CHAPTER III

F-CS-RICKART MODULES

In this chapter, we provide the concept of F-CS-Rickart modules. We would like to point out that the notion of F-CS-Rickart modules are extended from CS-Rickart modules by Abyzov and Nhan given in [1], and F-inverse split modules by Lee, Rizvi and Roman in [11]. We integrate the idea of being an essential submodule of some direct summand of $\ker f$ from CS-Rickart modules and the idea of being a direct summand of $f^{-1}(F)$ from F-inverse split modules for all $f \in \operatorname{End}(M)$.

Various properties of F-CS-Rickart modules and characterizations of those are investigated in Section 3.1. We show that the intersection of two submodules of an F-CS-Rickart module is essential in some direct summand where one of those two submodules contains F. Moreover, we study when a submodule of an F-CS-Rickart module is also an F'-CS-Rickart module where F' is a fully invariant submodule of that submodule. Relationships between F-CS-Rickart modules and F-inverse split modules, likewise, relationships between F-CS-Rickart modules and CS-Rickart modules are presented. Furthermore, we give a notion and a characterization of strongly F-CS-Rickart modules which is a special case of F-CS-Rickart modules. Observe that for F-CS-Rickart modules the inverse images of endomorphisms are considered. So, in Section 3.2, we extend to consider the inverse image of a homomorphism which is an essential submodule in some direct summand. In Section 3.3, we focus on specific fully invariant submodules, namely, singular submodules, second singular submodules and cosingular submodules. Finally, in Section 3.4, we concern any images of F-CS-Rickart projective modules satisfying C_2 condition. We obtain that they can be written as a direct sum of two submodules one of which is a projective module and the other one of which is contain in F^*. In addition, we define a right I-CS-Rickart ring for a given ideal I of R. Then the free R-module $R^{(n)}$ is an $I^{(n)}$-CS-Rickart module if and only if

$M_n(R)$ is a right $M_n(I)$-CS-Rickart ring where $R^{(n)}$ and $I^{(n)}$ are the finite direct sum of n copies of R and I, respectively.

3.1 Properties of F-CS-Rickart Modules

First, we examine relationships between F-CS-Rickart modules and F-inverse split modules, as well as, relationships between F-CS-Rickart modules and CS-Rickart modules. Next, we are interested in when a submodule N of an F-CS-Rickart module is also an F'-CS-Rickart module for some fully invariant submodule F' of N. Later, characterizations of F-CS-Rickart modules are provided. One of main results is that any F-CS-Rickart module can be written as a direct sum of two submodules one of which is an essential extension of F and the other one of which is a CS-Rickart module.

As we mentioned earlier, the concept of F-CS-Rickart modules are extended from CS-Rickart modules and F-inverse split modules. A module M is a *CS-Rickart module*, given in [1], if for any $f \in \text{End}(M)$, there is a direct summand M' of M such that $\ker f \leq_{ess} M'$; in addition, M is an *F-inverse split module*, given in [17], if for any $f \in \text{End}(M)$, $f^{-1}(F)$ is a direct summand of M. Now, we provide the definition of an F-CS-Rickart module by combining the main ideas of those as follows.

Definition 3.1.1. Let F be a fully invariant submodule of M. Then M is an *F-CS-Rickart module* if for any $f \in \text{End}(M)$, there is a direct summand M' of M such that $f^{-1}(F)$ is an essential submodule of M'.

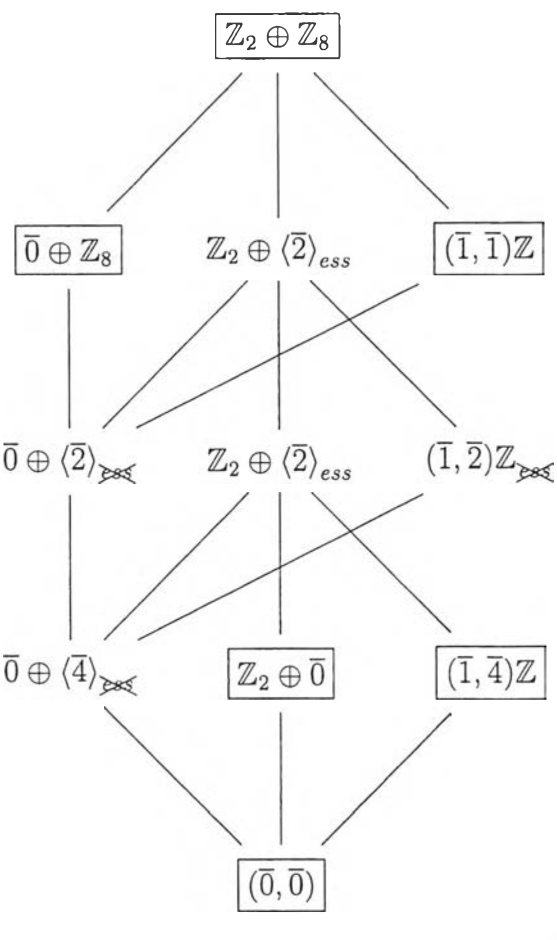
Note that M is a CS-Rickart module if and only if M is a 0-CS-Rickart module.

Proposition 3.1.2. *Any F-inverse split module is an F-CS-Rickart module.*

Proof. Let M be an F-inverse split module. Then, for each $f \in \text{End}(M)$, we obtain that $f^{-1}(F) \leq_{ess} f^{-1}(F) \leq^{\oplus} M$. Therefore, M is an F-CS-Rickart module. □

Observe that $f^{-1}(F)$ is a submodule of M containing F for any $f \in \text{End}(M)$. So we can conclude that M is an F-CS-Rickart module if and only if any submodule of M containing F is an essential submodule of a direct summand of M. The following example shows an F-CS-Rickart module which is not an F-inverse split module for some given fully invariant submodule F of M.

Example 3.1.3. Let M be the $\mathbb{Z}$-module $\mathbb{Z}_2 \oplus \mathbb{Z}_8$. Let $N = \overline{0} \oplus \langle \overline{2} \rangle$. Then N is a fully invariant submodule of M obtained directly from the definition. The following diagram describes all submodules of $Z_2 \oplus Z_8$. Each submodule contained in a box is a direct summand of M but the others are not direct summands of M. Furthermore, if a submodule N is an essential submodule of M, we write N_{ess}, otherwise; we write $N_{\cancel{ess}}$.



Observe that, all submodules of M containing N are N, $\overline{0} \oplus \mathbb{Z}_8$, $\mathbb{Z}_2 \oplus \langle \overline{2} \rangle$, $(\overline{1}, \overline{1})\mathbb{Z}$ and M. Among these, only $\overline{0} \oplus \mathbb{Z}_8$, $(\overline{1}, \overline{1})\mathbb{Z}$ and M are direct summands of M, i.e., they are essential submodules of themselves, and only $\mathbb{Z}_2 \oplus \langle \overline{2} \rangle$ is an essential submodule of M but N is not a direct summand and not an essential submodule of M. Moreover, N is an essential submodule of $\overline{0} \oplus \mathbb{Z}_8$ which is a direct summand of M because all proper submodules of $\overline{0} \oplus \mathbb{Z}_8$ contained in N. As mention above, we can conclude that any submodule of M containing N is an essential submodule of a direct summand of M. This shows that M is an N-CS-Rickart module. However, M is not an N-inverse split module because $1_S^{-1}(N) = N$ is not a direct summand of M.

Proposition 3.1.2 together with Example 3.1.3 guarantee that F-CS-Rickart modules actually generalized F-inverse split modules. We know that M is a CS-Rickart module if and only if M is a 0-CS-Rickart module. For a given fully invariant submodule F of M, "M is an F-CS-Rickart module" does not imply "M is a CS-Rickart module"; moreover, "M is a CS-Rickart module" does not imply "M is an F-CS-Rickart module". Example 3.1.3 shows that $\mathbb{Z}_2 \oplus \mathbb{Z}_8$ is a $\overline{0} \oplus \langle\overline{2}\rangle$-CS-Rickart module; however, $\mathbb{Z}_2 \oplus \mathbb{Z}_8$ is not a CS-Rickart module shown in the next example.

Example 3.1.4. Let M be the $\mathbb{Z}$-module $\mathbb{Z}_2 \oplus \mathbb{Z}_8$ and $N = \overline{0} \oplus \langle\overline{2}\rangle$. Then $\mathrm{End}(\mathbb{Z}_2 \oplus \mathbb{Z}_8) \cong \begin{pmatrix} \mathrm{End}(\mathbb{Z}_2) & \mathrm{Hom}(\mathbb{Z}_8, \mathbb{Z}_2) \\ \mathrm{Hom}(\mathbb{Z}_2, \mathbb{Z}_8) & \mathrm{End}(\mathbb{Z}_8) \end{pmatrix}$ from Proposition 2.1.13. Let $h = \begin{pmatrix} f_0 & g_1' \\ f_4' & g_2 \end{pmatrix}$ where f_0 is the zero homomorphism on $\mathbb{Z}_2$, $f_4'(\overline{x}) = \overline{4x}$, $g_1'(\overline{y}) = \overline{y}$ and $g_2(\overline{y}) = \overline{2y}$ for all $\overline{x} \in \mathbb{Z}_2$ and $\overline{y} \in \mathbb{Z}_8$. Then $\ker h = (\overline{1}, \overline{2})\mathbb{Z}$ which is not an essential submodule of all direct summands of M shown in the diagram. Thus $\mathbb{Z}_2 \oplus \mathbb{Z}_8$ is not a CS-Rickart module.

Next, we give an example of CS-Rickart modules which is not an F-CS-Rickart module for some fully invariant submodule F.

In [10], Lam provided that $Z(M) = \{x \in M \mid (0 :_R x) \leq_{ess} R\}$ and $Z_2(M) = \{x \in M \mid (Z(M) :_R x) \leq_{ess} R\}$ are submodules of M. Moreover, they are fully invariant submodules of M.

Example 3.1.5. Let P be the set of prime integers. Consider the $\mathbb{Z}$-module $M = \prod_p \mathbb{Z}_p$. For the fully invariant submodule $Z_2(M)$, we show later that M is a $Z_2(M)$-CS-Rickart module if and only if it is a $Z_2(M)$-inverse split module, see Proposition 3.3.2. Moreover, Example 2.12 in [18] shows that $Z(M) = Z_2(M) = \bigoplus_p \mathbb{Z}_p \neq 0$ and M is not a $Z_2(M)$-inverse split module but M is a Rickart module. Since M is not a $Z_2(M)$-inverse split module, M is not a $Z_2(M)$-CS-Rickart module. In addition, M is a CS-Rickart module because M is a Rickart module by Lemma 2.7 in [1]. Therefore, M is not a $Z_2(M)$-CS-Rickart module but M is a CS-Rickart module.

For a given fully invariant submodule F of M, unlike F-inverse split modules and F-CS-Rickart module, CS-Rickart modules and F-CS-Rickart modules do not imply each other obtaining from Example 3.1.4 and Example 3.1.5. Next, we present some properties of F-CS-Rickart modules.

Proposition 3.1.6. *Let M be an F-CS-Rickart module and P be a module. If M is isomorphic to P by isomorphism $\phi : M \to P$, then P is a $\phi(F)$-CS-Rickart module.*

Proof. Assume that ϕ is an isomorphism from P onto M. Let $f \in \mathrm{End}(P)$. So $\phi^{-1}f\phi \in \mathrm{End}(M)$. Let $y \in \phi(F)$. Then $y = \phi(x)$ for some $x \in F$. Thus $\phi^{-1}f(y) = \phi^{-1}f\phi(x) \in F$ because $F \leq_{fully} M$. It forces that $f(y) \in \phi(F)$. Hence $\phi(F) \leq_{fully} P$. Since M is an F-CS-Rickart module, $\left(\phi^{-1}f\phi\right)^{-1}(F) \leq_{ess} M'$ for some direct summand M' of M. Thus $\phi^{-1}f^{-1}\big(\phi(F)\big) \leq_{ess} M'$. Applying Proposition 2.2.6, $\phi\Big(\phi^{-1}f^{-1}\big(\phi(F)\big)\Big) \leq_{ess} \phi(M')$. Since M' is a direct summand of M, there is a submodule K of M such that $M = M' \oplus K$. This implies that, $P = \phi(M) = \phi(M') \oplus \phi(K)$ so that $\phi(M')$ is a direct summand of P. Thus $f^{-1}\big(\phi(F)\big) \leq_{ess} \phi(M')$. Therefore, P is a $\phi(F)$-CS-Rickart module. □

In general, the intersection of two direct summands may not be a direct summand. However, the intersection of two direct summands of M turns out to be a direct summand provided M is a Rickart module; moreover, the intersection of two direct summands of M is an essential submodule of some direct summand of M if M is a CS-Rickart module. Similarly, we focus on the intersections of two direct summands of an F-CS-Rickart module. Next example shows that there is the intersection of two direct summands of an F-CS-Rickart module which is not a direct summand but it is an essential submodule of some direct summand.

