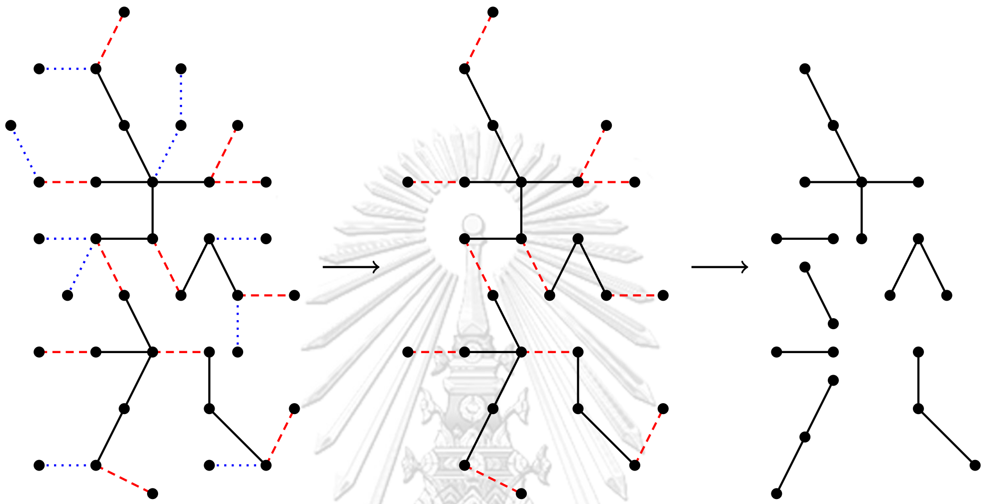


Figure 3.1: The strategy for Isolator in the Toucher-Isolator game on a tree and the modification of the graph, where the red dashed and blue dotted edges are Toucher and Isolator edges respectively.

Lemma 3.2. *For non-negative integers m, k and ℓ,*

$$\alpha(m, k, \ell) \geq \left\lfloor \frac{m + 4k - 3\ell + 4}{5} \right\rfloor.$$

The strategy for Isolator in the non-leaf Isolator-Toucher game is that he claims consecutive edges which immediately creates an untouched vertex in every move except the first one for as long as he can, and then he repeats in a different part of the forest. The key step is to determine which part of the forest is the most profitable for Isolator to play in. We do this by breaking down the gains and losses in each move of both players.

The rest of this chapter is organized as follows. Section 3.2 is devoted to proving Lemma 3.2 and then applying it to prove Theorem 3.1. In Section 3.3, we give some concluding remarks and mention related interesting questions.

3.2 Proofs of Theorem 3.1 and Lemma 3.2

Before proving Lemma 3.2 and Theorem 3.1, we give some definitions necessary for the proofs and make observations regarding how to modify the graph after deleting some edges, to keep the game equivalent to the original game, and how much Isolator gains in each move of both players.

For convenience, we first give some names to vertices and edges in a forest. A *leaf* is a vertex of degree 1. A *small vertex* is a vertex of degree 2. A *big vertex* is a vertex of degree at least 3. A *big edge* is an edge incident to a big vertex. A *leaf edge* is an edge incident to a leaf. An *internal vertex* of a subgraph is a vertex adjacent to no vertex outside the subgraph.

We also give some names to paths in a forest. A *path component* is a component of the forest which is a path. A *branch* is a path such that the non-endpoint vertices are internal and both endpoints are big. A *twig* is a path such that the non-endpoint vertices are internal and one endpoint is a leaf while the other is big.

Finally, we define some game related terms. A *Toucher edge* is an edge claimed by Toucher. An *Isolator edge* is an edge claimed by Isolator. An *Isolator subgraph* is a subgraph whose edges are Isolator edges. An *Isolator path* is an Isolator subgraph which is either a path component, a branch or a twig. A *partially played graph* is a graph where each edge is either a Toucher edge, an Isolator edge or an unclaimed edge.

