CHAPTER III

TRANSFORMATION SEMIGROUPS

Most of the main results of the thesis are in this chapter. We first characterize locally factorizable partial transformation semigroups. This result is applied to give characterizations of locally factorizable full transformation semigroups and locally factorizable 1-1 partial transformation semigroups, and to show that for any set X, the semigroup of all almost identical partial transformations of X is locally factorizable, and then we have the following results as corollaries : For any set X, the semigroup of all almost identical transformations of X and the semigroup of all almost identical 1-1 partial transformations of X are locally factorizable.

Throughout this chapter, the following notation are adopted : For a set X, let

T_X = the partial transformation semigroup on X,

$\mathcal{T}_X$ = the full transformation semigroup on X,

I_X = the symmetric inverse semigroup on X or the 1-1 partial transformation semigroup on X,

U_X = the semigroup of all almost identical partial transformations of X, ie.,

$U_X = \{\alpha \in T_X \mid |S(\alpha)| < \infty\}$ where $S(\alpha) = \{x \in \Delta\alpha \mid x\alpha \neq x\}$, the shift of α,

V_X = the semigroup of all almost identical transformations of X, ie.,

$V_X = \{\alpha \in \mathcal{J}_X \mid |S(\alpha)| < \infty\}$,

W_X = the semigroup of all almost identical 1-1 partial transformations of X, ie.,

$W_X = \{\alpha \in I_X \mid |S(\alpha)| < \infty\}$,

G_X = the symmetric group (the permutation group) on X.

The first theorem gives a characterization of locally factorizable partial transformation semigroups. To prove the theorem, the following lemma is required :

3.1 Lemma. Let X be a set and $\alpha, \beta \in T_X$. If $\alpha = \beta\gamma$ for some $\gamma \in T_X$, then $\Delta\alpha \subseteq \Delta\beta$ and for $x \in \Delta\alpha$, $x\pi_\alpha = \bigcup_{y \in x\pi_\alpha} y\pi_\beta$.

Proof : Let $\alpha, \beta \in T_X$, and assume that $\alpha = \beta\gamma$ for some $\gamma \in T_X$. Then $\Delta\alpha = \Delta\beta\gamma \subseteq \Delta\beta$. Next, let $x \in \Delta\alpha$. If $y \in x\pi_\alpha$ and $t \in y\pi_\beta$, then $x\alpha = y\alpha$, $y\beta = t\beta$, so $x\alpha = y\alpha = y\beta\gamma = (y\beta)\gamma = (t\beta)\gamma = t\beta\gamma = t\alpha$, and hence $t \in x\pi_\alpha$. This proves that $\bigcup_{y \in x\pi_\alpha} y\pi_\beta \subseteq x\pi_\alpha$. If $s \in x\pi_\alpha$, then $s \in s\pi_\beta \subseteq \bigcup_{y \in x\pi_\alpha} y\pi_\beta$. Hence $x\pi_\alpha = \bigcup_{y \in x\pi_\alpha} y\pi_\beta$. #

3.2 Theorem. The partial transformation semigroup on a set X is locally factorizable if and only if X is finite.

Proof : If T_X is locally factorizable, then T_X is factorizable since T_X has an identity, so X is finite [7, Theorem 3.1].

Assume X is a finite set. Let $\alpha \in E(T_X)$. To show that $\alpha T_X \alpha = H_\alpha E(\alpha T_X \alpha)$, let $\rho \in \alpha T_X \alpha$. If $\rho = 0$ then $\rho = \alpha 0 \in H_\alpha E(\alpha T_X \alpha)$. Assume $\rho \neq 0$. Since $\rho \in \alpha T_X \alpha$, $\rho = \alpha\rho'\alpha$ for some $\rho' \in T_X$. It follows that $\Delta\rho \subseteq \Delta\alpha$, $\nabla\rho \subseteq \nabla\alpha$. By Lemma 3.1, we have that for each $x \in \Delta\rho$, $x\pi_\rho$ =

$\bigcup_{y \varepsilon x\pi_\rho} y\pi_\alpha$. For each $a \varepsilon \nabla\rho$, if $x \varepsilon a\rho^{-1}$, then $a\rho^{-1} = x\pi_\rho = \bigcup_{y \varepsilon x\pi_\rho} y\pi_\alpha$ $= \bigcup_{y \varepsilon a\rho^{-1}} y\pi_\alpha$. Hence for each $a \varepsilon \nabla\rho$, $a\rho^{-1} = \bigcup_{y\varepsilon a\rho^{-1}} y\pi_\alpha$. Since X is finite, for each $a \varepsilon \nabla\rho$, there exist $d_a^1, d_a^2, \ldots, d_a^{n_a}$ in $a\rho^{-1}$ such that $a\rho^{-1} = \bigcup_{i=1}^{n_a} d_a^i\pi_\alpha$ is a disjoint union. Therefore $\Delta\rho = \bigcup_{a\varepsilon\nabla\rho} a\rho^{-1}$ $= \bigcup_{a\varepsilon\nabla\rho} (\bigcup_{i=1}^{n_a} d_a^i\pi_\alpha)$ which is a disjoint union. Because X is finite, $|\Delta\alpha/\pi_\alpha| = |\nabla\alpha| < \infty$. It follows that $|\Delta\alpha/\pi_\alpha \setminus \{d_a^1\pi_\alpha | a \varepsilon \nabla\rho\}| = |\nabla\alpha \setminus \nabla\rho|$. Then there exists a one-to-one map ψ from the set $\Delta\alpha/\pi_\alpha \setminus \{d_a^1\pi_\alpha | a \varepsilon \nabla\rho\}$ onto the set $\nabla\alpha \setminus \nabla\rho$. Define the map β from $\Delta\alpha$ into X as follows :

$$x\beta = \begin{cases} a & \text{if } x \varepsilon d_a^1\pi_\alpha,\ a \varepsilon \nabla\rho, \\ (x\pi_\alpha)\psi & \text{if } x \varepsilon \Delta\alpha \setminus \bigcup_{a\varepsilon\nabla\rho} d_a^1\pi_\alpha. \end{cases}$$

Then

$$x\beta = \begin{cases} a & \text{if } x \varepsilon d_a^1\pi_\alpha,\ a \varepsilon \nabla\rho, \\ (x\pi_\alpha)\psi & \text{if } x\pi_\alpha \varepsilon \Delta\alpha/\pi_\alpha \setminus \{d_a^1\pi_\alpha | a \varepsilon \nabla\rho\}. \end{cases}$$

Hence $\Delta\beta = \Delta\alpha$, $\nabla\beta = \nabla\alpha$ since ψ is onto , $\pi_\beta = \pi_\alpha$ since ψ is one-to-one. Therefore $\beta \varepsilon H_\alpha$. (Chapter I, page 8). Since $\nabla\psi = \nabla\alpha \setminus \nabla\rho$, we have that $\nabla\rho$ and $\{(d_a^i\pi_\alpha)\psi \mid a \varepsilon \nabla\rho,\ i = 2, 3, \ldots, n_a\}$ are disjoint sets. Define the map γ from $\nabla\rho \cup \{(d_a^i\pi_\alpha)\psi \mid a \varepsilon \nabla\rho,\ i = 2, 3, \ldots, n_a\}$ into $\nabla\rho$ as follows :

$$x\gamma = \begin{cases} x & \text{if } x \varepsilon \nabla\rho, \\ a & \text{if } x = (d_a^i\pi_\alpha)\psi,\ a \varepsilon \nabla\rho,\ i = 2, 3, \ldots, n_a. \end{cases}$$

Then $\nabla\gamma = \nabla\rho$ and $\Delta\gamma \subseteq \nabla\rho \cup \nabla\psi \subseteq \nabla\alpha \cup \nabla\alpha = \nabla\alpha$. Hence $\nabla\gamma \subseteq \Delta\gamma$ and if $x \varepsilon \nabla\gamma$, then $x \varepsilon \nabla\rho$, so $x\gamma = x$. Thus $\gamma \varepsilon E(T_X)$ (Chapter I, page 7). Since $\nabla\gamma = \nabla\rho \subsetneq \nabla\alpha \subseteq \Delta\alpha$, we have $\Delta\gamma\alpha = (\nabla\gamma \cap \Delta\alpha)\gamma^{-1} = (\nabla\gamma)\gamma^{-1} = \Delta\gamma$ and $x\gamma\alpha = x\gamma$ for all $x \varepsilon \Delta\gamma$ (because $y\alpha = y$ for all $y \varepsilon \nabla\alpha$). Therefore $\gamma\alpha = \gamma$. It follows that $(\alpha\gamma\alpha)^2 = \alpha\gamma\alpha^2\gamma\alpha = \alpha\gamma^2\alpha = \alpha\gamma\alpha$, hence $\alpha\gamma\alpha \varepsilon E(\alpha T_X\alpha)$.

