CHAPTER IV

NUMERICAL RESULTS

This chapter gives accuracies comparison between the explicit formulas and numerical simulations by considering the following ECIR process

$$dv_t = \kappa\left(\frac{\sigma_0^2 de^{2\sigma_1 t}}{4\kappa} - v_t\right)dt + \sigma_0 e^{\sigma_1 t}\sqrt{v_t}dW_t, \tag{4.1}$$

where $\kappa(t) = \kappa$, $\theta(t) = \frac{1}{4\kappa}\sigma_0^2 de^{2\sigma_1 t}$, $\sigma(t) = \sigma_0 e^{\sigma_1 t}$ and $d, \kappa, \sigma_0, \sigma_1$ are positive constants, see [5] for more details of these parameters. In numerical test, we illustrate only Theorems 3.1 and 3.5 by comparing the results between the explicit formulas and the MC simulations.

4.1 Parameters of ECIR

Higham and Mao [6] showed that the EM scheme is a simulation with correctly qualities for square root process such as the ECIR process. The discretizing N time steps, $\Delta t = \frac{T}{N}$ on $[0, T]$, the EM scheme for the process gives (see section 2.3 for details)

$$v_{t_{i+1}} = v_{t_i} + \kappa\left(\frac{\sigma_0^2 de^{2\sigma_1 t_i}}{4\kappa} - v_{t_i}\right)\Delta t + \sigma_0 e^{\sigma_1 t_i}\sqrt{v_{t_i}}\sqrt{\Delta t}\ N(0,1). \tag{4.2}$$

Our simulations generate 10,000 sample paths of v_t on $[0, T_i]$, for $T_i = 0.01, 5$, $i = 1, 2$, with parameters $\kappa = 0.03, d = 2, \sigma_0 = 0.01$ and $\sigma_1 = 0.02$. By observing the requirement of the Assumption in Chapter 3, we have

$$\delta(t) = \frac{4(\kappa)(\frac{1}{4\kappa}\sigma_0^2 de^{2\sigma_1 t})}{(\sigma_0 e^{\sigma_1 t})^2} = d \geq 2 \tag{4.3}$$

and bounded for all $t \in [0, T_i]$. This guarantees positive values of v_t for all $t \in [0, T_i]$.

From now on, $U_E^{M(\gamma,\alpha,\beta)}$ denotes the results obtained by MC simulations, an approximation of $U_E^{(\gamma,\alpha,\beta)}$ with parameters declared above.

4.2 Comparisons results

This section provides seven examples of the comparisions between the explicit formula and the MC simulations. Theorem 3.5 is compared with varied parameters α, β for $N = 0$ and 2, and Theorem 3.1 with $\gamma = \pm\frac{1}{2}$ and $K = 2$. In order to perform these examples, we compute $B(\tau)$ in (3.7), namely,

$$B_\alpha(\tau) = \frac{2\xi\alpha e^{-\kappa\tau}}{2\xi - \alpha\sigma_0^2 e^{2\sigma_1 T}(1 - e^{-\xi\tau})}, \tag{4.4}$$

where $\xi = \kappa + 2\sigma_1$. Notice that $B_0(\tau) = 0$. For $\alpha \neq 0$, we set

$$\omega_\alpha(\tau) = \alpha^{-1} e^{\kappa\tau} B_\alpha(\tau), \tag{4.5}$$

and obtain that $\lim_{\alpha \to 0^+} \omega_\alpha(\tau) = 1$. Without loss of generality, since $\mathbb{E}^{\mathbb{P}}\left[V_T^\gamma e^{\alpha V_T + \beta} \mid V_t = v\right] = e^\beta \mathbb{E}^{\mathbb{P}}\left[V_T^\gamma e^{\alpha V_T} \mid V_t = v\right]$, we will assume that $\beta = 0$. In the case of the Theorem 3.1, we define the finite sums $U_E^{(\gamma,\alpha,\beta,K)}$ as the approximations up to order $\gamma - K$ of $U_E^{(\gamma,\alpha,\beta)}$.

