CHAPTER III

THE COMPLEX SEQUENTIAL WIENER INTEGRAL FOR ANALYTIC AND HARMONIC FUNCTIONALS

In this chapter, we establish an existence theorem for the complex sequential Wiener integral for a restricted class of analytic and harmonic functionals.

Definition 3.1 A subset E of $C[a,b]$ will be called a universal null set if ρE is a Wiener null set in $C[a,b]$ for each positive real number ρ. By ρE we mean the set of all functions ρx, where $x \varepsilon E$. A statement involving an element $x \varepsilon C[a,b]$ will be said to be true almost universally (a.u.) if it is true everywhere in $C[a,b]$ except on a universal null set. For example, for fixed x in $C[a,b]$, the set of polygonal functions x_n such that $x_n \to x$ is a universal null set.

Theorem 3.2 Let $\sigma = \rho e^{i\theta}$, where $\rho > 0$ and $0 < \theta \leq \pi/4$, and let Λ be the open sector of complex numbers λ such that $0 < \arg \lambda < \theta$. Let $F(y)$ be a Borel functional defined for all y of the form $\lambda x(\cdot)$, where $\lambda \varepsilon \Lambda^*$ and $x \varepsilon C[a,b]$, and Λ^* denotes the closure of Λ with $\lambda = 0$ omitted. Suppose that F also satisfies the following four conditions:

1. $F(\lambda x)$ is analytic in λ on Λ for each x in $C[a,b]$.
2. $F(\lambda x)$ is a continuous function of λ on Λ^* for each x in $C[a,b]$.

3. $F(\sigma x)$ and $F(\sigma^* x)$ are continuous functions of x in the uniform topology a.u. in $C[a,b]$, where $\sigma^* = \rho e^{i\theta^*}$ and $0 < \theta^* < \theta$.

4. There is an $M > 0$ such that

$$|F(e^{i\gamma}x)| \leq M$$

for all x in $C[a,b]$ and all γ on $(0,\theta)$.

Then the sequential Wiener integral (with parameter σ) exists on $C[a,b]$ and we have

$$(3.2.1) \qquad \int_{C[a,b]}^{sw_\sigma} F(x)dx = \int_{C[a,b]} F(\sigma x)dW(x).$$

Moreover the following integrals exist and are equal

$$(3.2.2) \qquad \int_{C[a,b]}^{sw_\lambda} F(x)dx = \int_{C[a,b]} F(\lambda x)dW(x)$$

wherever λ is in the set $\mathcal{S}$ defined by

$$\mathcal{S} = \{\lambda : \lambda \neq 0,\ 0 < \arg \lambda < \theta \text{ and } |\lambda| < \rho\} .$$

Finally, both members of (3.2.2) are analytic functions of λ on $\mathcal{S}$ and they approach the members of (3.2.1) as $\lambda \to \sigma$ from inside $\mathcal{S}$.

<u>Proof</u>: We note from condition 2 that condition 4 holds for $0 \leq \gamma \leq \theta$, and hence we have that for all λ in Λ^* and all x in $C[a,b]$,

$$(3.2.3) \qquad |F(\lambda x)| = |F(\frac{\lambda}{|\lambda|} \cdot |\lambda| x)| = |F(e^{i\gamma}y)| \leq M.$$

Let $\mathcal{S}^*$ be the closure of $\mathcal{S}$ with the origin omitted.

Since the proof of this theorem is very long, it will be convenient to divide it into several steps.

STEP I. For each subdivision vector τ,

(3.2.4) $$\int_{\mathbb{R}^n} K_\lambda(\tau,\xi)F(\psi_{\tau,\xi})d\xi$$

and

(3.2.5) $$\int_{\mathbb{R}^n} K(\tau,\xi)F(\lambda\psi_{\tau,\xi})d\xi$$

exist for $\lambda \in \mathcal{S}^*$ and are analytic functions of λ on $\mathcal{S}$.