Example 3.1.7. Let M be the $\mathbb{Z}$-module $\mathbb{Z}_2 \oplus \mathbb{Z}_8$ and $N = \overline{0} \oplus \langle\overline{2}\rangle$. Then M is an N-CS-Rickart module, see Example 3.1.3. Moreover, $A = \overline{0} \oplus \mathbb{Z}_8$ and $B = (\overline{1}, \overline{1})\mathbb{Z}$ are direct summands of M. Then $A \cap B = \overline{0} \oplus \langle\overline{2}\rangle$ is not a direct summand of M but $A \cap B = \overline{0} \oplus \langle\overline{2}\rangle \leq_{ess} A$.

However, if M is an F-CS-Rickart module satisfying some conditions, then it guarantees that the intersection of direct summands is an essential submodule of a direct summand of M. Nevertheless, the following lemma is needed.

Lemma 3.1.8. *Let F be a fully invariant submodule of M. Let $h^2 = h, g^2 = g \in \mathrm{End}(M)$ and $F \subseteq gM$. Then $gM = (1-g)^{-1}(F)$. Moreover, $\big((1-g)h\big)^{-1}(F) = \big(hM \cap gM\big) \oplus (1-h)M$.*

Proof. It is clear that, $(1-g)gM = 0 \subseteq F$, so $gM \subseteq (1-g)^{-1}(F)$. Next, let $m \in (1-g)^{-1}(F)$. Then $(1-g)m \in F \subseteq gM$. Thus $(1-g)m \in gM \cap (1-g)M = 0$ leading to $m \in \ker(1-g) = gM$ from Proposition 2.1.3. This shows that $gM = (1-g)^{-1}(F)$.

Now, we let $x \in \big((1-g)h\big)^{-1}(F)$. Then $(1-g)h(x) \in F$ so that $h(x) \in (1-g)^{-1}(F) = gM$. Thus $x = h(x) + (1-h)(x) \in \big(hM \cap gM\big) \oplus (1-h)M$. For the reverse of inclusion, let $x + y \in \big(hM \cap gM\big) \oplus (1-h)M$ where $x \in hM \cap gM$ and $y \in (1-h)M$. So $x = h(x) = g(x)$ and $y = (1-h)(y)$. Then $(1-g)h(x+y) = (1-g)h(x) + (1-g)h(y) = 0 \in F$. Hence $x + y \in \big((1-g)h\big)^{-1}(F)$. Therefore, the second result follows. □

Proposition 3.1.9. *Let M be an F-CS-Rickart module. Then the following statements hold.*

(i) For any direct summands N and K of M, if $F \subseteq K$, then $N \cap K \leq_{ess} M'$ for some direct summand M' of M.

(ii) For any submodules N and K of M, if there are direct summands M_1 and M_2 of M such that $N \leq_{ess} M_1$ and $F \subseteq K \leq_{ess} M_2$, then $N \cap K \leq_{ess} M'$ for some direct summand M' of M.

(iii) For any $f_1, \dots, f_n \in \mathrm{End}(M)$, *there is a direct summand* M' *of* M *such that* $\bigcap_{i=1}^{n} f_i^{-1}(F) \leq_{ess} M'$.

Proof. (i) Assume that N and K are direct summands of M and $F \subseteq K$. Then $N = hM$ and $K = gM$ for some $h^2 = h, g^2 = g \in \mathrm{End}(M)$. Since $F \subseteq K = gM$, Lemma 3.1.8 gives

$$\big((1-g)h\big)^{-1}(F) = \big(hM \cap gM\big) \oplus (1-h)M.$$

Since M is an F-CS-Rickart module, $((1-g)h)^{-1}(F) \leq_{ess} eM$ for some $e^2 = e \in \mathrm{End}(M)$. Thus $(1-h)M \subseteq eM$. As $M = hM \oplus (1-h)M$ and $(1-h)M \subseteq eM$, we obtain $eM = M \cap eM = \big(hM \oplus (1-h)M\big) \cap eM = (hM \cap eM) \oplus (1-h)M$ by Modular Law. So $hM \cap eM \leq^{\oplus} eM$ and

$$\big(hM \cap gM\big) \oplus (1-h)M = \big((1-g)h\big)^{-1}(F) \leq_{ess} eM = (hM \cap eM) \oplus (1-h)M.$$

Therefore, $N \cap K = hM \cap gM \leq_{ess} hM \cap eM \leq^{\oplus} M$.

(ii) Assume that N and K are submodules of M and $N \leq_{ess} M_1$ and $K \leq_{ess} M_2$ for some direct summands M_1 and M_2 of M such that $F \subseteq K$. By (i), we obtain $M_1 \cap M_2 \leq_{ess} M'$ for some direct summand M' of M. Applying Proposition 2.2.4, $N \cap K \leq_{ess} M_1 \cap M_2 \leq_{ess} M'$. Therefore, $N \cap K \leq_{ess} M'$ by Proposition 2.2.3.

(iii) Let $f_i \in \mathrm{End}(M)$ for all $i \in \{1, \dots, n\}$. Since M is an F-CS-Rickart module, for each i, $F \subseteq f_i^{-1}(F) \leq_{ess} M_i$ for some direct summand M_i of M. Applying (ii) repeatedly, we obtain $\bigcap_{i=1}^{n} f_i^{-1}(F) \leq_{ess} M'$ for some direct summand M' of M. □

A module M is an *SIP-CS* module if the intersection of two direct summands is an essential submodule of a direct summand of M, see [1]. From the previous proposition, the intersection of two direct summands of an F-CS-Rickart module is essential in a direct summand of M when one of direct summands contains F.

Corollary 3.1.10. *Let* M *be an* F*-CS-Rickart module. Then* M *is an SIP-CS module provided that* F *is contained in all direct summands of* M.

Similar to CS-Rickart modules and F-inverse split modules, we investigate when a submodule N of an F-CS-Rickart module is also an F'-CS-Rickart module for some fully invariant submodule F' of this submodule N. We provide the following lemma using for obtaining the mentioned result.

Lemma 3.1.11. *Let N and F be fully invariant submodules of M. If each endomorphism of N can be extended to an endomorphism of M, then $N \cap F$ is a fully invariant submodule of N. Moreover, for any $g \in \text{End}(N)$, there is $f \in \text{End}(M)$ such that $g = f|_N$ and $g^{-1}(N \cap F) = N \cap f^{-1}(F)$.*

Proof. Assume that each $g \in \text{End}(N)$ can be extended to an $f \in \text{End}(M)$. Let $g \in \text{End}(N)$. Then there exists $f \in \text{End}(M)$ such that $g = f|_N$. Let $x \in N \cap F$. So $f|_N(x) = g(x) \in N$ and $f|_N(x) = f(x) \in F$. Thus $g(x) = f|_N(x) \in N \cap F$. Therefore, $N \cap F \leq_{fully} N$.

Moreover, we claim that $g^{-1}(N \cap F) = N \cap f^{-1}(F)$. Let $x \in g^{-1}(N \cap F)$. Then $x \in N$ and $f(x) = g(x) \in N \cap F$, so $x \in N \cap f^{-1}(F)$. Next, let $y \in N \cap f^{-1}(F)$. Then $g(y) = f(y) \in F$ and $g(y) \in N$. So $y \in g^{-1}(N \cap F)$. Therefore, $g^{-1}(N \cap F) = N \cap f^{-1}(F)$. □

Proposition 3.1.12. *Let M be an F-CS-Rickart module and N be a fully invariant submodule of M. If each endomorphism of N can be extended to an endomorphism of M, then N is an $(N \cap F)$-CS-Rickart module.*

Proof. Assume that each $g \in \text{End}(N)$ can be extended to an $f \in \text{End}(M)$. Let $g \in \text{End}(N)$. Then $f|_N = g$ for some $f \in \text{End}(M)$ and $g^{-1}(N \cap F) = N \cap f^{-1}(F)$. Since M is an F-CS-Rickart module, $f^{-1}(F) \leq_{ess} eM$ for some $e^2 = e \in \text{End}(M)$. Thus $N \cap f^{-1}(F) \leq_{ess} N \cap eM$. Since $N \leq_{fully} M$ and $(e|_N)^2 = e|_N \in \text{End}(N)$, $N \cap eM = e|_N(N) \leq^{\oplus} N$. So $g^{-1}(N \cap F) = N \cap f^{-1}(F) \leq_{ess} N \cap eM \leq^{\oplus} N$. Therefore, N is an $(N \cap F)$-CS-Rickart module. □

Observe that the intersection of fully invariant submodules N and F of M needs not be a fully invariant submodule of N. However, the intersection of a direct summand N of M and a fully invariant submodule F of M is always a fully invariant submodule of N from Proposition 2.1.8. So, now, we obtain one characterization of F-CS-Rickart modules.

Theorem 3.1.13. *A module M is an F-CS-Rickart module if and only if N is an $(N \cap F)$-CS-Rickart module for any direct summand N of M.*

Proof. The sufficiency is clear because M is always a direct summand of M itself.

For the necessity, let N be a direct summand of M. Then $N = eM$ for some $e^2 = e \in \mathrm{End}(M)$ and $N \cap F$ is a fully invariant submodule of N. Let $g \in \mathrm{End}(N)$ and $K = (1-e)M$. From Lemma 2.1.10, $g^{-1}(N \cap F) \oplus K = (g \oplus 0_K)^{-1}(F)$. Since M is an F-CS-Rickart module, $(g \oplus 0_K)^{-1}(F) \leq_{ess} M'$ for some direct summand M' of M. Since $M = N \oplus K$ and $K \subseteq (g \oplus 0_K)^{-1}(F) \subseteq M'$, we obtain that $M' = (N \cap M') \oplus K$. Thus $N \cap M' \leq^{\oplus} N$ because $N \cap M' \leq^{\oplus} M$ and $N \cap M' \subseteq N$. This forces that $g^{-1}(N \cap F) \leq_{ess} N \cap M'$ by Proposition 2.2.7. Therefore, N is an $(N \cap F)$-CS-Rickart module for any direct summand N of M. □

A direct sum of F-CS-Rickart modules where each summand is also a fully invariant submodule is studied in the following theorem.

Theorem 3.1.14. *Let M_j be a fully invariant submodule of $\bigoplus_{i=1}^{n} M_i$ and F_j be a fully invariant submodule of M_j for all $j \in \{1, \ldots, n\}$. Then $\bigoplus_{i=1}^{n} M_i$ is a $\bigoplus_{i=1}^{n} F_i$-CS-Rickart module if and only if M_j is an F_j-CS-Rickart module for all $j \in \{1, \ldots, n\}$.*

Proof. Assume that $\bigoplus_{i=1}^{n} M_i$ is a $\bigoplus_{i=1}^{n} F_i$-CS-Rickart. Since each $M_j \leq^{\oplus} \bigoplus_{i=1}^{n} M_i$, we obtain that each M_j is an $(M_j \cap \bigoplus_{i=1}^{n} F_i)$-CS-Rickart module by Theorem 3.1.13. Therefore, M_j is an F_j-CS-Rickart module because $M_j \cap \bigoplus_{i=1}^{n} F_i = F_j$ for all $j \in \{1, \ldots, n\}$.

