CHAPTER II

GENERALIZED CONTINUED FRACTION EXPANSIONS

In this chapter, an algorithm to construct continued fraction expansions, is given in the first section. We give detailed proofs of its convergence and uniqueness. In the second section, several well-known examples are shown to be special cases of our continued fraction expansions.

2.1 Algorithm

In this section, our algorithm for constructing continued fraction expansions of elements in a discrete non-archimedean valued field is given.

Let K be a complete field with respect to a discrete non-archimedean valuation $|\cdot|$ and $\mathcal{A} \subseteq \mathcal{O}$ be a set of representatives of $\mathcal{O}/\mathcal{P}$. By Theorem 1.14, each $\alpha \in K \setminus \{0\}$ is uniquely representable as

$$\alpha = \sum_{i=r}^{\infty} c_i \tau^i, \qquad (c_r \neq 0)$$

where $r \in \mathbb{Z}$, $c_i \in \mathcal{A}$ and $\tau \in \mathcal{O}$. Such representation is usually referred to as its *canonical representation.* The non-archimedean valuation of α is defined by

$$|\alpha| = 2^{-r}, \quad \text{with} \quad |0| := 0.$$

The *head part* $\langle\alpha\rangle$ of α is defined as the finite series

$$\langle\alpha\rangle = \sum_{i=r}^{0} c_i \tau^i \ \text{ if } r \leq 0, \text{ and } 0 \text{ otherwise.}$$

Denote the set of all head parts by

$$S := \{\langle\alpha\rangle \ : \ \alpha \in K\}.$$

We are now ready to introduce our continued fraction expansion algorithm.

Let $\{b_i\}_{i\geq 1}$ be a sequence in $K \smallsetminus \{0\}$, each of whose elements b_i is either fixed or is uniquely determined from α and previously known parameters b_j, a_j $(j < i)$ arising from the algorithm.

For convenience, we consider $\alpha \in K$ such that $|\alpha| < 1$. Let $A_1 := \alpha \neq 0$. Assume that $b_1 \in K \smallsetminus \{0\}$ is subject to the condition that

$$\left|\frac{b_1}{A_1}\right| \geq 1. \tag{2.1}$$

Define $a_1 = \left\langle \dfrac{b_1}{A_1} \right\rangle \in S \smallsetminus \{0\}$.

- If $a_1 = \dfrac{b_1}{A_1}$, then the process stops and we write

$$\alpha = A_1 = \frac{b_1}{a_1}.$$

- When $a_1 \neq \dfrac{b_1}{A_1}$, we have $0 < \left|\dfrac{b_1}{A_1} - a_1\right| < 1$. Assume that $b_2 \in K \smallsetminus \{0\}$ is subject to the condition that the element $A_2 = \dfrac{1}{b_2}\left(\dfrac{b_1}{A_1} - a_1\right) \neq 0$ satisfies

$$0 < |A_2| < 1. \tag{2.2}$$

  Thus,

$$\alpha = A_1 = \frac{b_1}{a_1 + b_2 A_2}.$$

Next, define $a_2 = \left\langle \dfrac{1}{A_2} \right\rangle \in S \smallsetminus \{0\}$.

- If $a_2 = \dfrac{1}{A_2}$, then the process stops and we write

$$\alpha = \frac{b_1}{a_1 + b_2 A_2} = \frac{b_1}{a_1 + \dfrac{b_2}{a_2}} = \frac{b_1}{a_1+} \frac{b_2}{a_2}.$$

- When $a_2 \neq \dfrac{1}{A_2}$, we have $0 < \left| \dfrac{1}{A_2} - a_2 \right| < 1$. Assume that $b_3 \in K \setminus \{0\}$ is subject to the condition that the element $A_3 = \dfrac{1}{b_3}\left(\dfrac{1}{A_2} - a_2\right) \neq 0$ satisfies

$$0 < |A_3| < 1. \tag{2.3}$$

Thus,

$$\alpha = \frac{b_1}{a_1 + b_2 A_2} = \frac{b_1}{a_1 + \dfrac{b_2}{a_2 + b_3 A_3}} = \frac{b_1}{a_1+} \frac{b_2}{a_2 + b_3 A_3}.$$

Continuing this process, if $A_i \neq 0$ for all $i \geq 2$ has already constructed with $0 < |A_i| < 1$, then define $a_i = \left\langle \dfrac{1}{A_i} \right\rangle \in S \setminus \{0\}$.

