CHAPTER IV

F-DUAL-CS-RICKART MODULES

In this chapter, we give the notions of F-dual-Rickart modules and F-dual-CS-Rickart modules. The concept of F-dual-Rickart modules are generalized from dual-Rickart modules given by Lee, Rizvi and Roman in [12]. We extend the idea of being a direct summand of $f(M)$ to $f(F)$ for all $f \in \text{End}(M)$ after that the ideas of F-dual Rickart modules and dual-CS-Rickart modules, defined by Abyzov and Nhan in [1], module are combined. We integrate the idea of being a direct summand of $f(F)$ from F-dual-Rickart modules and the idea of lying above some direct summand of $f(M)$ from dual-CS-Rickart modules for all $f \in \text{End}(M)$.

Several properties of F-dual-CS-Rickart modules and characterizations of those are investigated in Section 4.1. We show that the intersection of two direct summands one of which contained in F of an F-dual-CS-Rickart module lies above some direct summand. Moreover, we study when a submodule of F-dual-CS-Rickart module is also an F'-CS-Rickart module where F' is a fully invariant submodule of that submodule. Relationships between F-dual-CS-Rickart modules and F-dual-Rickart modules as well as relationships between F–dual-CS-Rickart modules and dual-CS-Rickart modules are presented. Furthermore, we give a notion and a characterization of strongly F-dual-CS-Rickart modules which is a special case of F-CS-Rickart modules. Observe that the idea of F-dual-CS-Rickart modules considers the images of endomorphism on itself. So, in Section 3.2, we extend this idea to consider an image of a homomorphism which lies above some direct summands.

4.1 Properties of F-dual-CS-Rickart Modules

First, we provide the definition of an F-dual-Rickart module. Then the notion of F-dual-CS-Rickart modules are given by extending the concept of F-dual-Rickart modules and dual-CS-Rickart modules. We show that the sum of two submodules of M which lie above some direct summands lies above a direct summand of M if M is an F-dual-CS-Rickart Module and one of those submodules is contained in F. One of main points is that any F-dual-CS-Rickart module can be written as a direct sum of two submodules one of which is contained in F and the other one of which is a dual-CS-Rickart module.

Lee, Rizvi and Roman provided in [12] the concept of dual-Rickart modules in 2011. A module M is a *dual-Rickart module* if $f(M)$ is a direct summand of M for any $f \in \mathrm{End}(M)$. Thus we are interested in when $f(F)$ is a direct summand of M for all $f \in \mathrm{End}(M)$ and we call the modules satisfying this condition F-dual-Rickart modules.

Definition 4.1.1. Let F be a fully invariant submodule of M. A module M is an *F-dual-Rickart module* if $f(F)$ is a direct summand of M for any $f \in \mathrm{End}(M)$.

Next, the notion of dual-CS-Rickart modules are introduced by Abyzov and Nhan in 2014. A module M is a *dual-CS-Rickart module* if $f(M)$ lies above direct summand of M for any $f \in \mathrm{End}(M)$. We combine the concepts of F-dual-Rickart modules and dual-CS-Rickart modules as follows.

Definition 4.1.2. Let F be a fully invariant submodule of M. A module M is an *F-dual-CS-Rickart module* if $f(F)$ lies above a direct summand of M for any $f \in \mathrm{End}(M)$.

Note that M is a dual-CS-Rickart module if and only if M is an M-dual-CS-Rickart module.

Proposition 4.1.3. *Let M be an F-dual-CS-Rickart module and $f \in \mathrm{End}(M)$. The the following statements are equivalent.*
(i) There is a direct summand N of M such that $N \subseteq f(F)$ and $f(F)/N \ll M/N$.

(ii) There is a direct summand N of M and a submodule K of M such that $N \subseteq f(F)$, $f(F) = N + K$ and $K \ll M$.

(iii) There is a decomposition $M = N \oplus K$ with $N \subseteq f(F)$ and $K \cap f(F) \ll K$.

(iv) $f(F) = eM \oplus (1-e)f(F)$ and $(1-e)f(F) \ll M$ for some $e^2 = e \in \operatorname{End}(M)$.

Proof. The proof follows from Proposition 2.3.7. □

For an F-dual-Rickart module M, any $f \in \operatorname{End}(M)$, $f(F)$ is a direct summand of M so that $f(F)$ lies above itself. Next, we show that any F-dual-Rickart module is always an F-dual-CS-Rickart module.