Proof: It follows from lemma 2.5 and (2.1.2) that the integrand of (3.2.4) is measurable in ξ, and in view of (2.1.2),(3.2.3) satisfies for $\lambda \in \mathcal{S}^*$ the inequalities

(3.2.6)
$$\begin{aligned} &|\lambda|^n\sqrt{(2\pi)^n(\tau_1-\tau_0)\dots(\tau_n-\tau_{n-1})}\,|K_\lambda(\tau,\xi)F(\psi_{\tau,\xi})| \\ &\le M \exp\left[-\operatorname{Re}(\lambda^{-2})\sum_{i=1}^{n}\frac{(\xi_i-\xi_{i-1})^2}{2(\tau_i-\tau_{i-1})}\right] \\ &\le M \exp\left[-\operatorname{Re}(\sigma^{-2})\sum_{i=1}^{n}\frac{(\xi_i-\xi_{i-1})^2}{2(\tau_i-\tau_{i-1})}\right]. \end{aligned}$$

Since the last member of (3.2.6) is integrable in ξ over $\mathbb{R}^n$, (3.2.4) exists for all λ in $\mathcal{S}^*$ and all subdivision vectors τ. To show that (3.2.4) is analytic in λ on $\mathcal{S}$, let Δ be any closed triangle in $\mathcal{S}$. Then we have

$$\int_{\partial\Delta} K_\lambda(\tau,\xi)F(\psi_{\tau,\xi})\,d\lambda = 0$$

since $K_\lambda(\tau,\xi)F(\psi_{\tau,\xi})$ is analytic in λ and $\partial\Delta$ denotes the boundary of Δ. Since (3.2.4) exists,

$$\int_{\partial\Delta}\left(\int_{\mathbb{R}^n} |K_\lambda(\tau,\xi)F(\psi_{\tau,\xi})|\,d\xi\right) d\lambda < \infty$$

thus we can exchange the order of integration by Fubini theorem and get

$$\int_{\partial\Delta}\left(\int_{\mathbb{R}^n} K_\lambda(\tau,\xi)F(\psi_{\tau,\xi})\,d\xi\right)d\lambda$$

$$= \int_{\mathbb{R}^n}\left(\int_{\partial\Delta} K_\lambda(\tau,\xi)F(\psi_{\tau,\xi})\,d\lambda\right)d\xi$$

$$= 0.$$

Hence, by Morera's theorem we have that (3.2.4) is an analytic function of λ in $\mathcal{S}$.

Next we show that for each τ, (3.2.5) exists for $\lambda \in \mathcal{S}^*$ and is an analytic function of λ in $\mathcal{S}$. The argument is very similar to the corresponding argument for (3.2.4). The inequality corresponding to (3.2.6) is

$$\sqrt{(2\pi)^n(\tau_1-\tau_0)\ldots(\tau_n-\tau_{n-1})}\,|K(\tau,\xi)F(\lambda\psi_{\tau,\xi})|$$

(3.2.7)

$$\leq M \exp\left[-\sum_{i=1}^{n} \frac{(\xi_i-\xi_{i-1})^2}{2(\tau_i-\tau_{i-1})}\right]$$

for λ in $\mathcal{S}^*$. Thus both (3.2.4) and (3.2.5) are analytic on $\mathcal{S}$.

<u>STEP II</u> For each λ in $\mathcal{S}^*$, (3.2.4) and (3.2.5) are equal, i.e.,

$$(3.2.8) \qquad \int_{\mathbb{R}^n} K_\lambda(\tau,\xi)F(\psi_{\tau,\xi})d\xi = \int_{\mathbb{R}^n} K(\tau,\xi)F(\lambda\psi_{\tau,\xi})d\xi.$$

<u>Proof</u>: By condition 2 and step I, the integrand of (3.2.4) is continuous in λ on $\mathcal{S}^*$ and is integrable in ξ over $\mathbb{R}^n$. Thus (3.2.4) is continuous in λ on $\mathcal{S}^*$, and so does (3.2.5). Moreover if $\lambda \in \mathcal{S}^*$ and λ is real, we may replace ξ by $\lambda^{-1}\xi$ in (3.2.5) and using (2.3.1) we find that the expression (3.2.4) is equal to the expression (3.2.5) on the real edge of $\mathcal{S}^*$. Let L denote the real edge of $\mathcal{S}^*$ and

