CHAPTER I

ON THE WIENER MEASURE

In this chapter, we recall without proof some facts about the Wiener measure and the Wiener integral, sufficiently for use in this thesis. The materials of this chapter are drawn from the reference [1].

A. The Wiener Space.

The Wiener space C of functions of one variable is the collection of real-valued continuous functions x defined on $[0,1]$ and satisfying $x(0) = 0$.

C is a separable Banach space with the norm $\|\cdot\|$ defined by

$$\|x\| = \max_{0 \le t \le 1} |x(t)| , \quad x \in C.$$

Let $\{t_1,\dots,t_n\}$ be a finite collection of numbers satisfying $0 < t_1 < t_2 < \dots < t_n \le 1$ and let E be a Borel set of the n-dimensional Euclidean space $\mathbb{R}^n$, i.e., $E \in \mathcal{B}(\mathbb{R}^n)$. A subset I of C defined by

$$(1.1) \qquad I = \{x \in C : (x(t_1),\dots, x(t_n)) \in E\}$$

will be called a quasi-interval in C. The points $t_1,\dots, t_n$ and the set E will be called the restriction points and the restricting set of I. In particular, if E is a rectangle in $\mathbb{R}^n$, then I which we will denote by I^o, will be called an interval in C; i.e.,

$$(1.2) \qquad I^o = \{x \in C : \alpha_i \le x(t_i) \le \beta_i, \ i = 1,\dots, n\}$$

or any set obtained by replacing any or all of the signs by <.

Let E be a set in $\mathbb{R}^n$ and let k be an integer, $1 \le k \le n$. We define

$$E \underset{k}{\boxtimes} \mathbb{R} = \{(\alpha_1, \ldots, \alpha_{k-1}, \alpha^*, \alpha_k, \ldots, \alpha_n) \in \mathbb{R}^{n+1} :$$

$(\alpha_1, \ldots, \alpha_{k-1}, \alpha_k, \ldots, \alpha_n) \in E$ and $\alpha^* \in \mathbb{R}\}$.

Then $E \underset{k}{\boxtimes} \mathbb{R} \in \mathcal{B}(\mathbb{R}^{n+1})$ whenever $E \in \mathcal{B}(\mathbb{R}^n)$.

Note that for a given collection of n restriction points $t_1, \ldots, t_n$ and a given restricting set E there is associated a unique subset I of C given by (1.1), but the converse is not true. For instance, for the subset I defined by (1.1) we may throw in a few more points in $(0,1]$, so that there are m additional points in $(0,1]$, let the restriction at each of the additional restriction points t_{k_j} be the trivial restriction, $-\infty < x(t_{k_j}) < \infty$, and let the restricting set be $E \underset{k_1}{\boxtimes} \mathbb{R} \underset{k_2}{\boxtimes} \cdots \underset{k_m}{\boxtimes} \mathbb{R}$, a Borel set in $\mathbb{R}^{n+m}$, where

$$E \underset{k_1}{\boxtimes} \mathbb{R} \underset{k_2}{\boxtimes} \cdots \underset{k_m}{\boxtimes} \mathbb{R} = \{(\alpha_1, \ldots, \alpha_{k_1-1}, \alpha_{k_1}, \alpha_{k_1+1}, \ldots, \alpha_{k_j-1}, \alpha_{k_j}, \alpha_{k_j+1}, \ldots, \alpha_{n+m}) \in \mathbb{R}^{n+m} : (\alpha_1, \ldots, \alpha_{k_1-1}, \alpha_{k_1+1}, \ldots, \alpha_{k_j-1}, \alpha_{k_j+1}, \ldots, \alpha_{n+m}) \in E \text{ and } \alpha_{k_1}, \ldots, \alpha_{k_j}, \ldots, \alpha_{k_m} \in \mathbb{R}\}.$$

Then the subset of C defined with $(n+m)$ restriction points and the restricting set $E \underset{k_1}{\boxtimes} \mathbb{R} \underset{k_2}{\boxtimes} \cdots \underset{k_m}{\boxtimes} \mathbb{R}$ is identical with the one defined by (1.1).

Let $\mathcal{I}$ be the collection of all quasi-intervals defined by (1.1) and $\mathcal{I}^{\circ}$ be the collection of all intervals defined by (1.2). Then $\mathcal{I}$ is an algebra of sets and $\mathcal{I}^{\circ}$ is a semialgebra of sets.

