CHAPTER II

LOCALLY FACTORIZABLE SEMIGROUPS

In this chapter, general properties of locally factorizable semigroups are investigated.

Recall that a semigroup S is <u>factorizable</u> if $S = GE(S)$ for some subgroup G of S where E(S) is the set of all idempotents of S, and a semigroup S is <u>locally factorizable</u> if each local subsemigroup of S is factorizable, that is, for each $e \in E(S)$, $eSe = GE(eSe)$ for some subgroup G of eSe.

It is easily seen that every group and every group with zero is factorizable and also locally factorizable. Observe from the definition of locally factorizable semigroups that for any semigroup S with identity, if S is locally factorizable, then S is factorizable. In general, a locally factorizable semigroup need not be factorizable, and vice versa. Some counter-examples are given below.

<u>Example</u>. It follows from the definitions that any semigroup without idempotents is locally factorizable but not factorizable.

Let S be a left zero semigroup such that $|S| > 1$. Then $ab = a$ for all $a, b \in S$ and $E(S) = S$. Since for $a \in E(S) = S$, $aSa = \{a\}$ which is factorizable, we have that S is locally factorizable. Because $E(S) = S$, a subset G of S is a subgroup of S if and only if $G = \{a\}$ for some $a \in S$. But for $a \in S$, $\{a\}E(S) = aS = \{a\} \neq S$ since $|S| > 1$, so S is not factorizable. Hence every left zero semigroup of cardinality > 1 is locally factorizable but not factorizable.

Let $\mathbb{N}$ and $\mathbb{Z}$ be the set of all positive integers and the set of all integers, respectivety. It has been shown by Chen and Hsieh in [1] that

$$A_{\mathbb{Z}} = \{ \alpha \in I_{\mathbb{Z}} \mid |\mathbb{Z} \setminus \Delta\alpha| = |\mathbb{Z} \setminus \nabla\alpha| \}$$

is a factorizable subsemigroup of $I_{\mathbb{Z}}$ where $I_{\mathbb{Z}}$ is the symmetric inverse semigroup on $\mathbb{Z}$. Let $1_{\mathbb{N}}$ be the identity map on $\mathbb{N}$. Then $1_{\mathbb{N}} \in E(A_{\mathbb{Z}})$, $I_{\mathbb{N}} = 1_{\mathbb{N}} I_{\mathbb{N}} 1_{\mathbb{N}} \subseteq 1_{\mathbb{N}} A_{\mathbb{Z}} 1_{\mathbb{N}}$. Since for $\alpha \in A_{\mathbb{Z}}$, $\Delta(1_{\mathbb{N}} \alpha 1_{\mathbb{N}}) \subseteq \Delta 1_{\mathbb{N}} = \mathbb{N}$ and $\nabla(1_{\mathbb{N}} \alpha 1_{\mathbb{N}}) \subseteq \nabla 1_{\mathbb{N}} = \mathbb{N}$, it follows that $1_{\mathbb{N}} A_{\mathbb{Z}} 1_{\mathbb{N}} \subseteq I_{\mathbb{N}}$. Hence $1_{\mathbb{N}} A_{\mathbb{Z}} 1_{\mathbb{N}} = I_{\mathbb{N}}$. Since $\mathbb{N}$ is an infinite set, $I_{\mathbb{N}}$ is not factorizable [1, Corollary of Theorem 3.1], so $1_{\mathbb{N}} A_{\mathbb{Z}} 1_{\mathbb{N}}$ is not factorizable. Hence $A_{\mathbb{Z}}$ is factorizable but not locally factorizable.

It was shown in [7, Theorem 2.4] that for a semigroup S with identity 1, if S is factorizable as $S = GE(S)$, then $G = H_1$ which is the maximum subgroup of S having 1 as its identity.

Let S be a semigroup and $e \in E(S)$. Then H_e (the $\mathcal{H}$- class of S containing e) is the maximum subgroup of S having e as its identity. Since $H_e = eH_ee \subseteq eSe$, we have that H_e is the maximum subgroup of eSe having e as its identity. But eSe is a subsemigroup of S having e as its identity, hence we have

2.1 <u>Proposition</u>. Let S be a semigroup. Then :

(1) For $e \in E(S)$, eSe is factorizable if and only if $eSe = H_eE(eSe)$.