For the converse, assume that M_j is an F_j-CS-Rickart module for all $j \in \{1, \ldots, n\}$. Let $f \in \mathrm{End}(\bigoplus_{i=1}^{n} M_i)$. Let $(x_1, \ldots, x_n) \in \bigoplus_{i=1}^{n} M_i$. Then

$$f(x_1, \ldots, x_n) = f(x_1, \ldots, 0) + \cdots + f(0, \ldots, x_n) = f_1(x_1) + \cdots + f_n(x_n)$$

where $f_j := fi_j : M_j \to \bigoplus_{i=1}^{n} M_i$ and i_j is the inclusion map from M_j into $\bigoplus_{i=1}^{n} M_i$ for all $j \in \{1, \ldots, n\}$. Since each $M_j \leq_{fully} \bigoplus_{i=1}^{n} M_i$, we get $f_j : M_j \to M_j$ and $f_j(F_j) \subseteq F_j$. Thus $f_j^{-1}(F_j) \leq_{ess} e_jM_j$ for some idempotent $e_j \in \text{End}(M_j)$ because each M_j is an F_j-CS-Rickart module. Applying Proposition 2.2.7, $\bigoplus_{i=1}^{n} f_i^{-1}(F_i) \leq_{ess} \bigoplus_{i=1}^{n} e_iM_i$. Note that

$$\begin{aligned}
f^{-1}(\bigoplus_{i=1}^{n} F_i) &= \Big\{(x_1, \ldots, x_n) \in \bigoplus_{i=1}^{n} M_i \mid f(x_1, \ldots, x_n) \in \bigoplus_{i=1}^{n} F_i\Big\} \\
&= \Big\{(x_1, \ldots, x_n) \in \bigoplus_{i=1}^{n} M_i \mid f_1(x_1) + \cdots + f_n(x_n) \in \bigoplus_{i=1}^{n} F_i\Big\} \\
&= \Big\{(x_1, \ldots, x_n) \in \bigoplus_{i=1}^{n} M_i \mid f_j(x_j) \in F_j \text{ for all } j \in \{1, \ldots, n\}\Big\} \\
&= \Big\{(x_1, \ldots, x_n) \in \bigoplus_{i=1}^{n} M_i \mid x_j \in f_j^{-1}(F_j) \text{ for all } j \in \{1, \ldots, n\}\Big\} \\
&= \bigoplus_{i=1}^{n} f_i^{-1}(F_i).
\end{aligned}$$

Hence $f^{-1}(F) = \bigoplus_{i=1}^{n} f_i^{-1}(F_i) \leq_{ess} \bigoplus_{i=1}^{n} e_iM_i$ and $\bigoplus_{i=1}^{n} e_iM_i$ is a direct summand of $\bigoplus_{i=1}^{n} M_i$. Therefore, $\bigoplus_{i=1}^{n} M_i$ is a $\oplus_{i=1}^{n} F_i$-CS-Rickart module. □

Next, other characterizations of F-CS-Rickart modules are given. Let N be a submodule of M and I be a nonempty subset of $\text{End}(M)$. Recall that $(N :_M I) = \{x \in M \mid f(x) \in N \text{ for any } f \in I\} = \bigcap_{f \in I} f^{-1}(N)$. Moreover, if I is a principal left ideal of $\text{End}(M)$ generated by f, then $(F :_M I) = (F :_M f) = f^{-1}(F)$.

Theorem 3.1.15. *The following statements are equivalent.*

(i) M is an F-CS-Rickart module.

(ii) For any finite nonempty subset I of $\text{End}(M)$, *$(F :_M I)$ is an essential submodule of M' for some direct summand M' of M.*

(iii) For any finitely generated left ideal I of $\text{End}(M)$, *$(F :_M I)$ is an essential submodule of M' for some direct summand M' of M.*

Proof. (i)→ (ii) Assume (i). Let I be a finite nonempty subset of $\text{End}(M)$. Thus $(F :_M I) = \bigcap_{f \in I} f^{-1}(F) \leq_{ess} M'$ for some direct summand M' of M by applying

Proposition 3.1.9 (iii).

(ii)$\to$ (i) This is clear.

(i)$\to$ (iii) Assume (i). Let $I = \langle f_1, \ldots, f_n\rangle$ be a finitely generated left ideal of End(M). We prove by induction on n. If $n = 1$, the statement clearly holds. Suppose that the statement holds for $n - 1$. Let $J = \langle f_1, \ldots, f_{n-1}\rangle$. We obtain that $(F :_M J) \leq_{ess} M_{n-1}$ for some direct summand M_{n-1} of M. It follows that $(F :_M I) = (F :_M J) \cap f_n^{-1}(F)$ and $f_n^{-1}(F) \leq_{ess} M_n$ for some direct summand M_n of M. Thus $(F :_M J) \cap f_n^{-1}(F) \leq_{ess} M_{n-1} \cap M_n$. Since $(F :_M J)$ and $f_n^{-1}(F)$ contains F, by Proposition 3.1.9 (ii), $(F :_M J) \cap f_n^{-1}(F) \leq_{ess} M'$ for some direct summand M' of M. Therefore, $(F :_M I) \leq_{ess} M'$.

(iii)$\to$ (i) This holds because for any $f \in$ End(M), $(F :_M I) = f^{-1}(F)$ where I is the principal left ideal of End(M) generated by f. □

We know that F-inverse split modules are F-CS-Rickart modules but the converse is not necessary true from Proposition 3.1.2 and Example 3.1.3. As a result, finding conditions that make the converse valid is our next interest. Observe that I is an ideal of a ring R if and only if I is a fully invariant submodule of the right R-module R. We let $F_S = \{f \in \text{End}(M) \mid f(M) \subseteq F\}$. Then F_S is an ideal of the ring End(M), so F_S is a fully invariant submodule of the module End(M). The set $\Delta(M) = \{f \in \text{End}(M) \mid \ker f \leq_{ess} M\}$ given in [7] is a left ideal of End(M) and M is a *$\mathcal{K}$-nonsingular module* if $\Delta(M) = \{0\}$ given in [16]. In this research, we extend $f^{-1}(\{0\}) = \ker f$ to $f^{-1}(F)$. So, we provide the set $\Delta_F(M) = \{f \in \text{End}(M) \mid f^{-1}(F) \leq_{ess} M\}$. Obviously, $\Delta_F(M)$ is a left ideal of End(M) and $F_S \subseteq \Delta_F(M)$. Next, we provide a generalization of $\mathcal{K}$-nonsingular module as follows.

Definition 3.1.16. *A module M is an F-$\mathcal{K}$-nonsingular module if $\Delta_F(M) = F_S$.*

One can see that, M is a *$\mathcal{K}$-nonsingular module* if and only if M is a 0-$\mathcal{K}$-nonsingular module.

Proposition 3.1.17. *If M is an F-inverse split module, then M is an F-$\mathcal{K}$-nonsingular module.*

Proof. Assume that M is an F-inverse split module. Let $f \in \Delta_F(M)$. Then $f \in \mathrm{End}(M)$ and then $f^{-1}(F) \leq^{\oplus} M$ and $f^{-1}(F) \leq_{ess} M$ so that $f^{-1}(F) = M$. That is $f(M) \subseteq F$. Therefore, M is an F-$\mathcal{K}$-nonsingular module. □

Next, we give an example of F-$\mathcal{K}$-nonsingular modules. However, a helpful lemma is given in order to show that a module M is an F-inverse split module, so that M is an F-$\mathcal{K}$-nonsingular module.

Lemma 3.1.18. ([17], Theorem 2.3) *A module M is an F-inverse split module if and only if $M = F \oplus K$ where K is a Rickart module.*

Example 3.1.19. Let $M = \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z} \end{pmatrix}$ be a module over itself. Then the submodule $N = \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & 0 \end{pmatrix}$ is both a fully invariant submodule and a direct summand of M. So $M = N \oplus K$ where $K = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{Z} \end{pmatrix} \cong \mathbb{Z}$. Note that $\mathbb{Z}$ is a Rickart module because, for any $f \in \mathrm{End}(\mathbb{Z})$ there exists $n \in \mathbb{Z}$ such that $f(x) = nx$ for all $x \in \mathbb{Z}$, so that $\ker f = 0$ or $\mathbb{Z}$ which both are direct summands of $\mathbb{Z}$. This forces that K is a Rickart module. By applying Lemma 3.1.18, M is an N-inverse split module. Thus M is an N-$\mathcal{K}$-nonsingular module.

Relationships between F-CS-Rickart modules and F-inverse split modules are ready to be investigated.

Theorem 3.1.20. *The following statements are equivalent.*
(i) M is an F-CS-Rickart module and an F-$\mathcal{K}$-nonsingular module.
(ii) M is an F-inverse split module.

Proof. (ii)→ (i) This follows from Proposition 3.1.2 and Proposition 3.1.17.

(i)→ (ii) Assume (i). Let $f \in \mathrm{End}(M)$. Then $f^{-1}(F) \leq_{ess} eM$ for some $e^2 = e \in \mathrm{End}(M)$. Thus $f^{-1}(F) \oplus (1-e)M \leq_{ess} eM \oplus (1-e)M = M$. Since $f^{-1}(F) \subseteq eM$ and $e(1-e)M = 0$, we obtain $fe\big(f^{-1}(F) \oplus (1-e)M\big) \subseteq F$. It forces that $f^{-1}(F) \oplus (1-e)M \subseteq (fe)^{-1}(F)$. Next, let $x \in (fe)^{-1}(F)$. Then $f(ex) =$

$fe(x) \in F$ so that $ex \in f^{-1}(F)$. Hence $x = ex + (1-e)x \in f^{-1}(F) \oplus (1-e)M$. Then $(fe)^{-1}(F) = f^{-1}(F) \oplus (1-e)M \leq_{ess} M$. Since M is an F-$\mathcal{K}$-nonsingular module, $fe(M) \subseteq F$. This implies that $eM \subseteq f^{-1}(F)$. Thus $f^{-1}(F) = eM$. Therefore, M is an F-inverse split module. □

The next example shows that there is an F-CS-Rickart module which is not an F-$\mathcal{K}$-nonsingular module.

Example 3.1.21. From Example 3.1.19, let $M = \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z} \end{pmatrix}$. A submodule $K = \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & n\mathbb{Z} \end{pmatrix}$ is a fully invariant submodule of M but is not a direct summand of M. By Lemma 3.1.18, M is not a K-inverse split module. Note that K is an essential submodule of M so that any submodule of M containing K is also an essential submodule of M applying Proposition 2.2.3. Thus M is a K-CS-Rickart module. By Theorem 3.1.20, M is not a K-$\mathcal{K}$-nonsingular module.

Observe from the definition that an F-CS-Rickart module M has a direct summand depending on each inverse image of F. In fact, there is a submodule N of M such that $M = N \oplus K$ where the inverse image of F is essential in N. Next, we focus on the inverse image of the identity endomorphism which is equal to F in the following result.

Theorem 3.1.22. *If M is an F-CS-Rickart module, then $M = N \oplus K$ where F is an essential submodule of N and K is a CS-Rickart module. The converse holds if N is a fully invariant submodule of M.*

Proof. First, assume that M is an F-CS-Rickart module. Then $F = 1_S^{-1}(F) \leq_{ess} N$ for some $N \leq^{\oplus} M$. So there is a submodule K of M such that $M = N \oplus K$. Since $K \leq^{\oplus} M$ and M is an F-CS-Rickart module, K is a $(K \cap F)$-CS-Rickart module by applying Theorem 3.1.13. Thus K is a CS-Rickart module because $K \cap F = 0$

To show that the converse is valid , assume that $M = N \oplus K$ where $F \leq_{ess} N$, K is a CS-Rickart module and N is a fully invariant submodule of M. Let $f \in$ End (M) and $\pi_K : M \to K$ be the projection homomorphism. Then

$\pi_K f|_K \in \operatorname{End}(K)$ and $f^{-1}(N) = N \oplus \ker(\pi_K f|_K)$ by Proposition 2.1.11. Since K is a CS-Rickart module, $\ker(\pi_K f|_K) \leq_{ess} K'$ for some direct summand K' of K. This forces that $N \oplus K'$ is a direct summand of M and

$$f^{-1}(F) \leq_{ess} f^{-1}(N) = N \oplus \ker(\pi_K f|_K) \leq_{ess} N \oplus K'.$$

Hence $f^{-1}(F) \leq_{ess} N \oplus K'$. Therefore, M is an F-CS-Rickart module. □

Now, F-CS-Rickart modules having two direct summands are considered.