Now we show how a partially played graph should be modified after deleting a Toucher edge or an Isolator subgraph, in order to keep the game equivalent to the original game. For a partially played graph G with a Toucher edge uv, we define $G \ominus uv$ to be the partially played graph obtained from G by

- deleting the vertices u and v, and all edges incident to them,
- adding new vertices $u_1, u_2, u_3, \ldots, u_{\deg(u)-1}$ and joining u_i to u'_i where $N_G(u) \setminus \{v\} = \{u'_1, u'_2, u'_3, \ldots, u'_{\deg(u)-1}\}$ such that if uu'_i has been claimed by a player,

then we let u_iu_i' be claimed by the same player,

- adding new vertices $v_1, v_2, v_3, \ldots, v_{\deg(v)-1}$ and joining v_i to v_i' where $N_G(v) \setminus \{u\} = \{v_1', v_2', v_3', \ldots, v_{\deg(v)-1}'\}$ such that if vv_i' has been claimed by a player, then we let v_iv_i' be claimed by the same player,

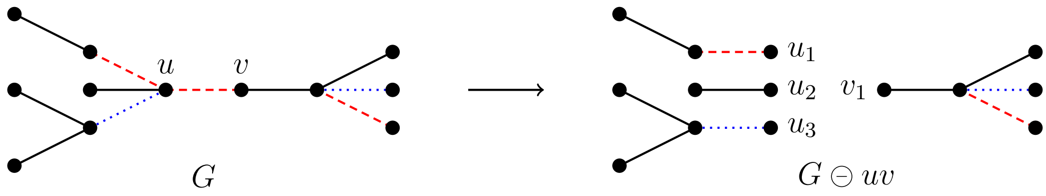


Figure 3.2: The partially played graph $G \ominus uv$, where the red dashed and blue dotted edges are Toucher and Isolator edges respectively.

For a partially played graph G with an Isolator subgraph H, we define $G \ominus H$ to be the partially played graph obtained from G by deleting the edges of H and the internal vertices of H.

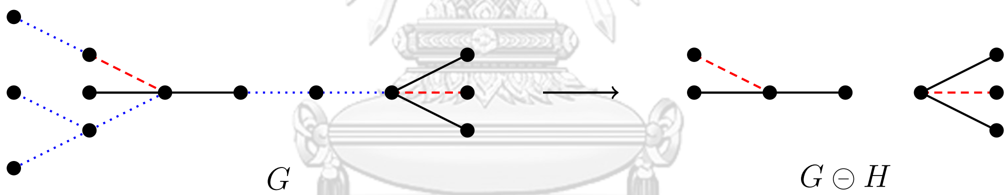


Figure 3.3: The partially played graph $G \ominus H$, where H is the subgraph induced by the set of Isolator edges, and the red dashed and blue dotted edges are Toucher and Isolator edges respectively.

Proposition 3.3. *(i) The non-leaf Isolator-Toucher game on a partially played graph G with a Toucher edge e is equivalent to that on $G \ominus e$.*

(ii) The Toucher-Isolator game on a partially played graph G with an Isolator subgraph H with r internal vertices is equivalent to that on $G \ominus H$ with an extra score of r. The non-leaf Isolator-Toucher game on a partially played graph G with the Isolator subgraph H with r non-leaf internal vertices is equivalent to that on $G \ominus H$ with an extra score of r.

(iii) The score of the non-leaf Isolator-Toucher game on a partially played graph G when both players play optimally is equal to that on $G - U$, where U is the set of vertices of path components of length 1 in G.

Proof. (i) Clearly, there is a bijection between the edges of $G - e$ and $G \ominus e$. The endpoints of the Toucher edge e in the game on G and the new leaves in the game on $G \ominus e$ are not counted in the score of each game.

(ii) Clearly, there is a bijection between the edges of $G - E(H)$ and $G \ominus H$. Deleting an Isolator edge does not change the touched/untouched status of its endpoints. An extra score of r comes from the (non-leaf) internal vertices on H.

(iii) A player gains nothing by claiming a path component of length 1 because its vertices are leaves which are not counted in the score. □

Next, in order to determine which part of the forest is the most profitable for Isolator to play in, it is useful to calculate the changes in the number of edges, components and leaves of the forest when deleting a Toucher edge or an Isolator path. Moreover, deleting path components of length 1 also produces a profit.