(2) The semigroup S is locally factorizable if and only if for

each $e \in E(S)$, $eSe = H_eE(eSe)$.

The next two propositions give another examples of locally factorizable semigroups.

2.2 <u>Proposition</u>. Every band is locally factorizable.

<u>Proof</u> : Let S be a band. Then $E(S) = S$. Let $a \in E(S) = S$. Then $E(aSa) = aSa$, $\{a\}$ is a subgroup of aSa and $aSa = \{a\}aSa = \{a\}E(aSa)$. Therefore aSa is factorizable. #

2.3 <u>Corollary</u>. Every semilattice, every left zero semigroup and every right zero semigroup is locally factorizable.

<u>Proof</u> : This follows from Proposition 2.2 and the fact that a semilattice, a left zero semigroup and a right zero semigroup are all bands. #

2.4 <u>Proposition</u>. Every left group and every right group is locally factorizable.

<u>Proof</u> : Let e be an idempotent of a left group S. Then eS is a subsemigroup of S having e as a left identity. For each $a \in eS$, $Sa = S$ since S is left simple, so there exists an $x \in S$ such that $xa = e$. Then $ex \in eS$ and $exa = ee = e$. Hence eS is a subgroup of S having e as its identity. Thus $eSe = eS$ is a group, so it is factorizable. Therefore a left group is locally factorizable. Dually, a right group is locally factorizable. #

The local factorizability of any semigroup S is equivalent to that of the semigroup S^0. However, we get only one implication for the case S^1.

2.5 Proposition. (1) For a semigroup S, S is locally factorizable if and only if S^0 is locally factorizable.

(2) For a semigroup S, if S^1 is locally factorizable, then so is S.

Proof : Let S be a semigroup.

(1) If S has a zero, then $S^0 = S$, and so we are done. Suppose S has no zero. First we note that for $e \in E(S)$, the $\mathcal{H}$ - class of S containing e, H_e, is the $\mathcal{H}$ - class of S^0 containing e (since in S^0, $H_0 = \{0\}$).

Assume that S is locally factorizable. To show that S^0 is locally factorizable, let $e \in E(S^0)$. If $e = 0$, then $eS^0e = 0S^00 = \{0\}$ which is factorizable. If $e \neq 0$, then $e \in E(S)$ and $eS^0e = eSe \cup \{0\}$ $= H_eE(eSe) \cup \{0\} = H_eE(eS^0e)$. Hence S^0 is locally factorizable.

For the converse, assume that S^0 is locally factorizable, let $e \in E(S)$. Then $e \in E(S^0)$, so $eS^0e = H_eE(eS^0e)$ where H_e is the $\mathcal{H}$ - class of S containing e which is also the $\mathcal{H}$ - class of S^0 containing e. Thus $eSe \cup \{0\} = eS^0e = H_eE(eS^0e) = H_eE(eSe) \cup \{0\}$. Since $0 \notin eSe$ and $0 \notin H_eE(eSe)$, it follows that $eSe = H_eE(eSe)$. This shows that S is locally factorizable.

(2) Assume that the semigroup S^1 is locally factorizable. If $e \in E(S)$, then $e^3 = e = e^2$ and therefore $eSe = eS^1e$ which is factorizable. Hence S is locally factorizable. #

The converse of Proposition 2.5 (2) is not true. An example is given as follows :

Example. Let S be the multiplicative semigroup of nonnegative even integers. Then $S = \{0, 2, 4, \dots\}$, $E(S) = \{0\}$ and $0S0 = \{0\}$ which is factorizable. Therefore S is a locally factorizable semigroup, but S^1 is not a locally factorizable semigroup since $S \neq H_1E(1S^11) = \{1\}\{1,0\} = \{1,0\}$.

A subsemigroup of a locally factorizable semigroup need not be locally factorizable.