- If $a_i = \dfrac{1}{A_i}$, then the process stops and we have a finite continued fraction expansion

$$\alpha = \frac{b_1}{a_1+} \frac{b_2}{a_2+} \cdots \frac{b_i}{a_i}.$$

- When $a_i \neq \dfrac{1}{A_i}$, we have $0 < \left| \dfrac{1}{A_i} - a_i \right| < 1$. Assume that $b_{i+1} \in K \setminus \{0\}$ is subject to the condition that the element $A_{i+1} = \dfrac{1}{b_{i+1}}\left(\dfrac{1}{A_i} - a_i\right) \neq 0$ satisfies

$$0 < |A_{i+1}| < 1, \tag{2.4}$$

and so

$$\alpha = \frac{b_1}{a_1+} \frac{b_2}{a_2+} \cdots \frac{b_{i-1}}{a_{i-1}+} \frac{b_i}{a_i + b_{i+1} A_{i+1}}.$$

Observe that $|a_1| = \dfrac{|b_1|}{|A_1|} > |b_1|$. Since $0 < |b_2 A_2| = \left| \dfrac{b_1}{A_1} - a_1 \right| < 1$ and

$0 < |b_{i+1}A_{i+1}| = \left|\frac{1}{A_i} - a_i\right| < 1$ for all $i \geq 2$, we have

$$|a_{i+1}| = \frac{1}{|A_{i+1}|} > |b_{i+1}| \qquad (i \geq 0). \tag{2.5}$$

Note that if the b_i's belong to $S \setminus \{0\}$, then the requirements (2.1), (2.2), (2.3) and (2.4) hold automatically.

Summing up, we see that the algorithm yields a continued fraction expansion of the form

$$\alpha = \frac{b_1}{a_1+}\,\frac{b_2}{a_2+}\cdots\frac{b_{n-1}}{a_{n-1}+}\,\frac{b_n}{a_n + b_{n+1}A_{n+1}}$$

where $a_i \in S \setminus \{0\}$ and b_i are subject to (2.5). If $a_1 = \frac{b_1}{A_1}$ or $a_n = \frac{1}{A_n}$ for some $n \geq 2$, then

$$\alpha = \frac{b_1}{a_1+}\,\frac{b_2}{a_2+}\cdots\frac{b_n}{a_n},$$

i.e., a continued fraction expansion of α is finite. If $a_1 \neq \frac{b_1}{A_1}$ and $a_n \neq \frac{1}{A_n}$ for all $n \geq 2$, we now proceed to show that this continued fraction expansion converges.

Define two sequences (C_n), (D_n) as follows:

$$C_{-1} = 1, \quad C_0 = 0, \quad C_{n+1} = a_{n+1}C_n + b_{n+1}C_{n-1} \quad \text{for all} \quad n \geq 0 \tag{2.6}$$

$$D_{-1} = 0, \quad D_0 = 1, \quad D_{n+1} = a_{n+1}D_n + b_{n+1}D_{n-1} \quad \text{for all} \quad n \geq 0. \tag{2.7}$$

Proposition 2.1. *For any $n \geq 0$, $\beta \in K \setminus \{0\}$, we have*

$$\frac{\beta C_n + b_{n+1}C_{n-1}}{\beta D_n + b_{n+1}D_{n-1}} = \frac{b_1}{a_1+}\,\frac{b_2}{a_2+}\cdots\frac{b_n}{a_n+}\,\frac{b_{n+1}}{\beta}.$$

Proof. For each $n \geq 0$, let $P(n)$ be the statement

$$\frac{\beta C_n + b_{n+1}C_{n-1}}{\beta D_n + b_{n+1}D_{n-1}} = \frac{b_1}{a_1+}\,\frac{b_2}{a_2+}\cdots\frac{b_n}{a_n+}\,\frac{b_{n+1}}{\beta}.$$

Since $\frac{\beta C_0 + b_1C_{-1}}{\beta D_0 + b_1D_{-1}} = \frac{b_1}{\beta}$, $P(0)$ is true.