Claim that $\rho = \beta\gamma$. Let $x \in \Delta\rho$, and let $a = x\rho$. Then $a \in \nabla\rho$, so $x \in x\pi_\rho = a\rho^{-1} = \bigcup_{i=1}^{n_a} d_a^i\pi_\alpha$. If $x \in d_a^1\pi_\alpha$, then $x\beta = a \in \nabla\rho$, and hence $x\beta\gamma = a\gamma = a = x\rho$. If $x \in d_a^i\pi_\alpha$ for some $i \in \{2, 3, \ldots, n_a\}$, then $x\beta\gamma = ((x\pi_\alpha)\psi)\gamma = ((d_a^i\pi_\alpha)\psi)\gamma = a = x\rho$. This shows that $\Delta\rho \subseteq \Delta\beta\gamma$ and $x\rho = x\beta\gamma$ for every $x \in \Delta\rho$. Next, let $y \in \Delta\beta\gamma$. Then $y\beta \in \Delta\gamma = \nabla\rho \cup \{(d_a^i\pi_\alpha)\psi \mid a \in \nabla\rho,\ i = 2, 3, \ldots, n_a\}$.

<u>Case $y\beta \in \nabla\rho$.</u> Then $(y\beta)\gamma = y\beta \in \nabla\rho$. From the definition of β, $y \in d_a^1\pi_\alpha$, $y\beta = a$ for some $a \in \nabla\rho$. Since $d_a^1\pi_\alpha \subseteq a\rho^{-1}$, $y \in a\rho^{-1}$ and so $y\rho = a$. Hence $y\beta\gamma = a\gamma = a = y\rho$.

<u>Case $y\beta = (d_a^i\pi_\alpha)\psi$ for some $a \in \nabla\rho$, $i \in \{2, 3, \ldots, n_a\}$.</u> Then $y\pi_\alpha = d_a^i\pi_\alpha$ since ψ is one-to-one. Hence $y\beta\gamma = ((y\pi_\alpha)\psi)\gamma = ((d_a^i\pi_\alpha)\psi)\gamma = a = y\rho$ since $y \in d_a^i\pi_\alpha \subseteq a\rho^{-1}$.

Hence $\beta\gamma = \rho$. But $\beta \in H_\alpha$ and $\gamma\alpha = \gamma$, it follows that $\rho = \beta\gamma = \beta\alpha\gamma\alpha \in H_\alpha E(\alpha T_X\alpha)$. Hence $\alpha T_X\alpha = H_\alpha E(\alpha T_X\alpha)$.

Therefore, the theorem is proved. #

Let X be a set, S the transformation semigroup $\mathcal{I}_X$ or I_X, and $\alpha \in E(S)$. We know that the $\mathcal{H}$ - class of T_X containing α, $H_\alpha = \{\beta \in T_X \mid \Delta\beta = \Delta\alpha,\ \nabla\beta = \nabla\alpha$ and $\pi_\alpha = \pi_\beta\}$. If $\Delta\alpha = X$, then for all $\beta \in H_\alpha$, $\Delta\beta = X$. If α is one-to-one, then π_α is the identity relation on $\Delta\alpha$, so for $\beta \in H_\alpha$, π_β is the identity relation on $\Delta\beta$ which implies β is one-to-one. Hence $H_\alpha \subseteq S$. Hence, for $\alpha \in S$, the $\mathcal{H}$ - class of S containing α is the $\mathcal{H}$ - class of T_X containing α. Using this result and Theorem 3.2, we have the following corollary.

3.3 <u>Corollary</u>. Let X be a set and let S be $\mathcal{I}_X$ or I_X. Then the transformation semigroup S is locally factorizable if and only if X is finite.

Proof : If S is locally factorizable, then S is factorizable since S has an identity, so X is finite [1, Corollary of Theorem 3.1 and 7, Theorem 3.2].

Assume that X is finite. Let $\alpha \in E(S)$. Then $\alpha \in E(T_X)$. Therefore $\alpha T_X \alpha = H_\alpha E(\alpha T_X \alpha)$ where H_α is the $\mathcal{H}$ - class of T_X (and S also) containing α, by Proposition 2.1 and Theorem 3.2. To show $\alpha S \alpha = H_\alpha E(\alpha S \alpha)$, let $\rho \in S$. Then $\alpha\rho\alpha \in \alpha T_X \alpha$. Since $\alpha T_X \alpha = H_\alpha E(\alpha T_X \alpha)$, we have $\alpha\rho\alpha = \beta\alpha\gamma\alpha$ for some $\beta \in H_\alpha$ and $\gamma \in T_X$ such that $\alpha\gamma\alpha \in E(\alpha T_X \alpha)$. Because $\alpha\rho\alpha \in S$ and $H_\alpha \subseteq S$, it follows that $\alpha\gamma\alpha = \alpha\alpha\gamma\alpha = \beta'\beta\alpha\gamma\alpha = \beta'\alpha\rho\alpha \in S$ where β' is the group inverse of β in the group H_α. Therefore $\alpha\gamma\alpha = \alpha(\alpha\gamma\alpha)\alpha \in E(\alpha S \alpha)$, and hence $\alpha\rho\alpha \in H_\alpha E(\alpha S \alpha)$. Thus $\alpha S \alpha = H_\alpha E(\alpha S \alpha)$. Therefore S is locally factorizable. #

Let T be a subsemigroup of a semigroup S. Let $e \in E(T)$. Then $e \in E(S)$, H_e is the maximum subgroup of S having e as its identity and $H_e \cap T$ is a subsemigroup of T having e as its identity. If $H_e \cap T$ is a subgroup of T, then it becomes the maximum subgroup of T having e as its identity (since every subgroup of T is a subgroup of S), and it then follows that $H_e \cap T$ is the $\mathcal{H}$ - class of T containing e.

3.4 Lemma. Let X be a set and S the transformation semigroup U_X, V_X or W_X. If α is an idempotent of S, then $H_\alpha \cap S$ is the $\mathcal{H}$ - class of S containing α where H_α is the $\mathcal{H}$ - class of T_X containing α.