Example. The additive group of real numbers $(\mathbb{R},+)$, is a locally factorizable semigroup, but the subsemigroup $(\mathbb{N} \cup \{0\},+)$ of $(\mathbb{R},+)$ is not locally factorizable since $0+(\mathbb{N} \cup \{0\})+0 = \mathbb{N} \cup \{0\}$ but $H_0 + E(0+(\mathbb{N} \cup \{0\})+0) = \{0\}+\{0\} = \{0\}$.

The next theorem shows that a subsemigroup which is either a filter, a left ideal , a right ideal or an ideal of a locally factorizable semigroup is always locally factorizable.

2.6 Theorem. If a semigroup S is locally factorizable, then every left [right] ideal and every filter of S is locally factorizable.

Proof : First, let A be a left [right] ideal of S. Let $e \in E(A)$. Then $e \in E(S)$. Since A is a left [right] ideal of S and $e \in A$, we have $Se \subseteq Ae$ [$eS \subseteq eA$] which implies $eAe = eSe$. Therefore eAe is factorizable since eSe is factorizable. Hence A is locally factorizable.

Next, let T be a filter of S and let $e \in E(T)$. Since S is locally factorizable, $eSe = H_eE(eSe)$ (Proposition 2.1). If $x \in H_e$, then $e = xy$ for some $y \in S^1$, so $x \in T$ since $xy = e \in T$ and T is a filter of S. Therefore $H_e \subseteq T$ and thus $H_eE(eTe) \subseteq T$. To show that $eTe = H_eE(eTe)$, let $t \in T$. Then $ete \in eTe \subseteq eSe = H_eE(eSe)$, so $ete = aebe$ for some $a \in H_e$ and $b \in S$ such that $ebe \in E(eSe)$. Since $aebe = ete \in T$ and T is a filter, it follows that $b \in T$, so $ebe \in E(eTe)$. Therefore $ete = aebe \in H_eE(eTe)$. Hence $eTe = H_eE(eTe)$. This proves that T is locally factorizable. #

2.7 Corollary. Every ideal of a locally factorizable semigroup is locally factorizable.

Proof : This follows from Theorem 2.6 and the fact that an ideal is a left ideal. #

Let S be a semigroup. If eS is locally factorizable for all $e \in E(S)$, then S is locally factorizable since for $e \in E(S)$, $e = ee \in eS$ and $eSe = e(eS)e$ which is factorizable. Dually, if Se is locally factorizable for all $e \in E(S)$, then S is locally factorizable. Also, if SeS is locally factorizable for all $e \in E(S)$, then S is locally factorizable since for $e \in E(S)$, $e = eee \in SeS$ and $eSe = eeeSe \subseteq e(SeS)e \subseteq eSe$ which implies $eSe = e(SeS)e$ which is factorizable.

Let S be a locally factorizable semigroup. For each $e \in E(S)$, SeS [eS,Se] is an ideal [a right ideal, a left ideal] of S, so SeS, eS, Se are locally factorizable semigroups.

Therefore the following theorem is obtained :

2.8 Theorem. Let S be a semigroup. Then the following are equivalent :

(1) S is locally factorizable.

(2) SeS is locally factorizable for all $e \in E(S)$.

(3) eS is locally factorizable for all $e \in E(S)$.

(4) Se is locally factorizable for all $e \in E(S)$.

A homomorphic image of a factorizable semigroup is clearly a factorizable semigroup since a homomorphic image of a group is a group and a homomorphism maps an idempotent to an idempotent. The following example shows that a homomorphic image of a locally factorizable semigroup need not be locally factorizable. The next two theorems show that this is true for locally factorizable inverse semigroups and for homomorphic image in the type of Rees quotient semigroups.

