CHAPTER V
CROSS PRISM OF C_n

In this chapter, we define a prism-like graph, called the cross prism of a cycle C_n. We can show that this graph is an edge-odd graceful graph for every $n \geq 3$.

Definition 5.1 Let $n \geq 3$ and C_n be an n-cycle $u_1u_2u_3 \cdots u_nu_1$. Let $C'_n = u'_1u'_2u'_3 \cdots u'_nu'_1$ be a copy of C_n. Define $\text{XPrism}(C_n)$, called the cross prism of C_n, by a graph that consists of $V(\text{XPrism}(C_n)) = V(C_n) \cup V(C'_n)$ and

$$E(\text{XPrism}(C_n)) = \begin{cases} \{u_1u'_1, u_2u'_2, u_3u'_3, \ldots, u_{j-1}u'_{j-1}, u_ju'_{j+1}, u_{j+1}u'_j, u_{j+2}u'_{j+2}, \\ u_{j+3}u'_{j+3}, \ldots, u_nu'_n\} \cup E(C_n) \cup E(C'_n), \\ \qquad \text{for } j \in \{1,2,3,\ldots,n-1\}, \\ \{u_1u'_n, u'_1u_n, u_2u'_2, u_3u'_3, u_4u'_4, \ldots, u_{n-1}u'_{n-1}\} \cup E(C_n) \cup \\ E(C'_n), \text{ for } j = n. \end{cases}$$

Let us call $u_ju'_{j+1}, u'_ju_{j+1}$ if $j \in \{1,2,3,\ldots,n-1\}$ and $u_1u'_n, u'_1u_n$ if $j = n$, cross bridges.

Example 5.1 From Definition 5.1, we have $\text{XPrism}(C_4)$, shown in Figure 5.1.

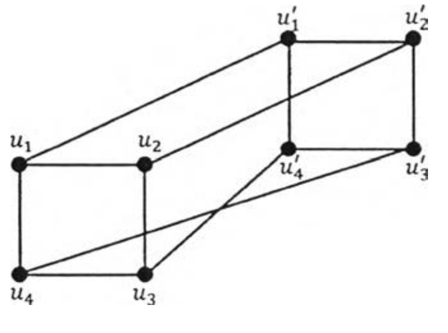

Figure 5.1 $\text{XPrism}(C_4)$.

The cycles C_n and C_n' in Definition 5.1 allow the "cross bridges" to be occurred at any corresponding consecutive pair. However, we can rename the vertices in such a way that it is easy for labeling. Thus, in our algorithms, we will fixed cross bridges at some given consecutive pair.

First, Figure 5.2 shows one example on edge-labeling of $\text{XPrism}(C_3)$.

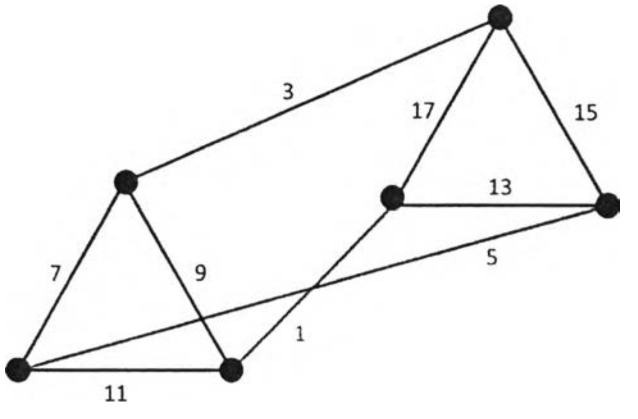

Figure 5.2 Edge-labeling for $\text{XPrism}(C_3)$.

Next, for any $n > 3$, we can label the edges of $\text{XPrism}(C_n)$ by using the following algorithm.

