CHAPTER IV

GREEN OPERATOR

In this chapter, we study the linear problem (3.1). We introduce the Green operator associated with the Cauchy problem. Then we prove the boundedness of the Green operator on weighted Lebesgue spaces with a logarithmic weight.

Let $u(\cdot, t) \in L_C^{\infty,\rho}(\mathbb{R}^n)$ be a solution of

$$\partial_t u = \int_{\mathbb{R}^n} J(x, y)u(y, t)dy - u(x, t). \tag{4.1}$$

By setting $v(\cdot, t) = e^t u(\cdot, t)$, $v(\cdot, t)$ is a solution of

$$v_t(x, t) = \int_{\mathbb{R}^n} J(x, y)v(y, t)dy = \mathfrak{J}v(x, t) \quad x \in \mathbb{R}^n, t \geq 0 \tag{4.2}$$

So it suffices to study (4.2) to get the solution of (4.1) in the form

$$u(\cdot, t) = e^{-t}v(\cdot, t).$$

Definition 4.1. Let $T > 0$ and $u_0 \in L_C^{\infty,\rho}(\mathbb{R}^n)$ for some $b \in \mathbb{R}$. A function $v : \mathbb{R}^n \times [0, T) \to \mathbb{R}$ is said to be a *solution* of the homogeneous Cauchy problem

$$\begin{cases} v_t(x, t) = \displaystyle\int_{\mathbb{R}^n} J(x, y)v(y, t)dy & x \in \mathbb{R}^n, 0 \leq t \leq T, \\ v(x, 0) = u_0(x) & x \in \mathbb{R}^n. \end{cases} \tag{4.3}$$

if $v \in C\left([0, T)\,; L_C^{\infty,\rho}(\mathbb{R}^n)\right)$ and it satisfies the equation

$$v(x, t) = u_0(x) + \int_0^t \mathfrak{J}v(x, s)ds \qquad \text{at each } x \in \mathbb{R}^n \text{ and } 0 \leq t < T. \tag{4.4}$$

Lemma 4.2. *Assume* $u_0 \in L_C^{\infty,\rho}(\mathbb{R}^n)$. *Then there exists* $v \in C\big([0,\infty); L_C^{\infty,\rho}(\mathbb{R}^n)\big)$ *solving the Cauchy problem* (4.3) *for all* $t \geq 0$. *Moreover,*

$$\|v(\cdot,t)\|_\rho \leq (\ln(e+et^2))^{|b|} e^{t\lambda(R,b,t)} \|u_0\|_\rho \text{ for all } t \geq 0.$$

Proof. Let $t \in [0,\infty)$. We shall prove that there exists a function satisfying

$$v(\cdot,t) = u_0 + \int_0^t \mathfrak{J} v(\cdot,s)ds. \tag{4.5}$$

Define a sequence $(v_k(\cdot,t))_{k\in\mathbb{N}}$ by $v_0 = u_0$ and

$$v_k(\cdot,t) = u_0 + \int_0^t \mathfrak{J} v_{k-1} ds = \sum_{j=0}^{k} \frac{t^j}{j!}\mathfrak{J}^j v_0 \qquad \forall k \geq 1.$$

Since $u_0 \in L_C^{\infty,\rho}(\mathbb{R}^n)$ and $L_C^{\infty,\rho}(\mathbb{R}^n)$ is a vector space, we have $u_k(\cdot,t) \in L_C^{\infty,\rho}(\mathbb{R}^n)$ for all $k \in \mathbb{N}$. Furthermore, $v_k(\cdot,t)$ can be estimated by,

$$\begin{aligned}
|v_k(x,t)\rho(x)| &= \left|\sum_{j=0}^{k} \frac{t^j}{j!}\mathfrak{J}^j u_0(x)\rho(x)\right| \\
&\leq \sum_{j=0}^{k} \frac{t^j}{j!} \left|\mathfrak{J}^j u_0(x)\rho(x)\right| \\
&\leq \sum_{j=0}^{k} \frac{t^j}{j!} \left\|\mathfrak{J}^j u_0\right\|_\rho \\
&\leq \sum_{j=0}^{k} \frac{t^j}{j!} \left\|\mathfrak{J}^j\right\| \|u_0\|_\rho \\
&\leq \sum_{j=0}^{k} \frac{t^j}{j!} \left(\ln(e+et^2)\right)^{|b|} \lambda(R,b,t)^j \|u_0\|_\rho \\
&\leq (\ln(e+et^2))^{|b|} e^{t\lambda(R,b,t)} \|u_0\|_\rho \qquad \text{for all } x \in \mathbb{R}^n.
\end{aligned}$$

Next, we claim the sequence $(v_k(\cdot,t))$ converges to a solution of (4.5). From the above calculations, $v_k(\cdot,t) = \sum_{j=0}^{k} \frac{t^j}{j!}\mathfrak{J}^j u_0$ converges absolutely and uniformly to a function $v(\cdot,t)$ by the Weierstrass M-test. So

$$v(x,t) = \lim_{k\to\infty} v_k(x,t) \qquad \forall x \in \mathbb{R}^n, \forall t \geq 0.$$