Proposition 4.1.4. *Any F-dual-Rickart module is an F-dual-CS-Rickart module.*

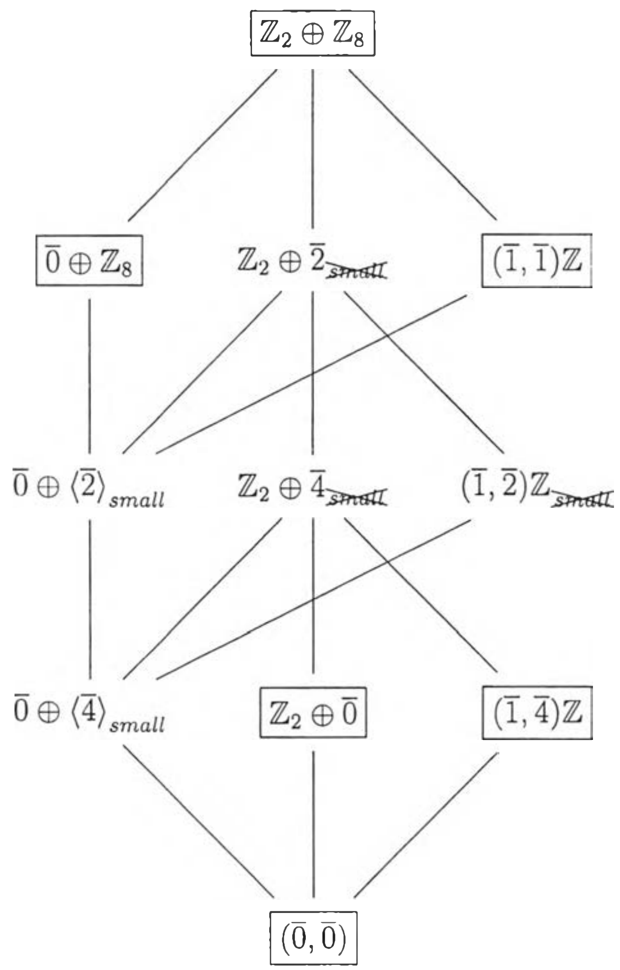
Proof. Let M be an F-dual-Rickart module. Then $f(F) = eM$ for some $e^2 = e \in \operatorname{End}(M)$. So $f(F)$ lies above eM. Therefore, M is an F-dual-CS-Rickart module. □

Observe that $f(F)$ is a submodule of M contained in F. So we can conclude that M is an F-dual-CS-Rickart module if and only if any submodule of M contained in F lies above a direct summand of M. Next, we give an example of F-dual-CS-Rickart modules which is not an F-dual-Rickart module for some given fully invariant submodule F of M.

Example 4.1.5. From Example 3.1.3, let M be the $\mathbb{Z}$-module $\mathbb{Z}_2 \oplus \mathbb{Z}_8$. Then the submodule $K = \mathbb{Z}_2 \oplus \langle\overline{4}\rangle$ is a fully invariant submodule of M. The following diagram describes all submodules of $Z_2 \oplus Z_8$. Each submodule contained in a box is a direct summand of M but the others are not direct summands of M. Furthermore, if a submodule N is a small submodule of M, we write N_{small}, otherwise; we write $N_{\cancel{small}}$.

Observe from the diagram that all submodules of M contained in K are $(\overline{0}, \overline{0})$, $\overline{0} \oplus \langle\overline{4}\rangle$, $\mathbb{Z}_2 \oplus \overline{0}$, $(\overline{1}, \overline{4})\mathbb{Z}$ and K. Among these, only $(\overline{0}, \overline{0})$, $\mathbb{Z}_2 \oplus \overline{0}$ and $(\overline{1}, \overline{4})\mathbb{Z}$ are direct summands of M, i.e., they lie above themselves, and only $\overline{0} \oplus \overline{4} \ll M$ but K is not a direct summand and not a small submodule of M. Moreover, $K = (\mathbb{Z}_2 \oplus \overline{0}) \oplus (\overline{0} \oplus \langle\overline{4}\rangle)$ lies above $\mathbb{Z}_2 \oplus \overline{0}$ because $(\overline{0} \oplus \langle\overline{4}\rangle) \ll M$ by applying

Proposition 2.3.7. We can see that any submodule of M contained in K lies above a direct summand of M. Thus M is a K-dual-CS-Rickart module. However, M is not a K-dual-Rickart module because $1_S(K) = K$ which is not a direct summand of M.



Proposition 4.1.4 together with Example 4.1.5 ensure that F-dual-CS-Rickart modules truly generalized F-dual-Rickart modules. We know that M is a dual-CS-Rickart module if and only if M is a M-dual-CS-Rickart module. For a given fully invariant submodule F of M, "M is an F-dual-CS-Rickart module" does not imply "M is a dual-CS-Rickart module". Example 4.1.5 shows that $\mathbb{Z}_2 \oplus \mathbb{Z}_8$ is a $(\mathbb{Z}_2 \oplus \langle \overline{4} \rangle)$-dual-CS-Rickart module; however, $\mathbb{Z}_2 \oplus \mathbb{Z}_8$ is not a dual-CS-Rickart module shown in the next example.

Example 4.1.6. From Example 4.1.5, let M be the $\mathbb{Z}$-module $\mathbb{Z}_2 \oplus \mathbb{Z}_8$. and $K = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \langle \overline{4} \rangle$. We obtain that M is a K-dual-CS-Rickart module. Let

$h = \begin{pmatrix} f_0 & g_1' \\ f_0' & g_2 \end{pmatrix} \in \begin{pmatrix} \mathrm{End}(\mathbb{Z}_2) & \mathrm{Hom}(\mathbb{Z}_8, \mathbb{Z}_2) \\ \mathrm{Hom}(\mathbb{Z}_2, \mathbb{Z}_8) & \mathrm{End}(\mathbb{Z}_8) \end{pmatrix} \cong \mathrm{End}(\mathbb{Z}_2 \oplus \mathbb{Z}_8)$ where f_0 is the zero homomorphism on $\mathbb{Z}_2$, f_0' is the zero homomorphism from $\mathbb{Z}_2$ into $\mathbb{Z}_8$, $g_1'(\overline{y}) = \overline{y}$ and $g_2(\overline{y}) = \overline{2y}$ for all $\overline{y} \in \mathbb{Z}_8$. Then $h(M) = (\overline{1}, \overline{2})\mathbb{Z}$ which does not lie above in all direct summands of M. Thus $\mathbb{Z}_2 \oplus \mathbb{Z}_8$ is not a dual-CS-Rickart module.