Algorithm 5.1

Let $n > 3$ be an odd integer. Let G denote $\text{XPrism}(C_n)$ where u_1u_2' and u_2u_1' are its cross bridges. Then, $q = 3n$. Define $f: E(G) \rightarrow \{1, 3, 5, \ldots, 6n-1\}$ by

1.1 $f(u_1u_2') = 1;$

1.2 $f(u_2u_1') = 2n-1;$

1.3 $f(u_iu_i') = 2n-2i+3$, for $i \in \{3, 4, 5, \ldots, n\};$

1.4 $f(u_1u_2) = 4n-1;$

1.5 $f(u_i u_{i+1}) = 2n + 2i - 3$, for $i \in \{2, 3, 4, \dots, n-1\}$;

1.6 $f(u_1 u_n) = 4n - 3$;

1.7 $f(u'_1 u'_2) = 6n - 1$;

1.8 $f(u'_i u'_{i+1}) = 4n + 2i - 1$, for $i \in \{2, 3, 4, \dots, n-1\}$;

1.9 $f(u'_1 u'_n) = 4n + 1$.

Example 5.2 From Algorithm 5.1, we can label each edge of $\mathrm{XPrism}(C_5)$ as shown in Figure 5.3.

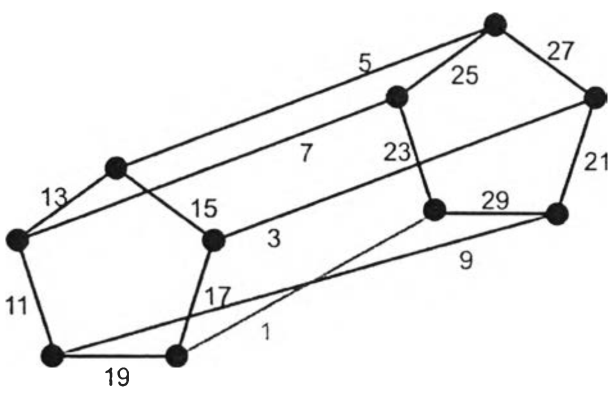

Figure 5.3 Edge-labeling for $\mathrm{XPrism}(C_5)$.

Algorithm 5.2

Let $n \geq 4$ be an even integer. Let G denote $\mathrm{XPrism}\,(C_n)$ where $u_{\frac{n}{2}+1} u'_{\frac{n}{2}+2}$ and $u_{\frac{n}{2}+2} u'_{\frac{n}{2}+1}$ are its cross bridges. Then, $q = 3n$. Define $f: E(G) \to \{1, 3, 5, \dots, 6n-1\}$ by

2.1 $f(u_1 u'_1) = 1$;

2.2 $f(u_i u'_i) = 2n - 2i + 3$,

for $i \in \left\{2, 3, 4, \dots, \frac{n}{2}\right\} \cup \left\{\frac{n}{2} + 3, \frac{n}{2} + 4, \frac{n}{2} + 5, \dots, n\right\}$;

2.3 $f\left(u_{\frac{n}{2}+1}u'_{\frac{n}{2}+2}\right) = n+1;$

2.4 $f\left(u_{\frac{n}{2}+2}u'_{\frac{n}{2}+1}\right) = n-1;$

2.5 $f(u_i u_{i+1}) = 2n+2i-1$, for $i \in \{1,2,3,\dots,n-1\}$;

2.6 $f(u_1 u_n) = 4n-1;$

2.7 $f(u'_1 u'_2) = 4n+1;$

2.8 $f\left(u'_{\frac{n}{2}+1}u'_{\frac{n}{2}+2}\right) = 4n+3;$

2.9 $f(u'_i u'_{i+1}) = 4n+2i+1$, for $i \in \left\{2,3,4,\dots,\frac{n}{2}\right\}$;

2.10 $f(u'_i u'_{i+1}) = 4n+2i-1$,

for $i \in \left\{\frac{n}{2}+2, \frac{n}{2}+3, \frac{n}{2}+4, \dots, n-1\right\}$;

2.11 $f(u'_1 u'_n) = 6n-1.$

Example 5.3 From Algorithm 5.2, we can label each edge of $\mathrm{XPrism}(C_6)$ as shown in Figure 5.4.