$$f(\lambda) = \int_{\mathbb{R}^n} K_\lambda(\tau,\xi)F(\psi_{\tau,\xi})d\xi ,$$

$$g(\lambda) = \int_{\mathbb{R}^n} K(\tau,\xi)F(\lambda\psi_{\tau,\xi})d\xi .$$

Thus we have $h(\lambda) = (f-g)(\lambda)$ is analytic in $\mathcal{S}$ and continuous on $\mathcal{S} \cup L$, hence by the Schwarz reflection principle, $h(\lambda)$ can be extended to a function which is analytic in $\mathcal{S} \cup L \cup \bar{\mathcal{S}}$, where $\bar{\mathcal{S}}$ denotes the reflection of $\mathcal{S}$. Since $h(\lambda) = 0$ for all λ in L and L has a limit point in $\mathcal{S} \cup L \cup \bar{\mathcal{S}}$, it follows that $h(\lambda) = 0$ for all λ in $\mathcal{S} \cup L \cup \bar{\mathcal{S}}$. Thus we have $f(\lambda) = g(\lambda)$ for all λ in $\mathcal{S}$ and hence by the continuity of $f(\lambda)$ and $g(\lambda)$, (3.2.8) holds for $\lambda \in \mathcal{S}^*$.

<u>STEP III</u> Let A denote the slanting edge of $\mathcal{S}^*$, and A^* the set of all λ in $\mathcal{S}^*$ in which $\arg\lambda = \theta^*$, i.e.,

$$A = \{\lambda : \lambda \neq 0, \arg\lambda = \theta \text{ and } |\lambda| \leq \rho\}$$

and

$$A^* = \{\lambda : \lambda \neq 0, \quad \arg \lambda = \theta^* \quad \text{and} \quad |\lambda| \leq \rho\}.$$

Then the following integrals exist and are equal

$$(3.2.9) \qquad \int_{C[a,b]}^{sw_\lambda} F(x)dx = \int_{C[a,b]} F(\lambda x)dW(x)$$

for $\lambda \in A \cup A^*$.

Proof: For each $\lambda \in A$,

$$\lambda = |\lambda| e^{i\theta} = \frac{|\lambda|}{\rho} (\rho e^{i\theta}) = (\frac{|\lambda|}{\rho}) \sigma.$$

Then by the continuity of F and of x and condition 3 we have that

$$F(\lambda x) = F(\frac{|\lambda|}{\rho} \sigma x)$$

is a continuous function of x in the uniform topology a.u. in $C[a,b]$. Similarly, this is true for λ in A^* Thus for each λ in $A \cup A^*$, the sequential Wiener integral and ordinary Wiener integral

$$(3.2.10) \qquad \int_{C[a,b]}^{sw} F(\lambda x)dx = \int_{C[a,b]} F(\lambda x)dW(x)$$

exist and are equal since the hypotheses of Theorem 2.7 are satisfied. Thus if $\{\tau_k\}$ is a sequence of subdivision vectors for which $\|\tau_k\| \to 0$ as $k \to \infty$, we have the right member of (3.2.8) approaching the left member of (3.2.10) as τ ranges over the sequence $\{\tau_k\}$. Hence, we have by (3.2.8) and (3.2.10) that

$$\overset{sw_\lambda}{\int_{C[a,b]}} F(x)dx = \lim_{\|\tau\|\to 0} \int_{\mathbb{R}^n} K_\lambda(\tau,\xi)F(\psi_{\tau,\xi})d\xi$$

$$= \lim_{\|\tau\|\to 0} \int_{\mathbb{R}^n} K(\tau,\xi)F(\lambda\psi_{\tau,\xi})d\xi$$

$$= \overset{sw}{\int_{C[a,b]}} F(\lambda x)dx$$

$$= \int_{C[a,b]} F(\lambda x)dW(x).$$

Thus we have shown that (3.2.9) holds for λ in $A \cup A^*$. In particular (3.2.9) holds for $\lambda = \sigma$ and (3.2.1) is established.

<u>STEP IV</u> For each λ in $\mathcal{S}$, the following integrals

$$(3.2.11) \qquad \overset{sw}{\int_{C[a,b]}} F(\lambda x)dx = \int_{C[a,b]} F(\lambda x)dW(x)$$

exist and are equal. Moreover the right member of (3.2.11) is analytic in $\mathcal{S}$ and is continuous in λ on $\mathcal{S}^*$.