4.2.1 Case $\alpha = 0$

As expected, Examples 1-3 (equations (4.6)-(4.8)) produce the same results expressed in Rujivan [11]. These examples are illustrated the agreement with [11].

Example 1: $N = 2$.

Applying (3.24) yields the closed-form formula of $\mathbb{E}^{\mathbb{P}}\left[V_T^2 \mid V_t = v\right]$ as

$$\begin{aligned} U_E^{(2,0,0)}(v,\tau) &= e^{-2\kappa\tau} v^2 \\ &\quad + e^{-\kappa\tau + 2\sigma_1 T - \xi\tau} \frac{(d+2)(e^{\xi\tau} - 1)\sigma_0^2}{2\xi} v \\ &\quad + e^{4\sigma_1 T - 2\xi\tau} \frac{d(d+2)(e^{\xi\tau} - 1)^2 \sigma_0^4}{16\xi^2} \end{aligned} \tag{4.6}$$

for all $v > 0$ and $\tau \geq 0$, where $\tau = T - t$.

Example 2: $\gamma = \frac{1}{2}$ and $K = 2$.

Applying (3.2) yields a second-order approximation of $\mathbb{E}^{\mathbb{P}}\left[V_T^{\frac{1}{2}} \,\middle|\, V_t = v\right]$ as

$$\begin{aligned} U_E^{(\frac{1}{2},0,0,2)}(v,\tau) &= e^{-\frac{\kappa}{2}\tau}v^{\frac{1}{2}} \\ &\quad + e^{\frac{\kappa}{2}\tau+2\sigma_1 T-\xi\tau}\frac{(d-1)(e^{\xi\tau}-1)\sigma_0^2}{8\xi}v^{-\frac{1}{2}} \\ &\quad - e^{\frac{3\kappa}{2}\tau+4\sigma_1 T-2\xi\tau}\frac{(d-3)(d-1)(e^{\xi\tau}-1)^2\sigma_0^4}{128\xi^2}v^{-\frac{3}{2}} \end{aligned} \tag{4.7}$$

for all $v > 0$ and $\tau \geq 0$, where $\tau = T - t$.

Example 3: $\gamma = -\frac{1}{2}$ and $K = 2$.

Applying (3.2) yields a second-order approximation of $\mathbb{E}^{\mathbb{P}}\left[V_T^{-\frac{1}{2}} \,\middle|\, V_t = v\right]$ as

$$\begin{aligned} U_E^{(-\frac{1}{2},0,0,2)}(v,\tau) &= e^{\frac{\kappa}{2}\tau}v^{-\frac{1}{2}} \\ &\quad - e^{\frac{3\kappa}{2}\tau+2\sigma_1 T-\xi\tau}\frac{(d-3)(e^{\xi\tau}-1)\sigma_0^2}{8\xi}v^{-\frac{3}{2}} \\ &\quad + e^{\frac{5\kappa}{2}\tau+4\sigma_1 T-2\xi\tau}\frac{3(d-5)(d-3)(e^{\xi\tau}-1)^2\sigma_0^4}{128\xi^2}v^{-\frac{5}{2}} \end{aligned} \tag{4.8}$$

for all $v > 0$ and $\tau \geq 0$, where $\tau = T - t$.

4.2.2 Testing Theorem 3.5

According to Theorem 3.5, the Examples 4 and 5 are presented to compared the results from the closed-form formulas and the MC simulations with initial $v = 0.1, 0.2, 0.3, \ldots, 2$, and, for convenience, we set $\alpha = 1$.

Example 4: $N = 0$.

Applying (3.24) yields the closed-form formula of $\mathbb{E}^{\mathbb{P}}\left[e^{V_T} \mid V_t = v\right]$ as

$$U_E^{(0,1,0)}(v,\tau) = (B_1(\tau))^{\frac{1}{2}d}e^{B_1(\tau)v+\frac{d\kappa}{2}\tau} \tag{4.9}$$

for all $v > 0$ and $\tau \geq 0$, where $\tau = T - t$.

The MC simulations with the same parameters are simulated using $10,000$ sample paths for final times $T_1 = 0.01$ and $T_2 = 5$, and compared with the closed-form formula

(4.9). The comparison values are displayed in Figure 4.1 showing that the closed-form formula agrees very well with the MC simulation.