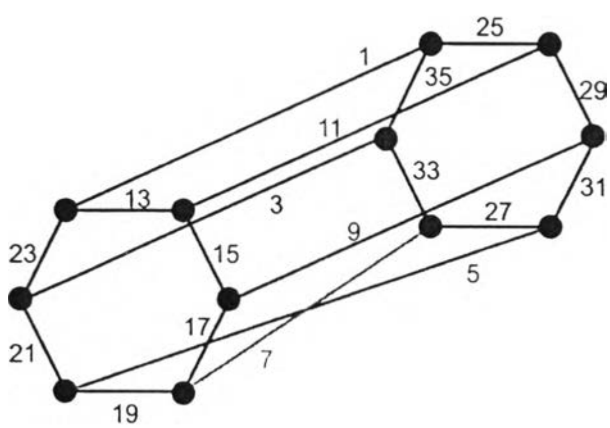

Figure 5.4 Edge-labeling for $\mathrm{XPrism}(C_6)$.

Lemma 5.1 *Let* $n \geq 3$ *be an odd integer.* $XPrism(C_n)$ *is an edge-odd graceful graph.*

Proof. From Figure 5.2, we can see immediately that the induced vertex-labeling is shown in Figure 5.5.

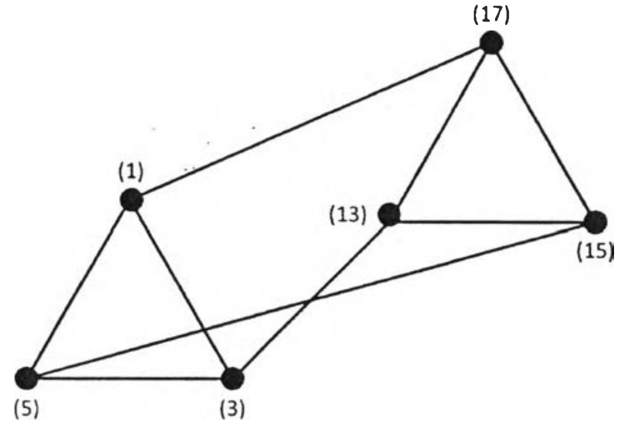

Figure 5.5 The vertex-labeling is induced from the edge-labeling in Figure 5.2.

Therefore, it is obvious from Figure 5.5 that $\mathrm{XPrism}(C_3)$ is an edge-odd graceful graph.

Let $n > 4$ be an odd integer. We first prove that the function f defined in Algorithm 5.1 is a bijection from $E(G)$ to $\{1, 3, 5, \dots, 6n - 1\}$. From Algorithm 5.1(1.1 and 1.2), we have

$$A \quad = \{f(u_1u_2'), f(u_2u_1')\} = \{1, 2n - 1\}.$$

From Algorithm 5.1(1.3), we have

$$B \quad = \{f(u_iu_i') \mid i \in \{3, 4, 5, \dots, n\}\} = \{2n - 3, 2n - 5, 2n - 7, \dots, 3\}.$$

From Algorithm 5.1(1.4, 1.5 and 1.6), we have

$$C \quad = \{f(u_1u_2)\} \cup \{f(u_iu_{i+1}) \mid i \in \{2, 3, 4, \dots, n - 1\}\} \cup \{f(u_1u_n)\}$$

$$= \{4n - 1\} \cup \{2n + 1, 2n + 3, 2n + 5, \dots, 4n - 5\} \cup \{4n - 3\}.$$

From Algorithm 5.1(1.7, 1.8 and 1.9), we have

$$D = \{f(u_1'u_2')\} \cup \{f(u_i'u_{i+1}') \mid i \in \{2,3,4,\dots,n-1\}\} \cup \{f(u_1'u_n')\}$$
$$= \{6n-1\} \cup \{4n+3, 4n+5, 4n+7, \dots, 6n-3\} \cup \{4n+1\}.$$

We can see clearly that A, B, C and D are disjoint and

$$f\left(E(\mathrm{XPrism}(C_n))\right) = A \cup B \cup C \cup D = \{1,3,5,\dots,6n-1\}.$$

