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CANCELLATION IDEALS AND MINIMAL CANCELLATION IDEALS OF SOME COMMUTATIVE RINGS WITH IDENTITY

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ถ้า R เป็นวงสลับที่ซึ่งมีเอกลักษณ์ และ J เป็นไอดีลของ R แล้ว เราเรียก J ว่าเป็นไอดีลการตัดออกของ R เมื่อ J มีสมบัติว่า $JA = JB$ ทำให้ได้ผลว่า $A = B$ ทุกๆไอดีล A, B ของ R ถ้า I เป็นไอดีลของ R และ J เป็นไอดีลการตัดออกของ R โดยที่ $I \subseteq J$ แล้ว เราเรียก J ว่าเป็นไอดีลการตัดออกพาดพิงกับ I

ในงานวิจัยนี้ เราได้ผลลัพธ์ที่สำคัญคือ ทฤษฎีบทต่อไปนี้

1) ให้ D เป็น unique factorization domain และ $a, b \in D \setminus \{0\}$ ซึ่ง $d = (a, b)$ จะได้ว่า $< a, b >$ เป็นไอดีลการตัดออกของ D ก็ต่อเมื่อ $< d > = < a, b >$
2) สำหรับทุกๆ $m \in N$, ถ้า J เป็นไอดีลการตัดออกของ $Z[x]$ พาดพิงกับ $< 2, x^m >$ แล้ว $J = Z[x]$

ภาควิชา **คณิตศาสตร์** ลายมือชื่อนิสิต..

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Let R be a commutative ring with identity. An ideal J of R is called a cancellation ideal if whenever $JA = JB$ for ideals A and B of R, then $A = B$. And J is called a cancellation ideal belonging to I if J is a cancellation ideal and $I \subseteq J$.

In this reserch, we obtain the important results as the two following theorems.

1) Let D be a unique factorization domain and $a,b \in D \setminus \{0\}$ such that $d = (a,b)$. Then $<a,b>$ is a cancellation ideal of D if and only if $<d> = <a,b>$.

2) For all $m \in N$, if J is a cancellation ideal belonging to $<2, x^m>$ of $Z[x]$, then $J = Z[x]$.

Department **Mathematics**	Student's signature..................................
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CHAPTER I

INTRODUCTION AND PRELIMINARIES

An introduction of cancellation ideals may be found in [2]. Characterization of cancellation ideals was given by D.D.Anderson and M.Roitman in [1], but checking a given ideal is a cancellation ideal, or not, is not easy to show by using their theorem. We can found some interesting notion for ideals in [3].

In this chapter, we give precise definitions, quoted results, and give some results for using in the next two chapters.

Definition. Let R be a commutative ring with identity. An ideal I of R is called a **cancellation ideal** if whenever $IB = IC$ for ideals B and C of R, then $B = C$.

Definition. An integral domain R is a **unique factorization domain** provided that:

(i) every nonzero nonunit element a of R can be written $a = c_1c_2\cdots c_n$, with $c_1, ..., c_n$ irreducible,

(ii) if $a = c_1c_2\cdots c_n$ and $a = d_1d_2\cdots d_m$ (c_i, d_i irreducible), then n=m and for some permutation σ of $\{1, 2, ..., n\}$, c_i and $d_{\sigma(i)}$ are associates for every i.

Definition. A ring R is called a **Boolean ring** if for every $a \in R$, $a^2 = a$.

Definition. A ring R is **Artinian** if R satisfies the descending chain condition on ideals.

The Theorem 1.1 is a well-know result.

Theorem 1.1. *If D is a unique factorization domain, then $D[x]$ is a unique factorization domain.*

Since $\mathbb{Z}$ is a unique factorization domain, $\mathbb{Z}[x]$ and $\mathbb{Z}[x, y]$ are unique factorization domain.

The next two results are given in [1]. The first lemma is easy to see and we always refer to it in the next chapter.

Lemma 1.2. *Let R be a commutative ring with identity and $a \in R$. Then $\langle a \rangle$ is a cancellation ideal of R if and only if a is not a zero divisor of R.*

Theorem 1.3. *Let R be a commutative ring with identity . An ideal I of R is a cancellation ideal of R if and only if I is locally a regular principal ideal.*

From Lemma 1.2 , we have that every ideal of $\mathbb{Z}$, except $\{0\}$, is a cancellation ideal of $\mathbb{Z}$. The following theorem is an interesting result.