Next, we provide some properties of F-dual-CS-Rickart modules.

Proposition 4.1.7. *Let M be an F-dual-CS-Rickart module and P be a module. If M is isomorphic to P by an isomorphism $\phi : M \to P$, then P is a $\phi(F)$-dual-CS-Rickart module.*

Proof. Assume that ϕ is an isomorphism from P to M. Let $f \in \mathrm{End}(P)$. Recall that $\phi(F) \leq_{fully} P$. So $\phi^{-1}f\phi \in \mathrm{End}(M)$. Since M is an F-dual-CS-Rickart module, $(\phi^{-1}f\phi)(F)$ lies above a direct summand of M. So there is a decomposition $M = N \oplus K$ with $N \subseteq (\phi^{-1}f\phi)(F)$ and $K \cap (\phi^{-1}f\phi)(F) \ll K$. Note that $P = \phi(M) = \phi(N) \oplus \phi(K)$ so that $\phi(N) \leq^{\oplus} P$ and $\phi(K) \leq^{\oplus} P$. It is clear that $\phi(N) \subseteq \phi\big((\phi^{-1}f\phi)(F)\big) \subseteq f\phi(F)$. Since $K \cap (\phi^{-1}f\phi)(F) \ll K$, it implies that $K \cap (\phi^{-1}f\phi)(F) \ll M$. By Proposition 2.3.6, $\phi\big(K \cap (\phi^{-1}f\phi)(F)\big) \ll \phi(M) = P$. Thus $\phi\big(K \cap (\phi^{-1}f\phi)(F)\big) \ll \phi(K)$ because $\phi\big(K \cap (\phi^{-1}f\phi)(F)\big) \subseteq \phi(K)$ and $\phi(K)$ is a direct summand of P. Since ϕ is an isomorphism, $\phi\big(K \cap (\phi^{-1}f\phi)(F)\big) = \phi(K) \cap f\phi(F)$. This forces that $\phi(K) \cap f\phi(F) \ll \phi(K)$. Therefore, $f\phi(F)$ lies above $\phi(N)$ from Proposition 4.1.3. □

The sum of any two direct summands may not be a direct summand, normally. However, dual-Rickart modules have property that the sum of two direct summands turns to be a direct summand; moreover, dual-CS-Rickart modules possess property that the sum of two direct summands lies above a direct summand. Similarly, we are interested in the sum of two direct summands of an F-dual-CS-Rickart module. Next example presents that there is the sum of two direct summands of an F-dual-CS-Rickart module which is not a direct summand but it lies above a direct summand.

Example 4.1.8. From Example 4.1.5, let $M = \mathbb{Z}_2 \oplus \mathbb{Z}_8$ and $K = \mathbb{Z}_2 \oplus \langle \overline{4} \rangle$. Recall that M is an K-dual-CS-Rickart module. Note that $A = (\overline{1}, \overline{4})\mathbb{Z}$ and $B = \mathbb{Z}_8 \oplus \overline{0}$ are direct summands of M. Then $A + B = \mathbb{Z}_2 \oplus \langle \overline{4} \rangle$ is not a direct summand of M but $A + B = \mathbb{Z}_2 \oplus \langle \overline{4} \rangle$ lies above $\mathbb{Z}_2 \oplus \overline{0}$.

Nevertheless, an F-dual-CS-Rickart module M satisfying some conditions confirms that the sum of two direct summands lies above a direct summand of M. The following lemma is necessary to prove this fact.

Lemma 4.1.9. *Let F be a fully invariant submodule of M. Let $h^2 = h, g^2 = g \in \mathrm{End}(M)$ and $gM \subseteq F$. Then $gF = gM$ and $hM + gM = hM \oplus (1-h)gM = hM \oplus (1-h)gF$.*

Proof. It is clear that $gF \subseteq gM$. Let $x \in gM \subseteq F$. Then $x = gx \in g(F)$ so that $gF = gM$.

Let $x \in hM + gM$. Then there are $u, v \in M$ such that $x = h(u) + g(v) = h(u) + \big(hg(v) + (1-h)g(v)\big) = h\big(u + g(v)\big) + (1-h)g(v) \in hM \oplus (1-h)gM$. Next, let $x + y \in hM \oplus (1-h)gM$ where $x \in hM$ and $y \in (1-h)gM$. Then $x = hx$ and $y = (1-h)g(w)$ for some $w \in M$. So $x + y = h(x) + (1-h)g(w) = h\big(x - g(w)\big) + g(w) \in hM + gM$. Thus $hM + gM = hM \oplus (1-h)gM$. Therefore, $hM + gM = hM \oplus (1-h)gF$ because $gF = gM$. □

Proposition 4.1.10. *Let M be an F-dual-CS-Rickart module. Then the following statements hold.*

(i) For any direct summands N and K of M, if $K \subseteq F$, then $N + K$ lies aboves M' for some direct summand M' of M.

(ii) For any submodules N and K of M, if there are direct summands M_1 and M_2 of M such that N lies above M_1 and K lies above M_2 with $M_2 \subseteq F$, then $N + K$ lies above M' for some direct summand M' of M.