Next, we will show that the induced vertex-labels from the edge-labels using Algorithm 5.1 are in $\{0,1,2,\dots,6n-1\}$ and all distinct. From Algorithm 5.1, we have

$$f^+(u_1) = \left(f(u_1u_2') + f(u_1u_2) + f(u_1u_n)\right) \pmod{6n}$$
$$= \left(1 + (4n-1) + (4n-3)\right) \pmod{6n}$$
$$= 2n-3;$$

$$f^+(u_2) = \left(f(u_2u_1') + f(u_1u_2) + f(u_2u_3)\right) \pmod{6n}$$
$$= \left((2n-1) + (4n-1) + (2n+1)\right) \pmod{6n}$$
$$= 2n-1;$$

$$f^+(u_i) = \left(f(u_iu_i') + f(u_{i-1}u_i) + f(u_iu_{i+1})\right) \pmod{6n}$$
$$= \left((2n-2i+3) + (2n+2i-5) + (2n+2i-3)\right) \pmod{6n}$$
$$= (6n+2i-5) \pmod{6n}$$
$$= 2i-5, \text{ for } i \in \{3,4,5,\dots,n-1\};$$

$$f^+(u_n) = \left(f(u_nu_n') + f(u_{n-1}u_n) + f(u_1u_n)\right) \pmod{6n}$$
$$= \left(3 + (4n-5) + (4n-3)\right) \pmod{6n}$$
$$= (8n-5) \pmod{6n}$$

$$= 2n - 5;$$

$$f^+(u_1') = \big(f(u_2u_1') + f(u_1'u_2') + f(u_1'u_n')\big) \pmod{6n}$$
$$= \big((2n-1) + (6n-1) + (4n+1)\big) \pmod{6n}$$
$$= (12n-1) \pmod{6n}$$
$$= 6n - 1;$$

$$f^+(u_2') = \big(f(u_1u_2') + f(u_1'u_2') + f(u_2'u_3')\big) \pmod{6n}$$
$$= \big(1 + (6n-1) + (4n+3)\big) \pmod{6n}$$
$$= (10n+3) \pmod{6n}$$
$$= 4n + 3;$$

$$f^+(u_i') = \big(f(u_iu_i') + f(u_{i-1}'u_i') + f(u_i'u_{i+1}')\big) \pmod{6n}$$
$$= \big((2n-2i+3) + (4n+2i-3) + (4n+2i-1)\big) \pmod{6n}$$
$$= (10n+2i-1) \pmod{6n}$$
$$= 4n + 2i - 1, \text{ for } i \in \{3,4,5,\dots,n-1\};$$

$$f^+(u_n') = \big(f(u_nu_n') + f(u_{n-1}'u_n') + f(u_1'u_n')\big) \pmod{6n}$$
$$= \big(3 + (6n-3) + (4n+1)\big) \pmod{6n}$$
$$= (10n+1) \pmod{6n}$$
$$= 4n + 1.$$

We can see that

$$\{f^+(u_i) \mid i \in \{1,2,3,\dots,n\}\}$$

$$= \{2n-3\} \cup \{2n-1\} \cup \{1,3,5,\dots,2n-7\} \cup \{2n-5\}$$

$$= \{1,3,5,\dots,2n-5,2n-3,2n-1\}$$

and

$$\{f^+(u_i') \mid i \in \{1,2,3,\dots,n\}\}$$

$$= \{6n-1\} \cup \{4n+3\} \cup \{4n+5,4n+7,4n+9,\dots,6n-3\} \cup \{4n+1\}$$

$$= \{4n+1,4n+3,4n+5,\dots,6n-3,6n-1\}.$$

It is clear that if $n \geq 4$ and n is odd, these two sets are disjoint and both are subsets of $\{0,1,2,\dots,6n-1\}$. Therefore, the function f defined in Algorithm 5.1 is an edge-odd graceful labeling. ■

Lemma 5.2 *Let $n \geq 4$ be an even integer. $XPrism(C_n)$ is an edge-odd graceful graph.*