Now, we show that v is a solution of (4.5). We first note that $v(\cdot,t) \in L_C^{\infty,\rho}(\mathbb{R}^n)$ because of the boundedness of $v_k(\cdot,t)$. It is clearly that $t \mapsto v(\cdot,t)$ is continuous hence $v \in C\left([0,\infty), L_C^{\infty,\rho}(\mathbb{R}^n)\right)$ for all $t > 0$. Next, consider

$$\begin{aligned}
&\left\| v(\cdot,t) - (u_0 + \int_0^t \mathfrak{J}v(\cdot,s)ds) \right\|_\rho \\
&= \left\| v(\cdot,t) - (v_{k+1}(\cdot,t) - \int_0^t \mathfrak{J}v_k(\cdot,s)ds) - \int_0^t \mathfrak{J}v(\cdot,s)ds \right\|_\rho \\
&= \left\| [v(\cdot,t) - v_{k+1}(\cdot,t)] + \int_0^t (\mathfrak{J}v_k(\cdot,s) - \mathfrak{J}v(\cdot,s)ds) \right\|_\rho \\
&\leq \|v(\cdot,t) - v_{k+1}(\cdot,t)\|_\rho + \left\| \int_0^t \mathfrak{J}(v_k(\cdot,s) - v(\cdot,s))ds \right\|_\rho \\
&\leq \|v(\cdot,t) - v_{k+1}(\cdot,t)\|_\rho + \int_0^t \|\mathfrak{J}(v_k(\cdot,s) - v(\cdot,s))\|_\rho \, ds \\
&\leq \|v(\cdot,t) - v_{k+1}(\cdot,t)\|_\rho + \int_0^t M \|v_k(\cdot,s) - v(\cdot,s)\|_\rho \, ds \to 0.
\end{aligned}$$

as $k \to \infty$.

Hence v is a solution of (4.5) which implies that it is a solution of (4.3) according to Definition 4.1. Now, we conclude that

$$v(\cdot,t) = \lim_{k\to\infty} v_k(\cdot,t) = \sum_{j=0}^{\infty} \frac{t^j}{j!}\mathfrak{J}^j u_0$$

is a solution of the Cauchy problem (4.3). Moreover, from the calculation, the following estimate holds:

$$\|v(\cdot,t)\|_\rho \leq \left(\ln(e + et^2)\right)^{|b|} e^{t\lambda(R,b,t)} \|u_0\|_\rho . \qquad \square$$

Theorem 4.3. *Assume* $u_0 \in L_C^{\infty,\rho}(\mathbb{R}^n)$. *Then there exists* $u(\cdot,t) \in L_C^{\infty,\rho}(\mathbb{R}^n)$ *solving the Cauchy problem* (3.1) *for all* $t \geq 0$. *Moreover,*

$$\|u(\cdot,t)\|_\rho \lesssim (\ln(e+et^2))^{|b|} \|u_0\|_\rho \text{ for all } t \geq 0.$$

Proof. By Lemma 4.2, there exists a solution $v(\cdot,t) \in L_C^{\infty,\rho}(\mathbb{R}^n)$ to the Cauchy problem (4.3). So by the transfomation $u = e^{-t}v$ that relates (3.1) and (4.3), we have that

$$u(\cdot,t) = e^{-t}v(\cdot,t) = e^{-t}\sum_{j=0}^{\infty}\frac{t^j}{j!}\mathfrak{J}^j u_0$$

is a solution to the Cauchy problem (3.1). In addition, the boundedness of $u(\cdot,t)$ follows from that of v by

$$\begin{aligned}\|u(\cdot,t)\|_\rho &= \left\|e^{-t}v(\cdot,t)\right\|_\rho \\ &\leq \left(\ln(e+et^2)\right)^{|b|} e^{-t}e^{t\lambda(R,b,t)} \|u_0\|_\rho \\ &\leq \left(\ln(e+et^2)\right)^{|b|} e^{(\lambda(R,b,t)-1)t} \|u_0\|_\rho .\end{aligned}$$

Next, we estimate the exponential term above. Consider

$$\begin{aligned}\lambda(R,b,t) &= \left(1+\ln\left(1+\frac{2R+R^2}{\sqrt{e+t^2}}\right)\right)^{|b|} \\ &\leq \left(1+\frac{2R+R^2}{\sqrt{e+t^2}}\right)^{|b|} \\ &\leq \left(1+\frac{R^2+2R}{\sqrt{1+t^2}}\right)^{|b|} \\ &\leq 1+\frac{C}{\langle t\rangle} \qquad \text{for some constant } C>0.\end{aligned}$$