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Let $n \geq 0$ and suppose $P(n)$ holds. Then

$$\begin{aligned}
\frac{\beta C_{n+1} + b_{n+2} C_n}{\beta D_{n+1} + b_{n+2} D_n} &= \frac{\beta\left(a_{n+1} C_n + b_{n+1} C_{n-1}\right) + b_{n+2} C_n}{\beta\left(a_{n+1} D_n + b_{n+1} D_{n-1}\right) + b_{n+2} D_n} \\
&= \frac{a_{n+1} C_n + b_{n+1} C_{n-1} + \dfrac{b_{n+2} C_n}{\beta}}{a_{n+1} D_n + b_{n+1} D_{n-1} + \dfrac{b_{n+2} D_n}{\beta}} \\
&= \frac{\left(a_{n+1} + \dfrac{b_{n+2}}{\beta}\right) C_n + b_{n+1} C_{n-1}}{\left(a_{n+1} + \dfrac{b_{n+2}}{\beta}\right) D_n + b_{n+1} D_{n-1}} \\
&= \frac{b_1}{a_1 +}\,\frac{b_2}{a_2 +} \cdots \frac{b_n}{a_n +}\,\frac{b_{n+1}}{a_{n+1} + \dfrac{b_{n+2}}{\beta}} \\
&= \frac{b_1}{a_1 +}\,\frac{b_2}{a_2 +} \cdots \frac{b_n}{a_n +}\,\frac{b_{n+1}}{a_{n+1} +}\,\frac{b_{n+2}}{\beta},
\end{aligned}$$

which verifies $P(n+1)$. □

From Proposition 2.1, we have

$$\frac{C_n}{D_n} = \frac{a_n C_{n-1} + b_n C_{n-2}}{a_n D_{n-1} + b_n D_{n-2}} = \frac{b_1}{a_1 +}\,\frac{b_2}{a_2 +} \cdots \frac{b_n}{a_n} \quad \text{for all} \quad n \geq 1,$$

and so $\dfrac{C_n}{D_n}$ is called the n^{th} convergent of continued fraction expansion of α.

Proposition 2.2. *For all $n \geq 1$, we have $C_n D_{n-1} - C_{n-1} D_n = (-1)^{n-1} b_1 b_2 \cdots b_n$.*

Proof. For each $n \geq 1$, let $P(n)$ be the statement

$$C_n D_{n-1} - C_{n-1} D_n = (-1)^{n-1} b_1 b_2 \cdots b_n.$$

Since $C_1 D_0 - C_0 D_1 = C_1 - 0 = a_1 C_0 + b_1 C_{-1} = b_1 = (-1)^{1-1} b_1$, $P(1)$ is true.

Let $n \geq 1$ and suppose $P(n)$ holds. Then

$$\begin{aligned}
C_{n+1} D_n - C_n D_{n+1} &= \left(a_{n+1} C_n + b_{n+1} C_{n-1}\right) D_n - C_n \left(a_{n+1} D_n + b_{n+1} D_{n-1}\right) \\
&= -b_{n+1}\left(C_n D_{n-1} - C_{n-1} D_n\right)
\end{aligned}$$

which proves $P(n+1)$. □

Proposition 2.3. $|C_1| = |b_1|,\ |C_n| = |b_1a_2a_3\cdots a_n|$ *for all* $n \geq 2$.

Proof. We have $|C_1| = |a_1C_0 + b_1C_{-1}| = |b_1|$.
For each $n \geq 2$, let $P(n)$ be the statement $|C_n| = |b_1a_2a_3\cdots a_n|$.
Since $|C_2| = |a_2C_1 + b_2C_0| = |a_2C_1| = |b_1a_2|$, $P(2)$ is true.
We have, by (2.5),

$$|a_3C_2| = |b_1a_2a_3| > |b_1a_2b_3| \geq |b_1b_3| = |b_3C_1|,$$

by Theorem 1.4, we get

$$|C_3| = |a_3C_2 + b_3C_1| = |a_3C_2| = |b_1a_2a_3|,$$

so $P(3)$ is true.
Suppose that $P(t)$ hold for all $3 \leq t \leq n$. We have

$$\begin{aligned} |a_{n+1}C_n| &= |b_1a_2\cdots a_{n-1}a_na_{n+1}| \\ &> |b_1a_2\cdots a_{n-1}a_nb_{n+1}| \\ &\geq |b_1a_2\cdots a_{n-1}b_{n+1}| = |b_{n+1}C_{n-1}|, \end{aligned}$$

by Theorem 1.4, we get

$$|C_{n+1}| = |a_{n+1}C_n + b_{n+1}C_{n-1}| = |a_{n+1}C_n| = |b_1a_2\cdots a_{n-1}a_na_{n+1}|,$$

and so $P(n+1)$ holds. □

Proposition 2.4. $|D_n| = |a_1a_2\cdots a_n| \neq 0$ *for all* $n \geq 1$.