Theorem 1.4. *Every proper ideal in $\mathbb{Z}_m$ is not a cancellation ideal of $\mathbb{Z}_m$.*

Proof. Let I be an ideal in $\mathbb{Z}_m$ such that $I \neq \mathbb{Z}_m$ and $I \neq \{\bar{0}\}$. Since $\mathbb{Z}_m$ is a principal ideal ring , $I = \langle \bar{k} \rangle$ for some $\bar{k} \in \mathbb{Z}_m \setminus \{\bar{0}\}$. Let d be the g.c.d. of k and m. Then $d \neq 1, d \mid k$ and $d \mid m$. There exist nonzero elements x and y of $\mathbb{Z}$ such that $k = dx$ and $m = dy$. Thus $ky = dxy = xdy = xm$, so $\bar{k}\bar{y} = \overline{ky} = \bar{0}$. Hence $\bar{k}$ is a zero divisor of $\mathbb{Z}_m$. By Lemma 1.1, $I = \langle \bar{k} \rangle$ is not a cancellation ideal of $\mathbb{Z}_m$.

Clearly, $\{\bar{0}\}$ is not a cancellation ideal of $\mathbb{Z}_m$ for $m \rangle 1$ and $\{\bar{0}\}$ is a cancellation ideal of $\mathbb{Z}_m$ for $m = 1$.

Next, we have to show that $\mathbb{Z}_m$ is a cancellation ideal of $\mathbb{Z}_m$ for $m > 1$. Let $\overline{n_1}$ and $\overline{n_2}$ be elements of $\mathbb{Z}_m$ such that $\mathbb{Z}_m\langle \overline{n_1} \rangle = \mathbb{Z}_m\langle \overline{n_2} \rangle$. Since $\bar{1}$ is the multiplicative identity of $\mathbb{Z}_m$, $\langle \overline{n_1} \rangle = \mathbb{Z}_m\langle \overline{n_1} \rangle = \mathbb{Z}_m\langle \overline{n_2} \rangle = \langle \overline{n_2} \rangle$.
Therefore, $\mathbb{Z}_m$ is a cancellation ideal of $\mathbb{Z}_m$. □

Theorem 1.5 is one that easy to prove but in order to check whether a given ideal is a cancellation ideal, is not practical.

Theorem 1.5. *Let R be a commutative ring and I an ideal of R such that I contains an element which is not a zero divisor of R. Then I is a cancellation ideal of R if and only if for every ideals A, B of R such that $A \cup B \subseteq I$, $IA = IB$ implies $A = B$.*

Proof. ($\rightarrow$) Clearly.
($\leftarrow$) Let A and B be ideals of R such that $IA = IB$ and k an element of I which is not a zero divisor of R. Then $\langle k \rangle IA = \langle k \rangle IB$, so $I\langle k \rangle A = I\langle k \rangle B$. Since $k \in I$, $\langle k \rangle A \cup \langle k \rangle B \subseteq I$, so $\langle k \rangle A = \langle k \rangle B$. By Lemma 1.2, $\langle k \rangle$ is a cancellation ideal of R, so $A = B$. □

We give the precise definition for a cancellation ideal belonging to an ideal which we consider in Chapter III here.

Definition. Let I be an ideal in the commutative ring R with identity.
A cancellation ideal J of R is said to be a **cancellation ideal belonging to ideal** I if $I \subseteq J$.

The following statements are facts about cancellation ideals of some familiar rings.
1. A maximal ideal in $\mathbb{Z}[x]$ need not be a cancellation ideal.
An example is the maximal ideal $\langle 2, x \rangle$ of $\mathbb{Z}[x]$ (see Chapter II for $\langle 2, x \rangle$ is not a cancellation ideal).
2. For any field F, $F[x]$ is a PID , so all ideals of $F[x]$, except $\{0\}$, are cancellation ideals of $F[x]$(by Lemma 1.2).
3.For $a \in \mathbb{Z}, |a| \geq 1$ and a is not prime, the ideal $\langle a \rangle$ is a cancellation ideal but not a maximal ideal of $\mathbb{Z}$ (by Lemma 1.2).

4. Let R be a subring of an integral domain T. If I is a cancellation ideal of R, then IT is a cancellation ideal of T. This fact is quoted from [1].

5. The ideal $I = \langle 2, x^2 \rangle$ of $\mathbb{Z}[x]$ is not a cancellation ideal and a cancellation ideal of $\mathbb{Z}[x]$ belonging to I must be $\mathbb{Z}[x]$, see Chapter III.