Proposition 3.1.23. *For every indecomposable F-CS-Rickart module M, either M is a CS-Rickart module or F is an essential submodule of M.*

Proof. Assume M is an indecomposable F-CS-Rickart module. Then $M = N \oplus K$ where $F \leq_{ess} N$ and K is a CS-Rickart module. Since M is an indecomposable module, $N = 0$ or $N = M$. In case $N = 0$, it follows that $F = 0$ so that M is a CS-Rickart module; otherwise, $N = M$, leading to $F \leq_{ess} M$. Therefore, either M is a CS-Rickart module or $F \leq_{ess} M$. □

Recall that M is a CS-Rickart module if and only if M is a 0-CS-Rickart module. Moreover, we gave an example of F-CS-Rickart modules which is not a CS-Rickart module in Example 3.1.4, likewise, we provided an example of CS-Rickart modules which is not an F-CS-Rickart module in Example 3.1.5. So we are interested in studying when an F-CS-Rickart module is a CS-Rickart module, as well as, a CS-Rickart module is an F-CS-Rickart module where $F \neq 0$. The following series of propositions provide relationships between F-CS-Rickart modules and CS-Rickart modules.

Proposition 3.1.24. *If M is an F-CS-Rickart module and* $\ker f$ *is an essential submodule of $f^{-1}(F)$ for any $f \in$* $\operatorname{End}(M)$ *which is not a monomorphism, then M is a CS-Rickart module.*

Proof. Assume that M is an F-CS-Rickart module and $\ker f \leq_{ess} f^{-1}(F)$ for any $f \in \operatorname{End}(M)$ which is not a monomorphism. Let $f \in \operatorname{End}(M)$. Then $f^{-1}(F) \leq_{ess} M'$ for some direct summand M' of M. Thus $\ker f \leq_{ess} M'$. Therefore, M is a CS-Rickart module. □

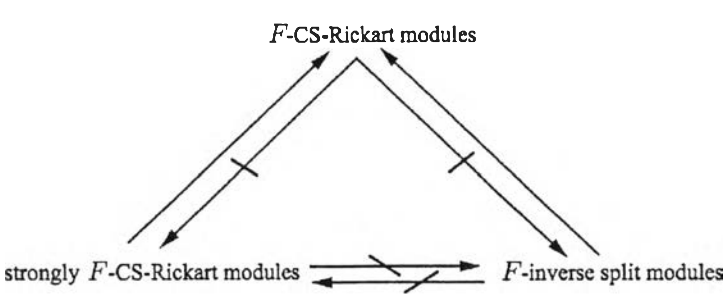
Proposition 3.1.25. *If M is a CS-Rickart module and F is an essential submodule of M' for some fully invariant direct summand M' of M, then M is an F-CS-Rickart module.*

Proof. Assume that M is a CS-Rickart module and $F \leq_{ess} M'$ for some fully invariant direct summand M' of M. Then $M = N \oplus K$ where K is a CS-Rickart module. As a consequence of the converse of Theorem 3.1.22, M is an F-CS-Rickart module. □

From Theorem 3.1.22, we obtain that if M is an F-CS-Rickart module, then $M = N \oplus K$ where $F \leq_{ess} N$ and K is a CS-Rickart module; in addition, the converse of this theorem holds if $N \leq_{fully} M$. One can see that being fully invariant submodule of M' is a necessary condition to force M to be an F-CS-Rickart module. So the inverse images of F which are essential submodules of a fully invariant direct summand are investigated.

Definition 3.1.26. A module M is a *strongly F-CS-Rickart module* if for any $f \in \text{End}(M)$ there is a fully invariant direct summand M' of M such that $f^{-1}(F)$ is an essential submodule of M'.

It is clear that strongly F-CS-Rickart modules and F-inverse split modules are F-CS-Rickart modules shown in the following diagram.



Next example presents a module M which is an F-CS-Rickart module but is not an F-inverse split module and not a strongly F-CS-Rickart module.

Example 3.1.27. Let $M = \mathbb{Z}_2 \oplus \mathbb{Z}_8$ and $N = \overline{0} \oplus \langle \overline{2} \rangle$ given in Example 3.1.3. Then M is an N-CS-Rickart module and M is not an N-inverse split module. Moreover, let $f = \begin{pmatrix} f_1 & g_1' \\ f_0' & g_2 \end{pmatrix} \in \mathrm{End}(M)$ where f_1 is the identity homomorphism, $f_0'(\overline{x}) = \overline{0}$, $g_1'(\overline{y}) = \overline{y}$ and $g_2(\overline{y}) = \overline{2y}$ for all $\overline{x} \in \mathbb{Z}_2$ and $\overline{y} \in \mathbb{Z}_8$. Then $f^{-1}(N) = (\overline{1}, \overline{1})\mathbb{Z}$ which is a direct summand of M but is not a fully invariant submodule of M. Note that submodules of M containing $f^{-1}(N)$ are $(\overline{1}, \overline{1})\mathbb{Z}$ and M. Since $(\overline{1}, \overline{1})\mathbb{Z}$ is a direct summand of M, it is not an essential submodule of M by applying Proposition 2.2.2. We can conclude that $f^{-1}(F)$ is not an essential submodule of all fully invariant direct summands of M. Thus M is not a strongly N-CS-Rickart module.

Likewise Theorem 3.1.13, we investigate that a direct summand of a strongly F-CS-Rickart module is also a strongly F'-CS-Rickart module for some fully invariant submodule F' of this direct summand.

Lemma 3.1.28. *Let M be a strongly F-CS-Rickart module. Then N is a strongly $(N \cap F)$-CS-Rickart module for any direct summand N of M.*

Proof. The proof is similar to one of Theorem 3.1.13. Let N be a direct summand of M. Then there is a submodule K of M such that $N \oplus K = M$. Let $f \in \mathrm{End}(N)$. Thus $f \oplus 0_K \in \mathrm{End}(M)$ and

$$f^{-1}(N \cap F) \oplus K = (f \oplus 0_K)^{-1}(F).$$

Since M is a strongly F-CS-Rickart module, $(f \oplus 0_K)^{-1}(F) \leq_{ess} M'$ for some fully invariant direct summand M' of M. So $M' = (N \cap M') \oplus K$ and $N \cap M'$ is a fully invariant direct summand of N by Proposition 2.1.8 (i). This forces that $f^{-1}(N \cap F) \leq_{ess} N \cap M'$. Therefore, N is a strongly $(N \cap F)$-CS-Rickart module. □

In the following theorem, we focus on the inverse image of the identity endomorphism which is equal to F and is an essential submodule of some direct summand of M. So each F-CS-Rickart module can be written as a direct sum depending on F. We also provide characterizations of strongly F-CS-Rickart modules.

Theorem 3.1.29. *The following statements are equivalent.*

(i) M is a strongly F-CS-Rickart module.

(ii) $M = N \oplus K$ where F is an essential submodule of a fully invariant submodule N of M and K is a strongly CS-Rickart module.

(iii) M is an F-CS-Rickart module and every direct summand of M containing F is a fully invariant submodule.

(iv) $M = N \oplus K$ where F is an essential submodule of a fully invariant submodule N of M and, for any $f \in \text{End}(M)$, $f^{-1}(F) \cap K$ is an essential submodule of a fully invariant direct summand of K .

Proof. (i)$\rightarrow$ (ii) Assume (i). Then $M = N \oplus K$ where $F = 1^{-1}(F) \leq_{ess} N$ for some fully invariant direct summand N of M. Thus K is a strongly CS-Rickart module by Lemma 3.1.28 because $K \leq^{\oplus} M$ and $K \cap F = 0$.

(ii)$\rightarrow$(i) The proof is similar to the proof of the converse of Theorem 3.1.22. Assume (ii). Let $f \in \text{End}(M)$. Since $N \leq_{fully} M$, by Proposition 2.1.11, $f^{-1}(N) = N \oplus \ker(\pi_K f|_K)$. Since K is a strongly CS-Rickart module, $\ker(\pi_K f|_K) \leq_{ess} K'$ for some fully invariant direct summand K' of K. Thus $f^{-1}(F) \leq_{ess} f^{-1}(N) = N \oplus \ker(\pi_K f|_K) \leq_{ess} N \oplus K'$ and $N \oplus K'$ is a fully invariant direct summand of M.

(i)$\rightarrow$ (iii) Assume (i). Then M is an F-CS-Rickart module. Next, let N be a direct summand of M and $F \subseteq N$. Then there is $e^2 = e \in \text{End}(M)$ such that $N = eM$. Let $x \in eM$. Then $(1-e)x = (1-e)ex = 0 \in F$. So $x \in (1-e)^{-1}(F)$. On the other hand, let $x \in (1-e)^{-1}(F)$. Then $(1-e)x \in F \subseteq eM$. This implies that $(1-e)x = 0$, so $x \in \ker(1-e) = eM$. Thus $eM = (1-e)^{-1}(F)$. By (i), $N = (1-e)^{-1}(F) \leq_{ess} M'$ for some fully invariant direct summand M' of M. Thus $N = M'$ because N is both an essential submodule and a direct summand of M'.

(iii)$\rightarrow$ (i) Assume (iii). For any $f \in \text{End}(M)$, we have $F \subseteq f^{-1}(F) \leq_{ess} M'$ for some direct summand M' of M. By assumption $M' \leq_{fully} M$. Thus M is a strongly F-CS-Rickart module.

(ii)$\rightarrow$(iv) Assume (ii). Let $f \in \text{End}(M)$ and $K = eM$ for some $e^2 =$

$e \in \text{End}(M)$. Then $f^{-1}(F) \leq_{ess} f^{-1}(N)$. So $f^{-1}(F) \cap K \leq_{ess} f^{-1}(N) \cap K$. From Proposition 2.1.11, $f^{-1}(N) \cap K = f^{-1}(N) \cap eM = \ker efe$. Since K is a strongly CS-Rickart module, $\ker efe \leq_{ess} K'$ for some fully invariant direct summand K' of K. Thus $f^{-1}(F) \cap K \leq_{ess} K'$.

(iv)$\rightarrow$ (ii) Assume (iv). Let $h \in \text{End}(K)$. Then $0|_N \oplus h \in \text{End}(M)$. Applying Lemma 2.1.10, $(0_{|_N} \oplus h)^{-1}(F) \cap K = h^{-1}(K \cap F) = \ker h$ because $K \cap F = 0$. By assumption, $(0_{|_N} \oplus h)^{-1}(F) \cap K \leq_{ess} K'$ for some fully invariant direct summand K' of K. This implies $\ker h \leq_{ess} K'$. Therefore, K is a strongly CS-Rickart module. □

In the following example, we provide fully invariant submodules F and F' of M such that M is both a strongly F-CS-Rickart module and an F-inverse split module; in addition, M is a strongly F'-CS-Rickart module but M is not an F'-inverse split module. We apply the previous theorem to prove next example.

Example 3.1.30. Let $M = \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z} \end{pmatrix}$ be a module over itself. Then the submodules $N = \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & 0 \end{pmatrix}$ and $K = \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & n\mathbb{Z} \end{pmatrix}$ are fully invariant submodules of M. So $M = N \oplus L$ where $L = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{Z} \end{pmatrix}$ and $L \cong \mathbb{Z}$ which is a Rickart module. Since N is a fully invariant direct summand of M, we obtain that M is both strongly N-CS-Rickart and N-inverse split form Theorem 3.1.29 and Proposition 3.1.18, respectively. Note that K is not a direct summand of M but $K \leq_{ess} M$. By Theorem 3.1.29, M is a strongly K-CS-Rickart module but M is not a K-inverse split module.

3.2 Relatively F-CS-Rickart modules

In this section, we extend $\text{End}(M)$ in F-CS-Rickart modules to $\text{Hom}(P, M)$ where P and M are modules and M is not necessary an F-CS-Rickart module. This leads us to define a relatively F-CS-Rickart module. Moreover, we show that a direct summand of relatively F-CS-Rickart modules is also a relatively F-CS-Rickart module.

Definition 3.2.1. Let P, M be modules and F be a fully invariant submodule of M. Then P is an *F-CS-Rickart module relative to M* (*relatively F-CS-Rickart module*) if for any $f \in \text{Hom}(P, M)$, there is a direct summand P' of P such that $f^{-1}(F) \leq_{ess} P'$.

It is clear that M is an F-CS-Rickart module if and only if M is an F-CS-Rickart module relative to M.

Equivalent to Theorem 3.1.13, we examine direct summands of relatively F-CS-Rickart modules.