Proof. Let $n \geq 4$ be an even integer. From Algorithm 5.2(2.1 and 2.2), we have

$$E = \left\{f(u_i u_i') \mid i \in \left\{2,3,4,\dots,\frac{n}{2}\right\} \cup \left\{\frac{n}{2}+3, \frac{n}{2}+4, \frac{n}{2}+5,\dots,n\right\}\right\} \cup \{f(u_1 u_1')\}$$

$$= \{2n-1,2n-3,2n-5,\dots,n+3,n-3,n-5,n-7,\dots,3\} \cup \{1\}.$$

From Algorithm 5.2(2.3 and 2.4), we have

$$F = \left\{f\left(u_{\frac{n}{2}+1} u_{\frac{n}{2}+2}'\right)\right\} \cup \left\{f\left(u_{\frac{n}{2}+2} u_{\frac{n}{2}+1}'\right)\right\} = \{n+1\} \cup \{n-1\}.$$

From Algorithm 5.2(2.5 and 2.6), we have

$$G = \{f(u_i u_{i+1}) \mid i \in \{1, 2, 3, \dots, n-1\}\} \cup \{f(u_1 u_n)\}$$

$$= \{2n + 1, 2n + 3, 2n + 5, \dots, 4n - 3\} \cup \{4n - 1\}.$$

From Algorithm 2.2(2.7, 2.8, 2.9), we have

$$H = \{f(u'_1 u'_2)\} \cup \left\{f\left(u'_{\frac{n}{2}+1} u'_{\frac{n}{2}+2}\right)\right\} \cup \left\{f(u'_i u'_{i+1}) \mid i \in \left\{2,3,4, \dots, \frac{n}{2}\right\}\right\}$$

$$= \{4n + 1\} \cup \{4n + 3\} \cup \{4n + 5, 4n + 7, 4n + 9, \dots, 5n + 1\}.$$

From Algorithm 2.2(2.10, 2.11), we have

$$I = \left\{f(u'_i u'_{i+1}) \mid i \in \left\{\frac{n}{2} + 2, \frac{n}{2} + 3, \frac{n}{2} + 4, \dots, n - 1\right\}\right\} \cup \{f(u'_1 u'_n)\}$$

$$= \{5n + 3, 5n + 5, 5n + 7, \dots, 6n - 3\} \cup \{6n - 1\}.$$

We can see clearly that E, F, G, H and I are disjoint and

$$f\left(E(\text{XPrism}(C_n))\right) = E \cup F \cup G \cup H \cup I = \{1, 3, 5, \dots, 6n - 1\}.$$

Next, we will show that the induced vertex-labels from the edge-labels using Algorithm 5.2 are in $\{0, 1, 2, \dots, 6n - 1\}$ and all distinct. From Algorithm 5.2, we have

$$f^+(u_1) = \left(f(u_1 u'_1) + f(u_1 u_2) + f(u_1 u_n)\right) \pmod{6n}$$

$$= (1 + (2n + 1) + (4n - 1)) \pmod{6n}$$

$$= (6n + 1) \pmod{6n}$$

$$= 1;$$

$$f^+(u_i) = \left(f(u_i u'_i) + f(u_{i-1} u_i) + f(u_i u_{i+1})\right) \pmod{6n}$$

$$= \left((2n - 2i + 3) + (2n + 2i - 3) + (2n + 2i - 1)\right)$$

$$\pmod{6n}$$

$$= 2i - 1,$$

for $i \in \left\{2, 3, 4, \dots, \frac{n}{2}\right\} \cup \left\{\frac{n}{2} + 3, \frac{n}{2} + 4, \frac{n}{2} + 5, \dots, n - 1\right\}$;

$$f^+\left(u_{\frac{n}{2}+1}\right) = \left(f\left(u_{\frac{n}{2}+1}u'_{\frac{n}{2}+2}\right) + f\left(u_{\frac{n}{2}}u_{\frac{n}{2}+1}\right) + f\left(u_{\frac{n}{2}+1}u_{+2}\right)\right) \pmod{6n}$$

$$= \left((n + 1) + (3n - 1) + (3n + 1)\right) \pmod{6n}$$

$$= (7n + 1) \pmod{6n}$$

$$= n + 1;$$

$$f^+\left(u_{\frac{n}{2}+2}\right) = \left(f\left(u_{\frac{n}{2}+2}u'_{\frac{n}{2}+1}\right) + f\left(u_{\frac{n}{2}+1}u_{\frac{n}{2}+2}\right) + f\left(u_{\frac{n}{2}+2}u_{\frac{n}{2}+3}\right)\right) \pmod{6n}$$