<u>Proof:</u> Let $\{\tau_k\}$ be a sequence of subdivision vectors such that $\|\tau_k\| \to 0$ as $k \to \infty$, and define

$$(3.2.12) \qquad f_k(\lambda) = \int_{\mathbb{R}^{n(k)}} K(\tau_k,\xi)F(\lambda\psi_{\tau_k,\xi})d\xi.$$

By step I and II, the functions $f_k(\lambda)$ are defined and continuous for $\lambda \in \mathcal{S}^*$ and are analytic in $\mathcal{S}$. Moreover from (3.2.12),(2.6.1),(3.2.3) we have for $\lambda \in \mathcal{S}^*$,

$$|f_k(\lambda)| = |\int_{\mathbb{R}^{n(k)}} K(\tau_k,\xi)F(\lambda\psi_{\tau_k,\xi})d\xi|$$

$$= |\int_{C[a,b]} F(\lambda x_{\tau_k})dW(x)|$$

$$\leq \int_{C[a,b]} |F(\lambda x_{\tau_k})| dW(x)$$

$$< \infty .$$

Thus the functions $f_k(\lambda)$ are uniformly bounded for $\lambda \in \mathcal{S}^*$. Moreover from the existence of the right member of (3.2.10), it follows that for λ in $A \cup A^*$ we have the existence of the limit

(3.2.13) $$\lim_{k\to\infty} f_k(\lambda) = \int_{C[a,b]} F(\lambda x)dx.$$

Since $\{f_k\}$ is a sequence of analytic functions in $\mathcal{S}$ and uniformly bounded on $\mathcal{S}^*$, it follows that $\{f_k\}$ is a normal family, i.e., every subsequence of $\{f_k\}$ contains a subsequence which converges uniformly on compact subsets of $\mathcal{S}$. Let K be any compact subset of $\mathcal{S}$, and let $\{f_{k_j}\}$ be a subsequence of $\{f_k\}$. Then there is a subsequence $\{f^*_{k_j}\}$ of $\{f_{k_j}\}$ such that $f^*_{k_j}$ converges uniformly, say to g, on K. Hence, g is analytic in $\mathcal{S}$ and also bounded on $\mathcal{S}$.

Let

(3.2.14) $$f(\lambda) = \int_{C[a,b]} F(\lambda x)dW(x) .$$

Then it follows from condition 1,2 and 4 that $f(\lambda)$ is analytic in $\mathcal{S}$ and continuous on $\mathcal{S}^*$. By (3.2.13), $f^*_{k_j}$ converges to f on A^*, and thus

$f = g$ on A^*. Since A^* has a limit point in $\mathcal{S}$, $f = g$ on $\mathcal{S}$ and hence $f^*_{k_j}$ converges uniformly to f on K. Since K is arbitrary, we have shown that every subsequence of $\{f_k\}$ contains a subsequence which converges to f uniformly on every compact subsets of $\mathcal{S}$. This implies that $\{f_k\}$ converges to f on every compact subsets of $\mathcal{S}$, and hence on $\mathcal{S}$ since for each λ in $\mathcal{S}$, $\{\lambda\}$ is compact in $\mathcal{S}$. Thus it follows from (3.2.14) that (3.2.13) holds for λ in $\mathcal{S}$ as well as on A. But since the limit of $f_k(\lambda)$ is independent of the choice of $\{\tau_k\}$, it follows from (2.1.1) that the sequential Wiener integral exists and (3.2.11) holds for $\lambda \varepsilon \mathcal{S}$.

STEP V. The sequential Wiener integral in (3.2.2) exists and is an analytic function of λ in $\mathcal{S}$ and (3.2.2) holds. Moreover both of members of (3.2.2) approach the members of (3.2.1) as $\lambda \to \sigma$ from inside $\mathcal{S}$.

Proof: It readily follows from (2.1.1), step II and step IV that for each $\lambda \varepsilon \mathcal{S}$,

$$\overset{sw_\lambda}{\int_{C[a,b]}} F(x)dx = \lim_{\|\tau\|\to 0} \int_{\mathbb{R}^n} K_\lambda(\tau,\xi)F(\psi_{\tau,\xi})d\xi$$

$$= \lim_{\|\tau\|\to 0} \int_{\mathbb{R}^n} K(\tau,\xi)F(\lambda\psi_{\tau,\xi})d\xi$$

$$= \overset{sw}{\int_{C[a,b]}} F(\lambda x)dx$$

$$= \int_{C[a,b]} F(\lambda x)dW(x).$$

Thus (3.2.2) is established and by the continuity of the right member of (3.2.2), both of members of (3.2.2) approach the members of

(3.2.1) as $\lambda \to \sigma$ from inside $\mathcal{S}$.