B. The Wiener Measure.

Let c be a positive constant and let $W_c : \mathcal{I} \to R \cup \{-\infty,\infty\}$. If $I \in \mathcal{I}$ is defined by (1.1) then $W_c(I)$ is given by

$$(1.3) \qquad W_c(I) = \int_E \cdots \int K\{t_1,\ldots,t_n,\xi_1,\ldots,\xi_n\}\, d\xi_1 \ldots d\xi_n$$

where

$$K\{t_1,\ldots,t_n,\xi_1,\ldots,\xi_n\}$$

$$(1.4) \qquad = \frac{1}{\sqrt{\pi^n c^n t_1(t_2-t_1)\ldots(t_n-t_{n-1})}} \exp\left[-\sum_{j=1}^{n} \frac{(\xi_j-\xi_{j-1})^2}{ct_j-ct_{j-1}}\right]$$

with the understanding that $\xi_0 = 0 = t_0$.

$W_c(I)$ is independent of the choice of restriction points and restricting set that decribe I.

W_c has the following properties :

(i) The value of $W_c(I)$ is non-negative for any $I \in \mathcal{I}$.

(ii) $W_c(\emptyset) = 0$.

(iii) $W_c(C) = 1$.

(iv) W_c is additive, i.e., if $I_1, I_2 \in \mathcal{I}$, $I_1 \cap I_2 = \emptyset$ and $I_1 \cup I_2 \in \mathcal{I}$, then $W_c(I_1 \cup I_2) = W_c(I_1) + W_c(I_2)$.

(v) W_c is countably additive , i.e., if a sequence of quasi-intervals $\{I_n\}$ in $\mathcal{I}$ is such that $I_i \cap I_j = \emptyset$ for $i \neq j$ and $I = \bigcup_{n=1}^{\infty} I_n \in \mathcal{I}$, then

$$W_c(I) = \sum_{n=1}^{\infty} W_c(I_n).$$

Then W_c is a measure on $\mathcal{I}$ and together with property (iii) it is a probability measure. Since c is an arbitrary positive number, W_c is a measure on $\mathcal{I}$ for all $c > 0$.

The outer measure of an arbitrary set $\Gamma \subseteq C$ is defined to be

$$W_c^*(\Gamma) = \inf \{ \sum_{k=1}^{\infty} W_c(I_k) \}$$

where $\{I_k\}$ ranges over all sequences from $\mathcal{I}$ such that $\Gamma \subseteq \bigcup_{k=1}^{\infty} I_k$.

A set $\Gamma \subseteq C$ is called Wiener measurable if for every set $A \subseteq C$ we have

$$W_c^*(A) = W_c^*(A \setminus \Gamma) + W_c^*(A \cap \Gamma).$$

The collection $\mathcal{J}_{W_c}$ of all Wiener measurable sets, the Caratheodory extension of $\mathcal{I}$, is a σ-algebra containing $\mathcal{I}$ and if we define for each $c > 0$

$$\bar{W}_c(\Gamma) = W_c^*(\Gamma) \qquad (\Gamma \in \mathcal{J}_{W_c})$$

then $\bar{W}_c$ is a measure on $\mathcal{J}_{W_c}$ and we denote this measure space by $(C, \mathcal{J}_{W_c}, \bar{W}_c)$.

The σ-algebra $\sigma[\mathcal{J}]$, generated by $\mathcal{J}$ is the collection $\mathcal{B}(C)$ of all Borel sets of C and we have $\sigma[\mathcal{J}^o] = \sigma[\mathcal{J}] = \mathcal{B}(C)$.

Since $\mathcal{J}^o$ is a semi-algebra of sets, if we define $W_c^o(I^o) = W_c(I^o)$ for every $I^o \in \mathcal{J}^o$, then according to the properties of W_c and the fact that $\mathcal{J}^o \subset \mathcal{J}$, we have that W_c^o has a unique extension to a measure on the algebra $\mathcal{A}$ generated by $\mathcal{J}^o$. If we extend W_c^o on $\mathcal{A}$ by the Caratheodory extension, we have (for each $c > 0$) a measure $\bar{W}_c^o$ on the σ-algebra $\mathcal{J}_{W_c^o}$ containing $\mathcal{J}^o$. Let us denote this measure space by $(C, \mathcal{J}_{W_c^o}, \bar{W}_c^o)$. Then it can be shown that $(C, \mathcal{J}_{W_c}, \bar{W}_c) = (C, \mathcal{J}_{W_c^o}, \bar{W}_c^o)$, this will enable us to express any Wiener measurable set in terms of members of $\mathcal{J}^o$.