Example. Let $S = \mathbb{N} \cup \mathbb{Z}_2$, where $\mathbb{N}$ is the set of all positive integers and $\mathbb{Z}_2$ is the set of all integers modulo 2. Define the operation $*$ on S by

$$m * n = m + n \quad \text{if } m, n \in \mathbb{N},$$
$$\bar{m} * \bar{n} = \bar{m} + \bar{n} \quad \text{if } \bar{m}, \bar{n} \in \mathbb{Z}_2,$$
$$m * \bar{n} = \bar{n} * m = \bar{n} \quad \text{if } m \in \mathbb{N}, \bar{n} \in \mathbb{Z}_2.$$

Then $(S,*)$ is a semigroup and $E(S) = \{\bar{0}\}$. S is locally factorizable since $\bar{0}*S*\bar{0} = \{\bar{0}, \bar{1}\} = \mathbb{Z}_2$ which is a group. Let $T = \mathbb{Z}_2 \cup \{e\}$ be a semilattice of groups having e as its identity. T is not a locally factorizable since $eTe = T \neq H_e E(eTe) = \{e\}\{e, \bar{0}\}$. Define a map ψ from S into T by

$$x\psi = \begin{cases} e & \text{if } x \in \mathbb{N}, \\ x & \text{if } x \in \mathbb{Z}_2. \end{cases}$$

It is easily seen that ψ is a homomorphism from S onto T.

2.9 Theorem. A homomorphic image of a locally factorizable inverse semigroup is locally factorizable.

Proof : Let T be a homomorphic image of a locally factorizable inverse semigroup S by a homomorphism ψ. Let $f \in E(T)$. Then there exists an element $e \in E(S)$ such that $e\psi = f$ (Chapter I, page 5). Since S is locally factorizable, $eSe = H_eE(eSe)$, which implies that $fTf = (e\psi)(S\psi)(e\psi) = (eSe)\psi = (H_eE(eSe))\psi = (H_e\psi)(E(eSe)\psi) \subseteq (H_e\psi)E(e\psi(S\psi)e\psi) = (H_e\psi)E(fTf)$. Since $H_e\psi$ is a subgroup of T and $H_e\psi = (eH_ee)\psi = (e\psi)(H_e\psi)(e\psi) = f(H_e\psi)f \subseteq fTf$, we have that $fTf = (H_e\psi)E(fTf)$ which is factorizable. Hence T is locally factorizable. #

2.10 Theorem. If S is a locally factorizable semigroup and I is an ideal of S, then the Rees quotient semigroup S/I is locally factorizable.

Proof : Let $a \in S$ be such that $(a\rho_I)^2 = a\rho_I$. Claim that $a\rho_I S/I a\rho_I = H_{a\rho_I}E(a\rho_I S/I a\rho_I)$. If $a \in I$ then $a\rho_I$ is the zero of S/I, so $a\rho_I S/I a\rho_I = \{a\rho_I\} = H_{a\rho_I}E(a\rho_I S/I a\rho_I)$. Now, assume that $a \notin I$. Then $a\rho_I = \{a\}$. Since $(a\rho_I)^2 = a\rho_I$, $a^2 = a$. Because S is locally factorizable, then $aSa = H_aE(aSa)$ where H_a is the $\mathcal{H}$ - class of S containing a. It follows that $a\rho_I S/I a\rho_I = (aSa)\rho_I = (H_aE(aSa))\rho_I = (H_a\rho_I)((E(aSa))\rho_I) \subseteq H_{a\rho_I}E(a\rho_I S/I a\rho_I)$. But $H_{a\rho_I} \subseteq a\rho_I S/I a\rho_I$, therefore $H_{a\rho_I}E(a\rho_I S/I a\rho_I) = a\rho_I S/I a\rho_I$.

Therefore the theorem is proved. #

Let I be an ideal of a semigroup S. By Corollary 2.7 and Theorem 2.10 we have that if S is locally factorizable, then so are I and S/I. However, the converse is not true in general.

Example. Let S be the multiplicative semigroup of nonnegative integers. Then $S = (\mathbb{N} \cup \{0\}, \cdot)$. Let $I = \{0, 2, 3, \ldots\}$. Then I is an ideal of S. Clearly I and $S/I = \{0\rho_I , 1\rho_I\}$ are locally factorizable. But S is not locally factorizable since $H_1E(1S1) = \{1\}\{1, 0\} = \{1, 0\} \neq 1S1$.