(iii) For any $f_1, \ldots, f_n \in \mathrm{End}(M)$, there is a direct summand M' of M such that $f_1(F) + \cdots + f_n(F)$ lies above M'.

Proof. (i) Assume that N and K are direct summands of M and $K \subseteq F$. Then $N = hM$ and $K = gM$ for some $h^2 = h, g^2 = g \in \mathrm{End}(M)$. From Lemma 4.1.9,

$N + K = hM + gM = hM \oplus (1-h)gF$. Since $(1-h)g \in \text{End}(M)$ and M is an F-dual-CS-Rickart module, $(1-h)g(F) = eM \oplus \Big((1-e)(1-h)g(F)\Big)$ and $(1-e)(1-h)g(F) \ll M$ by applying Proposition 4.1.3. Thus

$$N + K = hM \oplus eM \oplus \Big((1-e)(1-h)g(F)\Big).$$

Since $eM \subseteq (1-h)g(F) \subseteq (1-h)M$ and $M = eM \oplus (1-e)M$, by Modular Law $(1-h)M = eM \oplus \big((1-e)M \cap (1-h)M\big)$ and applying Proposition 2.1.5, $(1-e)M \cap (1-h)M = (1-e)(1-h)M$. We can conclude that $M = hM \oplus (1-h)M = hM \oplus eM \oplus \big((1-e)M \cap (1-h)M\big) = hM \oplus eM \oplus (1-e)(1-h)M$ so that $hM \oplus eM \leq^{\oplus} M$. Therefore, $N + K$ lies above $hM \oplus eM$.

(ii) Assume that N and K are submodules of M such that N lies above a direct summand hM of M and K lies above a direct summand gM of M with $gM \subseteq F$ for some $h^2 = h, g^2 = g \in \text{End}(M)$, respectively. From Proposition 2.3.7, we obtain $N = hM \oplus (1-h)N$ and $(1-h)N \ll M$; moreover, $K = gM \oplus (1-g)K$ and $(1-g)K \ll M$. As the results of (i), $hM + gM = eM \oplus (1-e)\big(hM + gM\big)$ and $(1-e)\big(hM + gM\big) \ll M$ for some $e^2 = e \in \text{End}(M)$. Thus

$$\begin{aligned} N + K &= \Big(hM \oplus (1-h)N\Big) + \Big(gM \oplus (1-g)K\Big) \\ &= \Big(hM + gM\Big) + \Big((1-h)N + (1-g)K\Big) \\ &= \Big(eM \oplus (1-e)\big(hM + gM\big)\Big) + \Big((1-h)N + (1-g)K\Big) \\ &= eM + \Big((1-e)\big(hM + gM\big) + (1-h)N + (1-g)K\Big). \end{aligned}$$

Moreover, $(1-e)\big(hM + gM\big) + \big((1-h)M \cap N\big) + \big((1-g)M \cap K\big) \ll M$ by applying Proposition 2.3.7. Therefore, $N + K$ lies above eM.

(iii) Let $f_i \in \text{End}(M)$ for all $i \in \{1, \ldots, n\}$. Since M is an F-CS-Rickart module, for each i, $f_i(F)$ lies above M_i for some direct summand M_i of M. Applying (ii) repeatedly, we obtain $f_1(F) + \cdots + f_n(F)$ lies above direct summand M' of M because $f_i(F) \subseteq F$ for all i. □

A module M is an *SSP-d-CS* module, given in [1], if the sum of two direct summands lies above a direct summand of M. The sum of two direct summands of an F-dual-CS-Rickart module lies above a direct summand of M when one of which contained in F shown from the previous proposition.

Corollary 4.1.11. *Let M be an F-dual-CS-Rickart module. Then M is an SSP-d-CS module provided that for all direct summands of M contained in F.*

Next, we show that a direct summand of an F-dual-CS-Rickart module is also an F'-dual-CS-Rickart module where F' is a fully invariant submodule of this direct summand. This result is similar to property in F-CS-Rickart modules.

Theorem 4.1.12. *A module M is an F-dual-CS-Rickart module if and only if N is an $(N \cap F)$-dual-CS-Rickart module for any direct summand N of M.*

Proof. The sufficiency is clear because M is always a direct summand of M itself.

For the necessity, let N be a direct summand of M. Then $N = eM$ for some $e^2 = e \in \text{End}(M)$ and $N \cap F$ is a fully invariant submodule of N. Let $K = (1-e)M$. Then $M = N \oplus K$. Since $M = N \oplus K$ and $F \leq_{fully} M$, by Proposition 2.1.8, $F = (N \cap F) \oplus (K \cap F)$. Let $g \in \text{End}(N)$. So $ge \in \text{End}(M)$. Since M is an F-dual-CS-Rickart module, $ge(F) = e_1M \oplus \big((1-e_1)ge(F)\big)$ and $(1-e_1)ge(F) \ll M$ for some $(e_1)^2 = e_1 \in \text{End}(M)$. Since $F \leq_{fully} M$, we have $N \cap F = \text{e}F$ so that $ge(F) = g(N \cap F) \subseteq N$. We obtain that $e_1M \leq^{\oplus} N$ because $e_1M \leq^{\oplus} M$ and $e_1M \subseteq ge(F) \subseteq N$. As $(1-e_1)ge(F) \ll M$ and $(1-e_1)ge(F) \subseteq ge(F) \subseteq N$ which is a direct summand of M, so $(1-e_1)ge(F) \ll N$ by applying Proposition 2.3.4. Thus $g(N \cap F) = e_1M \oplus \big((1-e_1)ge(F)\big)$ which $e_1M \leq^{\oplus} N$ and $(1-e_1)ge(F) \ll N$. This forces that $g(N \cap F)$ lies above e_1M. Therefore, N is an $(N \cap F)$-dual-CS-Rickart module. □

A direct sum of F-dual-CS-Rickart modules when each summand is also a fully invariant submodule is examined in the following result.