Proof : As mentioned above, to show that $H_\alpha \cap S$ is the $\mathcal{H}$ - class of S containing α, it suffices to show that $H_\alpha \cap S$ is a subgroup of S. Now, $H_\alpha \cap S$ is a subsemigroup of S having α as its identity. Let $\beta \in H_\alpha \cap S$. Since H_α is a subgroup of T_X having α as its identity, there exists $\gamma \in H_\alpha$ such that $\beta\gamma = \gamma\beta = \alpha$. Since γ, β and α are all

$\mathscr{H}$ - related, $\Delta\gamma = \Delta\beta = \Delta\alpha$. To show $S(\gamma)$ is finite, it suffices to show that $S(\gamma) \setminus S(\beta)$ is finite since $S(\gamma) \subsetneq (S(\gamma) \setminus S(\beta)) \cup S(\beta)$ and $S(\beta)$ is finite. For $x \in S(\gamma) \setminus S(\beta)$, we have $x \in \Delta\gamma = \Delta\beta = \Delta\alpha$, $x\gamma \neq x$ but $x\beta = x$. Hence for $x \in S(\gamma) \setminus S(\beta)$, we have $x\alpha = x\beta\gamma = (x\beta)\gamma = x\gamma \neq x$. Thus $S(\gamma) \setminus S(\beta) \subseteq S(\alpha)$. Since $S(\alpha)$ is finite, $S(\gamma) \setminus S(\beta)$ is finite, so $\gamma \in S$. This shows that $H_\alpha \cap S$ is a subgroup of S. Therefore $H_\alpha \cap S$ is the $\mathscr{H}$ - class of S containing α. #

Given a set X, $\alpha \in T_X$ and a subset A of X, let $\alpha|_{\Delta\alpha \cap A}$ denote the restriction of α to $\Delta\alpha \cap A$. Then for any set X, if $\alpha \in T_X$, then $\alpha|_{\Delta\alpha \cap A} \in T_X$ for all subsets A of X ; note that $\alpha|_{\Delta\alpha \cap A}$ need not belong to T_A.

3.5 Lemma. Let X be a set and $\alpha, \beta \in T_X$. If A is a subset of X such that $\alpha|_{\Delta\alpha \cap A}$, $\beta|_{\Delta\beta \cap A} \in T_A$, then $(\alpha|_{\Delta\alpha \cap A})(\beta|_{\Delta\beta \cap A}) = (\alpha\beta)|_{\Delta\alpha\beta \cap A}$.

Proof : Let $x \in \Delta(\alpha|_{\Delta\alpha \cap A})(\beta|_{\Delta\beta \cap A})$. Then $x \in \Delta\alpha \cap A$ and $x\alpha \in \Delta\beta \cap A$. Since $x \in \Delta\alpha$ and $x\alpha \in \Delta\beta$, we have that $x \in \Delta\alpha\beta$. Hence $x \in \Delta\alpha\beta \cap A$ and $x(\alpha|_{\Delta\alpha \cap A})(\beta|_{\Delta\beta \cap A}) = x\alpha\beta = x(\alpha\beta)|_{\Delta\alpha\beta \cap A}$.

Next, let $y \in \Delta(\alpha\beta)|_{\Delta\alpha\beta \cap A}$. Then $y \in \Delta\alpha\beta \cap A \subseteq A$, so $y \in \Delta\alpha \cap A$ and $y\alpha \in \Delta\beta$. Since $\alpha|_{\Delta\alpha \cap A} \in T_A$ and $y \in \Delta\alpha \cap A$, we have $y\alpha \in A$. Hence $y\alpha \in \Delta\beta \cap A$. Thus $y \in \Delta((\alpha|_{\Delta\alpha \cap A})(\beta|_{\Delta\beta \cap A}))$ and $y(\alpha\beta)|_{\Delta\alpha\beta \cap A} = y\alpha\beta = (y\alpha)(\beta|_{\Delta\beta \cap A}) = (y(\alpha|_{\Delta\alpha \cap A}))(\beta|_{\Delta\beta \cap A}) = y(\alpha|_{\Delta\alpha \cap A})(\beta|_{\Delta\beta \cap A})$.

Hence, we have that $(\alpha|_{\Delta\alpha \cap A})(\beta|_{\Delta\beta \cap A}) = (\alpha\beta)|_{\Delta\alpha\beta \cap A}$. #

Let X be a set and $Y \subseteq X$. Then $T_Y \subseteq T_X$. Let $\alpha \in T_Y$, and H_α and H'_α the $\mathscr{H}$ - class containing α of T_X and T_Y, respectively. Then

$$H_\alpha = \{\beta \in T_X \mid \Delta\beta = \Delta\alpha,\ \nabla\beta = \nabla\alpha,\ \pi_\beta = \pi_\alpha\},$$

$$H'_\alpha = \{\beta \in T_Y \mid \Delta\beta = \Delta\alpha,\ \nabla\beta = \nabla\alpha,\ \pi_\beta = \pi_\alpha\}.$$

Since $T_Y \subseteq T_X$, $H'_\alpha \subseteq H_\alpha$. If $\beta \in H_\alpha$, then $\Delta\beta = \Delta\alpha \subseteq Y$, $\nabla\beta = \nabla\alpha \subseteq Y$, $\pi_\beta = \pi_\alpha$, so $\beta \in H'_\alpha$. This proves that $H_\alpha = H'_\alpha$.

Hence, for a subset Y of a set X, if $\alpha \in T_Y$, then the $\mathcal{H}$ - class of T_Y containing α is the $\mathcal{H}$ - class of T_X containing α.

3.6 <u>Theorem</u>. For any set X, the semigroup of all almost identical partial transformations of X is locally factorizable.

<u>Proof</u> : Let X be a set and let $\alpha \in E(U_X)$. To show that $\alpha U_X \alpha = (H_\alpha \cap U_X)E(\alpha U_X \alpha)$ where H_α is the $\mathcal{H}$ - class of T_X containing α, let ρ belong to U_X. Set $A = S(\alpha) \cup (S(\alpha))\alpha \cup S(\rho) \cup (S(\rho))\rho$. Then $|A| < \infty$ since $|S(\alpha)| < \infty$ and $|S(\rho)| < \infty$. Let $\alpha' = \alpha|_{\Delta\alpha \cap A}$ and $\rho' = \rho|_{\Delta\rho \cap A}$. Then $\Delta\alpha'$, $\Delta\rho' \subseteq A$. For $x \in \Delta\alpha' = \Delta\alpha \cap A$, if $x \in S(\alpha)$, then $x\alpha' = x\alpha \in (S(\alpha))\alpha \subseteq A$, and if $x \notin S(\alpha)$, then $x\alpha' = x\alpha = x \in A$. Then $\nabla\alpha' \subseteq A$. Similarly, $\nabla\rho' \subseteq A$. Thus $\alpha', \rho' \in T_A$. Now $\nabla\alpha' \subseteq \nabla\alpha \cap A$. Since $\nabla\alpha \subseteq \Delta\alpha$, $\nabla\alpha \cap A \subseteq \Delta\alpha \cap A = \Delta\alpha'$. Then $\nabla\alpha' = (\Delta\alpha')\alpha' \supseteq (\nabla\alpha \cap A)\alpha' = (\nabla\alpha \cap A)\alpha = \nabla\alpha \cap A$ since $x\alpha = x$ for all $x \in \nabla\alpha$. Hence $\nabla\alpha' = \nabla\alpha \cap A$. Since $\alpha' = \alpha|_{\Delta\alpha \cap A} \in T_A$ and $\alpha^2 = \alpha$, by Lemma 3.5, we have that $(\alpha')^2 = (\alpha|_{\Delta\alpha \cap A})(\alpha|_{\Delta\alpha \cap A}) = \alpha^2|_{\Delta\alpha^2 \cap A} = \alpha|_{\Delta\alpha \cap A} = \alpha' \in E(T_A)$. By Lemma 3.5, we have $\alpha'\rho'\alpha' = (\alpha|_{\Delta\alpha \cap A})(\rho|_{\Delta\rho \cap A})(\alpha|_{\Delta\alpha \cap A}) = (\alpha\rho\alpha)|_{\Delta\alpha\rho\alpha \cap A}$, so $\Delta\alpha'\rho'\alpha' = \Delta\alpha\rho\alpha \cap A$. Since A is finite, T_A is locally factorizable by Theorem 3.2. But the $\mathcal{H}$ - class of T_A containing α' is the $\mathcal{H}$ - class $H_{\alpha'}$ of T_X containing α', so by Proposition 2.1, we have $\alpha' T_A \alpha' = H_{\alpha'} E(\alpha' T_A \alpha')$. Then $\alpha'\rho'\alpha' = \lambda\alpha'\gamma\alpha'$ for some $\lambda \in H_{\alpha'}$ and $\gamma \in T_A$ such that $\alpha'\gamma\alpha' \in E(\alpha' T_A \alpha')$. Since $\lambda \in H_{\alpha'}$, we have $\Delta\lambda = \Delta\alpha' = \Delta\alpha \cap A \subseteq \Delta\alpha$, $\nabla\lambda = \nabla\alpha' = \nabla\alpha \cap A$ and $\pi_\lambda = \pi_{\alpha'}$. Define a map $\bar{\lambda}$ from $\Delta\alpha$ into X as follows :