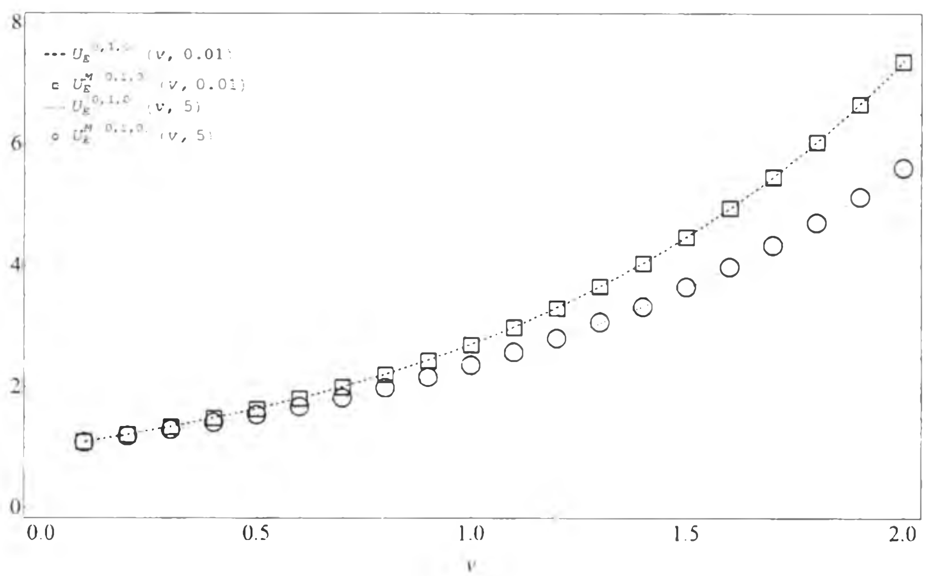

Figure 4.1: Comparison values of the exact formula $U_E^{(0,1,0)}$ and the MC simulations $U_E^{M(0,1,0)}$ for $T_1 = 0.01$ and $T_2 = 5$, respectively.

Example 5: $N = 2$.

Applying (3.24) yields the closed-form formula of $\mathbb{E}^{\mathbb{P}}\left[V_T^2 e^{V_T} \mid V_t = v\right]$ as

$$\begin{aligned} U_E^{(2,1,0)}(v,\tau) &= e^{-2\kappa\tau + B_1(\tau)v} \left(\omega_1(\tau)\right)^{\frac{1}{2}d+4} v^2 \\ &\quad + e^{-\kappa\tau + 2\sigma_1 T - \xi\tau + B_1(\tau)v} \frac{(d+2)(e^{\xi\tau}-1)\sigma_0^2}{2\xi} \left(\omega_1(\tau)\right)^{\frac{1}{2}d+3} v \\ &\quad + e^{4\sigma_1 T - 2\xi\tau + B_1(\tau)v} \frac{d(d+2)(e^{\xi\tau}-1)^2\sigma_0^4}{16\xi^2} \left(\omega_1(\tau)\right)^{\frac{1}{2}d+2} \end{aligned} \tag{4.10}$$

for all $v > 0$ and $\tau \geq 0$, where $\tau = T - t$.

Similarly, the MC simulations are provided to compare with the closed-form formula (4.10). The comparisons are displayed in Figure 4.2 showing that the closed-form formula agrees very well with the MC simulation.