Let S be a semilattice of groups. Then S is an inverse semigroup and $S = \bigcup_{e\varepsilon E(S)} H_e$ which is a semilattice E(S) of groups H_e. We give a characterization of a semilattice of groups to be locally factorizable in the next theorem. The following lemma is required.

2.11 Lemma. If S is a semilattice of groups, then for $e \varepsilon E(S)$, $eSe = \bigcup_{f\varepsilon E(S)} H_{ef}$ and $E(eSe) = \{ef \mid f \varepsilon E(S)\}$.

Proof : Let S be a semilattice of groups and let $e \varepsilon E(S)$. Then $S = \bigcup_{f\varepsilon E(S)} H_f$ is a semilattice E(S) of groups H_f. Then $eSe = e(\bigcup_{f\varepsilon E(S)} H_f)e = \bigcup_{f\varepsilon E(S)} eH_fe \subseteq \bigcup_{f\varepsilon E(S)} H_{ef}$ since $H_eH_f \subseteq H_{ef}$ and $H_fH_e \subseteq H_{ef}$ for all $f \varepsilon E(S)$. If $f \varepsilon E(S)$, then $H_{ef} = efH_{ef}fe \subseteq eSe$, therefore $eSe = \bigcup_{f\varepsilon E(S)} H_{ef}$, and hence $E(eSe) = \{ef \mid f \varepsilon E(S)\}$ since for each $f \varepsilon E(S)$, f is the only one idempotent in H_f. #

2.12 (A) Theorem. Let S be a semilattice of groups. Then S is locally factorizable if and only if for $e, f \varepsilon E(S)$, $H_ef = H_{ef}$.

Proof : Let S be a semilattice of groups. Then $S = \bigcup_{e\varepsilon E(S)} H_e$

is a semilattice $E(S)$ of groups H_e.

Assume S is locally factorizable. Let $e \in E(S)$. Then $eSe = H_eE(eSe)$, so by Lemma 2.11, $eSe = H_e\{ef \mid f \in E(S)\} = \bigcup_{f\in E(S)} H_eef = \bigcup_{f\in E(S)} H_ef$. But $eSe = \bigcup_{f\in E(S)} H_{ef}$ (Lemma 2.11), so $\bigcup_{f\in E(S)} H_{ef} = \bigcup_{f\in E(S)} H_ef$. Since $S = \bigcup_{f\in E(S)} H_f$ is a semilattice $E(S)$ of groups H_f, it follows that $H_ef \subseteq H_{ef}$ for all $f \in E(S)$. To show that $H_ef = H_{ef}$ for all $f \in E(S)$, let $f' \in E(S)$ and let $a \in H_{ef'}$. Then $a \in \bigcup_{f\in E(S)} H_{ef} = \bigcup_{f\in E(S)} H_ef$, so $a \in H_ef''$ for some $f'' \in E(S)$. Since $H_ef'' \subseteq H_{ef''}$, we have $a \in H_{ef'} \cap H_{ef''}$, so $ef' = ef''$. Since $a \in H_ef''$, $a = bf''$ for some $b \in H_e$. Therefore $a = bf'' = (be)f'' = b(ef'') = b(ef') = (be)f' = bf' \in H_ef'$. Thus $H_ef' = H_{ef'}$. This proves that $H_ef = H_{ef}$ for all $f \in E(S)$.

Conversely, assume that $H_ef = H_{ef}$ for all $e, f \in E(S)$. To show S is locally factorizable, let $e \in E(S)$. Then $eSe = \bigcup_{f\in E(S)} H_{ef}$ (Lemma 2.11), so by assumption, we have $eSe = \bigcup_{f\in E(S)} H_ef = \bigcup_{f\in E(S)} H_eef = H_e\{ef \mid f \in E(S)\} = H_eE(eSe)$ (Lemma 2.11). Hence S is locally factorizable. #

Let S be a semigroup. Consider the following statements :

(1) $H_ef = H_{ef}$ for all $e, f \in E(S)$.

(2) $H_ef = H_f$ for all $e, f \in E(S)$, $f \leqslant e$.