Therefore, by steps I, II, III, IV and V the theorem is proved.

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<u>Corollary 3.3</u>. The conclusion of the existence and equality of the members of (3.2.2) for all λ in $\mathcal{S}$ and their analyticity in $\mathcal{S}$ and their approach to the right member of (3.2.1) as $\lambda \to \sigma$ from inside $\mathcal{S}$ all remains valid if $F(\sigma x)$ in condition 3 of the hypothesis of Theorem 3.2 is replaced by $F(x)$.

A reexamination of the proof of Theorem 3.2 on the basis of the hypothesis of the above corollary will show that the corresponding conclusions hold.

If we replace the analyticity of $F(\lambda x)$ in condition 1 of the hypothesis of Theorem 3.2 by the harmonicity, then we get the generalization of Theorem 3.2 since every analytic function is harmonic, but the converse is false. For example, let $f(z) = \bar{z}$ where $z = x+iy$ and $\bar{z}$ is the conjugate of z., Then $f(z)$ is harmonic, but not analytic.

<u>Theorem 3.4</u> Let $\sigma = \rho e^{i\theta}$, where $\rho > 0$ and $0 < \theta \leq \pi/4$ and let Λ be the open sector of complex numbers λ such that $0 < \arg \lambda < \theta$. Let $H(y)$ be a Borel functional defined for all y of the form $\lambda x(\cdot)$, where $\lambda \in \Lambda^*$ and $x \in C[a,b]$, and Λ^* denotes the closure of Λ with $\lambda = 0$ omitted. Suppose that H also satisfies the following four conditions:

1. $H(\lambda x)$ is harmonic in λ on Λ for each x in $C[a,b]$.
2. $H(\lambda x)$ is a continuous function of λ on Λ^* for each x in $C[a,b]$.

3. $H(\sigma x)$ and $H(\sigma^* x)$ are continuous functions of x in the uniform topology a.u. in $C[a,b]$, where $\sigma^* = \rho e^{i\theta^*}$, $0 < \theta^* < \theta$.

4. There is an $M > 0$ such that

$$|H(e^{i\gamma}x)| \leq M$$

for all x in $C[a,b]$ and all γ on $(0,\theta)$.

Then the sequential Wiener integral (with parameter σ) exists on $C[a,b]$ and we have

$$(3.4.1) \qquad \int_{C[a,b]}^{sw_\sigma} H(x)dx = \int_{C[a,b]} H(\sigma x)dW(x).$$

Moreover the following integrals exist and are equal

$$(3.4.2) \qquad \int_{C[a,b]}^{sw_\lambda} H(x)dx = \int_{C[a,b]} H(\lambda x)dW(x)$$

whenever λ is in the set $\mathcal{S}$ defined by

$$\mathcal{S} = \{\lambda : \lambda \neq 0, \ 0 < \arg \lambda < \theta, \ |\lambda| < \rho\} .$$

Finally, both members of (3.4.2) are harmonic functions of λ on $\mathcal{S}$ and they approach the members of (3.4.1) as $\lambda \to \sigma$ from inside $\mathcal{S}$.

Since every complex function is harmonic if and only if its real part and its imaginary part are harmonic, we need only prove Theorem 3.4 for a real harmonic function $H(\lambda x)$.

<u>Proof</u>: We divide the proof into five steps:

STEP I For all λ in Λ^* and all x in $C[a,b]$, we let $H(\lambda,x) = H(\lambda x)$. Then for each x in $C[a,b]$ there exists an analytic function $F(\lambda,x)$ of λ on Λ such that $\text{Re}\,[F(\lambda,x)] = H(\lambda,x)$.