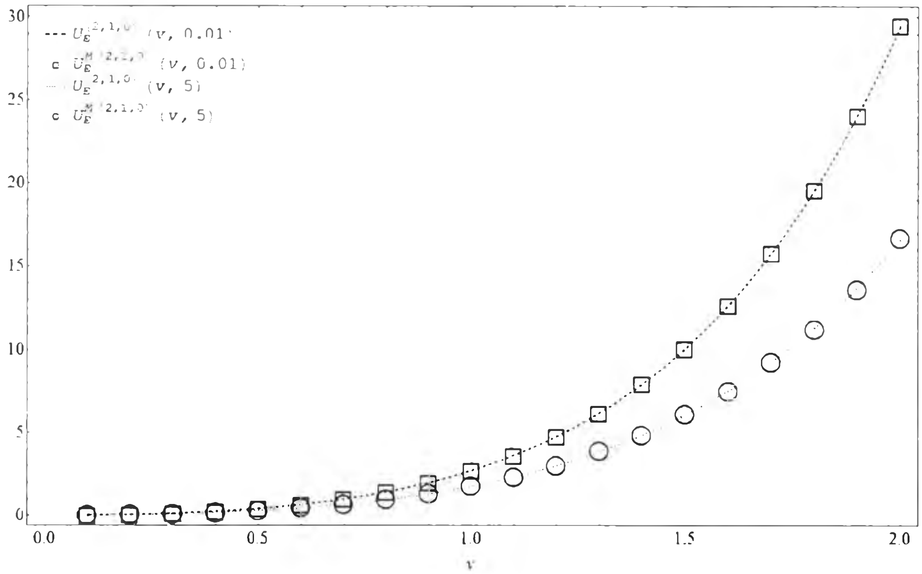

Figure 4.2: Comparison values of the exact formula $U_E^{(2,1,0)}$ and the MC simulations $U_E^{M(2,1,0)}$ for $T_1 = 0.01$ and $T_2 = 5$, respectively.

The Examples 4 and 5 above confirm that the results from the exact formula for Theorem 3.5 agree with the results from MC simulations for various values of parameters. This gives the advantage in using the exact formula instead of the MC simulations in terms of saving time-consuming, especially, for N that is not too large.

4.2.3 Testing Theorem 3.1

These examples illustrate the Theorem 3.1, where the closed-form formula, the infinite sums $U_E^{(\gamma,\alpha,\beta)}(v,\tau)$ are approximated by finite sums $U_E^{(\gamma,\alpha,\beta,K)}(v,\tau)$ using $K = 2$ for $\gamma = \pm\frac{1}{2}$. For the tests, the MC simulations are also produced with the same parameters followed Section 4.1 using initial $v = 0.1, 0.2, 0.3, \ldots, 2$.

Example 6: $\gamma = \frac{1}{2}$.

Applying (3.2) yields an approximation of $\mathbb{E}^{\mathbb{P}}\left[V_T^{\frac{1}{2}} e^{V_T} \,\middle|\, V_t = v\right]$ for $K = 2$ as

$$\begin{aligned} U_E^{(\frac{1}{2},1,0,2)}(v,\tau) &= e^{-\frac{\kappa}{2}\tau + B_1(\tau)v} (\omega_1(\tau))^{\frac{1}{2}d+1} v^{\frac{1}{2}} \\ &\quad + e^{\frac{\kappa}{2}\tau + 2\sigma_1 T - \xi\tau + B_1(\tau)v} \frac{(d-1)(e^{\xi\tau}-1)\sigma_0^2}{8\xi} (\omega_1(\tau))^{\frac{1}{2}d} v^{-\frac{1}{2}} \\ &\quad - e^{\frac{3\kappa}{2}\tau + 4\sigma_1 T - 2\xi\tau + B_1(\tau)v} \frac{(d-3)(d-1)(e^{\xi\tau}-1)^2\sigma_0^4}{128\xi^2} (\omega_1(\tau))^{\frac{1}{2}d-1} v^{-\frac{3}{2}} \end{aligned} \tag{4.11}$$

for all $v > 0$ and $\tau \geq 0$, where $\tau = T - t$.

The MC simulations using the same parameters are simulated for 10,000 sample paths for final times $T_1 = 0.01$ and $T = 5$ to compare with the approximate formula. The comparisons are displayed in Figure 4.3 showing that the approximate formula agrees with the MC simulation for varies initial v's for $T_1 = 0.01$ and $T_2 = 5$.