Proposition 3.4. *(i) Let G be a partially played graph which is a forest with m edges, k components and ℓ leaves, and let uv be a Toucher edge in G. Suppose that $G \ominus uv$ is a forest with $m + \Delta m$ edges, $k + \Delta k$ components and $\ell + \Delta \ell$ leaves. Then the change in $m + 4k - 3\ell$ is as shown in Table 3.1 and the profit $p_T(G, uv) = \Delta(m + 4k - 3\ell) + 3$ is non-negative.*

Toucher edge uv		Δm	Δk	$\Delta \ell$	$\Delta(m+4k-3\ell)$	$p_T(G,uv)$
u	v					
small	small	-1	1	2	-3	0
small	big	-1	$\deg(v) - 1$	$\deg(v)$	$\deg(v) - 5 \geq -2$	≥ 1
small	leaf	-1	0	0	-1	2
big	big	-1	$\deg(u) + \deg(v) - 3$	$\deg(u) + \deg(v) - 2$	$\deg(u) + \deg(v) - 7 \geq -1$	≥ 2
big	leaf	-1	$\deg(u) - 2$	$\deg(u) - 2$	$\deg(u) - 3 \geq 0$	≥ 3
leaf	leaf	-1	-1	-2	1	4

Table 3.1: The profit of deleting a Toucher edge.

(ii) Let G be a partially played graph which is a forest with m edges, k components and ℓ leaves, and let P be an Isolator path of length $r+1$ in G. Suppose that $G \ominus P$ is a forest with $m+\Delta m$ edges, $k+\Delta k$ components and $\ell+\Delta\ell$ leaves. Then the change in $m+4k-3\ell$ is as shown in Table 3.2 and the profit $p_I(G,P)=\Delta(m+4k-3\ell)+r-1$ is non-negative.

u,v-Isolator path		Δm	Δk	$\Delta\ell$	$\Delta(m+4k-3\ell)$	$p_I(G,P)$
u	v					
leaf	leaf	$-(r+1)$	-1	-2	$-r+1$	0
big	leaf	$-(r+1)$	0	-1	$-r+2$	1
big	big	$-(r+1)$	1	0	$-r+3$	2

Table 3.2: The profit of deleting an Isolator path.

(iii) Let G be a partially played graph which is a forest with m edges, k components, ℓ leaves, and let U be a set of q path components of length 1. Suppose that $G-U$ is a forest with $m+\Delta m$ edges, $k+\Delta k$ components and $\ell+\Delta\ell$ leaves. Then the change in $m+4k-3\ell$ is as shown in Table 3.3 and the profit $p_L(G,U)=\Delta(m+4k-3\ell)$ is equal to q.

Δm	Δk	$\Delta\ell$	$\Delta(m+4k-3\ell)$	$p_L(G,U)$
$-q$	$-q$	$-2q$	q	q

Table 3.3: The profit of deleting q path components of length 1.

Proof. The calculation steps are shown in the tables. The profit $p_T(G,uv)\geq 0$ since the term $+3$ in the definition of $p_T(G,uv)$ comes from (-1) times the minimum value of $\Delta(m+4k-3\ell)$ in Table 3.1. The profit $p_I(G,P)\geq 0$ since the term $+(r-1)$ in the definition of $p_I(G,uv)$ comes from (-1) times the minimum value of $\Delta(m+4k-3\ell)$ in Table 3.2. □

We are now ready to prove our main lemma which provides a lower bound for $\alpha(m,k,\ell)$ of the non-leaf Isolator-Toucher game on a forest.

Proof of Lemma 3.2. We use induction on the number of edges m in a forest. Let F be a forest with n vertices, m edges, k components, ℓ leaves, a small vertices and b big vertices. First, we suppose that all path components have lengths at most 2, all branches have lengths at most 2, and all twigs have lengths at most 1. In this case, we shall show that $\lfloor \frac{m+4k-3\ell+4}{5} \rfloor \leq 0$, and thus there is nothing to prove. Since $\sum_{v \in F} \deg(v) = 2m = 2(n-k)$, we have $\ell + 2a + \sum_{\deg(v)\geq 3} \deg(v) = 2\ell + 2a + 2b - 2k$. Then $\ell = \sum_{\deg(v)\geq 3} \deg(v) - 2b + 2k$ and thus $\ell \geq b + 2k$. Since every edge in a non-path component is adjacent to a big vertex and every path component contains at most 2 edges, it follows that

$$m \leq \sum_{\deg(v)\geq 3} \deg(v) + 2k = \ell + 2b \leq 3\ell - 4k$$

as required.