Theorem 4.1.13. *Let M_j be a fully invariant submodule of $\bigoplus_{i=1}^{n} M_i$ and F_j be a fully invariant submodule of M_j for all $j \in \{1, \ldots, n\}$. Then $\bigoplus_{i=1}^{n} M_i$ is a $\bigoplus_{i=1}^{n} F_i$-dual-CS-Rickart module if and only if M_j is an F_j-dual-CS-Rickart module for all $j \in \{1, \ldots. n\}$.*

Proof. For the necessity, assume that $\bigoplus_{i=1}^{n} M_i$ is a $\bigoplus_{i=1}^{n} F_i$-dual-CS-Rickart module. Since each $M_j \leq^{\oplus} \bigoplus_{i=1}^{n} M_i$, we obtain that M_j is an $(M_j \cap \bigoplus_{i=1}^{n} F_i)$-dual-CS-Rickart module by Theorem 4.1.12. Therefore, M_j is an F_j-dual-CS-Rickart module because $M_j \cap \bigoplus_{i=1}^{n} F_i = F_j$ for all $j \in \{1, \dots, n\}$.

To show the sufficiency, assume that M_j is an F_j-dual-CS-Rickart module for all $j \in \{1, \dots, n\}$. Let $f \in \mathrm{End}(\bigoplus_{i=1}^{n} M_i)$ and $(x_1, \dots, x_n) \in \bigoplus_{i=1}^{n} M_i$. Then $f(x_1, \dots, x_n) = f(x_1, \dots, 0) + \cdots + f(0, \dots, x_n) = f_1(x_1) + \cdots + f_n(x_n)$ where $f_j := fi_j : M_j \to \bigoplus_{i=1}^{n} M_i$ and i_j is the inclusion map from M_j into $\bigoplus_{i=1}^{n} M_i$ for all $j \in \{1, \dots, n\}$. As each $M_j \leq_{fully} \bigoplus_{i=1}^{n} M_i$, we obtain $f_j : M_j \to M_j$ and $f_j(F_j) \subseteq F_j$. Since each M_j is an F_j-dual-CS-Rickart module, $f_j(F_j)$ lies above $e_j M_j$ for some $e_j^2 = e_j \in \mathrm{End}(M_j)$. That is $f_j(F_j) = e_j M_j \oplus (1 - e_j) f_j(F_j)$ and $(1 - e_j) f_j(F_j) \ll M_j$ for all $j \in \{1, \dots, n\}$. Hence $f(\bigoplus_{i=1}^{n} F_i) = \bigoplus_{i=1}^{n} f_i(F_i) = \left(\bigoplus_{i=1}^{n} e_i M_i\right) \bigoplus \left(\bigoplus_{i=1}^{n} (1 - e_i) f_i(F_i)\right)$ and $\bigoplus_{i=1}^{n} (1 - e_i) f_i(F_i) \ll \bigoplus_{i=1}^{n} M_i$. Therefore, $\bigoplus_{i=1}^{n} M_i$ is a $\bigoplus_{i=1}^{n} F_i$-dual-CS-Rickart module. □

We know that F-dual-Rickart modules are F-dual-CS-Rickart modules but the converse is not necessary true from Example 4.1.5.

As a result, we are interested in finding conditions that make the converse valid true. A module M is a *$\mathcal{T}$-noncosingular module*, given in [1], if for any $f \in \mathrm{End}(M)$, if $f(M) = 0$ provided $f(M)$ is a small submodule of M. We give a generalization of $\mathcal{T}$-noncosingular modules as follows.

Definition 4.1.14. A module M is an *F-$\mathcal{T}$-noncosingular module*, if for any nonzero $f \in \mathrm{End}(M)$, if $f(F) = 0$ provided $f(F)$ is a small submodule of M.

Proposition 4.1.15. *If M is an F-dual-CS-Rickart module, then M is an F-$\mathcal{T}$-noncosingular module.*

Proof. Assume that M is an F-dual-CS-Rickart module. Let $f \in \mathrm{End}(M)$ and $f(F) \ll M$. So $f(F) = eM$ for some $e^2 = e \in \mathrm{End}(M)$. Thus $f(F) = 0$ by applying Proposition 2.3.1. Therefore, M is an F-$\mathcal{T}$-cononsingular module. □

Theorem 4.1.16. *The following statements are equivalent.*

(i) M is an F-dual-CS-Rickart module and an F-$\mathcal{T}$-cononsingular module.

(ii) M is an F-dual Rickart module.

Proof. (ii)$\rightarrow$ (i) This follows from Proposition 4.1.4 and Proposition 4.1.15.