Proof. For each $n \geq 1$, let $P(n)$ be the statement $|D_n| = |a_1a_2\cdots a_n|$.
Since $|D_1| = |a_1D_0 + b_1D_{-1}| = |a_1|$, $P(1)$ is true.
We have

$$|a_2D_1| = |a_1a_2| > |a_1b_2| \geq |b_2| = |b_2D_0|,$$

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by Theorem 1.4, we get

$$|D_2| = |a_2D_1 + b_2D_0| = |a_2D_1| = |a_1a_2|,$$

so $P(2)$ is ture.

Suppose that $P(t)$ hold for all $2 \leq t \leq n$. We have

$$\begin{aligned}|a_{n+1}D_n| &= |a_1a_2\cdots a_{n-1}a_na_{n+1}| \\ &> |a_1a_2\cdots a_{n-1}a_nb_{n+1}| \\ &\geq |a_1a_2\cdots a_{n-1}b_{n+1}| = |b_{n+1}D_{n-1}|,\end{aligned}$$

by Theorem 1.4, we get

$$|D_{n+1}| = |a_{n+1}D_n + b_{n+1}D_{n-1}| = |a_{n+1}D_n| = |a_1a_2\cdots a_{n-1}a_na_{n+1}|,$$

and so $P(n+1)$ holds. □

From the algorithm and Proposition 2.1, we see that

$$\alpha = \frac{b_1}{a_1+}\,\frac{b_2}{a_2+}\cdots\frac{b_{n-1}}{a_{n-1}+}\,\frac{b_n}{a_n + b_{n+1}A_{n+1}} = \frac{(a_n + b_{n+1}A_{n+1})\,C_{n-1} + b_nC_{n-2}}{(a_n + b_{n+1}A_{n+1})\,D_{n-1} + b_nD_{n-2}}.$$

Using Proposition 2.2, we get

$$\begin{aligned}\alpha - \frac{C_n}{D_n} &= \frac{(a_n + b_{n+1}A_{n+1})\,C_{n-1} + b_nC_{n-2}}{(a_n + b_{n+1}A_{n+1})\,D_{n-1} + b_nD_{n-2}} - \frac{C_n}{D_n} \\ &= \frac{(C_{n-1}D_n - C_nD_{n-1})\,(a_n + b_{n+1}A_{n+1}) + b_n\,(C_{n-2}D_n - C_nD_{n-2})}{D_n\,((a_n + b_{n+1}A_{n+1})\,D_{n-1} + b_nD_{n-2})} \\ &= \frac{(C_{n-1}D_n - C_nD_{n-1})\,(a_n + b_{n+1}A_{n+1}) + b_na_n\,(C_{n-2}D_{n-1} - C_{n-1}D_{n-2})}{D_n\,((a_n + b_{n+1}A_{n+1})\,D_{n-1} + b_nD_{n-2})} \\ &= \frac{(-1)^nb_1b_2\cdots b_n\,(a_n + b_{n+1}A_{n+1}) + (-1)^{n-1}b_1b_2\cdots b_{n-1}b_na_n}{D_n\,((a_n + b_{n+1}A_{n+1})\,D_{n-1} + b_nD_{n-2})} \\ &= \frac{(-1)^nb_1b_2\cdots b_nb_{n+1}A_{n+1}}{D_n\,((a_n + b_{n+1}A_{n+1})\,D_{n-1} + b_nD_{n-2})}.\end{aligned}$$

From

$$|a_n| \geq 1 > \frac{|b_{n+1}|}{|a_{n+1}|} = |b_{n+1}A_{n+1}|,$$

we get $|a_n + b_{n+1}A_{n+1}| = |a_n|$, and so $|\,(a_n + b_{n+1}A_{n+1})\,D_{n-1} + b_n D_{n-2}| = |a_n D_{n-1}|$. Thus,

$$\left|A_1 - \frac{C_n}{D_n}\right| = \frac{|b_1 b_2 \cdots b_{n+1}|}{|D_n||D_{n+1}|} \to 0 \quad \text{as} \quad n \to \infty.$$