Now, we suppose that there is either a path component of length at least 3, a branch of length at least 3, or a twig of length at least 2.

Isolator's strategy is to keep claiming consecutive edges, for as long as he can, to form an Isolator path. Therefore, he only plays within a path component, a branch, or a twig, say P. We label the edges of P by $e_1, e_2, e_3, \ldots, e_s$ respectively starting from a big edge (if exists). Note that we shall use this convention to label any path component, branch, or twig in this proof. Assuming he has claimed the edges $e_t, e_{t+1}, e_{t+2}, \ldots, e_{t+r}$, he then claims e_{t-1} or e_{t+r+1} if it is available, otherwise he stops. That is, he stops if ($t = 1$ or e_{t-1} is a Toucher edge) and ($t + r = s$ or e_{t+r+1} is a Toucher edge).

Suppose Isolator stops with edges $e_t, e_{t+1}, e_{t+2}, \ldots, e_{t+r}$. Then these edges form a path Q. So far, both players have claimed $r + 1$ edges each since Isolator plays first, and the score is r since Isolator creates an untouched vertex in every move except the first one. We note that the case where Toucher has claimed only r edges because all edges had been claimed, can be proved similarly. Let G be the partially played graph at this step. If $f_1, f_2, f_3, \ldots, f_{r+1}$ are the Toucher edges in G, then let $G_1 = G \ominus f_1 \ominus f_2 \ominus \cdots \ominus f_{r+1}$ be a forest with m_1 edges, k_1 components and ℓ_1 leaves, let $G_2 = G_1 \ominus Q$ be a forest with m_2 edges, k_2 components and

ℓ_2 leaves, and let $G_3 = G_2 - U$ be a forest with m_3 edges, k_3 components and ℓ_3 leaves, where U is the set of vertices of path components of length 1 in G_2.

By Proposition 3.3, the game on G is equivalent to the game on G_1 which is equivalent to the game on G_2 with an extra score of r, and the score of the game on G_2 when both players play optimally is equal to that on G_3. Therefore, it follows that

$$\begin{aligned}
\alpha(m,k,\ell) &\geq r + \alpha(m_3,k_3,\ell_3) \\
&\geq r + \left\lfloor \frac{m_3 + 4k_3 - 3\ell_3 + 4}{5} \right\rfloor \quad \text{(by the induction hypothesis)} \\
&= r + \left\lfloor \frac{m+4k-3\ell+4}{5} + \frac{\Delta_1(m+4k-3\ell)}{5} + \frac{\Delta_2(m+4k-3\ell)}{5} \right. \\
&\qquad \left. + \frac{\Delta_3(m+4k-3\ell)}{5} \right\rfloor \\
&= r + \left\lfloor \frac{m+4k-3\ell+4}{5} + \frac{\sum_{i=0}^{r}(-3 + p_T(G \ominus f_1 \ominus \cdots \ominus f_i, f_{i+1}))}{5} \right. \\
&\qquad \left. + \frac{-r+1+p_I(G_1,Q)}{5} + \frac{p_L(G_2,U)}{5} \right\rfloor \\
&\qquad \text{(by Proposition 3.4 since } Q \text{ is an Isolator path in } G_1\text{)} \\
&= r + \left\lfloor \frac{m+4k-3\ell+4}{5} + \frac{-3(r+1)+p_T}{5} + \frac{-r+1+p_I}{5} + \frac{p_L}{5} \right\rfloor \\
&= \left\lfloor \frac{m+4k-3\ell+4}{5} + \frac{r+p_T+p_I+p_L-2}{5} \right\rfloor,
\end{aligned}$$

where

$$\begin{aligned}
\Delta_1(m+4k-3\ell) &= (m_1+4k_1-3\ell_1) - (m+4k-3\ell), \\
\Delta_2(m+4k-3\ell) &= (m_2+4k_2-3\ell_2) - (m_1+4k_1-3\ell_1), \\
\Delta_3(m+4k-3\ell) &= (m_3+4k_3-3\ell_3) - (m_2+4k_2-3\ell_2),
\end{aligned}$$
$$p_T = \sum p_T(G \ominus f_1 \ominus \cdots \ominus f_i, f_{i+1}),\ p_I = p_I(G_1,Q) \text{ and } p_L = p_L(G_2,U).$$

Therefore, it suffices to show that $r + p_T + p_I + p_L \geq 2$. Since every term in the sum $r + \sum p_T(G \ominus f_1 \ominus \cdots \ominus f_i, f_{i+1}) + p_I + p_L$ is non-negative by Proposition 3.4, we shall find a subset of terms whose sum is at least 2. Recall that there is either a path component of length at least 3, a branch of length at least 3, or a twig of length at least 2. The proof is divided into five cases.