(i)$\rightarrow$ (ii) Assume (i). Let $f \in \mathrm{End}(M)$. Then $f(F) = eM \oplus (1-e)f(F)$ and $(1-e)f(F) \ll M$ for some $e^2 = e \in \mathrm{End}(M)$. Hence $(1-e)f(F) = 0$ because M is an F-$\mathcal{T}$-noncosingular module. Hence $f(F) = eM$. Therefore, M is an F-dual-Rickart module. □

Similar to F-CS-Rickart modules, each F-dual-CS-Rickart module M has a direct summand depending on each image of F. So, for any $f \in \mathrm{End}(M)$, there is a submodule N of M such that $M = N \oplus K$ where $f(F)$ lies above N.

Theorem 4.1.17. *If M is an F-dual-CS-Rickart module, then $M = N \oplus K$ where $N \subseteq F$, $K \cap F \ll M$ and N is a dual CS-Rickart module. The converse holds if N is a fully invariant submodule of M.*

Proof. Assume that M is an F-dual-CS-Rickart module. Then $F = 1_S(F)$ lies above N for some $N \leq^{\oplus} M$. So there is a submodule K of M such that $M = N \oplus K$, $F = N \oplus (K \cap F)$ and $K \cap F \ll M$. Since $N \leq^{\oplus} M$ and M is an F-dual-CS-Rickart module, N is an $(N \cap F)$-dual-CS-Rickart module by applying Theorem 4.1.12. Thus N is a dual-CS-Rickart module because $N \cap F = N$.

To show the converse is valid, assume that $M = N \oplus K$ where $N \subseteq F$, $K \cap F \ll M$, N is a dual-CS-Rickart module and N is a fully invariant submodule of M. Thus $F = N \oplus (K \cap F)$ because $N \subseteq F$. Let $f \in \mathrm{End}(M)$. Then $f(F) = f(N \oplus (K \cap F)) = f(N) + f(K \cap F)$ and $f(K \cap F) \ll M$ by Proposition 2.3.6. Since $N \leq_{fully} M$, we obtain $f_{|N} \in \mathrm{End}(N)$ so that $f_{|N}(N) = f(N)$. As N is a dual-CS-Rickart module, $f_{|N}(N) = N_1 \oplus \big(N_2 \cap f(N)\big)$ and $N_2 \cap f(N) \ll N$ where $N = N_1 \oplus N_2$. Thus

$$f(F) = f(N) + f(K \cap F) = N_1 + \Big(\big(N_2 \cap f(N)\big) + f(K \cap F)\Big)$$

where $N_1 \leq^{\oplus} M$ and $\big(N_2 \cap f(N)\big) + f(K \cap F) \ll M$. Hence $f(F)$ lies above N_1. Therefore, M is an F-dual-CS-Rickart module. □

Now, F-dual-CS-Rickart modules having two direct summands are considered.

Proposition 4.1.18. *For every indecomposable F-dual-CS-Rickart module M, either M is a dual-CS-Rickart module or $F \ll M$.*

Proof. Assume M is an indecomposable F-dual-CS-Rickart module. Then $M = N \oplus K$ where $N \subseteq F$, $K \cap F \ll M$ and N is a dual CS-Rickart module. Since M is an indecomposable module, $N = 0$ or $N = M$. In case $N = 0$, it follows that $F = 0$ so that $K = K \cap F \ll M$; otherwise, $N = M$ implying that M is a dual CS-Rickart module. Therefore, either M is a dual-CS-Rickart module of $F \ll M$. □

Recall that M is a dual-CS-Rickart module if and only if M is an M-dual-CS-Rickart module. Furthermore, we provide an example of F-dual-CS-Rickart modules which is not a CS-Rickart module in Example 4.1.6. So we are interested in studying when an F-dual-CS-Rickart module is a dual-CS-Rickart module, as well as, when a dual-CS-Rickart module is an F-dual-CS-Rickart module where $F \neq 0$. Relationships between F-CS-Rickart modules and CS-Rickart modules are provided in the following series of propositions.

Proposition 4.1.19. *If M is an F-dual-CS-Rickart module and $fM/fF \ll M/fF$ for all $f \in$ End(M), then M is a dual-CS-Rickart module.*

Proof. Assume that M is an F-dual-CS-Rickart module and, for any $f \in \text{End}(M)$, $f(M)/f(F) \ll M/f(F)$. Let $f \in \text{End}(M)$. Since M is an F-dual-CS-Rickart module, there is $e^2 = e \in \text{End}(M)$ such that $f(F) = eM \oplus (1-e)f(F)$ and $(1-e)f(F) \ll M$. It forces that $M = eM \oplus (1-e)M = f(F) + (1-e)M$. As $f(F) \subseteq f(M)$, we obtain that $eM \subseteq f(M)$ and $(1-e)f(M) = (1-e)M \cap f(M)$. Note that $M = f(F) + (1-e)M$ and $f(F) \subseteq f(M)$, applying Proposition 2.3.3, $\big(f(M) \cap (1-e)M\big)/\big(f(F) \cap (1-e)M\big) \ll M/\big(f(F) \cap (1-e)M\big)$. It follows that $(1-e)f(M)/(1-e)f(F) \ll M/(1-e)f(F)$. Since $(1-e)f(F) \ll M$, by Proposition 2.3.2, $(1-e)f(M) \ll M$. Therefore, M is a dual-CS-Rickart module. □

Proposition 4.1.20. *If M is a dual-CS-Rickart module and F lies above M' for some fully invariant direct summand M' of M, then M is an F-dual-CS-Rickart.*

Proof. Assume that M is a dual-CS-Rickart module and F lies above M' for some fully invariant direct summand M' of M. Then $M = M' \oplus N$ where $M' \subseteq F$ and $N \cap F \ll M$. Since $M' \leq^{\oplus} M$ and M is a dual-CS-Rickart module, M' is a dual-CS-Rickart module. As a consequence of the converse of Theorem 4.1.17, M is an F-dual-CS-Rickart module. □

Similar to F-CS-Rickart modules, the converse of Theorem 4.1.17, being fully invariant submodule of M' is a necessary condition to force M to be an F-dual-CS-Rickart module. So the images of F which lie above a fully invariant direct summand are investigated.