$$x\bar{\lambda} = \begin{cases} x\lambda & \text{if } x \in \Delta\lambda, \\ x & \text{if } x \in \Delta\alpha \setminus \Delta\lambda. \end{cases}$$

Note that $\Delta\lambda = \Delta\alpha \cap A \subseteq A$, $\Delta\alpha \setminus \Delta\lambda = \Delta\alpha \setminus (\Delta\alpha \cap A) = \Delta\alpha \setminus A$. Since $\nabla\alpha \subseteq \Delta\alpha$, $\nabla\alpha \setminus A \subseteq \Delta\alpha \setminus A$. If $x \in \Delta\alpha \setminus A$, then $x \notin S(\alpha)$ since $S(\alpha) \subseteq A$, so $x = x\alpha \in \nabla\alpha \setminus A$. Thus $\nabla\alpha \setminus A = \Delta\alpha \setminus A$. It follows that $\nabla\bar{\lambda} = \nabla\lambda \cup (\Delta\alpha \setminus \Delta\lambda) = (\nabla\alpha \cap A) \cup (\Delta\alpha \setminus A) = (\nabla\alpha \cap A) \cup (\nabla\alpha \setminus A) = \nabla\alpha$. Hence $\nabla\bar{\lambda} = \nabla\alpha$. Since $(\Delta\lambda)\bar{\lambda} = \nabla\lambda$ and $(\Delta\bar{\lambda} \setminus \Delta\lambda)\bar{\lambda} = (\Delta\alpha \setminus \Delta\lambda)\bar{\lambda} = \Delta\alpha \setminus \nabla\lambda$, it follows that $(\Delta\lambda)\bar{\lambda} \cap (\Delta\bar{\lambda} \setminus \Delta\lambda)\bar{\lambda} = \phi$. Thus for $x, y \in \Delta\bar{\lambda}$, $x\bar{\lambda} = y\bar{\lambda}$ implies $x, y \in \Delta\lambda$ or $x, y \notin \Delta\lambda$, and so $x\lambda = y\lambda$ or $x = y$. Let $a, b \in \Delta\alpha = \Delta\bar{\lambda}$ be such that $a\alpha = b\alpha$. Suppose $a \in \Delta\alpha'$ $(= \Delta\alpha \cap A)$ and $b \notin \Delta\alpha'$. Then $a \in A$ and $b \notin A$. Since $\nabla\alpha' \subseteq A$ and $S(\alpha) \subseteq A$, we have that $a\alpha' \in A$ and $b\alpha = b \notin A$. Hence $a\alpha' = a\alpha = b\alpha = b$ which is a contradiction. This proves that for $x,y \in \Delta\alpha$, $x\alpha = y\alpha$ implies either $x,y \in A$ or $x,y \notin A$, and so $x\alpha' = y\alpha'$ or $x = y$ since $S(\alpha) \subseteq A$. Hence, for $x, y \in \Delta\bar{\lambda} = \Delta\alpha$,

$$\begin{aligned}(x,y) \in \pi_{\bar{\lambda}} &\Longleftrightarrow x\bar{\lambda} = y\bar{\lambda}\\ &\Longleftrightarrow x\lambda = y\lambda \text{ or } x = y\\ &\Longleftrightarrow x\alpha' = y\alpha' \text{ or } x = y \quad (\text{since } \pi_\lambda = \pi_{\alpha'})\\ &\Longleftrightarrow x\alpha = y\alpha\\ &\Longleftrightarrow (x,y) \in \pi_\alpha.\end{aligned}$$

Therefore we have $\pi_{\bar{\lambda}} = \pi_\alpha$. Hence $\bar{\lambda} \in H_\alpha$ (Chapter I, page 8). Clearly, $S(\bar{\lambda}) = S(\lambda)$. Then $|S(\bar{\lambda})| = |S(\lambda)| \leqslant |A| < \infty$. Thus $\bar{\lambda} \in H_\alpha \cap U_X$. Since $\Delta\gamma \subseteq A$, we have that $\Delta\gamma \cup (\Delta\alpha\rho\alpha \setminus A)$ is a disjoint union. Define the map $\bar{\gamma}$ from $\Delta\gamma \cup (\Delta\alpha\rho\alpha \setminus A)$ into X as follows :

$$x\bar{\gamma} = \begin{cases} x\gamma & \text{if } x \in \Delta\gamma,\\ x & \text{if } x \in \Delta\alpha\rho\alpha \setminus A.\end{cases}$$

Then $|S(\bar{\gamma})| = |S(\gamma)| \leqslant |A| < \infty$, so $\bar{\gamma} \in U_X$. To show that $\alpha\bar{\gamma}\alpha = (\alpha\bar{\gamma}\alpha)^2$, first note that $\Delta\bar{\gamma} \cap A = (\Delta\gamma \cup (\Delta\alpha\rho\alpha \setminus A)) \cap A = \Delta\gamma \cap A = \Delta\gamma$ since $\Delta\gamma \subseteq A$. Also, $\alpha|_{\Delta\alpha \cap A} = \alpha' \in T_A$, and $\bar{\gamma}|_{\Delta\gamma} = \gamma \in T_A$. It then follows from Lemma 3.5 that $(\alpha\bar{\gamma}\alpha)|_{\Delta\alpha\bar{\gamma}\alpha \cap A} = \alpha'\gamma\alpha' \in T_A$. But $(\alpha'\gamma\alpha')^2 = \alpha'\gamma\alpha'$, so

$(\alpha\bar{\gamma}\alpha)|_{\Delta\alpha\bar{\gamma}\alpha\cap A} = ((\alpha\bar{\gamma}\alpha)|_{\Delta\alpha\bar{\gamma}\alpha\cap A})^2 = (\alpha\bar{\gamma}\alpha)^2|_{\Delta(\alpha\bar{\gamma}\alpha)^2\cap A}$ (Lemma 3.5).

