CHAPTER III
NONLOCAL OPERATOR

In this chapter, we let $J(x, y)$ satisfy (H1)-(H4) and investigate the linear nonlocal equation. We prove an estimate involving power weight functions. Then we transform the Cauchy problem of the homogeneous equation into an easy one. Specifically, we study the problem

$$\begin{cases} \partial_t u = \displaystyle\int_{\mathbb{R}^n} J(x,y)u(y,t)dy - u(x,t) & \forall x \in \mathbb{R}^n, t > 0 \\ u(x,0) = u_0(x) & \forall x \in \mathbb{R}^n \end{cases} \tag{3.1}$$

where $u_0 \in L_C^{\infty,\rho}(\mathbb{R}^n)$.

We begin with the following lemma.

Lemma 3.1. *Let $x, y \in \mathbb{R}^n$ and $R > 0$. If $|x - y| \leq R$ then*

$$\frac{e + |y|^2}{e + |x|^2} < 1 + \frac{2R + R^2}{\sqrt{e}}.$$

Proof. Let $x, y \in \mathbb{R}^n$ be such that $|x - y| \leq R$.

Case 1 $|y| \leq \frac{|x|}{2}$:

$$\frac{e + |y|^2}{e + |x|^2} \leq \frac{e + \frac{|x|^2}{4}}{e + |x|^2} \leq \frac{1}{4}\left(1 + \frac{3e}{e}\right) = 1 \leq 1 + \frac{2R + R^2}{\sqrt{e}}.$$

Case 2 $|y| > \frac{|x|}{2}$:

Subcase 2.1 $|x| = 0$:

$$\frac{e + |y|^2}{e + |x|^2} = \frac{e + |y|^2}{e} \leq 1 + \frac{(|x| + R)^2}{e} = 1 + \frac{R^2}{e} \leq 1 + \frac{2R + R^2}{\sqrt{e}}.$$

Subcase 2.2 $|x| \neq 0$;

$$\begin{aligned}
\frac{e+|y|^2}{e+|x|^2} &= \left(\frac{e+|y|^2}{e+|x|^2} - 1\right) + 1 \\
&= 1 + \frac{(|y|-|x|)(|y|+|x|)}{e+|x|^2} \\
&\leq 1 + \frac{R(|y|+|x|)}{e+|x|^2} \\
&\leq 1 + \frac{R(2|x|+R)}{e+|x|^2} \\
&= 1 + \frac{2|x|R}{e+|x|^2} + \frac{R^2}{e+|x|^2} \\
&\leq 1 + \frac{2|x|R}{\sqrt{e}|x|} + \frac{R^2}{e} \\
&= 1 + \frac{2R+R^2}{\sqrt{e}}.
\end{aligned}$$

Hence, we obtain the result that $\frac{e+|y|^2}{e+|x|^2} \leq 1 + \frac{2R+R^2}{\sqrt{e}}$. □

Lemma 3.2. *Let $x, y \in \mathbb{R}^n$ and $R > 0$. If $|x-y| \leq R$ then*

$$\frac{\ln(e+|x|^2)}{\ln(e+|y|^2)} \leq 1 + \ln\left(1 + \frac{2R+R^2}{\sqrt{e}}\right).$$

Proof. Let $x, y \in \mathbb{R}^n$ be such that $|x-y| \leq R$. Then

$$\begin{aligned}
\frac{\ln\left(e+|x|^2\right)}{\ln\left(e+|y|^2\right)} &= 1 + \left(\frac{\ln\left(e+|x|^2\right)}{\ln\left(e+|y|^2\right)} - 1\right) \\
&= 1 + \frac{\ln\left(\frac{e+|x|^2}{e+|y|^2}\right)}{\ln(e+|y|^2)} \\
&\leq 1 + \frac{\ln\left(1 + \frac{2R+R^2}{\sqrt{e}}\right)}{\ln(e)} \\
&= 1 + \ln\left(1 + \frac{2R+R^2}{\sqrt{e}}\right).
\end{aligned}$$