CHAPTER II

CANCELLATION IDEALS OF SOME RINGS

In this chapter, we consider ideals of arbitrary commutative ring in Theorem 2.1-2.3, and we consider ideals in some special forms of $\mathbb{Z}[x]$ and $\mathbb{Z}[x, y]$ in Theorem 2.4-2.16. Ideals of Boolean rings with identity and ideals of an Artinian rings with identity have considered in Theorem 2.17-2.18.

Theorem 2.1. *Let $I_1, I_2, ..., I_n$ be ideals of a commutative ring R.*
Then $I_1I_2...I_n$ is a cancellation ideal of R if and only if I_j is a cancellation ideal of R for each $j \in \{1, 2, ..., n\}$.

Proof. Assume that $I_1I_2...I_n$ is a cancellation ideal of R.
Let$j \in \{1, 2, ..., n\}$ and B and C be ideals such that $I_jB = I_jC$.
Then $I_jBI_1...I_{j-1}I_{j+1}...I_n = I_jCI_1...I_{j-1}I_{j+1}...I_n,$ so $I_1I_2...I_nB = I_1I_2...I_nC$. Since $I_1I_2...I_n$ is a cancellation ideal of R, $B = C$. Thus I_j is a cancellation ideal of R.
Next , assume that for all $j \in \{1, 2, ..., n\},$ I_j is a cancellation ideal of R.
Let B and C be ideals such that $I_1I_2...I_nB = I_1I_2...I_nC$.
Since I_1 is a cancellation ideal of R, $I_2I_3...I_nB = I_2I_3...I_nC$.
Since I_2 is a cancellation ideal of R, $I_3I_4...I_nB = I_3I_4...I_nC$. By the same argument, we must have $B = C$.
Thus $I_1I_2...I_n$ is a cancellation ideal of R. □

Theorem 2.2. *Let R be a commutative ring.*
(i) If $A, B$and C are ideals of R such that $A+B, A+C$ and $B+C$ are cancellation

ideals of R, then $A+B+C$ is also a cancellation ideal of R.

(ii) If every ideal generated by two elements of R is a cancellation ideal, then every finitely generated ideal of R is a cancellation ideal of R.

Proof. (i)Assume that A, B and C are ideals of R such that $A+B, A+C$ and $B+C$ are cancellation ideals of R . By Theorem 2.1 , $(A+B)(A+C)(B+C)$ is a cancellation ideal of R. Since $(A+B+C)(AB+AC+BC) = (A+B)(A+C)(B+C)$, $A+B+C$ is a cancellation ideal of R.

(ii)Assume that each ideal generated by two elements of R is a cancellation ideal. Let k be an integer greater than 1 and suppose that every ideal generated by a set of k elements is a cancellation ideal of R. Let $x_1, x_2, ..., x_{k+1}$ be arbitrary elements in R. We have that $\langle x_1, x_2, ..., x_{k+1}\rangle = \langle x_1\rangle + \langle x_2, ..., x_k\rangle + \langle x_{k+1}\rangle$ and by assumption $\langle x_1\rangle + \langle x_2, ..., x_k\rangle, \langle x_1\rangle + \langle x_{k+1}\rangle$ and

$\langle x_2, ..., x_k\rangle + \langle x_{k+1}\rangle$ are cancellation ideals of R. By (i) , $\langle x_1, x_2, ..., x_{k+1}\rangle$ is a cancellation ideal of R. □

Theorem 2.3. *Let I be a proper ideal of a commutative ring R with identity. If I is a cancellation ideal of R, then I is not a minimal ideal .*

Proof. Assume that I is a cancellation ideal of R. We have that $\{0\} \subseteq I^2 \subseteq I$. If $I^2 = \{0\}$,then $II = \{0\} = I\{0\}$, so $I = \{0\}$. A contradiction since $\{0\}$ is not a cancellation ideal of R. If $I^2 = I$, then $II = I = I\langle 1\rangle$, so $I = R$, a contradiction. Thus $\{0\} \subsetneq I^2 \subsetneq I$, so I is not a minimal ideal. □

Example. Every nonzero ideal of $\mathbb{Z}$ is a cancellation ideal of $\mathbb{Z}$, so it is not a minimal ideal.

Converse of Theorem 2.3 is not true . For example , $\{0\} \subsetneq \langle x\rangle \subsetneq \langle 2, x\rangle$ in $\mathbb{Z}[x]$ and $\langle 2, x\rangle$ is not a cancellation ideal of $\mathbb{Z}[x]$.