Hence $\Delta\alpha\bar{\gamma}\alpha\cap A = \Delta(\alpha\bar{\gamma}\alpha)^2\cap A$ and $(\alpha\bar{\gamma}\alpha)|_{\Delta\alpha\bar{\gamma}\alpha\cap A} = (\alpha\bar{\gamma}\alpha)^2|_{\Delta\alpha\bar{\gamma}\alpha\cap A}$. Let $x \in \Delta(\alpha\bar{\gamma}\alpha)$. If $x \in A$, then $x \in \Delta\alpha\bar{\gamma}\alpha\cap A$, so $x\alpha\bar{\gamma}\alpha = x(\alpha\bar{\gamma}\alpha)|_{\Delta\alpha\bar{\gamma}\alpha\cap A}$ $= x(\alpha\bar{\gamma}\alpha)^2|_{\Delta\alpha\bar{\gamma}\alpha\cap A} = x(\alpha\bar{\gamma}\alpha)^2$. If $x \notin A$, then $x \notin S(\alpha)$ since $S(\alpha)\subseteq A$, so $x = x\alpha \in \Delta\bar{\gamma}\setminus A$ which implies $x = x\alpha = (x\alpha)\bar{\gamma} \in \Delta\alpha\setminus A$, hence $x = x\alpha\bar{\gamma} = (x\alpha\bar{\gamma})\alpha = x\alpha\bar{\gamma}\alpha$. Thus $x\alpha\bar{\gamma}\alpha = x(\alpha\bar{\gamma}\alpha)^2$. This proves $\Delta\alpha\bar{\gamma}\alpha \subseteq \Delta(\alpha\bar{\gamma}\alpha)^2$ and $x\alpha\bar{\gamma}\alpha = x(\alpha\bar{\gamma}\alpha)^2$ for all $x \in \Delta\alpha\bar{\gamma}\alpha$. But $\Delta(\alpha\bar{\gamma}\alpha)^2 \subseteq \Delta\alpha\bar{\gamma}\alpha$, so we have $\alpha\bar{\gamma}\alpha = (\alpha\bar{\gamma}\alpha)^2 \in E(\alpha T_X\alpha)$.

Next, we claim that $\alpha\rho\alpha = \bar{\lambda}\alpha\bar{\gamma}\alpha$. We have that $\alpha|_{\Delta\alpha\cap A} = \alpha'$, $\rho|_{\Delta\rho\cap A} = \rho'$, $\bar{\lambda}|_{\Delta\bar{\lambda}\cap A} = \bar{\lambda}|_{\Delta\lambda} = \lambda$, $\bar{\gamma}|_{\Delta\bar{\gamma}\cap A} = \bar{\gamma}|_{\Delta\gamma} = \gamma$ and $\alpha', \rho', \lambda, \gamma$ are all in T_A. Also, we have $\alpha'\rho'\alpha' = \lambda\alpha'\gamma\alpha'$. Then by Lemma 3.5, we get $(\alpha\rho\alpha)|_{\Delta\alpha\rho\alpha\cap A} = \alpha'\rho'\alpha' = \lambda\alpha'\gamma\alpha' = (\bar{\lambda}\alpha\bar{\gamma}\alpha)|_{\Delta\bar{\lambda}\alpha\bar{\gamma}\alpha\cap A}$. Hence $\Delta\alpha\rho\alpha\cap A = \Delta\bar{\lambda}\alpha\bar{\gamma}\alpha\cap A$ and $(\alpha\rho\alpha)|_{\Delta\alpha\rho\alpha\cap A} = (\bar{\lambda}\alpha\bar{\gamma}\alpha)|_{\Delta\alpha\rho\alpha\cap A}$. Let $x \in \Delta\alpha\rho\alpha$. If $x \in A$, then $x \in \Delta\alpha\rho\alpha\cap A$, so $x\alpha\rho\alpha = x\bar{\lambda}\alpha\bar{\gamma}\alpha$. If $x \notin A$, then $x \notin S(\alpha)$, $x \notin S(\rho)$, $x \in \Delta\alpha\setminus A = \Delta\alpha\setminus\Delta\lambda$, $x \in \Delta\alpha\rho\alpha\setminus A$ and hence $x\alpha = x$, $x\alpha\rho = x\rho = x$, $x\bar{\lambda} = x$, $x\bar{\gamma} = x$ which implies $x\alpha\rho\alpha = x = x\bar{\lambda}\alpha\bar{\gamma}\alpha$. This proves that $\Delta\alpha\rho\alpha \subseteq \Delta\bar{\lambda}\alpha\bar{\gamma}\alpha$ and $x\alpha\rho\alpha = x\bar{\lambda}\alpha\bar{\gamma}\alpha$ for all $x \in \Delta\alpha\rho\alpha$. Next, let $y \in \Delta\bar{\lambda}\alpha\bar{\gamma}\alpha$. Then $y \in \Delta\bar{\lambda}$ $= \Delta\alpha$. If $y \in A$, then $y \in \Delta\bar{\lambda}\alpha\bar{\gamma}\alpha\cap A = \Delta\alpha\rho\alpha\cap A$ and thus $y\alpha\rho\alpha = y\bar{\lambda}\alpha\bar{\gamma}\alpha$. Assume $y \notin A$. Then $y \in \Delta\bar{\lambda}\setminus A = \Delta\alpha\setminus A$, so $y\bar{\lambda} = y$, $y\alpha = y$. Thus $y\bar{\lambda}\alpha\bar{\gamma}\alpha = y\bar{\gamma}\alpha$. Therefore $y \in \Delta\bar{\gamma}\setminus A = \Delta\alpha\rho\alpha\setminus A \subseteq \Delta\alpha\rho\alpha$. Hence $y\bar{\lambda}\alpha\bar{\gamma}\alpha = y\bar{\gamma}\alpha$ $= y\alpha = y$ since $y \notin S(\alpha)$. Also, $y \in \Delta\alpha\rho\alpha$ and $y\alpha\rho\alpha = y\rho\alpha = y\alpha = y$ since $y \notin S(\rho)$. It follows that $y\bar{\lambda}\alpha\bar{\gamma}\alpha = y = y\alpha\rho\alpha$. Hence we have proven $\alpha\rho\alpha = \bar{\lambda}\alpha\bar{\gamma}\alpha$, so we have the claim.

Therefore $\alpha\rho\alpha = \bar{\lambda}\alpha\bar{\gamma}\alpha \in (H_\alpha\cap U_X)E(\alpha U_X\alpha)$, so $\alpha U_X\alpha \subseteq (H_\alpha\cap U_X)E(\alpha U_X\alpha)$. Hence $\alpha U_X\alpha = (H_\alpha\cap U_X)E(\alpha U_X\alpha)$. This proves that U_X is locally factorizable for any set X (Lemma 3.4). #

Let X be a set and S the transformation semigroup V_X or W_X. Let $\alpha \in E(S)$. Then, by Lemma 3.4 $H_\alpha \cap U_X$ is the $\mathcal{H}$ - class of U_X containing α, so $H_\alpha \cap U_X$ is the maximum subgroup of U_X containing α. But $H_\alpha \cap U_X = \{\beta \in U_X \mid \Delta\beta = \Delta\alpha, \nabla\beta = \nabla\alpha, \pi_\alpha = \pi_\beta\}$, so if $\Delta\alpha = X$, then $\Delta\beta = X$ for all $\beta \in H_\alpha \cap U_X$, and if π_α is the identity relation on $\Delta\alpha$, then for each $\beta \in H_\alpha \cap U_X$, π_β is the identity relation on $\Delta\beta = \Delta\alpha$. Then $H_\alpha \cap U_X \subseteq S \subseteq U_X$, so $H_\alpha \cap U_X$ is the maximum subgroup of S having α as its identity, so it is the $\mathcal{H}$ - class of S containing α. Using this fact, we have the following corollary.

3.7 <u>Corollary</u>. For any set X, the semigroup of all almost identical transformations of X and the semigroup of all almost identical 1-1 partial transformations of X are locally factorizable.