Definition 4.1.21. A module M is a *strongly F-dual-CS-Rickart module* if for any $f \in \text{End}(M)$, there is a fully invariant direct summand M' of M such that $f(F)$ lies above M'.

It is clear that strongly F-dual-CS-Rickart modules are F-CS-Rickart modules. Next, we consider when a direct summand of a strongly F-dual-CS-Rickart module is also a strongly F'-dual-CS-Rickart module for some fully invariant submodule F' of this direct summand.

Lemma 4.1.22. *Let M be a strongly F-dual-CS-Rickart module. Then N is a strongly $(N \cap F)$-dual-CS-Rickart module for any direct summand N of M.*

Proof. The proof is similar to one of Theorem 4.1.12. Let N be a direct summand of M and $N = eM$ for some $e^2 = e \in \text{End}(M)$. Let $f \in \text{End}(N)$. Then $fe \in \text{End}(M)$. Since M is a strongly F-dual-CS-Rickart module, there is a fully invariant direct summand M' of M such that $fe(F) = e'M \oplus \big((1-e')M \cap feF\big)$ and $(1-e')M \cap feF \ll M$ where $M' = e'M$ for some $(e')^2 = e' \in \text{End}(M)$. Note that both $e'M$ and $(1-e')M \cap feF$ contained in N. This forces that $e'M$ is a fully invariant direct summand of N and $(1-e')M \cap feF \ll N$. Thus $f(N \cap F)$ lies above the fully invariant direct summand $e'M$. □

Finally, in this section, we focus on the image of the identity endomorphism of F which is equal to F and lies above some direct summand of M. So each F-dual-CS-Rickart module can be written as a direct sum depending on F.

Theorem 4.1.23. *The following statements are equivalent.*

(i) M is a strongly F-dual-CS-Rickart module.

(ii) $M = N \oplus K$ where $N \subseteq F$, $N \leq_{fully} M$, $K \cap F \ll M$ and N is a strongly dual-CS-Rickart module.

(iii) M is an F-dual-CS-Rickart module and every direct summand of M contained in F is fully invariant.

(iv) $M = N \oplus K$ where $N \subseteq F$, $N \leq_{fully} M$, $K \cap F \ll M$ and, for any $f \in \mathrm{End}(M)$, $f(F) \cap N$ lies above a fully invariant direct summand of N .

Proof. (i)→(ii) Assume (i). Then $M = N \oplus K$ where $N \subseteq F$, $K \cap F \ll M$. Thus N is a strongly dual-CS-Rickart module by Lemma 4.1.22 because $N \leq^{\oplus} M$ and $N \cap F = N$.

(ii)→(i) The proof is similar to the proof of the converse of Theorem 4.1.17. Assume (ii). Thus $F = N \oplus (K \cap F)$ because $N \subseteq F$. Let $f \in \mathrm{End}(M)$. Then $f(F) = f(N \oplus (K \cap F)) = f(N) + f(K \cap F)$ and $f(K \cap F) \ll M$ by Proposition 2.3.6. Since $N \leq_{fully} M$, we obtain $f_{|N} \in \mathrm{End}(N)$ so that $f_{|N}(N) = f(N)$. As N is a strongly dual-CS-Rickart module, $f_{|N}(N) = N_1 \oplus \big(N_2 \cap f(N)\big)$ and $N_2 \cap f(N) \ll N$ where $N = N_1 \oplus N_2$ and N_1 is a fully invariant submodule of N. Thus

$$f(F) = f(N) + f(K \cap F) = N_1 + \Big(\big(N_2 \cap f(N)\big) + f(K \cap F)\Big)$$

where $N_1 \leq^{\oplus} M$ and $N_1 \leq_{fully} M$ and $\big(N_2 \cap f(N)\big) + f(K \cap F) \ll M$. Hence $f(F)$ lies above N_1. Therefore, M is a strongly F-dual-CS-Rickart module.

(i)→(iii) Assume (i). Then M is an F-dual-CS-Rickart module. Next, let L be a direct summand of M and $L \subseteq F$. Then there is $e^2 = e \in \mathrm{End}(M)$ such that $L = eM$, so that $L = L \cap F = eM \cap F = eF$ because $F \leq_{fully} M$. Since M is a strongly F-dual-CS-Rickart module, $eF = N \oplus \big(K \cap e(F)\big)$ and $K \cap e(F) \ll M$ where N is a fully invariant direct summand of M and K is a submodule of M.

Since $K \cap e(F) \leq^{\oplus} eF = L$ and $L \leq^{\oplus} M$, we obtain that $K \cap e(F) \leq^{\oplus} M$. Thus $K \cap e(F) = 0$ because $K \cap e(F)$ is both a small submodule and a direct summand of M. Therefore, $N = eF = L$ which is a fully invariant direct summand of M.

