CHAPTER I

INTRODUCTION

Apart from the heat equations there is another type of equations that has many properties similar to the classical heat equation, the so-called nonlocal diffusion equations. A nonlocal equation is an equation of the form

$$\partial_t u(x,t) = \int_{\mathbb{R}^n} J(x,y)(u(y,t) - u(x,t))dy + F\left(u(x,t)\right) \tag{1.1}$$

with a given function $F : \mathbb{R} \to \mathbb{R}$ and the kernel $J : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$. These equations play a crucial role to model various phenomena such as protron scattering, elastic scattering, cell dispersal, population dispersal and diffusion processes, see [1], [3], [6], [9], [12], [13]. In the context of population dispersal problems, the author in [6] described the physical meaning of the homogeneous equation (1.1) concretely as follows. For the homogeneous dispersal phenomenon, it is assumed that $J(x,y) = J(x-y)$, and J is radially symmetric. The terms

- $u(x,t)$ represents the density of population at location x and time $t > 0$;
- $J(x,y) = J(x-y)$ is the probability distribution for population jumping from location y into x;
- $\int_{\mathbb{R}^n} J(x-y)u(y,t)dy$ is the rate such that population in the whole space arrives in the location x;
- $\int_{\mathbb{R}^n} J(x-y)u(x,t)dy$ is the rate such that the population at x jumps to all other locations;
- $F(u(x,t))$ represents the supplied/depletion of resources;

When J is radially symmetric and $f = 0$, the global well-posedness of solutions to the nonlocal equation (1.1) was proved by Chasseigne et al. [2] in 2006, under a broad assumption on the expansion of the kernel J in the Fourier variable. The authors were able to employ the Fourier transform technique owning to the radially symmetric of the kernel. The asymptotic behavior of solutions was also established by the Fourier splitting argument.

In 2007, Ignat and Rossi [10] investigated the existence and uniqueness of solutions for the Cauchy problem (1.1) with initial data in $L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and $F(u)$ is a nonlocal convection term of the form $G * f(u) - f(u)$. In addition to assuming the radial symmetry of J and G, they imposed that these kernel functions are rapidly decreasing. They began by analysing the linear problem, i.e. $G = 0$, using the Fourier transform technique. For the nonlinear problem, they applied the contraction mapping principle.

In 2010, Jorge and Fernando [8] investigated Eq. (1.1) when J is radially symmetric with $\|J\|_{L^1} = 1$ and $F(u) = u^p$ $(p > 1)$. After establishing the property of the linear semigroup $e^{\mathcal{A}t}$, where

$$\mathcal{A}u := J * u - u,$$

they proved the local existence of solution for Eq.(1.1) for all $p > 1$. Then using the standard first eigenvalue technique and the contraction mapping principle, they could obtain the Fujita critical exponent of (1.1) to be $p = 1 + 2/n$, which is the same as the heat equation.

In this study, we investigate the Cauchy problem (1.1) with non-radially symmetric J and the nonlinear term is inhomogeneous:

$$\begin{cases} \partial_t u = \displaystyle\int_{\mathbb{R}^n} J(x,y)u(y,t)dy - u(x,t) + (\ln(e + |x|^2))^\sigma u^p(x,t) & \forall x \in \mathbb{R}^n, t > 0 \\ u(x,0) = u_0(x) & \forall x \in \mathbb{R}^n. \end{cases} \tag{1.2}$$

Here. $\sigma > 0$ and $p > 0$.

Throughout this work, the kernel $J : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ is assumed to satisfy:

(H1) $\sup_{x \in \mathbb{R}^n} \int_{\mathbb{R}^n} J(x, y)dy \leq 1$ and $\sup_{y \in \mathbb{R}^n} \int_{\mathbb{R}^n} J(x, y)dx \leq 1$,

(H2) $J(x, y)$ is continuous in x,

(H3) there exists $R > 0$ such that for $x, y \in \mathbb{R}^n$ if $|x - y| > R$ then $J(x, y) = 0$.

(H4) there exists $g \in L^1(\mathbb{R}^n)$ such that $\sup_{x \in \mathbb{R}^n} J(x, y) \leq g(y)$ for all $y \in \mathbb{R}^n$.

The goal of this study is to establish the existence and uniqueness of solutions in weighted Lebesgue spaces for the Cauchy problem (1.2). The definition of weighted Lebesgue spaces $L_C^{\infty,\rho}(\mathbb{R}^n)$ is given in Chapter 2. In the last section of Chapter 2, we recall the semigroup of bounded linear operators on a Banach space. In Chapter 3, we introduce the nonlocal operator associated with Eq. (1.2) and prove basic properties of this operator including the linearity of any k-fold iterations and their boundedness on weighted Lebesgue spaces. Then in Chapter 4, we established the boundedness of the Green operator of the nonlocal operator introduced in the previous chapter. Finally, by employing by the Banach contraction mapping principle, together with the boundedness of the Green operator, we can prove the existence and uniqueness of local solutions to the nonlinear problem.