Case 1. There is a path component of length 3.

Isolator claims the edge e_2 in his first move. If Toucher claims the leaf edge e_1 or e_3 in some move, then $p_T \geq 2$ by Proposition 3.4. Otherwise, Isolator claims the edges e_1 and e_3, hence $r = 2$.

Case 2. There is a path component of length at least 4.

Isolator claims the edge e_3 in his first move. If Toucher claims the leaf edge e_1 in some move, then $p_T \geq 2$ by Proposition 3.4. If Toucher claims the edge e_2 in some move (but not e_1), then G_2 has a path component e_1 of length 1 and thus $p_L \geq 1$ by Proposition 3.4. Clearly, $r \geq 1$, hence it follows that $r + p_L \geq 2$. Otherwise, Isolator claims the edges e_1 and e_2, hence $r \geq 2$.

Case 3. There is a branch of length at least 3.

Isolator claims the edge e_2 in his first move. If Toucher claims the big edge e_1 in some move, then $p_T \geq 1$ by Proposition 3.4. Clearly, $r \geq 1$, hence it follows that $r + p_T \geq 2$. If Toucher claims the edge e_3 in some move, then $p_I \geq 1$ by Proposition 3.4 since Isolator claims the big edge e_1. Clearly, $r = 1$, hence it follows that $r + p_I \geq 2$. Otherwise, Isolator claims the edges e_1 and e_3, hence $r \geq 2$.

Case 4. There is a twig of length 2.

Isolator claims the edge e_1 in his first move. If Toucher claims the leaf edge e_2 in some move, then $p_T \geq 2$ by Proposition 3.4. Otherwise, Isolator claims the edge e_2, hence $p_I \geq 1$ by Proposition 3.4 since Isolator claims the big edge e_1. Clearly, $r = 1$, hence it follows that $r + p_I \geq 2$.

Case 5. There is a twig of length at least 3.

Isolator claims the edge e_2 in his first move. If Toucher claims the big edge e_1 in some move, then $p_T \geq 1$ by Proposition 3.4. Clearly, $r \geq 1$, hence it follows that $r + p_T \geq 2$. If Toucher claims the edge e_3 in some move, then $p_I \geq 1$ by Proposition 3.4 since Isolator claims the big edge e_1. Clearly, $r = 1$, it follows that $r + p_I \geq 2$. Otherwise, Isolator claims the edges e_1 and e_3, hence $r \geq 2$.

This completes the proof of Lemma 3.2. $\square$

We now prove Theorem 3.1 which improves the lower bound for $u(T)$ of the Toucher-Isolator game, by applying the result on the non-leaf Isolator-Toucher game in Lemma 3.2.

Proof of Theorem 3.1. Let T be a tree with $m \geq 2$ edges and ℓ leaves. We shall show that

$$u(T) \geq \left\lfloor \frac{m+4}{5} \right\rfloor.$$

For a partially played graph G, a *meta-leaf* in G is a leaf in the graph obtained from G by deleting all Isolator edges, and a *meta-leaf edge* in G is an edge incident to a meta-leaf in G.

Isolator's strategy is to keep claiming an edge which produces a new untouched vertex in every move, i.e., he claims a meta-leaf edge in the current partially played graph if it is available, otherwise he stops (see Figure 3.1: left). That is, he stops when all meta-leaf edges are Toucher edges. We note that he always obtains a score of one in every move because if he claims the edge uv where u is a meta-leaf, then all already played edges incident to u are Isolator edges, and thus u becomes untouched. If the process stops after Isolator's move, i.e., all edges have been claimed by both players, then Isolator obtains a score of $\lfloor \frac{m}{2} \rfloor \geq \lfloor \frac{m+4}{5} \rfloor$, as required. Therefore, we may assume that the process stops after Toucher's move, and in particular, $m \geq 3$.