Theorem 2.4. *$\langle 2, x\rangle$ is not a cancellation ideal of $\mathbb{Z}[x]$.*

Proof. We have that

$$\begin{aligned}\langle 2, x\rangle\langle 4, x^2\rangle &= \langle 8, 2x^2, 4x, x^3\rangle \\ &= \langle 2, x\rangle\langle 4, 2x, x^2\rangle.\end{aligned}$$

Suppose that $2x \in \langle 4, x^2\rangle$. Then there exist $f(x), g(x) \in \mathbb{Z}[x]$ such that

$$2x = 4f(x) + x^2 g(x).$$

Let $f(x) = \sum_{i=0}^{m} a_i x^i$ and $g(x) = \sum_{j=0}^{n} b_j x^j$ where $a_i, b_j \in \mathbb{Z}$ and $m, n \in \mathbb{N}$. Then

$$2x = \sum_{i=0}^{m} 4a_i x^i + \sum_{j=0}^{n} b_j x^{j+2}.$$

By comparing the coefficients, we get $2 = 4a_1$ which is impossible. Hence $2x \notin \langle 4, x^2\rangle$, so $\langle 4, x^2\rangle \neq \langle 4, 2x, x^2\rangle$. Therefore $\langle 2, x\rangle$ is not a cancellation ideal of $\mathbb{Z}[x]$. □

Theorem 2.5. *Let $a, b \in \mathbb{Z} \setminus \{0\}$. Then $\langle a, bx\rangle$ is a cancellation ideal of $\mathbb{Z}[x]$ if and only if $a \mid b$.*

Proof. Assume that $\langle a, bx\rangle$ is a cancellation ideal of $\mathbb{Z}[x]$. Since

$$\begin{aligned}\langle a, bx\rangle\langle a^2, abx, b^2x^2\rangle &= \langle a^3, a^2bx, ab^2x^2, b^3x^3\rangle \\ &= \langle a, bx\rangle\langle a^2, b^2x^2\rangle,\end{aligned}$$

$\langle a^2, abx, b^2x^2\rangle = \langle a^2, b^2x^2\rangle$. So $abx \in \langle a^2, b^2x^2\rangle$. There exist $f(x), g(x) \in \mathbb{Z}[x]$ such that

$$abx = a^2 f(x) + b^2x^2 g(x).$$

Then $f(x) = \sum_{i=0}^{m} a_i x^i$ *and* $g(x) = \sum_{j=0}^{n} b_j x^j$ for some $a_i, b_j \in \mathbb{Z}$ and $m, n \in \mathbb{N}$. Thus

$$abx \;=\; \sum_{i=0}^{m} a^2 a_i x^i + \sum_{j=0}^{n} b^2 b_j x^{j+2}\;.$$

By comparing the coefficients, we get $ab \;=\; a^2 a_1$, and so $b = aa_1$, that is $a \mid b$.

Assume that $a \mid b$. We have $\langle a, bx\rangle \;=\; \langle a\rangle$ which is a cancellation ideal of $\mathbb{Z}[x]$ by Lemma 1.2. $\square$

In Theorem 2.4, we consider an ideal generated by two elements of $\mathbb{Z}[x]$ which have no nonunit common factor. Next we will consider an ideal generated by two elements of $\mathbb{Z}[x]$ which have a nonunit common factor.

Example. $\langle (x-1)^2, x^2-1\rangle$ is not a cancellation ideal of $\mathbb{Z}[x]$.

Proof. We have

$$\begin{aligned}
&\langle (x-1)^2, x^2-1\rangle\langle (x-1)^4, (x^2-1)(x-1)^2, (x^2-1)^2\rangle\\
&=\; \langle (x-1)^6, (x^2-1)(x-1)^4, (x-1)^2(x^2-1)^2, (x^2-1)^3\rangle\\
&=\; \langle (x-1)^2, x^2-1\rangle\langle (x-1)^4, (x^2-1)^2\rangle.
\end{aligned}$$

Suppose that $(x^2-1)(x-1)^2 \in\; \langle (x-1)^4, (x^2-1)^2\rangle$. Then $(x^2-1)(x-1)^2$ $=\; f(x)(x-1)^4 + g(x)(x^2-1)^2$ for some $f(x), g(x) \in \mathbb{Z}[x]$. So

$$\begin{aligned}
(x-1)(x+1) \;&=\; f(x)(x-1)^2 + g(x)(x+1)^2,\\
(x-1)((x+1) - f(x)(x-1)) \;&=\; g(x)(x+1)^2. \qquad (2.1)
\end{aligned}$$

Since $x-1$ and $(x+1)^2$ are relatively prime , $x-1 \mid g(x)$. There exists $h_1(x) \in \mathbb{Z}[x]$ such that $g(x) \;=\; h_1(x)(x-1)$.