This shows that $\dfrac{C_n}{D_n}$ converges to α, which enables us to write

$$\alpha = \frac{b_1}{a_1+}\,\frac{b_2}{a_2+}\,\cdots\,\frac{b_n}{a_n+}\cdots.$$

To prove the uniqueness, suppose that $\alpha \in K \setminus \{0\}$, $|\alpha| < 1$, has two such continued fraction expansions

$$\frac{b_1}{a_1+}\,\frac{b_2}{a_2+}\,\cdots\,\frac{b_n}{a_n+}\cdots = \alpha = \frac{b_1'}{a_1'+}\,\frac{b_2'}{a_2'+}\,\cdots\,\frac{b_n'}{a_n'+}\cdots,$$

where $a_i, a_i' \in S \setminus \{0\}$ and the b_i, b_i' are subject to the same requirements as elaborated above. Observe that

$$\left|\frac{b_i}{a_i+}\,\frac{b_{i+1}}{a_{i+1}+}\cdots\right| \leq \frac{|b_i|}{|a_i|} < 1 \quad \text{for all } i \geq 1 \tag{2.8}$$

with the same relations for b_i', a_i' for all $i \geq 1$. From the construction requirement, we have $b_1 = b_1'$ which implies that

$$a_1 + \frac{b_2}{a_2+}\,\frac{b_3}{a_3+}\cdots = a_1' + \frac{b_2'}{a_2'+}\,\frac{b_3'}{a_3'+}\cdots.$$

Since $a_1, a_1' \in S$, using (2.8), we get

$$a_1 = a_1' \quad \text{and} \quad \frac{b_2}{a_2+}\,\frac{b_3}{a_3+}\,\frac{b_4}{a_4+}\cdots = \frac{b_2'}{a_2'+}\,\frac{b_3'}{a_3'+}\,\frac{b_4'}{a_4'+}\cdots.$$

Since $a_1 = a_1'$, from the definition, we have $b_2 = b_2'$. Continuing in the same manner, we get $a_i = a_i'$, $b_i = b_i'$ for all i. The following theorem summarizes our results so far obtained.

Theorem 2.5. *Let K be a complete field with respect to a discrete non-archimedean valuation $|\cdot|$. Then each $\alpha \in K \setminus \{0\}$ with $|\alpha| < 1$, can be represented uniquely by a continued fraction expansion of the form*

$$\alpha = \frac{b_1}{a_1+} \frac{b_2}{a_2+} \cdots \frac{b_n}{a_n+} \cdots,$$

where $a_i \in S \setminus \{0\}$ and the sequence $\{b_i\}_{i=1}^{\infty} \subseteq K \setminus \{0\}$ is either fixed or is uniquely determined from α and previously known parameters b_j, a_j $(j < i)$. Moreover, the partial numerators and denomintors are subject to the condition, which will henceforth be referred to as the **ab-condition**,

$$|a_i| > |b_i| \quad \text{for all} \quad i \geq 1. \tag{2.9}$$

Definition 2.6. A continued fraction expansion as in Theroem 2.5 is called a *JR-continued fraction expansion.*

Remark 2.7. *In the case that $|\alpha| \geq 1$, the JR-continued fraction expansion of α is of the form*

$$\alpha = a_0 + \frac{b_1}{a_1+} \frac{b_2}{a_2+} \cdots \frac{b_n}{a_n+} \cdots,$$

where $a_0 = \langle \alpha \rangle$, $a_i \in S \setminus \{0\}$ and b_i are subject to the ab-condition.

2.2 Examples

We turn now to specific examples. For the first three examples, let K be a complete field with respect to a discrete non-archimedean valuation $|\cdot|$.

Example 2.8. Taking $b_i = 1$ for all $i \geq 1$ in Theorem 2.5, we deduce that every $\alpha \in K \setminus \{0\}$ with $|\alpha| < 1$, has a unique regular continued fraction expansion of the form

$$\alpha = \frac{1}{a_1+} \frac{1}{a_2+} \cdots \frac{1}{a_n+} \cdots,$$

where $a_i \in S \setminus \{0\}$ are subject to the ab-condition, i.e., $|a_i| > 1$ for all $i \geq 1$. This is the well-known classical regular continued fraction expansion.

Example 2.9. Taking $b_1 = 1$ and $b_{i+1} = a_i$ for all $i \geq 1$ in Theorem 2.5, we deduce that every $\alpha \in K \setminus \{0\}$ with $|\alpha| < 1$, has a unique continued fraction expansion of the form

$$\alpha = \frac{1}{a_1+}\frac{a_1}{a_2+}\cdots\frac{a_{n-1}}{a_n+}\cdots,$$

where $a_i \in S \setminus \{0\}$ for all $i \geq 1$ are subject to the ab-condition, i.e., $|a_1| > 1$ and $|a_{i+1}| > |b_{i+1}| = |a_i|$ for all $i \geq 1$. This continued fraction expansion may be regarded as a non-archimedean analogue of the real Engel continued fraction expansion due to Hartono-Kraaikamp-Schweiger [12].