Since for $e, f \in E(S)$, $f \leqslant e$ implies $f = ef$, we have (1) implies (2). But for $e, f \in E(S)$, $H_ef = H_e(ef)$ and $ef \leqslant e$, so we have (2) implies (1). Hence the statements (1) and (2) are equivalent.

Therefore the next theorem is obtained.

2.12 (B) Theorem. Let S be a semilattice of groups. Then S is locally factorizable if and only if $H_e f = H_f$ for all e, f ε E(S), f ⩽ e.

We remark from Theorem 2.12 (B) that if S is a semilattice of groups and it is locally factorizable, then for e, f ε E(S) such that f ⩽ e, we have $|H_f| \leqslant |H_e|$ since $|H_e f| \leqslant |H_e|$.

Let S be a semilattice of groups. Then $E(S) \subseteq C(S)$ [2, Lemma 4.8] where for any semigroup T, $C(T) = \{a \in T \mid ax = xa \text{ for all } x \in T\}$ which is called the center of T. Thus ea = ae for all e ε E(S), a ε S. Let e, f ε E(S) be such that f ⩽ e. Then for a ε H_e, we have af ε $H_e f \subseteq H_{ef} = H_f$. Define $\Phi_{e,f} : H_e \to H_f$ by $a\Phi_{e,f} = af$. Since $E(S) \subseteq C(S)$, we have that $\Phi_{e,f}$ is a (group) homomorphism, and the image of $\Phi_{e,f}$ is $H_e f$. We call the homomorphisms $\Phi_{e,f}$ (e,f ε E(S), f ⩽ e) the corresponding homomorphisms of S. Now, we can see that $H_e f = H_f$ for all e,f ε E(S) such that f ⩽ e if and only if all of the corresponding homomorphisms of S are epimorphisms. Thus we have the following corollary.

2.13 Corollary. Let S be a semilattice of groups. Then S is locally factorizable if and only if all of the corresponding homomorphisms of S are epimorphisms.

Let $\{S_\alpha\}_{\alpha \in A}$ be a nonempty family of semigroups. The semigroup S defined on the cartesian product of the sets S_α with coordinatewise multiplication, ie. $(x_\alpha)(y_\alpha) = (x_\alpha y_\alpha)$, is the direct product of the semigroups $\{S_\alpha\}_{\alpha \in A}$ and is denoted by $S = \prod_{\alpha \in A} S_\alpha$. For β ε A, the map $p_\beta : S \to S_\beta$ defined by $(x_\alpha)p_\beta = x_\beta$ for all $(x_\alpha) \in S$ is the projection homomorphism of S onto the β - component S_β. Since a homomorphic image of a group is also a group, it follows that if G is a subgroup of S, then Gp_β is a subgroup of S_β for all $\beta \varepsilon A$. If for each $\alpha \varepsilon A$, G_α is a subgroup of S_α, then $\prod_{\alpha\varepsilon A} G_\alpha$ is clearly a subgroup of S. It is easily seen that $E(\prod_{\alpha\varepsilon A} S_\alpha) = \prod_{\alpha\varepsilon A}(E(S_\alpha))$.

The last theorem of this chapter shows that the direct product of locally factorizable semigroups is locally factorizable. We need the following lemma.

2.14 Lemma. Let $\{S_\alpha\}_{\alpha\varepsilon A}$ be a family of semigroups. Then S_α is factorizable for all $\alpha \varepsilon A$ if and only if $\prod_{\alpha\varepsilon A} S_\alpha$ is factorizable.

Proof : Assume S_α is factorizable for all $\alpha \varepsilon A$. Then for $\alpha \varepsilon A$, $S_\alpha = G_\alpha E(S_\alpha)$ where G_α is a subgroup of S_α. It follows that

$$\prod_{\alpha\varepsilon A} S_\alpha = \prod_{\alpha\varepsilon A}(G_\alpha E(S_\alpha)) = (\prod_{\alpha\varepsilon A} G_\alpha)(\prod_{\alpha\varepsilon A} E(S_\alpha)) = (\prod_{\alpha\varepsilon A} G_\alpha)(E(\prod_{\alpha\varepsilon A} S_\alpha)),$$

hence $\prod_{\alpha\varepsilon A} S_\alpha$ is factorizable.