Theorem 3.2.2. *Let P, M be modules and F be a fully invariant submodule of M. Then P is an F-CS-Rickart module relative to M if and only if for any direct summand P_1 of P and any direct summand M_1 of M, P_1 is an $(M_1 \cap F)$-CS-Rickart module relative to M_1.*

Proof. The sufficiency is obvious because P and M are direct summands of itself.

Assume that P is an F-CS-Rickart module relative to M. Let P_1 and M_1 be direct summands of P and M, respectively. Then $P_1 \oplus P_2 = P$ for some submodule P_2 of P. Let $g \in \text{Hom}(P_1, M_1)$. Then $f := g \oplus 0_P \in \text{Hom}(P, M)$. So $f^{-1}(F) = g^{-1}(M_1 \cap F) \oplus P_2$. Since P is an F-CS-Rickart module relative to M, $f^{-1}(F) \leq_{ess} P'$ for some direct summand P' of P. It follows that $P' = (P_1 \cap P') \oplus P_2$ because $P_2 \subseteq f^{-1}(F) \subseteq P'$. Hence $g^{-1}(M_1 \cap F) \oplus P_2 \leq_{ess} (P_1 \cap P') \oplus P_2$ and $P_1 \cap P'$ is a direct summand of P_1. Thus $g^{-1}(M_1 \cap F) \leq_{ess} P_1 \cap P'$ by Proposition 2.2.7. Therefore, P_1 is an $(M_1 \cap F)$-CS-Rickart module relative to M_1. □

If $P = M$ in Theorem 3.2.2, we obtain the following corollary.

Corollary 3.2.3. *The following statements are equivalent.*

(i) M is an F-CS-Rickart module.

(ii) For any direct summands N and K of M, N is an $(K \cap F)$-CS-Rickart module relative to K.

(iii) For any direct summands N and K of M, for any $f \in Hom(M, K)$ there is a direct summand N' of N such that $f|_N^{-1}(K \cap F) \leq_{ess} N'$.

Proof. (i) $\leftrightarrow$ (ii) This follows from Theorem 3.2.2 because M is an F-CS-Rickart module relative to M.

(ii) $\rightarrow$ (iii) Assume (ii). Let N and K be direct summands of M and $f \in \text{Hom}(M, K)$. Then $f|_N \in \text{Hom}(N, K)$. So $f|_N^{-1}(K \cap F) \leq_{ess} N'$ for some direct summand N' of N by the definition of relatively F-CS-Rickart modules.

(iii) $\rightarrow$ (i) This is clear because $N = M = K$. □

3.3 $Z(M)$, $Z_2(M)$ and $Z^*(M)$-CS-Rickart Modules

In this section, we focus on particular fully invariant submodules which are $Z(M)$, $Z_2(M)$ and $Z^*(M)$. The first subsection shows relationship between $Z(M)$-CS-Rickart modules and $Z_2(M)$-CS-Rickart modules. The other subsection shows specific properties of $Z^*(M)$-CS-Rickart modules.

3.3.1 $Z(M)$ and $Z_2(M)$-CS-Rickart modules

Recall that Lam provided, in [10], that

$$Z(M) = \{x \in M \mid (0 :_R x) \leq_{ess} R\}$$

is the singular submodule of M and

$$Z_2(M) = \{x \in M \mid (Z(M) :_R x) \leq_{ess} R\}$$

is the second singular submodule of M.

A module M is a *singular module* if $Z(M) = M$, and a *nonsingular module* if $Z(M) = 0$, given in [10]. Lam showed that the submodules $Z(M)$ and $Z_2(M)$ are fully invariant submodules of M; in addition, $Z_2(M)$ is a maximal essential extension of $Z(M)$, that is, $Z(M) \leq_{ess} Z_2(M)$ and for any submodule N of M, if $Z(M) \leq_{ess} N$ and $Z_2(M) \subseteq N$, then $Z_2(M) = N$.

By Proposition 3.1.2, $Z(M)$-inverse split modules are $Z(M)$-CS-Rickart modules; in addition, $Z_2(M)$-inverse split modules are $Z_2(M)$-CS-Rickart modules.

However, we can show that $Z_2(M)$-CS-Rickart modules are $Z_2(M)$-inverse split modules in the following proposition.

Lemma 3.3.1. *For any $f \in \mathrm{End}(M)$, $f^{-1}(Z_2(M))$ is a maximal essential extension of $f^{-1}(Z(M))$.*

Proof. Let $f \in \mathrm{End}(M)$. Note that $Z_2(M)$ is a maximal essential extension of $Z_2(M)$. Thus $f^{-1}(Z(M)) \leq_{ess} f^{-1}(Z_2(M))$ from Proposition 2.2.6. Next, let N be a submodule of M such that $f^{-1}(Z(M)) \leq_{ess} N$ and $f^{-1}(Z_2(M)) \subseteq N$. Let $x \in N$. If $f(x) = 0$, then $\big(Z(M) :_R f(x)\big) = R \leq_{ess} R$ so that $f(x) \in Z_2(M)$, i.e., $x \in f^{-1}(Z_2(M))$. Assume that $f(x) \neq 0$. Let $a \in R$ and $a \neq 0$. If $f(x)a = 0 \in Z(M)$, then $a1 = a \in \big(Z(M) :_R f(x)\big)$. Assume that $f(x)a \neq 0$. Then $xa \neq 0$ and $xa \in N$. Since $f^{-1}(Z(M)) \leq_{ess} N$, there is $r \in R$ such that $0 \neq xar \in f^{-1}(Z(M))$. So $f(x)ar = f(xar) \in Z(M)$. Then $ar \in \big(Z(M) :_R f(x)\big)$. This implies that $\big(Z(M) :_R f(x)\big) \leq_{ess} R$. Thus $f(x) \in Z_2(M)$ so that $x \in f^{-1}(Z_2(M))$. Hence $f^{-1}(Z_2(M)) = N$. Therefore, $f^{-1}(Z_2(M))$ is a maximal essential extension of $f^{-1}(Z(M))$. □

Proposition 3.3.2. *A module M is a $Z_2(M)$-CS-Rickart module if and only if M is a $Z_2(M)$-inverse split module.*

Proof. The necessary condition is clear from Proposition 3.1.2.

Next, assume that M is a $Z_2(M)$-CS-Rickart module. Let $f \in \mathrm{End}(M)$. Then $f^{-1}(Z_2(M)) \leq_{ess} M'$ for some direct summand M' of M. Since $f^{-1}(Z_2(M))$ is a maximal essential extension of $f^{-1}(Z(M))$, we obtain that $f^{-1}(Z_2(M)) = M'$. Therefore, M is a $Z_2(M)$-CS-Rickart module. □

Unger, Halicioglu and Harmanci, in [17], presented that $Z(M)$-inverse split modules are $Z_2(M)$-inverse split modules and the converse is not true in general. A ring R is a *right singular ring* if $Z(R) = R$ as a right R-module, and a *right nonsingular ring* if $Z(R) = 0$.

Lemma 3.3.3. ([17], Proposition 5.5) *If M is a $Z(M)$-inverse split module, then M is a $Z_2(M)$-inverse split module. The converse holds if R is a right nonsingular ring.*

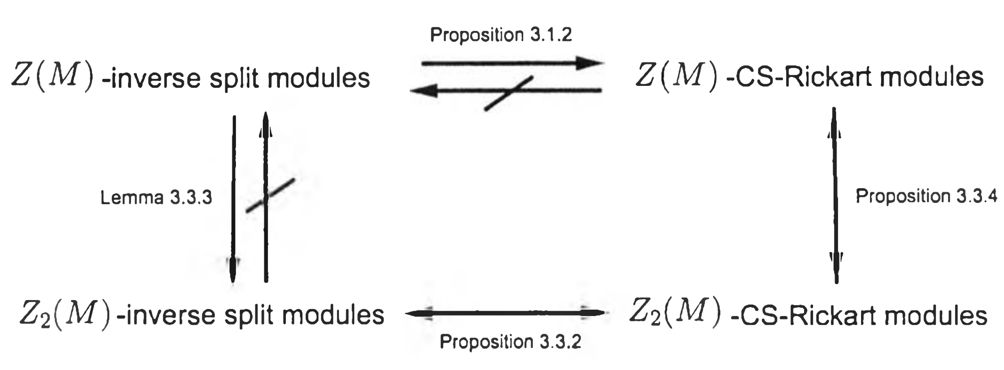
Next, we provide a relationship between $Z(M)$-CS-Rickart modules and $Z_2(M)$-CS-Rickart modules. Note that Lam showed in [10] that $Z(M) \cap N = Z(N)$ and $Z_2(M) \cap N = Z_2(N)$ for any submodule N of M.

Proposition 3.3.4. *A module M is a $Z(M)$-CS-Rickart module if and only if M is a $Z_2(M)$-CS-Rickart module.*

Proof. First, assume that M is a $Z(M)$-CS-Rickart module. Then $M = N \oplus K$ where $Z(M) \leq_{ess} N$ and K is a CS-Rickart module by applying Theorem 3.1.22. Thus $Z(M) = Z(M) \cap N = Z(N)$, so $Z(N) \leq_{ess} N$. Hence $Z_2(N) = N$ because $Z_2(N)$ is a maximal essential extension of $Z(N)$. Clearly, $Z_2(N) \subseteq Z_2(M)$. Since $Z(M) \subseteq N \subseteq Z_2(M)$ and $Z(M) \leq_{ess} Z_2(M)$, we obtain $N \leq_{ess} Z_2(M)$ by applying Proposition 2.2.3. Thus $N \leq^{\oplus} Z_2(M)$ because $N \leq^{\oplus} M$ and $N \subseteq Z_2(M)$. Since N satisfies both $N \leq_{ess} Z_2(M)$ and $N \leq^{\oplus} Z_2(M)$, it follows that $N = Z_2(M)$. Thus $M = Z_2(M) \oplus K$ where $Z_2(M) \leq_{ess} Z_2(M)$ and $Z_2(M)$ is a fully invariant direct summand of M and K is a CS-Rickart module. Therefore, M is a $Z_2(M)$-CS-Rickart module from the converse of Theorem 3.1.22.

Conversely, assume that M is a $Z_2(M)$-CS-Rickart module. Let $f \in \mathrm{End}(M)$. Then there is a direct summand M' of M such that $f^{-1}(Z_2(M)) \leq_{ess} M'$. Thus $f^{-1}(Z_2(M)) = M'$ from Lemma 3.3.1 so that $f^{-1}(Z(M)) \leq_{ess} M'$. Therefore, M is a $Z(M)$-CS-Rickart module. □

The following is a diagram presenting a relationship among $Z(M)$-inverse split modules, $Z_2(M)$-inverse split modules, $Z(M)$-CS-Rickart modules and $Z_2(M)$-CS-Rickart modules.



3.3.2 $Z^*(M)$-CS-Rickart modules

Let EM denote the injective hull of M. Unger, Halicioglu and Harmanci defined in [18] that

$$Z^*(M) = \{m \in M \mid mR \ll EM\}$$

is the cosingular submodule of M.

A module M is a *cosingular module* if $Z^*(M) = M$, and a *noncosingular module* if $Z^*(M) = 0$ provided in [18]. Unger, Halicioglu and Harmanci also presented that the consingular submodule $Z^*(M)$ is a fully invariant submodule of M and $Z^*(M) \cap N = Z^*(N)$ for any submodule N of M. In addition, a ring R is a *right cosingular ring* if $Z^*(R) = R$ as a right R-module, and a *right nonsingular ring* if $Z^*(R) = 0$.

Proposition 3.3.5. *If M is a $Z^*(M)$-CS-Rickart module, then $M = N \oplus K$ where $Z^*(M) \leq_{ess} N$ and K is a noncosingular CS-Rickart module.*

Proof. From Theorem 3.1.22, $M = N \oplus K$ where $Z^*(M) \leq_{ess} N$ and K is a CS-Rickart module. As $Z^*(K) = Z^*(M) \cap K = 0$, so K is a noncosingular module. □

Next, we consider when M is both an indecomposable module and a $Z^*(M)$-CS-Rickart module.