The last inequality can be justified as follows. Define $f\colon [0,1] \to \mathbb{R}$ by $f(x) = (1+Cx)-(1+Kx)^{|b|}$, where $K>0$ is a constant. We have that $f(0)=0$ and $f'(x) = C-|b|K(1+Kx)^{|b|-1}$. So $f'(x) \geq 0$ provided $C := \max\{K|b|, |b|K(1+K)^{|b|-1}\}$. Hence f is increasing and we obtain $f(x) \geq 0$ i.e. $(1+Kx)^{|b|} \leq 1+Cx$

for all $x \in \big[0,1\big]$. So we have

$$\begin{aligned}\|u(\cdot,t)\|_\rho &= \big(\ln(e+et^2)\big)^{|b|} e^{(1+\frac{C}{\langle t\rangle}-1)t} \|u_0\|_\rho \\ &\leq \big(\ln(e+et^2)\big)^{|b|} D \|u_0\|_\rho \quad \text{where } D = e^C \\ &\lesssim \big(\ln(e+et^2)\big)^{|b|} \|u_0\|_\rho .\end{aligned}$$

The Theorem is proved. □

The Cauchy problem (3.1) has the solution in the form

$$u(\cdot,t) = \mathcal{G}(t)u_0 := e^{-t}\sum_{j=0}^{\infty}\frac{t^j}{j!}\mathfrak{J}^j u_0 \qquad \forall t \geq 0. \tag{4.6}$$

Definition 4.4. $\mathcal{G}(t)$ is called the *Green operator* for the Cauchy problem (3.1).

Theorem 4.5 (The boundedness of $\mathcal{G}(t)$). *Let $b \in \mathbb{R}$. Then there exists a constant $C = C(b.t)$ such that*

$$\|\mathcal{G}(t)\psi\|_\rho \leq C\|\psi\|_\rho \quad \forall \psi \in L_C^{\infty,\rho}(\mathbb{R}^n).$$

Proof. Already proved in the preceding corollary. □

Theorem 4.6. *$\{\mathcal{G}(t)\}_{t\geq 0}$ in (4.6) is a uniformly continuous semigroup on $L_C^{\infty,\rho}(\mathbb{R}^n)$.*

Proof. It is clear that $\mathcal{G}(0) = I$. where I is the identity operator on $L_C^{\infty,\rho}(\mathbb{R}^n)$. Now, we show the semigroup property. let $s,t \in [0,\infty)$. To show $\mathcal{G}(t+s) = \mathcal{G}(t)\mathcal{G}(s)$, let $u \in L_C^{\infty,\rho}(\mathbb{R}^n)$ and we apply the absolutely convergence of the Green operator to rearrange terms in the following double series.

$$\begin{aligned}
\mathcal{G}(t+s)u &= e^{-(t+s)}\sum_{k=0}^{\infty}\frac{(t+s)^k}{k!}\mathfrak{J}^k u \\
&= e^{-(t+s)}\sum_{k=0}^{\infty}\sum_{i=0}^{k}\frac{t^i s^{k-i}k!}{(k-i)!i!}\frac{1}{k!}\mathfrak{J}^k u \\
&= e^{-(t+s)}\sum_{k=0}^{\infty}\sum_{i=0}^{k}\frac{t^i s^{k-i}}{(k-i)!i!}\mathfrak{J}^i(\mathfrak{J}^{k-i}u) \\
&= e^{-(t+s)}\sum_{i=0}^{\infty}\sum_{k=i}^{\infty}\frac{t^i s^{k-i}}{(k-i)!i!}\mathfrak{J}^i(\mathfrak{J}^{k-i}u) \\
&= e^{-t}\sum_{i=0}^{\infty}\frac{t^i}{i!}\mathfrak{J}^i\left(e^{-s}\sum_{k=i}^{\infty}\frac{s^{k-i}}{(k-i)!}\mathfrak{J}^{k-i}u\right) \\
&= e^{-t}\sum_{i=0}^{\infty}\frac{t^i}{i!}\mathfrak{J}^i(\mathcal{G}(s)u) \\
&= \mathcal{G}(t)\mathcal{G}(s)u.
\end{aligned}$$

To prove the uniform continuity of the semigroup, we consider

$$\begin{aligned}
\|\mathcal{G}(t)-I\| &= \left\|e^{-t}I+\sum_{k=1}^{\infty}\frac{(t\mathfrak{J})^k}{k!}-I\right\| \\
&\le \left\|e^{-t}I-I\right\| + \left\|\sum_{k=1}^{\infty}\frac{(t\mathfrak{J})^k}{k!}\right\| \\
&\le \left\|e^{-t}I-I\right\| + \sum_{k=1}^{\infty}\frac{t^k\|\mathfrak{J}\|^k}{k!} \\
&\le \left\|e^{-t}I-I\right\| + e^{t\|\mathfrak{J}\|}-1.
\end{aligned}$$

Hence, we have $\lim_{t\to 0^+}\|\mathcal{G}(t)-I\| = 0$, so $\{\mathcal{G}(t)\}_{t\ge 0}$ is a uniformly continuous semigroup. □