CHAPTER IV

THE LIMITING SEQUENTIAL WIENER AND FEYNMAN INTEGRALS

Corollary 3.3 suggests a natural extension of the sequential Wiener integral, since under its hypotheses the sequential Wiener integral becomes an analytic function of its parameter and approaches a limit as $\lambda \to \sigma$ from within $\mathcal{S}$. It is natural in this case to call the limiting value of the sw_λ integral as $\lambda \to \sigma$ from within $\mathcal{S}$ the "limiting sw_σ integral."

<u>Definition 4.1</u> Let σ be a fixed non-vanishing complex number such that $0 < \arg \sigma \leq \pi/4$. We define the <u>limiting sequential Wiener integral with parameter σ</u> as follows :

$$(4.1.1) \qquad \int_{C[a,b]}^{\to sw_\sigma} F(x)dx = \lim_{\varepsilon \to 0^+} \int_{C[a,b]}^{sw_{\sigma\exp(-i\varepsilon)}} F(x)dx$$

whenever $F(x)$ is a functional such that the right member exists. In particular if $\sigma = p\sqrt{i}$, we define the <u>limiting Feynman integral</u> as follows :

$$(4.1.2) \qquad \int_{C[a,b]}^{\to f_p} F(x)dx = \lim_{\varepsilon \to 0^+} \int_{C[a,b]}^{sw_{p\exp i(\pi/4-\varepsilon)}} F(x)dx$$

whenever the right member has meaning.

It is clear that Corollary 3.3 gives an existence theorem for the limiting sequential Wiener integral and, in particular, the limiting Feynman integral.

<u>Theorem 4.2</u> Let $p > 0$ and the interval $[a,b]$ be given, and let Λ be the open sector of numbers λ such that $0 < \arg \lambda < \pi/4$. Let $F(y)$ be a Borel functional defined for all y of the form $\lambda x(\cdot)$, where $\lambda \in \Lambda^*$ and $x \in C[a,b]$, and Λ^* denotes the closure of Λ with $\lambda = 0$ omitted. Let F also satisfy the following four conditions:

1. $F(\lambda x)$ is analytic in λ on Λ for each x in $C[a,b]$.

2. $F(\lambda x)$ is a continuous function of λ on Λ^* for each x in $C[a,b]$.

3. $F(x)$ and $F(\sigma^* x)$ are continuous functions of x in the uniform topology a.u. in $C[a,b]$, where $\sigma^* = p\, e^{i\theta^*}$, $0 < \theta^* < \pi/4$.

4. For all x in $C[a,b]$ and all γ in $(0, \pi/4)$,

$$|F(e^{i\gamma}x)| \leq M$$

for some $M > 0$.

Then the limiting p-Feynman integral exists on $C[a,b]$ and

$$(4.2.1) \qquad \overset{\rightarrow f_p}{\int_{C[a,b]}} F(x)dx = \int_{C[a,b]} F(\sqrt{i}px)\, dx.$$

Moreover both members of (3.2.2) exist and are continuous and the equality holds on the set H defined by

$$H = \{\lambda : \lambda \neq 0,\ 0 \leq \arg \lambda < \pi/4,\ |\lambda| < p\},$$

and the members of (3.2.2) are analytic in λ inside H and approach the members of (4.1.2) as $\lambda \to \sqrt{i}p$ from within H.

Proof: As in the proof of Theorem 3.2, we have (3.2.4) and (3.2.5) exist on H, and they are equal for real λ in H. Moreover it follows as before that (3.2.4) and (3.2.5) are continuous on H and analytic in the interior of H, and hence that (3.2.8) holds on H. Also for real positive λ and for λ such that $\arg \lambda = \theta^*$, $0 < |\lambda| \leq p$, (3.2.10) holds since the hypotheses of Theorem 2.7 are satisfied. We can readily see that the right member of (3.2.8) is uniformly bounded in every compact subset of H, and thus the argument given in Theorem 3.2 can be used to establish the existence and equality of the members of (3.2.2) on H and their continuity on H and analyticity in the interior of H. Moreover from condition 4 of the hypothesis we see that the right member of (4.1.2) exists, and from condition 1,2,4 we see that the right member of (3.2.2) approaches the right member of (4.1.2) as $\lambda \to \sqrt{i}p$ from within H. Thus Theorem 4.2 is established. #

Corollary 4.3 The conclusions of Theorem 4.2 remain valid if the analyticity of $F(\lambda x)$ in condition 1 of the hypothesis of Theorem 4.2 is replaced by the harmonicity.