Example 2.10. Taking $b_1 = 1$ and $b_{i+1} = a_i^2 - a_i + 1$ for all $i \geq 1$ in Theorem 2.5, we deduce that every $\alpha \in K \setminus \{0\}$ with $|\alpha| < 1$, has a unique continued fraction expansion of the form

$$\alpha = \frac{1}{a_1+}\frac{a_1^2 - a_1 + 1}{a_2+}\cdots\frac{a_{n-1}^2 - a_{n-1} + 1}{a_n+}\cdots,$$

where $a_i \in S \setminus \{0\}$ for all $i \geq 1$ are subject to the ab-condition, i.e., $|a_1| > 1$ and $|a_{i+1}| > |b_{i+1}| = |a_i^2 - a_i + 1|$ for all $i \geq 1$. This continued fraction expansion may be regarded as a non-archimedean analogue of the real Sylvester continued fraction expansion due to A. H. Fan, B. W. Wang and J. Wu [10].

Example 2.11. (The field of p-adic numbers, $\mathbb{Q}_p$)
Let $K = \mathbb{Q}_p$ be the field of p-adic numbers, i.e., the completion of $\mathbb{Q}$ with respect to the p-adic valuation, $|\cdot|_p$ normalized so that $|p|_p = p^{-1}$. Here, the ring of p-adic integers is $\mathcal{O} = \mathbb{Z}_p$. Each $\alpha \in p\mathbb{Z}_p \setminus \{0\}$ is uniquely representable in the form

$$\alpha = \sum_{i=r}^{\infty} c_i p^i \qquad (r \in \mathbb{N},\ c_i \in \{0, 1, \ldots, p-1\},\ c_r \neq 0).$$

There are two well-known p-adic continued fraction expansions, due respectively to Ruban ([23]) and Schneider ([25]).

(1) For $\alpha \in p\mathbb{Z}_p \setminus \{0\}$, since $|\alpha^{-1}|_p > 1$, let its p-adic representation be

$$\alpha^{-1} = c_{-m}p^{-m} + c_{-m+1}p^{-m+1} + \cdots + c_{-1}p^{-1} + c_0 + c_1 p + c_2 p^2 + \cdots,$$

where $m \in \mathbb{N}$, $c_i \in \{0, 1, \ldots, p-1\}$ $(i \geq -m)$ and $c_{-m} \neq 0$. Let

$$a_1 := \langle \alpha^{-1} \rangle = c_{-m}p^{-m} + c_{-m+1}p^{-m+1} + \cdots + c_{-1}p^{-1} + c_0 \quad \text{and}$$
$$A_1 := \alpha^{-1} - a_1,$$

so that $\alpha^{-1} = a_1 + A_1$. If $A_1 = 0$, the process stops and we write $\alpha = \dfrac{1}{a_1}$. If $A_1 \neq 0$ and since $|A_1|_p < 1$, then by repeating the step just described, we uniquely write

$$A_1^{-1} = a_2 + A_2, \quad \text{where} \quad a_2 := \langle A_1^{-1} \rangle, \quad A_2 := A_1^{-1} - a_2.$$

Again if $A_2 = 0$, then the process stops and we write $\alpha = \dfrac{1}{a_1+} \dfrac{1}{a_2}$. Otherwise, we proceed in the same manner with A_2 replacing A_1, etc. Thus, each $\alpha \in p\mathbb{Z}_p \setminus \{0\}$ has a p-adic Ruban continued fraction expansion of the form

$$\alpha = \frac{1}{a_1+} \frac{1}{a_2+} \cdots \frac{1}{a_n+} \cdots,$$

where the a_i's are nonconstant elements in S. This is a JR-continued fraction expansion with all $b_i = 1$. The ab-condition holds trivially.