Proposition 3.3.6. *If M is an indecomposable $Z^*(M)$-CS-Rickart module, then either M is a noncosingular CS-Rickart module or $Z^*(M) \leq_{ess} M$.*

Proof. Assume that M is an indecomposable $Z^*(M)$-CS-Rickart module. Then $M = N \oplus K$ where $Z^*(M) \leq_{ess} N$ and K is a CS-Rickart module. Since M is an indecomposable module, $N = 0$ or $N = M$. If $N = 0$, then $Z^*(M) = 0$. So M is a noncosingular CS-Rickart module. If $N = M$, then $Z^*(M) \leq_{ess} M$. Therefore, either M is a noncosingular CS-Rickart module or $Z^*(M) \leq_{ess} M$. □

3.4 Projective F-CS-Rickart Modules

Throughout this section, let P and M be modules, $S = \text{End}(M)$ and $\text{Hom}(P, M)$ be the set of all homomorphisms from P into M. For each submodule N of M, Lam provided in [10],

$$N^* = \{x \in M \mid (N :_R x) \leq_{ess} R\}.$$

It is clear that $N \subseteq N^*$. Note that $\{0\}^* = Z(M)$ and $\big(Z(M)\big)^* = Z_2(M)$.

In current section, we investigate being an F-CS-Rickart module of a projective module. Moreover, we provide a notion of right F-CS-Rickart ring R where F is a fully invariant submodule of the right module R over itself. Recall that all rings are projective right modules over itself.

The following lemma shows a nice relationship on projective modules between essential submodules and singular modules.

Lemma 3.4.1. ([13], Lemma 2.10) *Let P be a projective module and K be a submodule of M. Then $K \leq_{ess} P$ if and only if P/K is a singular module. In particular, if P is both a projective module and a singular module, then $P = 0$.*

For a submodule N of M, we provide a relationship between N^* and singular submodule of M/N.

Proposition 3.4.2. *Let N and L be submodules of M and $N \subseteq L$. Then $L \subseteq N^*$ if and only if L/N is a singular module.*

Proof. First, assume that $L \subseteq N^*$. Let $x + N \in L/N$ where $x \in L$. Then $(N :_R x) \leq_{ess} R$ because $L \subseteq N^*$. Thus $\big(\{N\} :_R x + N\big) = (N :_R x) \leq_{ess} R$. Hence $x + N \in Z(L/N)$. Therefore, L/N is a singular module

Next, assume that L/N is a singular module. Then $Z(L/N) = L/N$. Let $x \in L$. Then $x + N \in Z(L/N)$, i.e., $\big(\{N\} :_R x + N\big) \leq_{ess} R$. Note that $(N :_R x) = \big(\{N\} :_R x + N\big)$. Thus $(N :_R x) \leq_{ess} R$ which implies that $x \in N^*$. Therefore, $L \subseteq N^*$. □

Before we present the further main point of this section, the helpful properties are provided.

Proposition 3.4.3. *Let P and M be modules and F be a fully invariant submodule of M. Let $f : P \to M$ be a homomorphism and $f^{-1}(F) \subseteq eP$ for some $e^2 = e \in$* End(P). *Then the following statements hold:*

(i) $fP = f(1-e)P \oplus feP$,

(ii) $(fP+F)/F = (f(1-e)P+F)/F \oplus (feP+F)/F$,

(iii) $(f(1-e)P+F)/F \cong (1-e)P \cong f(1-e)P$, *and*

(iv) $eP/f^{-1}(F) \cong (feP+F)/F$.

Proof. Note that $P = (1-e)P \oplus eP$.

(i) Notice that $fP = f(1-e)P + feP$. Since $\ker f \subseteq f^{-1}(F) \subseteq eP$, it follows that $fP = f(1-e)P \oplus feP$.

(ii) It is clear that $(fP+F)/F = (f(1-e)P+F)/F + (feP+F)/F$. Let $m+F \in (f(1-e)P+F)/F \bigcap (feP+F)/F$. Then $m+F = f(1-e)x+F = fey+F$ for some $x, y \in P$. Thus $f((1-e)x - ey) = f(1-e)x - fey \in F$. Then $(1-e)x - ey \in f^{-1}(F) \subseteq eP$, so $(1-e)x \in (1-e)P \cap eP = 0$. This implies that $m+F = f(1-e)x + F = F$. Therefore, $(fP+F)/F = (f(1-e)P+F)/F \oplus (feP+F)/F$.

(iii) Define $\phi : (1-e)P \to (f(1-e)P+F)/F$ by $\phi(x) = f(x)+F$ for all $x \in (1-e)P$. It is clear that ϕ is well-defined. Then, ϕ is an epimorphism and

$$\begin{aligned}\ker\phi &= \{x \in (1-e)P \mid \phi(x) = F\} = \{x \in (1-e)P \mid f(x)+F = F\} \\ &= \{x \in (1-e)P \mid f(x) \in F\} = \{x \in (1-e)P \mid x \in f^{-1}(F)\} \\ &= (1-e)P \cap f^{-1}(F) = 0.\end{aligned}$$

By the first isomorphism theorem, $(1-e)P \cong (f(1-e)P+F)/F$. Moreover, define $\theta : (1-e)P \to f(1-e)P$ by $\theta(x) = f(x)$ for all $x \in (1-e)P$. The proof is similar to the first part, we can conclude that $(1-e)P \cong f(1-e)P$.

(iv) Define $\beta : eP \to (feP+F)/F$ by $\beta(x) = f(x)+F$ for all $x \in eP$. Then β is an epimorphism and

$$\begin{aligned}\ker\beta &= \{x \in eP \mid \beta(x) = F\} = \{x \in eP \mid f(x)+F = F\} \\ &= \{x \in eP \mid f(x) \in F\} = \{x \in eP \mid x \in f^{-1}F\} \\ &= eP \cap f^{-1}(F) = f^{-1}(F).\end{aligned}$$

By the first isomorphism theorem, $eP/f^{-1}(F) \cong \big(feP + F\big)/F$. $\square$

For a relatively F-CS-Rickart module P, we obtain that the inverse image of F is essential in a direct summand of P. Note that any direct summands of projective modules are projective modules. Hence if P is both a projective module and an relatively F-CS-Rickart module, then the inverse image of F is essential in a direct summand which is also a projective module. Thus, we are interested in studying the image of each relatively F-CS-Rickart projective module.

Theorem 3.4.4. *Let P be a projective module, M be a module with a fully invariant submodule F. Then the following statements are equivalent.*
(i) P is an F-CS-Rickart module relative to M.
(ii) For any $f \in \mathrm{Hom}(P, M)$, $(fP + F)/F = N/F \oplus K/F$ where N/F is a projective module and K/F is a singular module.

Proof. (i)→ (ii) Assume (i). Let $f \in \mathrm{Hom}(P, M)$. Then $f^{-1}(F) \leq_{ess} eP$ for some $e^2 = e \in \mathrm{End}(P)$. So $P = eP \oplus (1 - e)P$ and by Proposition 3.4.3 (ii),

$$\big(fP + F\big)/F = \big(f(1 - e)P + F\big)/F \oplus \big(feP + F\big)/F.$$

From Proposition 3.4.3 (iv), we also obtain that $\big(f(1 - e)P + F\big)/F \cong (1 - e)P$ which is a projective module and $\big(feP + F\big)/F \cong eP/f^{-1}(F)$ which is a singular module because $f^{-1}(F) \leq_{ess} eP$.

(ii)→ (i) Assume (ii). Let $f \in \mathrm{Hom}(P, M)$. Then $\big(fP + F\big)/F = N/F \oplus K/F$ where N/F is a projective module and K/F is a singular module. Define $g : P \to \big(fP + F\big)/F$ by $g(x) = f(x) + F$ for any $x \in P$. Then g is an epimorphism and $\ker g = f^{-1}(F)$. Since N/F is a projective module and πg is an epimorphism where π is the projection homomorphism from $\big(fP + F\big)/F \to N/F$, applying Proposition 2.4.4, leads to $\ker \pi g = eP$ for some $e^2 = e \in \mathrm{End}(P)$. Next, define $h : eP \to K/F$ by $h(x) = f(x) + F$ for any $x \in eP$. Then $\ker h = eP \cap f^{-1}(F) = f^{-1}(F)$. So $eP/f^{-1}(F) \cong K/F$ which is a singular module. This implies that $f^{-1}(F) \leq_{ess} eP$. Therefore, P is an F-CS-Rickart module relative to M. $\square$

The next corollary is an immediate consequence of Theorem 3.4.4 in case $P = M$.

Corollary 3.4.5. *Let M be a projective module. Then the following statements are equivalent.*

(i) M is an F-CS-Rickart module.

(ii) For any $f \in \mathrm{End}(M), (fM+F)/F = N/F \oplus K/F$ where N/F is a projective module and K/F is a singular module.

The following corollary is a consequence of Corollary 3.4.5 when $F = 0$.

Corollary 3.4.6. ([2], Proposition 3.3) *Let M be a projective module. Then M is a CS-Rickart module if and only if every $f \in \mathrm{End}(M), fM = N \oplus K$ where N is a projective module and K is a singular module.*

Next, we investigate the image of each relatively F-CS-Rickart projective module.

Proposition 3.4.7. *Let P be a projective module, M be a module with a fully invariant submodule F. If P is an F-CS-Rickart module relative to M, then for any $f \in \mathrm{Hom}(P, M)$, $fP = N \oplus K$ where N is a projective module and $K \subseteq F^*$.*

Proof. Assume that P is an F-CS-Rickart module relative to M. Moreover, let $f \in \mathrm{Hom}(P, M)$. Then $f^{-1}(F) \leq_{ess} eP$ for some $e^2 = e \in \mathrm{End}(P)$. Applying Proposition 3.4.3 (i), $fP = f(1-e)P \oplus feP$. Observe that $f(1-e)P \cong (1-e)P$ which is a projective module. Moreover, from Proposition 3.4.3 (iv), $(feP+F)/F \cong eP/f^{-1}(F)$ which is a singular module because $f^{-1}(F) \leq_{ess} eP$. Since $(feP+F)/F$ is a singular module, $feP+F \subseteq F^*$ by Proposition 3.4.2. Therefore, $feP \subseteq F^*$ because $F \subseteq F^*$. □

Next, a relationship between a projective module and an F-CS-Rickart module via the idea of relatively F-CS-Rickart modules when $P = M$ is examined.

Corollary 3.4.8. *Let M be a projective module. If M is an F-CS-Rickart module, then, for any $f \in \mathrm{End}(M)$, $fM = N \oplus K$ where N is a projective module and $K \subseteq F^*$.*

Proof. The proof is similar to the proof of Proposition 3.4.7. Assume that M is an F-CS-Rickart module. Let $f \in \mathrm{End}(M)$. Then there is $e^2 = e \in \mathrm{End}(M)$ such that $fM = f(1-e)M \oplus feM$ where $f(1-e)M \cong (1-e)M$ which is a projective module and $feM \subseteq F^*$. □

In the proof of Corollary 3.4.8, $f(1-e)M$ is isomorphic to a direct summand of M. So, we are interested in when $f(1-e)M$ is actually a direct summand of M. A module M *satisfies* C_2 *condition*, given in [17], if any submodule N of M such that $N \cong M'$ for some direct summand M' of M is a direct summand.

Corollary 3.4.9. *Let M be a projective module. If M is an F-CS-Rickart module satisfying C_2 condition, then every $f \in \mathrm{End}(M), fM = eM \oplus K$ where $e^2 = e \in \mathrm{End}(M)$ and $K \subseteq F^*$.*

Proof. Assume that M is an F-CS-Rickart module satisfying C_2 condition. Since $f(1-e)M \cong (1-e)M$ where $(1-e)^2 = (1-e) \in \mathrm{End}(M)$ and M satisfies C_2 condition, $f(1-e)M$ is a direct summand of M. □

For $a \in R$, we denote l_a the module homomorphism from R into R with left multiplication by a, i.e., $l_a(r) = ar$ for all $r \in R$.