Proof: Since Λ is simply connected, the unit disc U (i.e., $U = D(0,1)$) and Λ are conformally equivalent, and hence there is a one-one conformal mapping ψ from Λ onto U. For each x in $C[a,b]$, let

$$H^*(z,x) = H(\psi^{-1}(z),x) \qquad (z \in U).$$

Then $H^*(z,x)$ is a real harmonic function of z on U and continuous in z on $\bar{U}$ ($\bar{U}$ denotes the closure of U). Thus (in U), $H^*(z,x)$ is the real part of the analytic function

$$(3.4.3) \qquad F^*(z,x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} H^*(e^{it},x)dt \qquad (z \in U).$$

For each x in $C[a,b]$, let

$$F(\lambda,x) = F^*(\psi(\lambda),x) \qquad (\lambda \in \Lambda).$$

Then $F(\lambda,x)$ is analytic in λ on Λ and we have

$$\text{Re}[F(\lambda,x)] = \text{Re}[F^*(\psi(\lambda),x)] = H^*(\psi(\lambda),x)$$

$$= H(\psi^{-1}(\psi(\lambda)),x) = H(\lambda,x).$$

Hence, for each x in $C[a,b]$ we have $H(\lambda,x)$ is the real part of $F(\lambda,x)$, the analytic function of λ on Λ.

STEP II. $F(\lambda,x)$ is a continuous function of λ on Λ^* for each x in $C[a,b]$.

Proof: For each z in U, $z = re^{i\theta}$, $0 \le r < 1$, θ is real, we have from (3.4.3) that

$$F^*(z,x) = H^*(z,x) + iG^*(z,x)$$

where

$$H^*(z,x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1-r^2}{1-2r\cos(t-\theta)+r^2} H^*(e^{it},x)dt$$

and

$$G^*(z,x) = -\frac{1}{\pi} \int_{-\pi}^{\pi} \frac{r\sin(t-\theta)}{1-2r\cos(t-\theta)+r^2} H^*(e^{it},x)dt.$$

We shall show that the limit

$$\lim_{z\to e^{i\theta}} F^*(z,x)$$

exists and is continuous on T, the boundary of U. Since $H^*(z,x)$ is continuous on $\bar{U}$,

$$\lim_{z\to e^{i\theta}} H^*(z,x) = H^*(e^{i\theta},x)$$

exists and is continuous on T. Then we need only show that,

$$\lim_{z\to e^{i\theta}} G^*(z,x)$$

exists and is continuous on T. We define

$$f(t,x) = H^*(e^{it},x) \qquad -\pi \le t \le \pi.$$

Then $f(t,x)$ is continuous on $[-\pi,\pi]$. Let

$$\psi(t,x) = f(\theta+t,x) - f(\theta-t,x).$$

Thus by conditions 2 and 4 of the hypothesis, we have for all z in $\bar{U}$ and all x in $C[a,b]$ that there is an $M \ge 0$ such that

$$|H^*(z,x)| \le M ,$$

and hence

$$|\psi(t,x)| \le K$$

for some $K > 0$. Then we have

$$G^*(z,x) = -\frac{1}{\pi}\int_{-\pi}^{\pi} \frac{r\sin(t-\theta)}{1-2r\cos(t-\theta)+r^2} f(t,x)dt$$

$$= -\frac{1}{\pi}\int_{0}^{\pi} \frac{r\sin t}{1-2r\cos t+r^2} \psi(t,x)dt .$$

Since $f(t,x)$ is continuous on $[-\pi,\pi]$, for every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that if $|t_1-t_2| < \delta$, then $|f(t_1,x) - f(t_2,x)| < \varepsilon$. Then for $\varepsilon = (1-r)^3$, there exists a $\delta = \delta(\varepsilon) > 0$ such that if $|t_1-t_2| < \delta$, then $|f(t_1,x) - f(t_2,x)| < \varepsilon = (1-r)^3$, so that if $|t| < \delta/2$, then $|(\theta+t)-(\theta-t) = 2|t| < \delta$, and hence $|\psi(t,x)| = |f(\theta+t,x) - f(\theta-t,x)| < (1-r)^3$. Thus

$$G^*(z,x) = \frac{1}{\pi}\int_{0}^{\pi} \frac{r\sin t}{1-2r\cos t+r^2} \psi(t,x)dt$$

$$= -\frac{1}{\pi}\int_{0}^{\delta/2} - \frac{1}{\pi}\int_{\delta/2}^{\pi}$$

$$= I + II ,$$

and obtain $$|I| \le \frac{1}{\pi}\int_0^{\delta/2} \frac{r}{(1-r)^2} |\psi(t,x)|dt \le r(1-r)\frac{\delta}{2\pi} ,$$