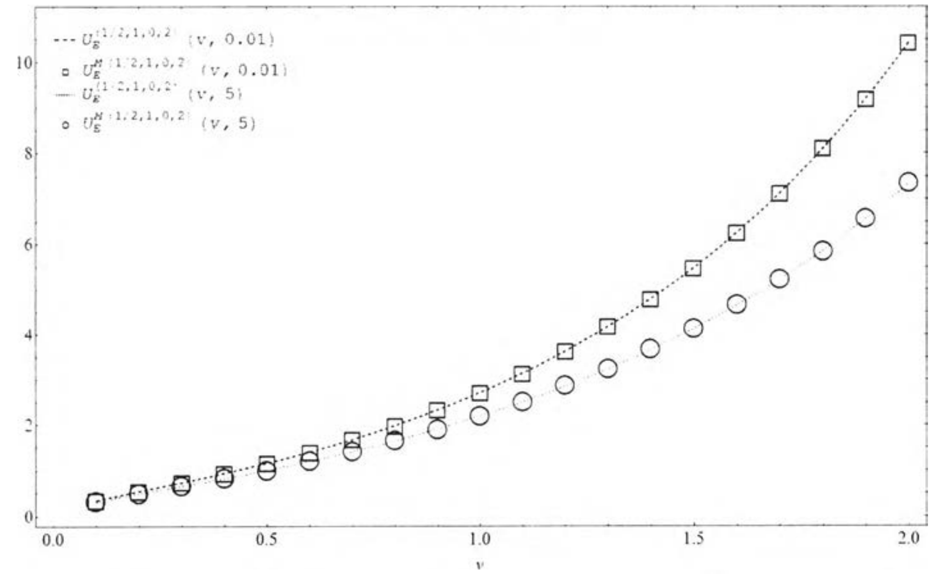

Figure 4.3: Comparison values of the approximate formula $U_E^{(\frac{1}{2},1,0,2)}$ and the MC simulations $U_E^{M(\frac{1}{2},1,0)}$ for $T_1 = 0.01$ and $T_2 = 5$, respectively.

Example 7: $\gamma = -\frac{1}{2}$.

Applying (3.2) yields an approximation of $\mathbb{E}^{\mathbb{P}}\left[V_T^{-\frac{1}{2}} e^{V_T} \,\middle|\, V_t = v\right]$ for $K = 2$ as

$$
\begin{aligned}
U_E^{(-\frac{1}{2},1,0,2)}(v,\tau) &= e^{\frac{\kappa}{2}\tau + B_1(\tau)v}\left(\omega_1(\tau)\right)^{\frac{1}{2}d-1} v^{-\frac{1}{2}} \\
&\quad - e^{\frac{3\kappa}{2}\tau + 2\sigma_1 T - \xi\tau + B_1(\tau)v}\frac{(d-3)(e^{\xi\tau}-1)\sigma_0^2}{8\xi}\left(\omega_1(\tau)\right)^{\frac{1}{2}d-2} v^{-\frac{3}{2}} \qquad (4.12)\\
&\quad + e^{\frac{5\kappa}{2}\tau + 4\sigma_1 T - 2\xi\tau + B_1(\tau)v}\frac{3(d-5)(d-3)(e^{\xi\tau}-1)^2\sigma_0^4}{128\xi^2}\left(\omega_1(\tau)\right)^{\frac{1}{2}d-3} v^{-\frac{5}{2}}
\end{aligned}
$$

for all $v > 0$ and $\tau \geq 0$, where $\tau = T - t$.

Similarly, the MC simulations are simulated for 10,000 sample paths to compare with the approximate formula (4.12). The comparisons are displayed in Figure 4.4 showing that the formula agrees with the MC simulations.