From (2.1), we get

$$\begin{aligned}
(x-1)((x+1) - f(x)(x-1)) \;&=\; h_1(x)(x-1)(x+1)^2,\\
(x+1) - f(x)(x-1) \;&=\; h_1(x)(x+1)^2,\\
(x+1)(1 - h_1(x)(x+1)) \;&=\; f(x)(x-1). \qquad (2.2)
\end{aligned}$$

Since $x-1$ and $x+1$ are relatively prime, $x+1 \mid f(x)$. There exists $h_2(x) \in \mathbb{Z}[x]$ such that $f(x) = h_2(x)(x+1)$.

From (2.2), we get

$$(x+1)(1-h_1(x)(x+1)) = h_2(x)(x+1)(x-1),$$
$$1 = h_1(x)(x+1) + h_2(x)(x-1).$$

Let $h_1(x) = \sum_{i=0}^{m} a_i x^i$ and $h_2(x) = \sum_{j=0}^{n} b_j x^j$ where $a_i, b_j \in \mathbb{Z}$ and $a_m, b_n \neq 0$ and $m, n \in \mathbb{N}$.

Since $1 = (\sum_{i=0}^{m} a_i x^i)(x+1) + (\sum_{j=0}^{n} b_j x^j)(x-1)$, $m = n$ and

$$1 = (a_0 - b_0) + (a_0 + a_1 + b_0 - b_1)x + (a_1 + a_2 + b_1 - b_2)x^2 + ... + (a_{n-1} + a_n + b_{n-1} - b_n)x^n + (a_n + b_n)x^{n+1}.$$

Thus

$$a_0 - b_0 = 1 ,$$
$$a_0 + b_0 + a_1 - b_1 = 0 ,$$
$$a_1 + b_1 + a_2 - b_2 = 0 ,$$
$$\cdots$$
$$a_{n-2} + b_{n-2} + a_{n-1} - b_{n-1} = 0 ,$$
$$a_{n-1} + b_{n-1} + a_n - b_n = 0 ,$$
$$a_n + b_n = 0 ,$$

so $2(a_0 + a_1 + ... + a_n) = 2a_0 + 2a_1 + ... + 2a_n = 1$, a contradiction. Then $(x^2-1)(x-1)^2 \notin \langle (x-1)^4, (x^2-1)^2 \rangle$.

Therefore, $\langle (x-1)^2, x^2-1 \rangle$ is not a cancellation ideal of $\mathbb{Z}[x]$. $\square$

Theorem 2.6 gives a necessary and sufficient condition for ideals generated by two nonzero elements of a unique factorization domain to be cancellation ideals.

Theorem 2.6. *Let R be a unique factorization domain, $a,\ b \in\ R \setminus \{0\}$ and d the greatest common divisor of a and b. Then $\langle a,b\rangle$ is a cancellation ideal of R if and only if $\langle a,b\rangle\ =\ \langle d\rangle$.*

Proof. Assume $\langle a,b\rangle = \langle d\rangle$. By Lemma 1.2, $\langle d\rangle$ is a cancellation ideal of R. Thus $\langle a,b\rangle$ is cancellation ideal of R.

Next, assume that $\langle a,b\rangle$ is a cancellation ideal of R. Since d is the greatest common divisor of a and b, $a = h_1d$ and $b = h_2d$ for some $h_1, h_2 \in\ R$ and h_1 and h_2 have no common factor. We have

$$
\begin{aligned}
&\langle a,b\rangle\langle a^2,b^2\rangle \\
&\quad = \langle a^3, a^2b, ab^2, b^3\rangle \\
&\quad = \langle a,b\rangle\langle a^2, ab, b^2\rangle.
\end{aligned}
$$

Since $\langle a,b\rangle$ is a cancellation ideal of R, $ab \in \langle a^2,b^2\rangle$. Thus $ab = \alpha a^2 + \beta b^2$ for some $\alpha, \beta \in R$. So

$$
\begin{aligned}
d^2h_1h_2 &= \alpha d^2h_1^2 + \beta d^2h_2^2, \\
h_1h_2 &= \alpha h_1^2 + \beta h_2^2 \ \text{, since } d \neq 0, \\
h_1(h_2 - \alpha h_1) &= \beta h_2^2. \qquad (2.3)
\end{aligned}
$$

Since h_1 and h_2 have no nonunit common factor , $h_1|\ \beta$. There exists $B \in R$ such that $\beta = h_1B$.