So we have the conclusion as desired. □

Proposition 3.3. *Let $x, y \in \mathbb{R}^n$, $b \in \mathbb{R}$ and $R > 0$. If $|x - y| \leq R$, then*

$$\frac{\rho(x)}{\rho(y)} \leq \left(1 + \ln\left(1 + \frac{2R + R^2}{\sqrt{e}}\right)\right)^{|b|} =: \lambda(R, b).$$

Proof. Let $x, y \in \mathbb{R}^n$ and $b \in \mathbb{R}$. Assume that $|x - y| \leq R$. By Lemma 3.2, we have that

$$\frac{1}{1 + \ln\left(1 + \frac{2R+R^2}{\sqrt{e}}\right)} \leq \frac{\ln(e + |x|^2)}{\ln(e + |y|^2)} \leq 1 + \ln\left(1 + \frac{2R + R^2}{\sqrt{e}}\right).$$

If $b \geq 0$, then

$$\begin{aligned}\frac{\rho(x)}{\rho(y)} &= \left(\frac{\ln(e + |x|^2)}{\ln(e + |y|^2)}\right)^b \\ &\leq \left(1 + \ln\left(1 + \frac{2R + R^2}{\sqrt{e}}\right)\right)^b.\end{aligned}$$

If $b < 0$, then

$$\begin{aligned}\frac{\rho(x)}{\rho(y)} &= \left(\frac{\ln(e + |y|^2)}{\ln(e + |x|^2)}\right)^{-b} \\ &\leq \left(1 + \ln\left(1 + \frac{2R + R^2}{\sqrt{e}}\right)\right)^{-b}.\end{aligned}$$

Hence, we have that for any $b \in \mathbb{R}$,

$$\frac{\rho(x)}{\rho(y)} \leq \left(1 + \ln\left(1 + \frac{2R + R^2}{\sqrt{e}}\right)\right)^{|b|} =: \lambda(R, b).$$ □

Now, we define a *nonlocal operator* $\mathfrak{J}$ on $L_C^{\infty,\rho}(\mathbb{R}^n)$ by

$$\mathfrak{J}\psi(x) = \int_{\mathbb{R}^n} J(x, y)\psi(y)dy \qquad \forall \psi \in L_C^{\infty,\rho}(\mathbb{R}^n).$$

Proposition 3.4. *$\mathfrak{J}$ is a self-map on $L_C^{\infty,\rho}(\mathbb{R}^n)$.*

Proof. Let $\psi \in L_C^{\infty,\rho}(\mathbb{R}^n)$. First, we will verify that $\mathfrak{J}\psi$ is continuous. Let $x \in \mathbb{R}^n$ and (x_k) be a sequence converging to x. Since

$$\begin{aligned}
&J(x_k, y)\psi(y) \to J(x, y)\psi(y) && \text{as } n \to \infty,\\
&|J(x_k, y)\psi(y)| \leq \|\psi\|_\infty J(x_k, y) && \text{for all } n \in \mathbb{N},\\
&\|\psi\|_\infty J(x_k, y) \to \|\psi\|_\infty J(x, y) && \text{as } n \to \infty,\\
&J(x_k, y) \leq g(y) && \text{for all } k \in \mathbb{N},
\end{aligned}$$

by the Generalized Dominated Convergence Theorem and the assumptions of J, we have

$$\begin{aligned}
\lim_{k\to\infty} \mathfrak{J}\psi(x_k) &= \lim_{k\to\infty} \int_{\mathbb{R}^n} J(x_k, y)\psi(y)dy\\
&= \int_{\mathbb{R}^n} \lim_{k\to\infty} J(x_k, y)\psi(y)dy\\
&= \int_{\mathbb{R}^n} J(x, y)\psi(y)dy\\
&= \mathfrak{J}\psi(x).
\end{aligned}$$

This conclude $\mathfrak{J}\psi$ is continuous on $\mathbb{R}^n$.

Finally, we prove that $\|\mathfrak{J}\psi\|_\rho < \infty$.