<u>Proof</u> : Let X be a set and let S be V_X or W_X. Then $S \subseteq U_X$. Let $\alpha \in E(S)$. It follows from Lemma 3.4 and Theorem 3.6 that $\alpha U_X \alpha = (H_\alpha \cap U_X)E(\alpha U_X \alpha)$. Now $H_\alpha \cap U_X$ is the $\mathcal{H}$ - class of S having α as its identity. To show $\alpha S \alpha = (H_\alpha \cap U_X)E(\alpha S \alpha)$, let $\rho \in S$. Then $\alpha\rho\alpha \in \alpha U_X \alpha$, and hence $\alpha\rho\alpha = \beta\alpha\gamma\alpha$ for some $\beta \in H_\alpha \cap U_X$ and $\gamma \in U_X$ such that $\alpha\gamma\alpha \in E(\alpha U_X \alpha)$. It follows that $\alpha\gamma\alpha = \alpha\alpha\gamma\alpha = \beta'\beta\alpha\gamma\alpha = \beta'\alpha\rho\alpha \in S$ where β' is the group inverse of β in the group $H_\alpha \cap U_X$. Hence $\alpha\gamma\alpha = \alpha(\alpha\gamma\alpha)\alpha \in E(\alpha S \alpha)$. Therefore $\alpha S \alpha \subseteq (H_\alpha \cap U_X)E(\alpha S \alpha)$. It follows that $\alpha S \alpha = (H_\alpha \cap U_X)E(\alpha S \alpha)$, so $\alpha S \alpha$ is factorizable. #

Let X be a set and ξ a cardinal number, $1 \leqslant \xi \leqslant |X|$. Let R_ξ, $\bar{R}_\xi$, D_ξ and $\bar{D}_\xi$ denote the following transformation semigroups :

$$R_\xi = \{\alpha \in T_X \mid |\nabla\alpha| < \xi\},$$

$$\bar{R}_\xi = \{\alpha \in T_X \mid |\nabla\alpha| \leqslant \xi\},$$

$$D_\xi = \{\alpha \in T_X \mid |\Delta\alpha| < \xi\},$$

$$\bar{D}_\xi = \{\alpha \in T_X \mid |\Delta\alpha| \leq \xi\}.$$

Then $R_\xi \subseteq \bar{R}_\xi$ and $D_\xi \subseteq \bar{D}_\xi$. Since $|\nabla\alpha| \leq |\Delta\alpha|$ for all $\alpha \in T_X$, it follows that $D_\xi \subseteq R_\xi$ and $\bar{D}_\xi \subseteq \bar{R}_\xi$.

Let S be R_ξ, $\bar{R}_\xi$, D_ξ or $\bar{D}_\xi$, and let $\alpha \in S$. Then $H_\alpha = \{\beta \in T_X \mid \Delta\beta = \Delta\alpha, \nabla\beta = \nabla\alpha$ and $\pi_\alpha = \pi_\beta\}$, the $\mathcal{H}$ - class of T_X containing α. It follows that $H_\alpha \subseteq S$, so H_α is also the $\mathcal{H}$ - class of S containing α. Thus for $\alpha \in E(S)$, $\alpha S\alpha$ is factorizable if and only if $\alpha S \alpha = H_\alpha E(\alpha S\alpha)$ (proposition 2.1).

3.8 <u>Theorem</u>. Let X be a set and $1 \leq \xi \leq |X|$. Then :

(1) R_ξ is locally factorizable if and only if $\xi \in \mathbb{N} \cup \{\aleph_0\}$, where $\mathbb{N}$ denotes the set of all positive integers and $\aleph_0$ denotes the cardinality of a denumerable set.

(2) $\bar{R}_\xi$ is locally factorizable if and only if $\xi \in \mathbb{N}$.

<u>Proof</u> : (1) Suppose ξ is a cardinal number and $\xi \in \mathbb{N} \cup \{\aleph_0\}$. Then if $\alpha \in R_\xi$, then $\nabla\alpha$ is a finite set. Let $\alpha \in E(R_\xi)$. Then $\nabla\alpha \subseteq \Delta\alpha$ and $x\alpha = x$ for all $x \in \nabla\alpha$. To show that $\alpha R_\xi \alpha = H_\alpha E(\alpha R_\xi \alpha)$ where H_α is the $\mathcal{H}$ - class of T_X containing α, and hence of R_ξ, let $\rho \in R_\xi$. Set $A = \nabla\alpha \cup (\nabla\alpha \cap \Delta\rho)\rho$. Then $|A| < \infty$. Let $\alpha' = \alpha|_{\Delta\alpha \cap A}$ and $\rho' = \rho|_{\Delta\rho \cap \nabla\alpha}$. Then $\Delta\alpha' \subseteq A$, $\Delta\rho' \subseteq A$, $\nabla\alpha' \subseteq \nabla\alpha \subseteq A$ and $\nabla\rho' = (\nabla\alpha \cap \Delta\rho)\rho' = (\nabla\alpha \cap \Delta\rho)\rho \subseteq A$. Therefore $\alpha', \rho' \in T_A$. If $x \in \nabla\alpha$, then $x \in \Delta\alpha \cap A = \Delta\alpha'$ and $x\alpha' = x\alpha = x$, so $x \in \nabla\alpha'$. Thus $\nabla\alpha = \nabla\alpha'$. Since $\alpha' = \alpha|_{\Delta\alpha \cap A} \in T_A$ and $\alpha^2 = \alpha$, by Lemma 3.5, we have $\alpha'^2 = \alpha'$, therefore $\alpha' \in E(T_A)$. Thus $\nabla\alpha' \subseteq \Delta\alpha'$. But A is finite, by Theorem 3.2 and the fact that the $\mathcal{H}$ - class of T_A containing α' is the $\mathcal{H}$ - class of T_X containing α', we then have $\alpha' T_A \alpha' = H_{\alpha'} E(\alpha' T_A \alpha')$, where $H_{\alpha'}$ is the $\mathcal{H}$ - class of T_X containing α'. Therefore $\alpha'\rho'\alpha' = \lambda\alpha'\gamma\alpha'$ for some $\lambda \in H_{\alpha'}$, $\gamma \in T_A$ such that $\alpha'\gamma\alpha' \in E(\alpha' T_A \alpha')$. Then

$\nabla\lambda = \nabla\alpha' = \nabla\alpha$, $\Delta\lambda = \Delta\alpha' \subsetneq \Delta\alpha$, and $\nabla\alpha'\gamma\alpha' \subseteq \Delta\alpha'\gamma\alpha'$. Since $\nabla\alpha \subseteq \Delta\alpha$ and $x\alpha = x$ for all $x \in \nabla\alpha$, it follows that $\Delta\alpha = \bigcup_{x\in\nabla\alpha} x\pi_\alpha$ is a disjoint union. Define the map $\bar{\lambda}$ from $\Delta\alpha$ into $\nabla\lambda$ (= $\nabla\alpha$) by

$$x\bar{\lambda} \quad = \quad y\lambda \iff x \in y\pi_\alpha,\ y \in \nabla\alpha.$$