Conversely, assume $\prod_{\alpha\varepsilon A} S_\alpha = GE(\prod_{\alpha\varepsilon A} S_\alpha)$ for some subgroup G of $\prod_{\alpha\varepsilon A} S_\alpha$. If $\beta \varepsilon A$, then $S_\beta = (\prod_{\alpha\varepsilon A} S_\alpha)p_\beta = (GE(\prod_{\alpha\varepsilon A} S_\alpha))p_\beta$ $= (Gp_\beta)(\prod_{\alpha\varepsilon A} E(S_\alpha))p_\beta = (Gp_\beta)(E(S_\beta))$ and Gp_β is a subgroup of S_β, hence S_β is factorizable. #

2.15 Theorem. Let $\{S_\alpha\}_{\alpha\varepsilon A}$ be a family of semigroups. If S_α is locally factorizable for all $\alpha \varepsilon A$, then $\prod_{\alpha\varepsilon A} S_\alpha$ is locally factorizable. Morever, if $E(S_\alpha) \neq \phi$ for all $\alpha \varepsilon A$, then the converse is true

Proof : Assume each S_α is locally factorizable. Let (e_α) $\varepsilon E(\prod_{\alpha\varepsilon A} S_\alpha)$. Then $e_\alpha \varepsilon E(S_\alpha)$ for all $\alpha \varepsilon A$ and $(e_\alpha)(\prod_{\alpha\varepsilon A} S_\alpha)(e_\alpha)$

$= \prod_{\alpha \in A} (e_\alpha S_\alpha e_\alpha)$. Since $e_\alpha S_\alpha e_\alpha$ is factorizable for all $\alpha \in A$, by Lemma 2.14, $\prod_{\alpha \in A} (e_\alpha S_\alpha e_\alpha)$ is factorizable. Hence $\prod_{\alpha \in A} S_\alpha$ is locally factorizable.

Assume that $\prod_{\alpha \in A} S_\alpha$ is locally factorizable and suppose that $E(S_\alpha) \neq \phi$ for all $\alpha \in A$. Let $\beta \in A$, and $e \in E(S_\beta)$. Since $E(S_\alpha) \neq \phi$ for all $\alpha \in A$, there exists $(e_\alpha) \in \prod_{\alpha \in A} E(S_\alpha)$ such that $(e_\alpha)p_\beta = e_\beta = e$. Then $(e_\alpha) \in E(\prod_{\alpha \in A} S_\alpha)$. Since $\prod_{\alpha \in A} S_\alpha$ is locally factorizable, $(e_\alpha) \prod_{\alpha \in A} S_\alpha (e_\alpha) = GE((e_\alpha) \prod_{\alpha \in A} S_\alpha (e_\alpha))$ for some subgroup G of $(e_\alpha) \prod_{\alpha \in A} S_\alpha (e_\alpha)$. Then $\prod_{\alpha \in A} (e_\alpha S_\alpha e_\alpha) = GE(\prod_{\alpha \in A} (e_\alpha S_\alpha e_\alpha)) = G \prod_{\alpha \in A} (E(e_\alpha S_\alpha e_\alpha))$. Thus $eS_\beta e = e_\beta S_\beta e_\beta = (Gp_\beta)E(e_\beta S_\beta e_\beta)$ and Gp_β is a subgroup of S_β. This shows that S_β is locally factorizable. #

If $E(S_\alpha) = \phi$ for some $\alpha \in A$, then the converse of the theorem is not necessarily true.

<u>Example</u>. $(\mathbb{Z}, \cdot) \times (\mathbb{N}, +)$ is locally factorizable since $(\mathbb{Z}, \cdot) \times (\mathbb{N}, +)$ has no idempotent. But $(\mathbb{Z}, \cdot)$ is not locally factorizable because $1\mathbb{Z}1 = \mathbb{Z}$ is not factorizable under multiplication.