Consider,

$$\begin{aligned}
\sup_{x\in\mathbb{R}^n} |\rho(x)\mathfrak{J}\psi(x)| &= \sup_{x\in\mathbb{R}^n} \left|\int_{\mathbb{R}^n} \rho(x)J(x,y)\psi(y)dy\right| \\
&\le \sup_{x\in\mathbb{R}^n} \int_{\mathbb{R}^n} \rho(x)J(x,y)\,|\psi(y)|\,dy \\
&= \sup_{x\in\mathbb{R}^n} \int_{\overline{B(x,R)}} \rho(x)J(x,y)\,|\psi(y)|\,dy \\
&= \sup_{x\in\mathbb{R}^n} \int_{\overline{B(x;R)}} \rho(x)J(x,y)\left|\frac{\psi(y)\rho(y)}{\rho(y)}\right| dy \\
&\le \|\psi\|_\rho \sup_{x\in\mathbb{R}^n} \int_{\overline{B(x;R)}} \frac{\rho(x)}{\rho(y)} J(x,y)dy \\
&\le \|\psi\|_\rho \lambda(R,b) \sup_{x\in\mathbb{R}^n} \int_{\overline{B(x;R)}} J(x,y)dy \\
&\le \|\psi\|_\rho \lambda(R,b).
\end{aligned}$$

So $\|\mathfrak{J}\psi\|_\rho = \sup_{x\in\mathbb{R}^n} |\rho(x)\mathfrak{J}\psi(x)| < \infty$ which means $\mathfrak{J}\psi \in L_C^{\infty,\rho}(\mathbb{R}^n)$ and hence $\mathfrak{J}$ is a map from $L_C^{\infty,\rho}(\mathbb{R}^n)$ into $L_C^{\infty,\rho}(\mathbb{R}^n)$. □

In addition, we demonstrate that $\mathfrak{J}$ and the self-compositions of $\mathfrak{J}$ are bounded linear operators on $L_C^{\infty,\rho}(\mathbb{R}^n)$.

Proposition 3.5. *$\mathfrak{J}$ is a bounded linear operator on $L_C^{\infty,\rho}(\mathbb{R}^n)$.*

Proof. Let u and v be in $L_C^{\infty,\rho}(\mathbb{R}^n)$ and $c \in \mathbb{R}$. To show $\mathfrak{J}(cu+v) = c\mathfrak{J}u + \mathfrak{J}v$, let $x \in \mathbb{R}^n$. We have

$$\begin{aligned}
\mathfrak{J}\,(cu+v)\,(x) &= \int_{\mathbb{R}^n} J(x,y)\,(cu(y)+v(y))\,dy \\
&= \int_{\mathbb{R}^n} J(x,y)(cu(y)) + J(x,y)v(y)dy \\
&= c\int_{\mathbb{R}^n} J(x,y)u(y)dy + \int_{\mathbb{R}^n} J(x,y)v(y)dy \\
&= c\mathfrak{J}u(x) + \mathfrak{J}v(x) \\
&= c\mathfrak{J}u + \mathfrak{J}v(x).
\end{aligned}$$

Hence $\mathfrak{J}(cu+v) = c\mathfrak{J}u + \mathfrak{J}v$ which means $\mathfrak{J}$ is linear. The boundedness part follows from Theorem 3.4. ☐

Convention 3.6. $\mathfrak{J}^k := \underbrace{\mathfrak{J} \circ \mathfrak{J} \circ \ldots \circ \mathfrak{J}}_{k \text{ copies}} = \mathfrak{J} \circ \mathfrak{J}^{k-1}$ and $\mathfrak{J}^0 = I$.

Theorem 3.7. *For each $k \in \mathbb{N}$, $\mathfrak{J}^k$ is a bounded linear operator on the weight Lebesgue space $L_C^{\infty,\rho}(\mathbb{R}^n)$. Moreover, $\|\mathfrak{J}^k\| \leq (\ln(e + et^2))^{|b|}\lambda(R, b, t)^k$.*

To prove Theorem 3.7, we need some estimation of the ratio involving logarithmic functions.