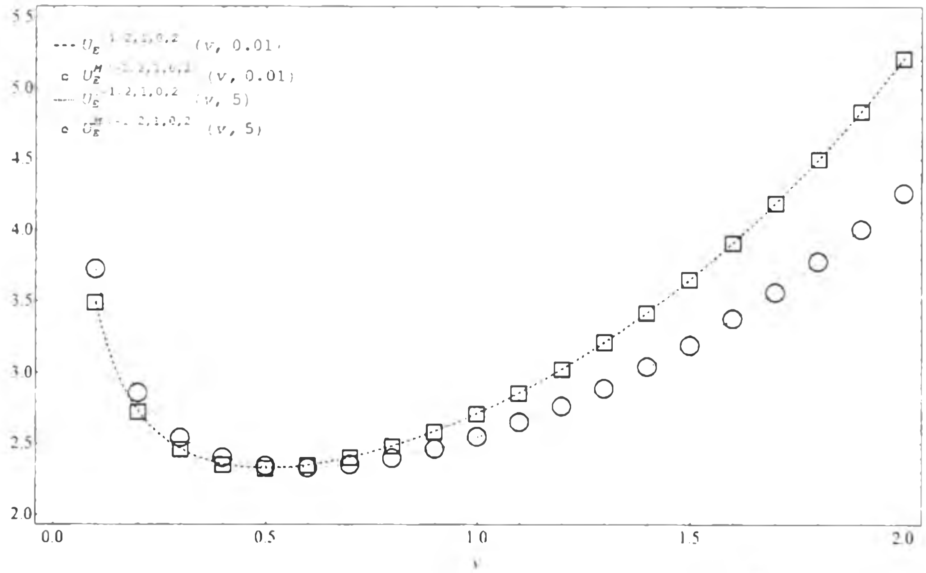

Figure 4.4: Comparison values of the approximate formula $U_E^{(-\frac{1}{2},1,0,2)}$ and the MC simulations $U_E^{M(-\frac{1}{2},1,0)}$ for $T_1 = 0.01$ and $T_2 = 5$, respectively.

As shown in the Figures 4.3 and 4.4, the results obtained from the approximate formulas (4.11) and (4.12) are confirmed with the results from the and MC simulations, for $K = 2$. Note that, since (3.2) involves infinite series as function of (v, τ), the series might not converge for some values of (v, τ). In this case, we cannot use finite sums to approximate (3.2). To study more effectively in this case, the analytical approach to approximate the explicit formula (3.2) is presented by using absolute relative error in the next section.

4.3 Approximation results

It is more concern to study the accuracy of the explicit formula when $\gamma \notin \{0\} \cup \mathbb{N}$. Since (4.11) and (4.12) in Example 6 and 7 are the approximations of $U_E^{(\frac{1}{2},1,0)}$ and $U_E^{(-\frac{1}{2},1,0)}$, respectively, this section gives analytical approaches for the explicit formulas in these examples. Note that, the power series (3.2) in Theorem 3.1 may not converge. One can understand that the increasing of K in $U_E^{(\gamma,\alpha,\beta,K)}(v, \tau)$ may not improve the accuracy of the explicit formula. In order to determine the level of accuracy of the formulas in practice, we construct a sequence on $K \in \mathbb{N}$ of absolute relative errors as

$$\varepsilon_K^{(\gamma,\alpha,\beta)}(v,\tau) := \left| \frac{U_E^{(\gamma,\alpha,\beta,K)}(v,\tau) - U_E^{(\gamma,\alpha,\beta,K-1)}(v,\tau)}{U_E^{(\gamma,\alpha,\beta,K)}(v,\tau)} \right| \tag{4.13}$$

for $(v,\tau) \in \mathbb{R}^+ \times [0,T]$. For convenience, the approximation $U_E^{(\gamma,\alpha,\beta,K)}(v,\tau)$ is correct up to at least 16 significant digits, whereas $\varepsilon_K^{(\gamma,\alpha,\beta)}(v,\tau) < 0.5 \times 10^{-16}$. We shall give approximations of Examples 6 and 7 up to order $\gamma - 5$, with initial $v = 0.0001, 0.001, 0.01, 0.1, 1, 2$ and $T_1 = 0.01, T_2 = 5$.