$$= ((n - 1) + (3n + 1) + (3n + 3)) \pmod{6n}$$

$$= (7n + 3) \pmod{6n}$$

$$= n + 3;$$

$$f^+(u_n) = \left(f(u_n u'_n) + f(u_{n-1}u_n) + f(u_1 u_n)\right) \pmod{6n}$$

$$= \left(3 + (4n - 3) + (4n - 1)\right) \pmod{6n}$$

$$= (8n - 1) \pmod{6n}$$

$$= 2n - 1;$$

$$f^+(u'_1) = \left(f(u_1 u'_1) + f(u'_1 u'_2) + f(u'_1 u'_n)\right) \pmod{6n}$$

$$= (1 + (4n + 1) + (6n - 1)) \pmod{6n}$$

$$= (10n + 1) \pmod{6n}$$

$$= 4n + 1;$$

$$
\begin{aligned}
f^+(u'_2) &= \left(f(u_2u'_2) + f(u'_2u'_3) + f(u'_1u'_2)\right) \pmod{6n}\\
&= ((2n-1) + (4n+5) + (4n+1)) \pmod{6n}\\
&= (10n+5) \pmod{6n}\\
&= 4n+5;
\end{aligned}
$$

$$
\begin{aligned}
f^+(u'_i) &= \left(f(u_iu'_i) + f(u'_{i-1}u'_i) + f(u'_iu'_{i+1})\right) \pmod{6n}\\
&= \left((2n-2i+3) + \left(4n+2i-1\right) + (4n+2i+1)\right)\\
&\quad \pmod{6n}\\
&= (10n+2i+3) \pmod{6n}\\
&= 4n+2i+3, \text{ for } i \in \left\{3,4,5,\ldots,\frac{n}{2}\right\};
\end{aligned}
$$

$$
\begin{aligned}
f^+\left(u'_{\frac{n}{2}+1}\right) &= \left(f\left(u_{\frac{n}{2}+2}u'_{\frac{n}{2}+1}\right) + f\left(u'_{\frac{n}{2}}u'_{\frac{n}{2}+1}\right) + f\left(u'_{\frac{n}{2}+1}u'_{\frac{n}{2}+2}\right)\right) \pmod{6n}\\
&= \left((n-1) + (5n+1) + (4n+3)\right) \pmod{6n}\\
&= (10n+3) \pmod{6n}\\
&= 4n+3;
\end{aligned}
$$

$$
\begin{aligned}
f^+\left(u'_{\frac{n}{2}+2}\right) &= \left(f\left(u_{\frac{n}{2}+1}u'_{\frac{n}{2}+2}\right) + f\left(u'_{\frac{n}{2}+2}u'_{\frac{n}{2}+3}\right) + f\left(u'_{\frac{n}{2}+1}u'_{\frac{n}{2}+2}\right)\right)\\
&\quad \pmod{6n}\\
&= ((n+1) + (5n+3) + (4n+3)) \pmod{6n}\\
&= (10n+7) \pmod{6n}\\
&= 4n+7;
\end{aligned}
$$

$$
\begin{aligned}
f^+(u_i') &= \big(f(u_iu_i') + f(u_{i-1}'u_i') + f(u_i'u_{i+1}')\big) \pmod{6n} \\
&= \big((2n-2i+3) + (4n+2i-3) + (4n+2i-1)\big) \\
&\quad \pmod{6n} \\
&= \big(10n + 2i - 1\big) \pmod{6n} \\
&= 4n + 2i - 1, \text{ for } i \in \left\{\tfrac{n}{2}+3, \tfrac{n}{2}+4, \tfrac{n}{2}+5, \dots, n-1\right\};
\end{aligned}
$$

$$
\begin{aligned}
f^+(u_n') &= \big(f(u_nu_n') + f(u_{n-1}'u_n') + f(u_1'u_n')\big) \pmod{6n} \\
&= \big(3 + (6n-3) + (6n-1)\big) \pmod{6n} \\
&= (12n-1) \pmod{6n} \\
&= 6n - 1.
\end{aligned}
$$