Suppose that Isolator stops after r moves. Let G be the partially played graph at this step. Then G has $r+1$ Toucher edges and r Isolator edges since Toucher plays first. Let H be the Isolator subrgaph of G formed by all Isolator edges, and let $G_1 = G \ominus H$ be a forest with m_1 edges, k_1 components and ℓ_1 leaves (see Figure 3.1: middle). Since Isolator claimed only meta-leaf edges and all meta-leaf edges in G are Toucher edges, G_1 is a tree all of whose leaves are touched, and $k_1 = 1$. By $m \geq 3$, each leaf of G_1 is incident to a distinct Toucher edge, and so $r+1 \geq \ell_1$. Let $f_1, f_2, f_3, \dots, f_{r+1}$ be the Toucher edges in G, and let $G_2 = G_1 \ominus f_1 \ominus \cdots \ominus f_{r+1}$ be the forest with m_2 edges, k_2 components and ℓ_2 leaves (see Figure 3.1: right).

By Proposition 3.3 and the fact that the leaves in G_1 are touched, the Toucher-Isolator game on G where Isolator plays first is equivalent to the non-leaf Isolator-Toucher game on G_1 which is equivalent to the non-leaf Isolator-Toucher game on G_2 with an extra score of r. Therefore, it follows that

$$
\begin{aligned}
u(T) &\geq r + \alpha(m_2, k_2, \ell_2) \\
&\geq r + \left\lfloor \frac{m + 4(1) - 3\ell + 4}{5} + \frac{\Delta_1(m + 4 - 3\ell)}{5} + \frac{\Delta_2(m + 4k - 3\ell)}{5} \right\rfloor \\
&\qquad \text{(by Lemma 3.2)} \\
&= r + \left\lfloor \frac{m + 4 - 3\ell + 4}{5} + \frac{(m_1 - m) + 4(k_1 - 1) - 3(\ell_1 - \ell)}{5} \right. \\
&\qquad \left. + \frac{\sum_{i=0}^{r}(-3 + p_T(G_1 \ominus f_1 \ominus \cdots \ominus f_i, f_{i+1}))}{5} \right\rfloor \\
&\geq r + \left\lfloor \frac{m - 3\ell + 8}{5} + \frac{(-r) + 4(0) - 3(\ell_1 - \ell)}{5} + \frac{-3(r+1) + 2\ell_1}{5} \right\rfloor \\
&\qquad \text{(by Proposition 3.4 since } G_1 \text{ has } \ell_1 \text{ leaf edges)} \\
&= \left\lfloor \frac{m + r - \ell_1 + 5}{5} \right\rfloor \\
&\geq \left\lfloor \frac{m + 4}{5} \right\rfloor, \quad (r + 1 \geq \ell_1)
\end{aligned}
$$

where

$$
\begin{aligned}
\Delta_1(m + 4k - 3\ell) &= (m_1 + 4k_1 - 3\ell_1) - (m + 4 - 3\ell) \text{ and} \\
\Delta_2(m + 4k - 3\ell) &= (m_2 + 4k_2 - 3\ell_2) - (m_1 + 4k_1 - 3\ell_1).
\end{aligned}
$$

□

3.3 Concluding Remarks

As a result of Theorem 3.1, for any tree T on $n \geq 3$ vertices,

$$u(P_n) \leq u(T) \leq u(S_n),$$

where S_n is a star on n vertices. Moreover, Theorem 3.1 implies that, for a forest with k trees, $u(F) \geq \sum_{i=1}^{k} \left\lfloor \frac{n_i+3}{5} \right\rfloor$, where n_i is the number of vertices of the i^{th} tree in F because, in each move, Isolator can play optimally on the tree Toucher just played. However, the lower bound of $\left\lfloor \frac{n+3k}{5} \right\rfloor$ is not possible because for example, $u(kP_3) = k$ where kP_3 is the disjoint union of k copies of P_3. Many interesting questions about the Toucher-Isolator game are still open (see [9]). For example, find a 3-regular graph G with n vertices that maximizes $u(G)$. Dowden, Kang, Mikalački and Stojaković [9] showed that the largest proportion of untouched vertices for a 3-regular graph is between $\frac{1}{24}$ and $\frac{1}{8}$.

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