For $t \geq 0$, define $\rho_t : \mathbb{R}^n \to \mathbb{R}$ by

$$\rho_t(x) = \left(\ln(e + t^2 + |x|^2)\right)^b \qquad \forall x \in \mathbb{R}^n.$$

Proposition 3.8. *Let $x, y \in \mathbb{R}^n$, $t > 0$ and R be a fixed positive real number. If $|x - y| \leq R$ then*

$$\frac{\rho_t(x)}{\rho_t(y)} \leq \lambda(R, b, t),$$

where $\lambda(R, b, t) := \left(1 + \ln\left(1 + \frac{2R + R^2}{\sqrt{e + t^2}}\right)\right)^{|b|}$.

Proof. The proof is the same as that of Proposition 3.3 by simply replacing e with $e + t^2$. ☐

Lemma 3.9. *Let $t > 0$ and $b \in \mathbb{R}$. Then*

$$\int_{\mathbb{R}^n} J(x, y)\rho_t(y)dy \leq \lambda(R, b, t)\rho_t(x) \quad \forall x \in \mathbb{R}^n.$$

Proof. Let $t > 0$ and $x \in \mathbb{R}^n$. Consider

$$
\begin{aligned}
&\int_{\mathbb{R}^n} J(x,y)\big(\ln(e+t^2+|y|^2)\big)^b dy \\
&\quad = \int_{\overline{B(x;R)}} J(x,y)\big(\ln(e+t^2+|y|^2)\big)^b \left(\frac{\ln(e+t^2+|x|^2)}{\ln(e+t^2+|x|^2)}\right)^b dy \\
&\quad \le \big(\ln(e+t^2+|x|)\big)^b \int_{\overline{B(x;R)}} J(x,y)\left(\frac{\ln(e+t^2+|y|^2)}{\ln(e+t^2+|x|^2)}\right)^b dy \\
&\quad \le \big(\ln(e+t^2+|x|)\big)^b \int_{\overline{B(x;R)}} J(x,y)\left(1+\ln\left(1+\frac{2R+R^2}{\sqrt{e+t^2}}\right)\right)^{|b|} dy \\
&\quad \le \big(\ln(e+t^2+|x|)\big)^b \left(1+\ln\left(1+\frac{2R+R^2}{\sqrt{e+t^2}}\right)\right)^{|b|} \\
&\quad = \lambda(R,b,t)\big(\ln(e+t^2+|x|)\big)^b.
\end{aligned}
$$

The proof is completed. □

Lemma 3.10. *For all $x \in \mathbb{R}^n$ and for all $t > 0$.*

$$
\frac{\rho(x)}{\ln(e+et^2)^{|b|}} \le \rho_t(x) \le \ln(e+et^2)^{|b|}\rho(x)
$$

Proof. Let $x \in \mathbb{R}^n$ and $t > 0$. If $b \ge 0$, then we have

$$
\begin{aligned}
\frac{\rho_t(x)}{\rho(x)} &\le \left(\frac{\ln\left((1+t^2)(e+|x|^2)\right)}{\ln(e+|x|^2)}\right)^b \\
&= \left(1+\frac{\ln(1+t^2)}{\ln(e+|x|^2)}\right)^b \\
&\le \left(1+\frac{\ln(1+t^2)}{\ln(e)}\right)^b \\
&= \big(\ln(e+et^2)\big)^b.
\end{aligned}
$$

If $b < 0$, then and we have

$$\begin{aligned}\frac{\rho_t(x)}{\rho(x)} &= \left(\frac{\ln(e+|x|^2)}{\ln(e+t^2+|x|^2)}\right)^{-b} \qquad\qquad (-b>0)\\ &\le \left(\frac{\ln(e+t^2+|x|^2)}{\ln(e+|x|^2)}\right)^{-b}\\ &\le \left(\ln(e+et^2)\right)^{-b}.\end{aligned}$$

Thus, we have $\rho_t(x) \le \left(\ln(e+et^2)\right)^{|b|}\rho(x)$ for all $x \in \mathbb{R}^n$ and all $t > 0$. By the same argument, it is easy to show the another side of inequality is true. □