We can see that

$\{f^+(u_i) \mid i \in \{1,2,3,\dots,n\}\}$

$= \{1\} \cup \{3,5,7,\dots,n-1,n+5,n+7,n+9,\dots,2n-3\} \cup \{n+1\} \cup \{n+3\}$

$\cup \{2n-1\}$

$= \{1,3,5,\dots,2n-1\}$

and

$\{f^+(u'_i) \mid i \in \{1,2,3,\dots,n\}\}$

$= \{4n+1\} \cup \{4n+5\} \cup \{4n+9,4n+11,4n+13,\dots,5n+3\} \cup \{4n+3\}$

$\cup \{4n+7\} \cup \{5n+5,5n+7,5n+9,\dots,6n-3\} \cup \{6n-1\}$

$= \{4n+1,4n+3,4n+5,\dots,6n-5,6n-3,6n-1\}.$

It is clear that if $n \geq 4$ and n is even, these two sets are disjoint and both are subsets of $\{0, 1, 2, \ldots, 6n-1\}$. Therefore, the function f defined in Algorithm 5.2 is an edge-odd graceful labeling. ■

Hence, from Lemmas 5.1 and 5.2, we conclude our result as in the following theorem.

Theorem 5.1 *For $n \geq 3$, $XPrism(C_n)$ is an edge-odd graceful graph.*

Example 5.4 From the edge-labeling in Example 5.2, the induced vertex-labeling of $\mathrm{XPrism}(C_5)$ is shown in Figure 5.6.

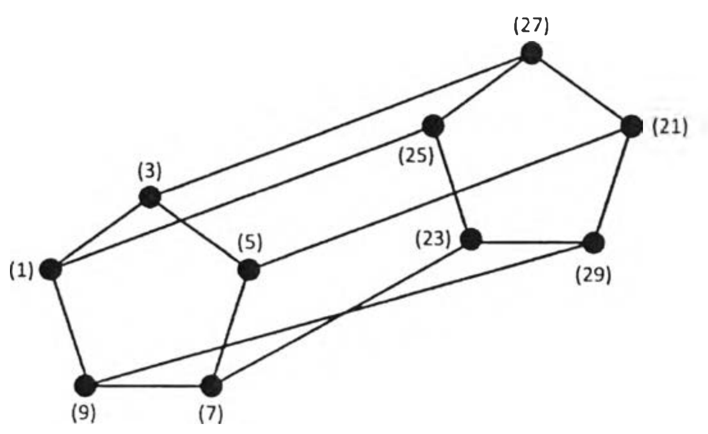

Figure 5.6 The vertex-labeling is induced from the edge-labeling in Example 5.2.

Example 5.5 From the edge-labeling in Example 5.3, the induced vertex-labeling of $\mathrm{XPrism}(C_6)$ is shown in Figure 5.7.

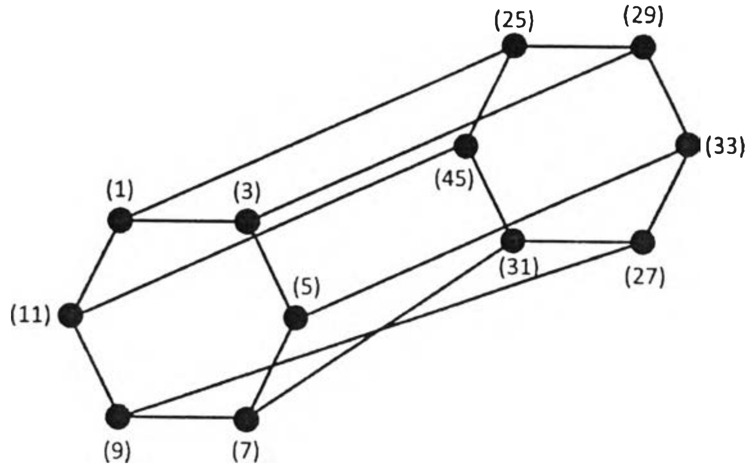

Figure 5.7 The vertex-labeling is induced from the edge-labeling in Example 5.3.