We simply write $W_c(\Gamma)$ instead of $\bar{W}_c(\Gamma)$ or $\bar{W}_c^o(\Gamma)$ even for set Γ in $\mathcal{J}_{W_c}$. We call W_c a <u>Wiener measure</u> in C. In case $c = 1$, we will denote W_1 by W. The integral in C with respect to W_c is called a <u>Wiener integral</u>. If F is a Wiener measurable functional on C, its integral will be denoted by

$$\int_C F(x)\, d\, W_c(x)\ .$$

<u>Theorem 1</u>. Let $0 < t_1 < \ldots < t_n \leq 1$ and $H(\xi_1, \ldots, \xi_n)$ be a Borel measurable function of n real variables $\xi_1, \ldots, \xi_n$. Then the functional $H(y(t_1), \ldots, y(t_n))$ defined on C is Wiener measurable and for each $c > 0$

$$(1.5) \qquad \int_C H(y(t_1), \ldots, y(t_n))\, dW_c(y)$$

$$= \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} H(\xi_1, \ldots, \xi_n) K\{t_1, \ldots, t_n, \xi_1 \ldots, \xi_n\} d\xi_1 \ldots d\xi_n$$

where $K\{t_1,\ldots,t_n,\xi_1,\ldots,\xi_n\}$ is defined by (1.4) and the existence of one side implies that of the other and the validity of the equality.

Example. Let t and s be any two points in $[0,1]$.

Then

$$\int_C (x(t) - x(s))^2 d\, W_c(x) = \frac{c}{2}\,|t-s|.$$

Solution. Assume $t < s$.

Case 1. If $t = 0$, then according to (1.5) we have

$$(*) \qquad \int_C (x(s))^2 dW_c(x) = \frac{1}{\sqrt{\pi cs}} \int_{-\infty}^{\infty} \xi^2 \exp(-\frac{\xi^2}{cs})\, d\xi \, .$$

Let $\eta = \frac{\xi}{\sqrt{cs}}$. Then (*) becomes

$$\int_C (x(s))^2 dW_c(x) = \frac{cs}{\sqrt{\pi}} \int_{-\infty}^{\infty} \eta^2 \exp(-\eta^2)\, d\eta$$

$$= \frac{cs}{\sqrt{\pi}} \int_0^{\infty} \eta \exp(-\eta^2) d\, \eta^2$$

$$= \frac{cs}{2} \qquad = \frac{c}{2}\,|t-s|,$$

since $\int_0^{\infty} \eta \exp(-\eta^2) d\eta^2 = \frac{\sqrt{\pi}}{2}$ and $t = 0$.

Case 2. If $t \neq 0$, then according to (1.5) we have

$$\int_C (x(t)-x(s))^2 dW_c(x) = \frac{1}{\sqrt{\pi^2 c^2 t(s-t)}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\xi_1-\xi_2)^2 \exp\left[\frac{-\xi_1^2}{ct} - \frac{(\xi_2-\xi_1)^2}{(cs-ct)}\right] \times d\xi_1 d\xi_2 .$$

Let $\eta_1 = \frac{\xi_1}{\sqrt{ct}}$, $\eta_2 = \frac{\xi_2-\xi_1}{\sqrt{cs-ct}}$. Then

$$\frac{\partial(\eta_1,\eta_2)}{\partial(\xi_1,\xi_2)} = \begin{vmatrix} \frac{1}{\sqrt{ct}} & 0 \\ \frac{-1}{\sqrt{cs-ct}} & \frac{1}{\sqrt{cs-ct}} \end{vmatrix} = \frac{1}{\sqrt{c^2t(s-t)}}$$

and hence

$$\int_C (x(t)-x(s))^2 dW_c(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \eta_2^2 (cs-ct) \exp(-\eta_1^2-\eta_2^2) d\eta_1 d\eta_2$$

$$= \frac{1}{\pi} (cs-ct) \int_{-\infty}^{\infty} \exp(-\eta_1^2) d\eta_1 \int_{-\infty}^{\infty} \eta_2^2 \exp(-\eta_2^2) d\eta_2 .$$

Since $\int_{-\infty}^{\infty} \exp(-\eta_1^2) d\eta_1 = \sqrt{\pi}$ and

$\int_{-\infty}^{\infty} \eta_2^2 \exp(-\eta_2^2) d\eta_2 = \frac{\sqrt{\pi}}{2}$, we have

$$\int_C (x(t)-x(s))^2 dW_c(x) = \frac{c}{2} (s-t) = \frac{c}{2} |t-s| . \quad \#$$