This is well-defined because $\nabla\alpha = \nabla\alpha' \subseteq \Delta\alpha' = \Delta\lambda$ and $\Delta\alpha = \bigcup_{y\in\nabla\alpha} y\pi_\alpha$ is a disjoint union. Now $\nabla\lambda = \nabla\alpha \subseteq \Delta\alpha = \Delta\bar{\lambda}$ and $\nabla\bar{\lambda} \subseteq \nabla\lambda$. Claim that $\bar{\lambda} \in H_\alpha$ (that is, $\Delta\bar{\lambda} = \Delta\alpha$, $\nabla\bar{\lambda} = \nabla\alpha$ and $\pi_{\bar{\lambda}} = \pi_\alpha$). First, we show that $\bar{\lambda}|_{\Delta\lambda} = \lambda$. Let $x \in \Delta\lambda$. Then $x\bar{\lambda} = y\lambda$ for some $y \in \nabla\alpha$ such that $x \in y\pi_\alpha$, so $x\alpha = y\alpha$. Since $y \in \nabla\alpha$ and $x \in \Delta\lambda = \Delta\alpha'$, we have $x\bar{\lambda} = y\lambda = (y\alpha)\lambda = (x\alpha)\lambda = (x\alpha')\lambda = x\alpha'\lambda = x\lambda$ since $\lambda \in H_{\alpha'}$ and $H_{\alpha'}$ is a group having α' as its identity. Thus $\bar{\lambda}|_{\Delta\lambda} = \lambda$. This implies that $\nabla\lambda \subsetneq \nabla\bar{\lambda}$ and hence $\nabla\bar{\lambda} = \nabla\lambda = \nabla\alpha' = \nabla\alpha$. Let $x_1, x_2 \in \Delta\alpha = \Delta\bar{\lambda}$. If $(x_1,x_2) \in \pi_\alpha$, then $x_1\alpha = x_2\alpha = x_2\alpha^2 = (x_2\alpha)\alpha$, so we have $x_1, x_2 \in (x_2\alpha)\pi_\alpha$ and $x_2\alpha \in \nabla\alpha$ which implies $x_1\bar{\lambda} = (x_2\alpha)\lambda = x_2\bar{\lambda}$, and hence $(x_1,x_2) \in \pi_{\bar{\lambda}}$. For the reverse inclusion, assume $(x_1,x_2) \in \pi_{\bar{\lambda}}$. Then $x_1\bar{\lambda} = x_2\bar{\lambda}$ and there exist $y_1, y_2 \in \nabla\alpha$ such that $x_1 \in y_1\pi_\alpha$, $x_2 \in y_2\pi_\alpha$. Thus $y_1\lambda = y_2\lambda$ (from the definition of $\bar{\lambda}$), so $(y_1,y_2) \in \pi_\lambda = \pi_{\alpha'}$. Hence $x_1\alpha = y_1\alpha = y_1\alpha' = y_2\alpha' = y_2\alpha = x_2\alpha$, so $(x_1,x_2) \in \pi_\alpha$. Hence we have the claim. Let $\gamma' = \gamma|_{\Delta\gamma\cap\nabla\alpha}$. Then $|\nabla\gamma'| \leqslant |\Delta\gamma'| \leqslant |\nabla\alpha| < \xi$, hence $\gamma' \in R_\xi$. But $\nabla\alpha = \nabla\alpha'$, $\Delta\gamma' \subseteq \Delta\gamma$, $\nabla\alpha'\gamma \subseteq \nabla\gamma \subseteq A$, so it follows that $\nabla\alpha\gamma'\alpha = (\nabla\alpha\gamma' \cap \Delta\alpha)\alpha = (((\nabla\alpha \cap \Delta\gamma')\gamma') \cap \Delta\alpha)\alpha \subseteq (((\nabla\alpha' \cap \Delta\gamma)\gamma) \cap \Delta\alpha)\alpha = (\nabla\alpha'\gamma \cap \Delta\alpha)\alpha = (\nabla\alpha'\gamma \cap (\nabla\alpha'\gamma \cap \Delta\alpha))\alpha \subseteq (\nabla\alpha'\gamma \cap (A \cap \Delta\alpha))\alpha = (\nabla\alpha'\gamma \cap \Delta\alpha')\alpha = (\nabla\alpha'\gamma \cap \Delta\alpha')\alpha' = \nabla\alpha'\gamma\alpha'$. Next, let $x \in \Delta\alpha'\gamma\alpha'$. Then $x\alpha' = x\alpha$ and $x\alpha' \in \Delta\gamma$, so $x\alpha = x\alpha' \in \Delta\gamma \cap \nabla\alpha = \Delta\gamma'$. Therefore $x\alpha\gamma'\alpha' = x\alpha\gamma'\alpha$, so $x \in \Delta\alpha\gamma'\alpha$. Hence $\Delta\alpha'\gamma\alpha' \subseteq \Delta\alpha\gamma'\alpha$ and $x\alpha'\gamma\alpha' = x\alpha\gamma'\alpha$ for all $x \in \Delta\alpha'\gamma\alpha'$. But $\nabla\alpha\gamma'\alpha \subseteq \nabla\alpha'\gamma\alpha' \subseteq \Delta\alpha'\gamma\alpha'$, so $\nabla\alpha\gamma'\alpha \subseteq \Delta\alpha\gamma'\alpha$. Let $y \in \nabla\alpha\gamma'\alpha$. Then $y \in \nabla\alpha'\gamma\alpha' \subseteq \Delta\alpha'\gamma\alpha'$. Therefore $y\alpha\gamma'\alpha = y\alpha'\gamma\alpha' = y$ since $\alpha'\gamma\alpha'$ is an idempotent. This proves that $\alpha\gamma'\alpha \in E(\alpha R_\xi\alpha)$.

Next, we shall show that $\alpha\rho\alpha = \bar{\lambda}\alpha\gamma'\alpha$. Let $x \in \Delta\alpha\rho\alpha$. Then $x\alpha \in \nabla\alpha \cap \Delta\rho = \Delta\rho'$, $x\alpha\rho \in \Delta\alpha$ and $x\alpha\rho \in (\nabla\alpha \cap \Delta\rho)\rho \subseteq A$. Therefore $x\alpha\rho \in A \cap \Delta\alpha = \Delta\alpha'$ and $x\alpha \in \nabla\alpha = \nabla\alpha' \subseteq \Delta\alpha'$. It follows that $(x\alpha\rho)\alpha = x\alpha\rho\alpha' = x\alpha\rho'\alpha' = x\alpha\alpha\rho'\alpha' = x\alpha(\alpha'\rho'\alpha') = x\alpha(\lambda\alpha'\gamma\alpha') = (x\alpha\bar{\lambda})\alpha'\gamma\alpha' = x\bar{\lambda}(\alpha'\gamma\alpha') = x\bar{\lambda}\alpha\gamma'\alpha$. The last equality follows from the fact that $\Delta\alpha'\gamma\alpha' \subseteq \Delta\alpha\gamma'\alpha$ and $t\alpha'\gamma\alpha' = t\alpha\gamma'\alpha$ for all $t \in \Delta\alpha'\gamma\alpha'$. Next, let $y \in \Delta\bar{\lambda}\alpha\gamma'\alpha$. Then $y\bar{\lambda}\alpha\gamma'\alpha = y\alpha\bar{\lambda}\alpha\gamma'\alpha$ since $\bar{\lambda} \in H_\alpha$. Since $y\alpha \in \nabla\alpha = \nabla\alpha' \subseteq \Delta\alpha' = \Delta\lambda$, we have $y\alpha\bar{\lambda} = y\alpha\lambda \in \nabla\lambda \cap \Delta\alpha \subseteq \Delta\alpha \cap A = \Delta\alpha'$, and so $y\alpha\lambda\alpha' = y\alpha\bar{\lambda}\alpha$. Hence $y\alpha\lambda\alpha'\gamma = y\alpha\bar{\lambda}\alpha\gamma' \in \Delta\alpha \cap \nabla\gamma' \subseteq \Delta\alpha \cap A = \Delta\alpha'$. It follows that $y\bar{\lambda}\alpha\gamma'\alpha = y\alpha\bar{\lambda}\alpha\gamma'\alpha = y\alpha\lambda\alpha'\gamma'\alpha' = y\alpha(\lambda\alpha'\gamma'\alpha') = y\alpha(\alpha'\rho'\alpha') = y\alpha\alpha\rho\alpha = y\alpha\rho\alpha$. This proves that $\alpha\rho\alpha = \bar{\lambda}\alpha\gamma'\alpha$, so $\alpha\rho\alpha \in H_\alpha E(\alpha R_\xi \alpha)$. Hence $\alpha R_\xi \alpha = H_\alpha E(\alpha R_\xi \alpha)$. Therefore R_ξ is locally factorizable.