		$\varepsilon_K^{(\frac{1}{2},1,0)}(v,\tau)$	$T_1 = 0.01$			
K	$v = 0.0001$	$v = 0.001$	$v = 0.01$	$v = 0.1$	$v = 1.0$	$v = 2.0$
1	0.12 E-02	0.12 E-03	0.12 E-04	0.13 E-05	0.13 E-06	0.63 E-07
2	0.78 E-06	0.78 E-08	0.78 E-10	0.78 E-12	0.78 E-14	0.20 E-14
3	0.29 E-08	0.29 E-11	0.29 E-14	0*	0	0
4	0.23 E-10	0.23 E-14	0	0	0	0
5	0.28 E-12	0	0	0	0	0
		$\varepsilon_K^{(\frac{1}{2},1,0)}(v,\tau)$	$T_2 = 5$			
K	$v = 0.0001$	$v = 0.001$	$v = 0.01$	$v = 0.1$	$v = 1.0$	$v = 2.0$
1	0.43	0.70 E-01	0.74 E-02	0.75 E-03	0.75 E-04	0.37 E-04
2	0.14	0.26 E-02	0.28 E-04	0.28 E-06	0.28 E-08	0.70 E-09
3	0.24	0.58 E-03	0.62 E-06	0.63 E-09	0.63 E-12	0.79 E-13
4	0.53	0.27 E-03	0.29 E-07	0.29 E-11	0.29 E-15	0
5	0.79	0.20 E-03	0.21 E-08	0.22 E-13	0	0

*Remark: Zeros signify $\varepsilon_K^{(\gamma,\alpha,\beta)}(v,\tau) < 0.5 \times 10^{-16}$.

Table 4.1: Absolute relative errors $\varepsilon_K^{(\frac{1}{2},1,0)}(v,\tau)$, using the explicit formula (3.2) based on the ECIR process (4.1) with $T_1 = 0.01$ and $T_2 = 5$.

			$\varepsilon_K^{(-\frac{1}{2},1,0)}(v,\tau)$	$T_1 = 0.01$		
K	$v = 0.0001$	$v = 0.001$	$v = 0.01$	$v = 0.1$	$v = 1.0$	$v = 2.0$
1	0.12 E-02	0.13 E-03	0.13 E-04	0.13 E-05	0.13 E-06	0.63 E-07
2	0.70 E-05	0.70 E-07	0.70 E-09	0.70 E-11	0.70 E-13	0.18 E-13
3	0.73 E-07	0.73 E-10	0.73 E-13	0*	0	0
4	0.11 E-08	0.11 E-12	0	0	0	0
5	0.23 E-10	0.23 E-15	0	0	0	0
			$\varepsilon_K^{(-\frac{1}{2},1,0)}(v,\tau)$	$T_2 = 5$		
K	$v = 0.0001$	$v = 0.001$	$v = 0.01$	$v = 0.1$	$v = 1.0$	$v = 2.0$
1	0.43	0.70 E-01	0.74 E-02	0.75 E-03	0.75 E-04	0.37 E-04
2	0.59	0.23 E-01	0.25 E-03	0.25 E-05	0.25 E-07	0.63 E-08
3	0.79	0.14 E-01	0.16 E-04	0.16 E-07	0.16 E-10	0.20 E-11
4	0.88	0.13 E-01	0.14 E-05	0.14 E-09	0.14 E-13	0.90 E-15
5	0.91	0.15 E-01	0.17 E-06	0.17 E-11	0	0

*Remark: Zeros signify $\varepsilon_K^{(\gamma,\alpha,\beta)}(v,\tau) < 0.5 \times 10^{-16}$.

Table 4.2: Absolute relative errors $\varepsilon_K^{(-\frac{1}{2},1,0)}(v,\tau)$, using the explicit formula (3.2) based on the ECIR process (4.1) with $T_1 = 0.01$ and $T_2 = 5$.

Note that the speed of convergence is more apparent in the explicit formula (3.2) when the initial v is larger away from 0. Moreover, since the power series (3.2) is approximated by $U_E^{(\gamma,\alpha,\beta,K)}$ which is dominated by the term $v^{\gamma-k}$, thus, the series $U_E^{(\gamma,\alpha,\beta,K)}$ will diverges when $v < 1$ as K increasing.

In particular, it is difficult to derive the convergent domain of $U_E^{(\gamma,\alpha,\beta)}(v,\tau)$ because the series is too complicate. In conclusion, by suitably choosing of v and τ, small T and large v, can be inferred that the accuracy of the explicit formula improves when K increases.