(2) For $\alpha \in p\mathbb{Z}_p \setminus \{0\}$, let its p-adic representation be

$$\alpha = c_n p^n + c_{n+1}p^{n+1} + \cdots,$$

where $n \in \mathbb{N}$, $c_i \in \{0, 1, \ldots, p-1\}$ $(i \geq n)$, $c_n \neq 0$. Now write

$$\alpha_1 := \alpha = b_1 u_1^{-1}, \quad \text{where} \quad b_1 := p^n, \quad u_1 := b_1\alpha^{-1} \quad \text{with} \quad |u_1|_p = 1.$$

Write $u_1 = c_{1,0} + c_{1,1}p + c_{1,2}p^2 + \cdots$, with $c_{1,i} \in \{0, 1, \ldots, p-1\}$ for all $i \geq 0$, $c_{1,0} \neq 0$. Thus,

$$\alpha = \frac{b_1}{u_1} = \frac{b_1}{a_1 + \alpha_2},$$

where $a_1 := c_{1,0}$, $\alpha_2 := u_1 - a_1 = c_{1,r_1}p^{r_1} + c_{1,r_1+1}p^{r_1+1} + \cdots$, with r_1 be the least positive integer such that $c_{1,r_1} \neq 0$.

If $\alpha_2 = 0$, then the process stops and we have $\alpha = \dfrac{b_1}{a_1}$.

If $\alpha_2 \neq 0$, then we write

$$\alpha_2 = b_2 u_2^{-1}, \quad \text{where } b_2 := p^{r_1}, \ u_2 := b_2 \alpha_2^{-1} \ \text{ with } \ |u_2|_p = 1.$$

Write $u_2 = c_{2,0} + c_{2,1}p + c_{2,2}p^2 + \cdots$, with $c_{2,i} \in \{0, 1, \ldots, p-1\}$ for all $i \geq 0$, $c_{2,0} \neq 0$. Thus,

$$\alpha = \frac{b_1}{a_1 + \alpha_2} = \frac{b_1}{a_1 +} \frac{b_2}{a_2 + \alpha_3},$$

where $a_2 := c_{2,0}$, $\alpha_3 := u_2 - a_2 = c_{2,r_2}p^{r_2} + c_{2,r_2+1}p^{r_2+1} + \cdots$, with r_2 be the least positive integer such that $r_{2,r_2} \neq 0$.

The process continues in this manner. Thus, each $\alpha \in p\mathbb{Z}_p \setminus \{0\}$ has a p-adic Schneider continued fraction expansion of the form

$$\alpha = \frac{b_1}{a_1 +} \frac{b_2}{a_2 +} \cdots \frac{b_n}{a_n +} \cdots,$$

where $a_i \in \{0, 1, \ldots, p-1\}$, $b_1 = |\alpha|_p^{-1}$, and each b_{i+1} is of the form p^{r_i} $(r_i, i \in \mathbb{N})$ and is uniquely determined form α and previously known a_j, b_j $(j < i)$. This is a JR-continued fraction expansion with all $a_i \in \{0, 1, \ldots, p-1\}$, b_i being positive powers of p. The ab-condition holds trivially.

Example 2.12. (The function field with respect to the degree valuation, $F\left((x^{-1})\right)$) Let F be a field and let

$$F\left((x^{-1})\right) := \Big\{ c_r x^r + c_{r-1}x^{r-1} + \cdots + c_1 x + c_0 + \frac{c_{-1}}{x} + \frac{c_{-2}}{x^2} + \cdots : \\ r \in \mathbb{Z} \text{ and } c_i \in F \text{ for all } i \leq r \Big\}$$

be the completion of the rational function field $F(x)$ with respect to the non-archimedean degree valuation, $|\cdot|_\infty$, normalized so that $|x^{-1}|_\infty = 2^{-1}$. There are at least three kinds of continued fraction expansions in $F\left((x^{-1})\right)$ as we now elaborate.

(1) Let $\{b_i\}_{i=1}^{\infty}$ be a fixed sequence in $F[x] \setminus \{0\}$. By Theorem 2.5, each $\alpha \in F((x^{-1})) \setminus \{0\}$, $|\alpha|_\infty < 1$, has a unique JR-continued fraction expansion of the form

$$\alpha = \frac{b_1}{a_1+}\ \frac{b_2}{a_2+}\ \cdots \frac{b_n}{a_n+} \cdots,$$

where $a_i \in F[x] \setminus \{0\}$ are subject to the ab-condition, i.e., $|a_i|_\infty > |b_i|_\infty$ for all $i \geq 1$. The JR-continued fraction expansion in this case is indeed the non-regular continued fraction expansion constructed in [16].