Conversely, assume ξ is an infinite cardinal number such that $\xi > \aleph_0$. Then there is a subset A of X such that $|A| = \aleph_0 < \xi$. Then $T_A \subseteq R_\xi$. Let 1_A be the identity map on A. Then 1_A is an idempotent of R_ξ. Since $1_A R_\xi 1_A \subseteq T_A$, we have that $1_A R_\xi 1_A = T_A$. But T_A is not factorizable [7, Theorem 3.1]. Therefore R_ξ is not locally factorizable.

Hence R_ξ is locally factorizable if and only if $\xi \in \mathbb{N} \cup \{\aleph_0\}$.

(2) Assume $\xi \in \mathbb{N}$. If $\xi = |X|$ then $\bar{R}_\xi = T_X$ is locally factorizable since X is finite. If $\xi < |X|$ then $\bar{R}_\xi = R_{\xi+1}$ is locally factorizable by (1). Hence if $\xi \in \mathbb{N}$, then $\bar{R}_\xi$ is locally factorizable.

Conversely, assume ξ is an infinite cardinal number. Then there is a subset A of X such that $|A| = \aleph_0 \leqslant \xi$. Hence 1_A, the identity map on A, is an idempotent of $\bar{R}_\xi$, and $T_A \subseteq \bar{R}_\xi$. But $1_A \bar{R}\, 1_A \subseteq T_A$, so $1_A \bar{R}\, 1_A = T_A$. Since A is infinite, T_A is not factorizable.

Therefore, the theorem is proved. #

3.9 Corollary. Let X be a set and $1 \leqslant \xi \leqslant |X|$. Then

(1) D_ξ is locally factorizable if and only if $\xi \in \mathbb{N} \cup \{\aleph_0\}$.

(2) $\bar{D}_\xi$ is locally factorizable if and only if $\xi \in \mathbb{N}$.

Proof : Let $\xi \in \mathbb{N} \cup \{\aleph_0\}$. Let $\alpha \in E(D_\xi)$. Since $D_\xi \subsetneq R_\xi$, we have $\alpha \in E(R_\xi)$ and $\alpha D_\xi \alpha \subseteq \alpha R_\xi \alpha$. If $\beta \in R_\xi$, then $|\Delta\alpha\beta\alpha| \leqslant |\Delta\alpha| < \xi$, so $\alpha\beta\alpha \in D_\xi$ and thus $\alpha\beta\alpha = \alpha(\alpha\beta\alpha)\alpha \in \alpha D_\xi \alpha$.

This proves that $\alpha D_\xi \alpha = \alpha R_\xi \alpha$ for all $\alpha \in E(D_\xi)$. Similarly, if $\xi \in \mathbb{N}$, then we also have $\alpha \bar{D}_\xi \alpha = \alpha \bar{R}_\xi \alpha$ for all $\alpha \in E(\bar{D}_\xi)$. By Theorem 3.8, we have that if $\xi \in \mathbb{N} \cup \{\aleph_0\}$, then D_ξ is locally factorizable, and if $\xi \in \mathbb{N}$, then $\bar{D}_\xi$ is locally factorizable.

Let ξ and $\bar{\xi}$ be cardinal numbers, $\xi \notin \mathbb{N} \cup \{\aleph_0\}$ and $\bar{\xi} \notin \mathbb{N}$. Then there exists a subset A of X such that $|A| = \aleph_0$. Then $|A| < \xi$ and $|A| \leqslant \bar{\xi}$. Thus $T_A \subseteq D_\xi$ and $T_A \subsetneq \bar{D}_{\bar{\xi}}$. Let 1_A be the identity map on A. Then $1_A \in E(D_\xi)$ and $1_A \in E(\bar{D}_{\bar{\xi}})$, $1_A D_\xi 1_A \subseteq T_A$, $1_A \bar{D}_{\bar{\xi}} 1_A \subseteq T_A$, so $1_A D_\xi 1_A = T_A = 1_A \bar{D}_{\bar{\xi}} 1_A$. Since A is infinite, T_A is not factorizable. Hence D_ξ and $\bar{D}_{\bar{\xi}}$ are not locally factorizable. #

Let X be a set, and let E_X and M_X denote the semigroup of all mappings from X onto X and the semigroup of all 1-1 mappings from X into X, respectivety. Then

$$E_X = \{\alpha : X \to X \mid \alpha \text{ is onto}\} = \{\alpha \in \mathcal{T}_X \mid \nabla\alpha = X\},$$

$$M_X = \{\alpha : X \to X \mid \alpha \text{ is 1-1}\} = \{\alpha \in I_X \mid \Delta\alpha = X\}.$$

Note that G_X (the symmetric group on X) is the unit group of E_X and of M_X. It is easily seen that if X is finite, then $E_X = M_X = G_X$. If $E_X = G_X$, then X is finite. Also, if $M_X = G_X$, then X is finite.

To prove this, suppose X is infinite. Let $a \in X$. Then $|X \setminus \{a\}| = |X|$. Let α be a 1-1 map from $X \setminus \{a\}$ onto X, and let β be a 1-1 from X onto $X \setminus \{a\}$. Then $\beta \in M_X \setminus G_X$. Define the map $\bar{\alpha}$ from X into X as follows :

$$x\bar{\alpha} = \begin{cases} a & \text{if } x = a, \\ x\alpha & \text{if } x \neq a. \end{cases}$$

Then $\bar{\alpha}$ is onto but not 1-1. Therefore $\bar{\alpha} \in E_X \setminus G_X$. Hence $E_X \neq G_X$ and $M_X \neq G_X$. This proves that $E_X = G_X$ if and only if X is finite and $M_X = G_X$ if and only if X is finite.

Let $\alpha \in E(E_X)$. If $x \in X$, then $x \in \nabla\alpha$, so $x\alpha = x$. Therefore α is the identity map on X, and hence $E(E_X) = \{1\}$.

Let $\alpha \in E(M_X)$. If $x \in X$, then $x\alpha \in \nabla\alpha$ and hence $(x\alpha)\alpha = x\alpha$, so $x\alpha = x$ since α is 1-1. Thus α is the identity map on X, hence $E(M_X) = \{1\}$.

3.10 Proposition. Let X be a set and let S be E_X or M_X. Then S is locally factorizable if and only if X is finite.

Proof : Assume S is locally factorizable. Because G_X is the maximum subgroup of S having 1 as the identity, then $1S1 = S = G_X\{1\} = G_X$. Therefore X is finite.

Conversely, if X is finite then $S = G_X$ which is locally factorizable since every group is locally factorizable. #

Let X be a set. For a nonempty subset A of X and for $x \in X$, let A_x denote the partial transformation of X with $\Delta A_x = A$ and $\nabla A_x = \{x\}$. Let C_X and F_X denote the following transformation semigroups on X :

$$C_X = \{A_x \mid \phi \neq A \subseteq X,\ x \in X\} \cup \{0\},$$

$$F_X = \{X_x \mid x \in X\} \quad \text{if } X \neq \phi$$

and

$$F_X = \{0\} \quad \text{if } X = \phi.$$

Let A and B be nonempty subsets of X and $x, y \in X$. Then

$$A_x B_y = \begin{cases} A_y & \text{if } x \in B, \\ 0 & \text{if } x \notin B. \end{cases}$$

Hence $A_x^2 = A_x$ if and only if $x \in A$. Then F_X is a band, so it is locally factorizable by Proposition 2.2. If $x \in A$, then we clearly have $A_x C_X A_x = \{0, A_x\} = \{A_x\}\{A_x, 0\}$ which is factorizable.

Hence, we have

3.11 Proposition. For a set X, the transformation semigroups C_X and F_X are locally factorizable.