Proposition 3.4.10. *Let R be a ring. Then $R \cong \mathrm{End}(R)$.*

Proof. Define $\theta : R \to \mathrm{End}(R)$ by $\theta(a) \to l_a$ for all $a \in R$. It is clear that θ is well-defined and then is a module homomorphism. Let $f \in \mathrm{End}(R)$. Then $f(1) \in R$ and $l_{f(1)}(r) = f(1)r = f(r)$ for all $r \in R$. So θ is an epimorphism. Moreover,

$$\begin{aligned} \ker\theta &= \{a \in R \mid \theta(a) = 0_S\} = \{a \in R \mid l_a = 0_S\} \\ &= \{a \in R \mid l_a(r) = 0_R\} = \{a \in R \mid ar = 0_R \text{ for all } r \in R\} = 0_R \end{aligned}$$

where 0_S is the zero homomorphism of $\mathrm{End}(R)$ and 0_R is the zero element of R. By the first isomorphism theorem, $R \cong \mathrm{End}(R)$. □

For now on, we let $S = \mathrm{End}(M)$. Then $\mathrm{End}(S) \cong S$. Recall from Section 3.1 that $F_S = \{f \in S \mid f(M) \subseteq F\}$. Then S is a right module over itself and F_S is a fully invariant submodule of S so that we apply Proposition 3.4.3 as follows.

Proposition 3.4.11. *Let $\theta \in S$ and $(F_S :_S \theta) \subseteq eS$ for some $e^2 = e \in S$. Then the following statements hold:*

(i) $\theta S = \theta(1- e)S \oplus \theta eS$,

(ii) $(\theta S + F_S)/F_S = (\theta(1-e)S + F_S)/F_S \oplus (\theta eS + F_S)/F_S$,

(iii) $(\theta(1-e)S + F)/F_S \cong (1-e)S \cong \theta(1-e)S$, *and*

(iv) $eS/(F_S :_S \theta) \cong (\theta eS + F_S)/F_S$.

Proof. Note that $\theta \in S \cong \text{End}(S)$ and $(F_S :_S \theta) \subseteq eS$ for some $e^2 = e \in S$. We obtain $l_\theta : S \to S$ defined by $l_\theta(g) = \theta g$. Observe that

$$(l_\theta)^{-1}(F_S) = \{g \in S \mid l_\theta(g) \in F_S\} = \{g \in S \mid \theta g \in F_S\} = (F_S :_S \theta) \subseteq eS.$$

Moreover, $l_\theta S = \theta S$, $l_\theta(1-e)S = \theta(1-e)S$ and $l_\theta(e)S = \theta eS$. By applying Proposition 3.4.3 and the later statements, we can conclude (i), (ii), (iii) and (iv).

□

Next, we provide a relationship between projective F-CS-Rickart modules and their endomorphisms. Recall from Section 3.1 that $\Delta_F(M) = \{f \in \text{End}(M) \mid f^{-1}(F) \leq_{ess} M\}$. Note that, for any $f \in S = \text{End}(M)$, if there is $e^2 = e \in S$ such that $f(M) \subseteq eM$, then $f = ef \in eS$.

Theorem 3.4.12. *Let M be a projective module. If M is an F-CS-Rickart module, then for any $f \in S, (fS+F_S)/F_S = N/F_S \oplus K/F_S$ where N/F_S is a projective module and $K \subseteq \Delta_F(M)$.*

Proof. Assume that M is an F-CS-Rickart module. Let $f \in S$. Then $F \subseteq f^{-1}(F) \leq_{ess} eM$ for some $e^2 = e \in S$. Note that $(F_S :_S f) = \{g \in S \mid fg \in F_S\} = \{g \in S \mid fg(M) \subseteq F\} = \{g \in S \mid g(M) \subseteq f^{-1}(F)\}$. Since $g(M) \subseteq f^{-1}(F) \subseteq eM$ for each $g \in (F_S :_S f)$, it forces that $g = eg \in eS$ so that $(F_S :_S f) \subseteq eS$. Applying Proposition 3.4.11, we obtain that $(fS+F_S)/F_S = (f(1-e)S+F_S)/F_S \oplus (feS+F_S)/F_S$ and $(f(1-e)S+F_S)/F_S \cong (1-e)S$ which is a projective module. Next, define $\phi : eM \to (feM + F)/F$ by $\phi(x) = f(x) + F$ for all $x \in eM$. Then ϕ is an epimorphism and $\ker\phi = f^{-1}(F)$, so $eM/f^{-1}(F) \cong (feM + F)/F \cong M/(fe)^{-1}(F)$. Thus $(fe)^{-1}(F) \leq_{ess} M$ because $f^{-1}(F) \leq_{ess} eM$. Therefore, $fe \in \Delta_F(M)$ so that $feS + F_S \subseteq \Delta_F(M)$. □

Proposition 3.4.13. *Let M be a projective module. If M is an F-CS-Rickart module, then for every $f \in S, fS = N \oplus K$ where N is a projective right ideal of S and K is a right ideal of S with $K \subseteq \Delta_F(M)$.*

Proof. The proof follows from Theorem 3.4.12 □

The next corollary follows from the previous proposition by taking $F = 0$. Recall that $\Delta(M) = \{f \in \text{End}(M) \mid \ker f \leq_{ess} M\}$.

Corollary 3.4.14. ([2], Proposition 3.3) *Let M be a projective module. If M is a CS-Rickart module, then for every $f \in S, fS = N \oplus K$ where N is a projective right ideal of S and K is a right ideal of S with $K \subseteq \Delta(M)$.*

Note that I is an ideal of a ring R if and only if I is a fully invariant submodule of R as a right R-module R. Next, we give the definition of a right I-CS-Rickart ring. Since $\text{End}(R) \cong R$, for any $\theta \in \text{End}(R)$, there exists $a \in R$ such that $\theta = l_a$ so that $\theta^{-1}(I) = \{r \in R \mid \theta(r) \in I\} = \{r \in R \mid ar \in I\} = (I :_R a)$. As a result, we define a right I-CS-Rickart ring as follows.

Definition 3.4.15. Let I be an ideal of a ring R. Then R is a *right I-CS-Rickart ring* if for any $a \in R$ there is a direct summand R' of R such that $(I :_R a) \leq_{ess} R'$.

A right 0-CS-Rickart ring R is also called a *right ACS-ring*, given in [13]. The following corollary is obtained from Corollary 3.4.5.

Corollary 3.4.16. *Let I be an ideal of a ring R. Then R is a right I-CS-Rickart ring if and only if for any $a \in R, (aR + I)/I = N/I \oplus K/I$ where N/I is a projective right module and K/I is a singular right module.*

Let I be an ideal of R and $J(R)$ be the Jacobson radical of R, that is, the intersection of all maximal right ideals of R. A ring R is a *right I-semiregular ring*, given in [13], if for any $a \in R$, $aR = eR \oplus A$ where $e^2 = e \in R$ and $A \subseteq I$ is a right ideal of R; moreover, R is a *left I-semiregular ring* if for any $a \in R$, $Ra = Re \oplus A$ where $e^2 = e \in R$ and $A \subseteq I$ is a left ideal of R. In particular, A ring R is a *semiregular ring* if R is a right $J(R)$-semiregular ring and a left

$J(R)$-semiregular ring. A ring R *satisfies right C_2 condition* if the R module over itself satisfies C_2 condition.

Lemma 3.4.17. ([13], Proposition 1.4) *Let I be an ideal of R and $I \subseteq J(R)$. Then R is a right I-semiregular ring if and only if R is a left I-semiregular ring.*

Lemma 3.4.18. ([13], Proposition 2.3) *If R satisfies right C_2 condition, then $Z(R) \subseteq J(R)$.*

Recall that R is a right ACS-ring if for any $a \in R$ there is a direct summand R' of R such that $(0 :_R a) \leq_{ess} R'$. Nichoson and Yousif characterized right ACS-rings satisfying right C_2 condition in [13].

Lemma 3.4.19. ([13], Theorem 2.4) *The following statements are equivalent.*
(i) R is a semiregular ring and $J(R) = Z(R)$.
(ii) R is a right $Z(R)$-semiregular ring.
(iii) For any $a \in R$, there is $e^2 = e \in R$ such that $aR = eR \oplus K$ where K is a singular module.
(iv) R is a right ACS-ring and every principal projective right ideal of R is a direct summand of R.
(v) R is a right ACS-ring satisfying right C_2 condition.

We now consider when R is a right I-CS-Rickart ring and R/I satisfies right C_2 condition and apply the following lemma as a main idea.

Theorem 3.4.20. *Let I be an ideal of a ring R. Then the following statements are equivalent.*
(i) R/I is a semiregular ring and $J(R/I) = Z(R/I)$.
(ii) R/I is a right $Z(R/I)$-semiregular ring.
(iii) For any $a \in R$, there is $(e+I)^2 = e + I \in R/I$ such that $(aR+I)/I = (e+I)(R/I) \oplus K/I$ where K/I is a singular module.
(iv) R is a right I-CS-Rickart ring and every principal projective right ideal of R/I is a direct summand of R/I.
(v) R is a right I-CS-Rickart ring and R/I satisfies right C_2 condition.

Proof. (i)→(ii)→ (iii) These follow from Lemma 3.4.19.

(iii) → (iv) Assume (iii). Let $a \in R$. Then there is $(e+I)^2 = e+I \in R$ such that $(aR+I)/I = (e+I)(R/I) \oplus K/I$ where K/I is a singular module. Since $(e+I)(R/I) \leq^{\oplus} R/I$ and R/I is a projective module, $(e+I)(R/I)$ is a projective module. Then R is a right I-CS-Rickart ring because of Corollary 3.4.16. Next, let L/I be a principal projective right ideal of R/I generated by $a+I$. Then $(a+I)(R/I) = (e+I)(R/I) \oplus K/I$ where $e^2+I = e+I \in R/I$ and K/I is a singular module. Since $K/I \leq^{\oplus} (a+I)(R/I)$ and $(a+I)(R/I)$ is a projective module, K/I is also a projective module. Thus $K/I = I$ because K/I is both a singular module and a projective module. Thus $(a+I)(R/I) = (e+I)(R/I)$. Therefore, L/I is a direct summand of R/I.

(iv)→ (v) Assume (iv). Let K/I be a right ideal of R/I such that K/I is isomorphic to a direct summand of R/I. Then K/I is a principal projective right ideal of R/I. Thus K/I is a direct summand.

(v)→(i) Assume (v). Let $a+I \in R/I$. Since R is a right I-CS-Rickart ring, by Corollary 3.4.16, $(aR+I)/I = N/I \oplus K/I$ where N/I is a projective module and K/I is a singular module, i.e., $K/I \subseteq Z(R/I)$. So N/I is isomorphic to a direct summand of R/I because N/I is a projective module. Since R/I satisfies right C_2 condition, $N/I = (e+I)(R/I)$ where $(\mathrm{e}+I)^2 = e+I \in R/I$ and $Z(R/I) \subseteq J(R/I)$ from Lemma 3.4.18. Thus $(aR+I)/I = (e+I)(R/I) \oplus K/I$ and $K/I \subseteq J(R/I)$, this forces that R/I is a right $J(R/I)$-semiregular ring. Applying Lemma 3.4.19, R/I is a left $J(R/I)$-semiregular ring so that R/I is a semiregular ring. Next, let $b+I \in J(R/I) \subseteq R/I$. Since R is a right I-CS-Rickart ring, $(b+I)(R/I) = (\mathrm{e}'+I)(R/I) \oplus K'/I$ where $(e'+I)^2 = e'+I \in R/I$ and $K'/I \subseteq Z(R/I) \subseteq J(R/I)$. Hence $\mathrm{e}'+I \in J(R/I)$. Since $J(R/I)$ does not contain any nonzero idempotents, $e'+I = I$. Thus $(b+I)(R/I) = K'/I \subseteq Z(R/I)$, so $b+I \in Z(R/I)$. □

Let $R_i = R$ and $I_i = I$ be an ideal of R for all $i \in \{1, \ldots, n\}$. Let $R^{(n)} = \bigoplus_{i=1}^{n} R_i$ and $I^{(n)} = \bigoplus_{i=1}^{n} I_i$ and $M_n(R)$ be the $n \times n$ matrix ring over R. From Proposition 2.1.14, $\mathrm{End}(R^{(n)}) \cong M_n(S_R)$ where $S_R = \mathrm{End}(R)$. Recall that

$\text{End}(R) \cong R$ so that $\text{End}(R^{(n)}) \cong M_n(R)$. Next, we consider $I_S = \{f \in \text{End}(R^{(n)}) \mid f(R^{(n)}) \subseteq I^{(n)}\}$ which is isomorphic to the matrix ring over I.