(iii)$\rightarrow$ (i) Assume (iii). Let $f \in \text{End}(M)$. Then $f(F) \subseteq F$ and $f(F)$ lies above M' for some direct summand M' of M. By assumption, $M' \leq_{fully} M$. Therefore, M is a strongly F-dual-CS-Rickart module.

(ii)$\rightarrow$(iv) Assume (ii) So $F = N \oplus (K \cap F)$. Let $f \in \text{End}(M)$. Thus $f|_N \in \text{End}(N)$ and $f(F) \cap N = f|_N(N \cap F) = f|_N(N)$ because $N \leq_{fully} M$ and $N \subseteq F$. Since N is a strongly dual CS-Rickart module, $f|_N(N)$ lies above a fully invariant direct summand of N. Thus $f(F) \cap N$ lies above a fully invariant direct summand of N.

(iv)$\rightarrow$ (ii) Assume (iv). Thus $F = N \oplus (K \cap F)$. Let $g \in \text{End}(N)$. Then $g \oplus 0_K \in \text{End}(M)$. Hence $(g \oplus 0_K)(F) \cap N = (g \oplus 0_K)(N \oplus (K \cap F)) \cap N = (g(N) + 0_K(K \cap F)) \cap N = g(N) \cap N = g(N \cap F)$. By assumption, $(g \oplus 0_N)(F) \cap N$ lies above a fully invariant direct summand of N. This implies that $g(N \cap F)$ lies above a fully invariant direct summand of N. Therefore, N is a strongly dual-CS-Rickart module. $\square$

4.2 Relatively F-Dual-CS-Rickart Modules

In this section, we provide a notion of relatively F-dual-CS-Rickart modules which is generalized form F-dual-CS-Rickart modules by extended $\text{End}(M)$ to $\text{Hom}(P, M)$ where P and M are modules and M is not necessary an F-dual-CS-Rickart module. Furthermore, a direct summand of relatively F-dual-CS-Rickart modules be a relatively F-dual-CS-Rickart module are proved.

Definition 4.2.1. Let P, M be modules and F be a fully invariant submodule of P. Then P is an *F-dual-CS-Rickart module relative to M (relatively F-dual-CS-Rickart module)* if for any $f \in \text{Hom}(P, M)$, there is a direct summand M' of M such that $f(F)$ lies above M'.

It is clear that M is an F-dual-CS-Rickart module if and only if M is an

F-dual-CS-Rickart module relative to M; moreover, P is a P-dual-CS-Rickart module relative to M if and only if P is a dual-CS-Rickart module relative to M given in [1]. Equivalent to Theorem 4.1.12, we examine direct summands of relatively F-dual-CS-Rickart modules.

Theorem 4.2.2. *Let P, M be modules and F be a fully invariant submodule of P. Then P is an F-dual-CS-Rickart module relative to M if and only if for any direct summand P_1 of P and any direct summand M_1 of M, P_1 is an $(P_1 \cap F)$-dual-CS-Rickart module relative to M_1.*

Proof. The sufficiency is obvious because P and M are direct summands of itself.

Assume that P is an F-dual-CS-Rickart module relative to M. Let P_1 and M_1 be direct summands of P and M, respectively. Then $P_1 \oplus P_2 = P$ for some submodule P_2 of P. Let $g \in \text{Hom}(P_1, M_1)$. Then $f := g \oplus 0_{P_2} \in \text{Hom}(P, M)$. Since $F \leq_{fully} M$, it follows that $F = (P_1 \cap F) \oplus (P_2 \cap F)$. So $f(F) = \big(g \oplus 0_{P_2}\big)\big((P_1 \cap F) \oplus (P_2 \cap F)\big) = g(P_1 \cap F) \subseteq M_1$. Since P is an F-CS-Rickart module relative to M, $f(F) = eM \oplus (1-e)f(F)$ and $(1-e)f(F) \ll M$ for some $e^2 = e \in \text{End}(M)$. Since $f(F) \subseteq M_1$, we obtain $eM \leq^{\oplus} M_1$ and $(1-e)f(F) \ll M_1$. Thus $g(P_1 \cap F)$ lies above eM. Therefore, P_1 is an $(P_1 \cap F)$-dual-CS-Rickart module relative to M_1. $\square$

If $P = M$ in Theorem 4.2.2, the following corollary is obtained.

Corollary 4.2.3. *The following statements are equivalent.*
(i) M is an F-dual-CS-Rickart module.
(ii) For any direct summands N and K of M, N is an $(N \cap F)$-dual-CS-Rickart module relative to K.
(iii) For any direct summands N and K of M, for any $f \in \text{End}(M)$ there is a direct summand K' of K such that $f|_N(N \cap F)$ lies above K'.

Proof. (i) $\leftrightarrow$ (ii) This follows from Theorem 4.2.2 because M is an F-dual-CS-Rickart module relative to M.

(ii) $\rightarrow$ (iii) Assume (ii). Let N and K be direct summands M and $f \in \text{Hom}(M, K)$. Then $f|_N \in \text{Hom}(N, K)$. So $f|_N^{-1}(N \cap F) \leq_{ess} K'$ for some direct summand K' of K by the definition of relatively F-CS-Rickart modules.

(iii) $\rightarrow$ (i) This is clear because $N = M = K$. □