(2) The x^{-1}-Ruban continued fraction expansion (see e.g. [24]) of $\alpha \in x^{-1}F((x^{-1})) \setminus \{0\}$ is of the form

$$\alpha = \frac{1}{a_1+}\ \frac{1}{a_2+} \cdots \frac{1}{a_n+} \cdots,$$

where the a_i's belong to the set of head parts $F[x] \setminus F$. This is a JR-continued fraction expansion with all $b_i = 1$. The ab-condition holds trivially.

(3) The x^{-1}-Schneider continued fraction expansion of $\alpha \in F((x^{-1})) \setminus \{0\}$ is of the form

$$\alpha = \frac{b_1}{a_1+}\ \frac{b_2}{a_2+} \cdots \frac{b_n}{a_n+} \cdots,$$

where the partial denominators and numerators are as the following:

$$a_i \in F \setminus \{0\},\ b_i = x^{-s_i},\ s_i \in \mathbb{N} \ \text{ for all } \ i \geq 1$$

and each b_i is uniquely determined form α and previously known a_j, b_j $(j < i)$. The ab-condition holds trivially.

Example 2.13. (The function field with respect to a π-adic valuation, $F((\pi(x)))$) Let F be a field and let $\pi(x)$ be a monic irreducible polynomial in $F[x]$. The field

$$F((\pi(x))) = \Big\{ \frac{c_{-r}}{\pi^r} + \frac{c_{-r+1}}{\pi^{r-1}} + \cdots + \frac{c_{-1}}{\pi} + c_0 + c_1\pi + c_2\pi^2 + \cdots : \\ r \in \mathbb{Z} \text{ and } c_i \in F[x] \text{ with } \deg c_i < \deg \pi \text{ for all } i \geq r \Big\}$$

of all formal Laurent series in $\pi(x)$ is the completion of $F(x)$ with respect to the π-adic valuation, $|\cdot|_\pi$, normalized so that $|\pi(x)|_\pi = 2^{-\deg \pi}$. Its ring of integers is the set of formal power series

$$F\left[[\pi(x)]\right] := \left\{c_0 + c_1\pi + c_2\pi^2 + \cdots : \quad c_i \in F[x],\ \deg c_i < \deg \pi\right\},$$

and the set of head parts is

$$S := \left\{\frac{c_{-r}}{\pi^r} + \frac{c_{-r+1}}{\pi^{r-1}} + \cdots + \frac{c_{-1}}{\pi} + c_0 : \ r \geq 0,\ c_i \in F[x],\ \deg c_i < \deg \pi\right\}.$$

By Theorem 2.5, each $\alpha \in \pi(x)F\left[[\pi(x)]\right] \setminus \{0\}$ is uniquely represented as a JR-continued fraction expansion of the form

$$\alpha = \frac{b_1}{a_1+}\,\frac{b_2}{a_2+}\cdots\frac{b_n}{a_n+}\cdots, \tag{2.10}$$

where $a_i \in S \setminus \{0\}$ and b_i are subject to the ab-condition. There are various particular examples of JR-continued fraction expansions in this setting. Let us mention two specific ones.

(1) The π-adic Ruban continued fraction expansion is constructed in exactly the same manner as the p-adic Ruban continued fraction expansion mentioned in Example 2.11 (1), i.e., each $\alpha \in \pi(x)F\left[[\pi(x)]\right] \setminus \{0\}$ is uniquely representable as

$$\alpha = \frac{1}{a_1+}\,\frac{1}{a_2+}\cdots\frac{1}{a_n+}\cdots,$$

where the a_i's are nonconstant elements in S. This is a JR-continued fraction expansion with all $b_i = 1$. The ab-condition holds trivially.

(2) The π-adic Schneider continued fraction expansion is constructed in exactly the same manner as the p-adic Schneider continued fraction expansion mentioned in Example 2.11 (2), i.e., each $\alpha \in \pi(x)F\left[[\pi(x)]\right] \setminus \{0\}$ is uniquely representable as

$$\alpha = \frac{b_1}{a_1+}\,\frac{b_2}{a_2+}\cdots\frac{b_n}{a_n+}\cdots,$$

where the partial denominators and numerators are as the following:

$$a_i \in F[x] \setminus \{0\}, \ \deg a_i < \deg \pi, \ b_i = \pi^{s_i}, \ s_i \in \mathbb{N} \text{ for all } i \geq 1$$

and each b_i is uniquely determined form α and previously known a_j, b_j $(j < i)$.