Lemma 3.4.21. *Let R be a ring and I be an ideal of R. Then the following statements holds.*

(i) $Hom(R, I) \cong I$.

(ii) $I_S \cong M_n(I)$.

Proof. (i) Observe that $\text{Hom}(R, I) = \{f \in \text{End}(R) \mid f(R) \subseteq I\} \cong \{a \in R \mid aR \subseteq I\} = I$ because $\text{End}(R) \cong R$.

(ii) Let $g \in I_S$. Then $g \in \text{End}(R^{(n)})$ and $g(R^{(n)}) \subseteq I^{(n)}$. Let $S_I = \text{Hom}(R, I)$. Define $\phi : I_S \to M_n(S_I)$ by

$$\phi(g) \to \begin{pmatrix} \pi_1 g i_1 & \dots & \pi_1 g i_n \\ \vdots & \pi_i g i_j & \vdots \\ \pi_n g i_1 & \dots & \pi_n g i_n \end{pmatrix}$$

where $\pi_i g i_j : R \to I$ for all $i, j \in \{1, \dots, n\}$. Then ϕ is an isomorphism so that $I_S \cong M_n(S_I)$. Therefore, $I_S \cong M_n(I)$ because $\text{Hom}(R, I) \cong I$. ☐

Observe that the set of all endomorphisms of $R^{(n)}$ and $M_n(R)$ are concerned as well as I_S and $M_n(I)$ are isomorphic. So we characterize the right $M_n(I)$-CS-Rickart rings and $M_n(R)$ for some given ideal I of R.

Theorem 3.4.22. *Let I be an ideal of a ring R and $n \in \mathbb{N}$. Then the following statements are equivalent.*

(i) The free R-module $R^{(n)}$ is an $I^{(n)}$-CS-Rickart module.

(ii) $\text{End}(R^{(n)})$ is a right I_S-CS-Rickart ring.

(iii) $M_n(R)$ is a right $M_n(I)$-CS-Rickart ring.

(iv)For any n-generated right ideal A of R, $(A + I)/I = N/I \oplus K/I$ where N/I is a projective module and K/I is a singular ring.

(v) The R-module $R^{(n)}$ is an I-CS-Rickart module relative to R.

(vi) For any n-generated submodule L of $R^{(n)}$, $(L + I^{(n)})/I^{(n)} = N_1/I^{(n)} \oplus \dots \oplus N_n/I^{(n)} \oplus K/I^{(n)}$ where each $N_i/I^{(n)}$ is a projective module and K/I is a singular module.

Proof. We let $S = \text{End}(R^{(n)})$.

(i)$\to$ (ii) Assume that the free R-module $R^{(n)}$ is an $I^{(n)}$-CS-Rickart module with basis $\{a_1, \ldots, a_n\}$. Let $f \in S$. Then $f^{-1}(I^{(n)}) \leq_{ess} eR^{(n)}$ for some $e^2 = e \in S$. Let $g \in (I_S :_S f)$. So $fg \in I_S$ that is $fg(R^{(n)}) \subseteq I^{(n)}$. Hence $g(R^{(n)}) \subseteq f^{-1}(I^{(n)}) \subseteq eR^{(n)}$ so that $g = eg$. Thus $(I_S :_S f)$ is a submodule of eS. Next, let $eh \in eS$ and $eh \neq 0$. Then $eh(R^{(n)}) \neq 0$. Since $f^{-1}(I^{(n)}) \leq_{ess} eR^{(n)}$, we get $eh(R^{(n)}) \cap f^{-1}(I^{(n)}) \neq 0$. There is $x \neq 0$ such that $x = eh(y)$ for some $y \in R^{(n)}$ and $f(x) \in I^{(n)}$. We define a homomorphism $\theta \in \text{End}(R^{(n)})$ by $\theta(a_1r_1 + \cdots + a_nr_n) = yr_1$ for all $r_1, \ldots, r_n \in R$. Then $eh\theta(a_1) = eh(y) = x$ and $\theta\big(R^{(n)}\big) = yR$, this forces that $eh\theta \neq 0$ and $feh\theta\big(R^{(n)}\big) = feh(yR) = f(x)R \subseteq I^{(n)}R \subseteq I^{(n)}$. So $feh\theta \in I_S$, that is $eh\theta \in (I_S :_S f)$. Hence $(I_S :_S f) \leq_{ess} eS$. Therefore, $\text{End}(R^{(n)})$ is a right I_S-CS-Rickart ring.

(ii)$\to$(i) Assume (ii). Let $f \in S$. Then $(I_S :_S f) \leq_{ess} eS$ for some $e^2 = e \in S$. Let $x \in f^{-1}(I^{(n)})$. Then $f(x) \in I^{(n)}$. Similar to the argument of the proof (i) $\to$ (ii), there is a homomorphism $\theta \in S$ such that $\theta\big(R^{(n)}\big) = xR$. So $f\big(\theta R^{(n)}\big) = f(xR) \subseteq I^{(n)}$, we obtain that $f\theta \in I_S$ that is $\theta \in (I_S :_S f)$. Thus $\theta = e\theta$ because $(I_S :_S f) \subseteq eS$. Then $x \in xR = \theta\big(R^{(n)}\big) = e\theta\big(R^{(n)}\big) \subseteq eR^{(n)}$. This implies that $f^{-1}(I^{(n)}) \subseteq eR^{(n)}$. Next, let $m \in eR^{(n)}$ and $m \neq 0$. Then $m = em$ so that $mR = emR$. So there is a nonzero homomorphism $h \in S$ such that $hR^{(n)} = mR = emR$, similar to the technique of the proof (i) $\to$ (ii). Since $(I_S :_S f) \leq_{ess} eS$, there is $g \in S$ such that $hg \neq 0$ and $fhg \in I_S$. So $0 \neq hg(R^{(n)})$ and $fhg(R^{(n)}) \subseteq I^{(n)}$. Hence $0 \neq hg(R^{(n)}) \subseteq f^{-1}(I^{(n)}$. This forces that $f^{-1}(I^{(n)}) \leq_{ess} eR^{(n)}$. Therefore, the free R-module $R^{(n)}$ is an $I^{(n)}$-CS-Rickart module.

(i)$\to$(v) This follows from Theorem 3.2.2.

(ii)$\leftrightarrow$(iii) This is clear because $\text{End}(R^{(n)}) \cong M_n(R)$ and $I_S \cong M_n(I)$.

(iv)$\to$(v) Assume (iv). Let $f \in \text{Hom}(R^{(n)}, R)$. Then for any $x_i \in R$,

$$f(x_1, \ldots, x_n) = f(1, \ldots, 0)x_1 + \cdots + f(0, \ldots, 1)x_n.$$

So $f(R^{(n)})$ is generated by $\{f(1, \ldots, 0), \ldots, f(0, \ldots, 1)\}$. By assumption, $\big(f(R^{(n)}) + I\big)/I = N/I + K/I$ where N/I is a projective module and K/I is a singular module.

From Theorem 3.4.4, $R^{(n)}$ is an I-CS-Rickart module relative to R.

(v)→(iv) Assume (v). Let A be an n-generated right ideal of R such that $A = a_1R+\cdots+a_nR$ where $a_1,\ldots,a_n \in R$. Define $\phi : R^{(n)} \to R$ by $\phi(x_1,\ldots,x_n) = a_1x_1+\cdots+a_nx_n$ for any $x_1,\ldots,x_n \in R$. Then ϕ is a module homomorphism and $\phi(R^{(n)}) = a_1R+\cdots+a_nR$. Therefore, $\big(A+I\big)/I = \big(\phi(R^{(n)})+I\big)/I = N/I+K/I$ where N/I is a projective module and K/I is a singular module because $R^{(n)}$ is an I-CS-Rickart module relative to R.

(v)→(vi) Assume (v). Let L be an n-generated submodule of $R^{(n)}$. Then $L = (x_1)R+\cdots+(x_n)R$ where $(x_1),\ldots,(x_n) \in R^{(n)}$ and $(x_i) = (x_{1i},\ldots,x_{ni})$ for all $i \in \{1,\ldots,n\}$. So

$$\begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix}\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} x_{11} \\ \vdots \\ x_{n1} \end{pmatrix} a_1+\cdots+\begin{pmatrix} x_{1n} \\ \vdots \\ x_{nn} \end{pmatrix} a_n \in (x_1)R+\cdots+(x_n)R.$$

Let $f = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix}\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \in M_n(R) \cong \mathrm{End}(R^{(n)})$. Then $f \in S$ and $f(R^{(n)}) = L$. Let π_i be the projection map from $R^{(n)}$ to its i-th component and θ_i be the inclusion map form R to $R^{(n)}$ for any $i \in \{1,\ldots,n\}$. Since the R-module $R^{(n)}$ is an I-CS-Rickart module relative to R and $\pi_1 f \in \mathrm{Hom}(R^{(n)}, R_1)$, we obtain that $f^{-1}(I^{(n)}) \subseteq (\pi_1 f)^{-1}(I) \leq_{ess} e_1R^{(n)}$ for some $e_1^2 = e_1 \in S$. So $R^{(n)} = P_1 \oplus e_1R^{(n)}$ and P_1 is a projective module because $P_1 \leq^{\oplus} R^{(n)}$ and $R^{(n)}$ is a projective module. Next, we consider the homomorphism $\pi_1 f_{|e_1R^{(n)}}$. Since $e_1R^{(n)} \leq^{\oplus} R^{(n)}$ and $\pi_2 f_{|e_1R^{(n)}} \in \mathrm{Hom}(e_1R^{(n)}, R_2)$, applying Theorem 3.2.2, $f^{-1}(I^{(n)}) \subseteq (\pi_2 f_{|e_1R^{(n)}})^{-1}(I) \leq_{ess} e_2R^{(n)}$ for some $e_2^2 = e_2 \in S$. Since $e_2R^{(n)} \leq^{\oplus} R^{(n)}$ and $e_2R^{(n)} \subseteq e_1R^{(n)}$, from Proposition 2.1.4, $e_2R^{(n)} \leq^{\oplus} e_1R^{(n)}$. Thus $e_1R^{(n)} = P_2 \oplus e_2R^{(n)}$ and P_2 is a projective module because $P_2 \leq^{\oplus} e_1R^{(n)}$. Hence $R^{(n)} = P_1 \oplus P_2 \oplus e_2R^{(n)}$. So we get $e_3,\ldots,e_n$ such that $f^{-1}(I^{(n)}) \subseteq \pi_j f_{|e_{j-1}R^{(n)}} \leq_{ess} e_jR^{(n)}$ for all $j \in \{3,\ldots,n\}$. Thus $f^{-1}(I^{(n)}) = (\pi_1 f)^{-1}(I) \cap \cdots \cap (\pi_n f)^{-1}(I) \leq_{ess} e_1R^{(n)} \cap \cdots \cap e_nR^{(n)} = e_nR^{(n)}$. Now, $R^n = P_1 \oplus \cdots \oplus P_n \oplus e_nR^{(n)}$ where each P_i is a projective module. Hence $(K + I^{(n)})/I^{(n)} = (f(R^{(n)}) + I^{(n)})/I^{(n)} =$

$(f(P_1)+I^{(n)})/I^{(n)} \oplus \cdots \oplus (f(P_n)+I^{(n)})/I^{(n)} \oplus f(e_nR^{(n)})+I^{(n)})/I^{(n)}$ where each $(f(P_i)+I^{(n)})/I^{(n)} \cong P_i$ which is a projective module and $f(e_nR^{(n)})+I^{(n)})/I^{(n)} \cong e_nR^{(n)}/f^{-1}(I^{(n)})$ which is a singular module.

□

Consequently, we obtain the following corollary when $F = 0$.

Corollary 3.4.23. ([1], Theorem 4.3) *Let $n \in \mathbb{N}$. Then the following statements are equivalent.*

(i) The free R-module $R^{(n)}$ is a CS-Rickart module.

(ii) $M_n(R)$ is a right CS-Rickart ring.

(iii) For any n-generated right ideal A of R, $A = N \oplus K$ where N is a projective module and K is a singular module.

(iv) The R-module $R^{(n)}$ is a CS-Rickart module relative to R.

(v) For any n-generated submodule L of $R^{(n)}$, $L = N_1 \oplus \cdots \oplus N_n \oplus K$ where each N_i is a projective module and K is a singular module.