Corollary 3.11. *For all $x \in \mathbb{R}^n$, $\displaystyle\int_{\mathbb{R}^n} J(x,y)\frac{1}{\rho_t(y)}dy \le \lambda(R,b,t)\frac{1}{\rho_t(x)}$.*

Proof. Let $x \in \mathbb{R}^n$ be arbitrary. We have

$$\begin{aligned}\int_{\mathbb{R}^n} J(x,y)\frac{1}{\rho_t(y)}dy &= \int_{\overline{B(x,R)}} J(x,y)\frac{1}{\rho_t(y)}\frac{\rho_t(x)}{\rho_t(x)}dy\\ &\le \int_{\overline{B(x,R)}} J(x,y)\frac{\lambda(R,b,t)}{\rho_t(x)}dy\\ &\le \int_{\overline{B(x,R)}} J(x,y)\frac{\lambda(R,b,t)}{\rho_t(x)}dy\\ &\le \frac{\lambda(R,b,t)}{\rho_t(x)}\end{aligned}$$

The proof is complete. □

Poof of Theorem 3.7. Let $k \in \mathbb{N}, \psi \in L_C^{\infty,\rho}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$. By mathematical induction, we can show that $\mathfrak{J}^k$ is a linear map. Now, we notice that

$$\begin{aligned}
\mathfrak{J}^k\psi(x) &= \int_{\mathbb{R}^n} J(x,x_1)\mathfrak{J}^{k-1}\psi(x_1)dx_1 \\
&= \int_{\mathbb{R}^n} J(x,x_1)\int_{\mathbb{R}^n} J(x_1,x_2)\mathfrak{J}^{k-2}\psi(x_2)dx_2dx_1 \\
&\vdots \\
&= \int_{\mathbb{R}^n} J(x,x_1)\int_{\mathbb{R}^n} J(x_1,x_2)\dots\int_{\mathbb{R}^n} J(x_{k-2},x_{k-1})\mathfrak{J}\psi(x_{k-1})dx_{k-1}\dots dx_1 \\
&= \int_{\mathbb{R}^n} J(x,x_1)\int_{\mathbb{R}^n} J(x_1,x_2)\dots\int_{\mathbb{R}^n} J(x_{k-1},x_k)\psi(x_k)dx_k\dots dx_1
\end{aligned}$$

We estimate the composition of $\mathfrak{J}$.

If $b \geq 0$ then

$$\begin{aligned}
\left|\rho(x)\mathfrak{J}^k\psi(x)\right| &= \left|\rho(x)\int_{\mathbb{R}^n} J(x,x_1)\dots\int_{\mathbb{R}^n} J(x_{k-1},x_k)\psi(x_k)dx_k\dots dx_1\right| \\
&\leq \rho(x)\int_{\mathbb{R}^n} J(x,x_1)\dots\int_{\mathbb{R}^n} J(x_{k-1},x_k)\left|\psi(x_k)\frac{\rho_t(x_k)}{\rho_t(x_k)}\right| dx_k\dots dx_1 \\
&\leq \rho(x)\int_{\mathbb{R}^n} J(x,x_1)\dots\int_{\mathbb{R}^n} J(x_{k-1},x_k)\left|\psi(x_k)\frac{(\ln(e+et^2))^{|b|}\rho(x_k)}{\rho_t(x_k)}\right| dx_k\dots dx_1 \\
&\leq \|\psi\|_\rho \rho(x)\int_{\mathbb{R}^n} J(x,x_1)\dots\int_{\mathbb{R}^n} J(x_{k-1},x_k)\frac{(\ln(e+et^2))^{|b|}}{\rho_t(x_k)}dx_k\dots dx_1 \\
&\leq \left(\ln(e+et^2)\right)^{|b|}\|\psi\|_\rho \rho(x)\int_{\mathbb{R}^n} J(x,x_1)\dots\int_{\mathbb{R}^n} J(x_{k-2},x_{k-1})\frac{\lambda(R,b,t)}{\rho_t(x_{k-1})}dx_{k-1}\dots dx_1 \\
&\leq \left(\ln(e+et^2)\right)^{|b|}\|\psi\|_\rho \rho(x)\int_{\mathbb{R}^n} J(x,x_1)\dots\int_{\mathbb{R}^n} J(x_{k-3},x_{k-2})\frac{\lambda(R,b,t)^2}{\rho_t(x_{k-2})}dx_{k-2}\dots dx_1 \\
&\vdots \\
&\leq \left(\ln(e+et^2)\right)^{|b|}\|\psi\|_\rho \rho(x)\int_{\mathbb{R}^n} J(x,x_1)\frac{\lambda(R,b,t)^{k-1}}{\rho_t(x_1)}dx_1 \\
&\leq \left(\ln(e+et^2)\right)^{|b|}\lambda(R,b,t)^{k-1}\|\psi\|_\rho \int_{\mathbb{R}^n} J(x,x_1)\frac{\rho_t(x)}{\rho_t(x_1)}dx_1 \\
&\leq \left(\ln(e+et^2)\right)^{|b|}\lambda(R,b,t)^{k-1}\|\psi\|_\rho \int_{\mathbb{R}^n} J(x,x_1)\lambda(R,b,t)dx_1 \\
&\leq \left(\ln(e+et^2)\right)^{|b|}\lambda(R,b,t)^{k}\|\psi\|_\rho .
\end{aligned}$$