hence $\lim_{r\to 1} I = 0$. In II, since $1-2r\cos t+r^2 \ge 4r\sin^2(t/2)$

and $\sin t = 2\sin(t/2)\cos(t/2)$,

$$\left| \frac{r \sin t}{1-2r\cos t+r^2} \psi(t,x) \right| \le K \cot(t/2).$$

Since $\cot(t/2)$ is integrable on $[\delta/2,\pi]$ and $\frac{r \sin t}{1-2r\cos t+r^2}$ is continuous on $[\delta/2,\pi]$, it follows that II exists and is continuous on $[\delta/2,\pi]$ and thus

$$\begin{aligned} \lim_{z\to e^{i\theta}} G^*(z,x) &= \lim_{r\to 1} (I+II) \\ &= \lim_{r\to 1} II \\ &= -\frac{1}{2\pi}\int_{\delta/2}^{\pi} \frac{\sin t}{1-\cos t}\psi(t,x)dt. \end{aligned}$$

By the same proof as before, we have that the last member of the equalities above exists and is continuous on $[\delta/2,\pi]$, hence the limit of $G^*(z,x)$ as $z \to e^{i\theta}$ exists and is continuous for all $e^{i\theta}$ on T, and thus

$$\lim_{z\to e^{i\theta}} F^*(z,x) = \lim_{z\to e^{i\theta}} H(z,x) + i\lim_{z\to e^{i\theta}} G^*(z,x)$$

exists and is continuous on T. Then it can be extended to a continuous function on $\bar{U}$, and hence $F(\lambda,x) = F^*(\psi(\lambda),x)$ is a continuous function of λ on Λ^*.

STEP III. $F(\sigma,x)$ and $F(\sigma^*,x)$ are continuous functions of x in the uniform topology a.u. in $C[a,b]$.

Proof: Let A and A^* be defined as in step III of Theorem 3.2 Then by the same proof as in step III of Theorem 3.2, $H(\lambda,x)$ is a continuous of x in the uniform topology a.u. in $C[a,b]$ for all λ in $A \cup A^*$. Thus for each z in $\psi(A) \cup \psi(A^*)$ we have $H^*(z,x) = H(\psi^{-1}(z),x)$ is a continuous function of x in the uniform topology a.u. in $C[a,b]$, and hence $F^*(z,x)$ is also a continuous function of x in the uniform topology a.u. in $C[a,b]$, so that for each λ in $A \cup A^*$, $F(\lambda,x) = F^*(\psi(\lambda),x)$ is a continuous function of x in the uniform topology a.u. in $C[a,b]$. In particular, this is true for $\lambda = \sigma$ and $\lambda = \sigma^*$.

STEP IV There is an $M > 0$ such that

$$|F(\lambda,x)| \leq M$$

for all λ in A^* and all x in $C[a,b]$.

Proof: It readily follows from (3.4.3) and step III that there is an $M > 0$ such that

$$|F^*(z,x)| \leq M$$

for all z in $\bar{U}$ and all x in $C[a,b]$. Hence,

$$|F(\lambda,x)| = |F^*(\psi(\lambda),x)| \leq M$$

for all λ in A^* and all x in $C[a,b]$.

STEP V The sequential Wiener integral in (3.4.1) exists and (3.4.1) holds. Moreover the integrals in (3.4.2) exist and (3.4.2) holds for λ in $\mathcal{S}$. Finally, both members of (3.4.2) are harmonic functions of λ on $\mathcal{S}$, and they approach the members of (3.4.1) as $\lambda \to \sigma$ from inside $\mathcal{S}$.

Proof: We first note that since $H(\lambda,x) = H(\lambda x)$, we have by virtue of a formal formula given by Ahlfors for determining a harmonic conjugate we can simply drop the comma sign from $F(\lambda,x)$.

By step I, II, III and IV, $F(\lambda x)$ satisfies the hypothesis of Theorem 3.2, and thus the conclusions of Theorem 3.2 hold for F. Since H is the real part of F, step V follows. #

A reexamination of the proof of step III in Theorem 3.4 and by Corollary 3.3, we obtain the following corollary:

Corollary 3.5 The conclusion of the existence and equality of the members of (3.4.2) for all λ in $\mathcal{S}$ and their analyticity in $\mathcal{S}$ and their approach to the right member of (3.4.1) as $\lambda \to \sigma$ from inside $\mathcal{S}$ all remains valid if $H(\sigma x)$ in condition 3 of the hypothesis of Theorem 3.4 is replaced by $H(x)$.