If $b < 0$ then

$$
\begin{aligned}
\left|\rho(x)\mathfrak{J}^k\psi(x)\right| &= \left|\rho(x)\int_{\mathbb{R}^n} J(x,x_1)\dots\int_{\mathbb{R}^n} J(x_{k-1},x_k)\psi(x_k)dx_k\dots dx_1\right| \\
&\leq \rho(x)\int_{\mathbb{R}^n} J(x,x_1)\dots\int_{\mathbb{R}^n} J(x_{k-1},x_k)\left|\psi(x_k)\frac{\rho_t(x_k)}{\rho_t(x_k)}\right| dx_k\dots dx_1 \\
&\leq \rho(x)\int_{\mathbb{R}^n} J(x,x_1)\dots\int_{\mathbb{R}^n} J(x_{k-1},x_k)\left|\psi(x_k)\frac{\rho(x_k)}{\rho_t(x_k)}\right| dx_k\dots dx_1 \\
&\leq \|\psi\|_\rho \rho(x)\int_{\mathbb{R}^n} J(x,x_1)\dots\int_{\mathbb{R}^n} J(x_{k-1},x_k)\frac{1}{\rho_t(x_k)}dx_k\dots dx_1 \\
&\leq \left(\ln(e+et^2)\right)^{|b|}\|\psi\|_\rho \rho(x)\int_{\mathbb{R}^n} J(x,x_1)\dots\int_{\mathbb{R}^n} J(x_{k-2},x_{k-1})\frac{\lambda(R,b,t)}{\rho_t(x_{k-1})}dx_{k-1}\dots dx_1 \\
&\ \vdots \\
&\leq \|\psi\|_\rho \rho(x)\int_{\mathbb{R}^n} J(x,x_1)\frac{\lambda(R,b,t)^{k-1}}{\rho_t(x_1)}dx_1 \\
&\leq \lambda(R,b,t)^{k-1}\|\psi\|_\rho \rho(x)\int_{\mathbb{R}^n} J(x,x_1)\frac{1}{\rho_t(x_1)}dx_1 \\
&\leq \lambda(R,b,t)^{k}\|\psi\|_\rho \frac{\rho(x)}{\rho_t(x)} \\
&\leq \left(\ln(e+et^2)\right)^{|b|}\lambda(R,b,t)^{k}\|\psi\|_\rho .
\end{aligned}
$$

Hence, $\left\|\mathfrak{J}^k\right\|$is bounded for all $k \in \mathbb{N}$.

Moreover we also have $\left\|\mathfrak{J}^k\right\| \leq \left(\ln(e+et^2)\right)^{|b|}\lambda(R,b,t)^k$. □