From (2.3),we get

$$
\begin{aligned}
h_1(h_2 - \alpha h_1) &= h_1Bh_2^2, \\
h_2 - \alpha h_1 &= Bh_2^2 \ \text{, since } h_1 \neq 0, \text{and so} \\
h_2(1 - Bh_2) &= \alpha h_1. \qquad (2.4)
\end{aligned}
$$

Since h_1 and h_2 have no nonunit common factor , $h_2|\ \alpha$. There exists $A \in R$ such that $\alpha\ =\ h_2A$.

From (2.4),we get

$$\begin{aligned} h_2(1 - Bh_2) = h_2Ah_1, & \\ 1 = Ah_1 + Bh_2 & \text{ ,since } h_2 \neq 0, \text{and so} \\ d = Ah_1d + Bh_2d & \\ = Aa + Bb. & \end{aligned}$$

Hence $\langle a, b\rangle = \langle d\rangle$. □

Corollary 2.7. *Let $f(x)$, $g(x) \in \mathbb{Z}[x] \setminus \{0\}$ and $d(x)$ the greatest common divisor of $f(x)$and $g(x)$. Then $\langle f(x), g(x)\rangle$ is a cancellation ideal of $\mathbb{Z}[x]$ if and only if $\langle f(x), g(x)\rangle = \langle d(x)\rangle$.*

Example. The ideal $\langle f(x)^n, f(x)^{n-1}g(x), ..., f(x)g(x)^{n-1}, g(x)^n\rangle$ is not a cancellation ideal of $\mathbb{Z}[x]$ for all $f(x), g(x) \in \mathbb{Z}[x]$ such that $\langle f(x), g(x)\rangle$ is not a principal ideal. This is because

$$\begin{aligned} &\langle f(x)^n, f(x)^{n-1}g(x), ..., f(x)g(x)^{n-1}, g(x)^n\rangle \\ &= \underbrace{\langle f(x), g(x)\rangle \cdots \langle f(x), g(x)\rangle}_{n \text{ copies}} \end{aligned}$$

and $\langle f(x), g(x)\rangle$ is not a cancellation ideal of $\mathbb{Z}[x]$ by Corollary 2.7 for all $f(x), g(x) \in \mathbb{Z}[x]$ such that $\langle f(x), g(x)\rangle$ is not a principal ideal.

Theorem 2.8. *Let $f(x), g(x), h(x) \in \mathbb{Z}[x] \setminus \{0\}$ be such that ax^m, bx^n and cx^l are the minimum degree monomials in $f(x), g(x), h(x)$,respectively. Suppose that $a \neq 0, a \nmid b$ and $0 \leq m < n \leq l$. Then $\langle f(x), g(x), h(x)\rangle$ is not a cancellation ideal of $\mathbb{Z}[x]$.*

Proof. We have

$$\begin{aligned}
&\langle f(x), g(x), h(x)\rangle\langle f(x)^2, g(x)^2, h(x)^2, g(x)h(x)\rangle \\
&= \langle f(x)^3, f(x)g(x)^2, f(x)h(x)^2, f(x)g(x)h(x), g(x)f(x)^2, g(x)^3, g(x)h(x)^2, \\
&\quad g(x)^2h(x), f(x)^2h(x), h(x)^3\rangle \\
&= \langle f(x), g(x), h(x)\rangle\langle f(x)^2, g(x)^2, h(x)^2, g(x)h(x), f(x)g(x)\rangle.
\end{aligned}$$

Suppose that $f(x)g(x) \in \langle f(x)^2, g(x)^2, h(x)^2, g(x)h(x)\rangle$. Then there exist $f_1(x)$, $f_2(x), f_3(x), f_4(x) \in \mathbb{Z}[x]$ such that

$f(x)g(x) = f_1(x)f(x)^2 + f_2(x)g(x)^2 + f_3(x)h(x)^2 + f_4(x)g(x)h(x)$.

Note that the minimum degree monomial in $f(x)g(x)$ is abx^{m+n}.

Since each nonzero term in $f_2(x)g(x)^2 + f_3(x)h(x)^2 + f_4(x)g(x)h(x)$, if exist, has degree at least $2n$, we have that abx^{m+n} is a term in $f_1(x)f(x)^2$. Since the minimum degree monomial in $f(x)^2$ is a^2x^{2m} and $a \neq 0$, the minimum degree monomial in $f_1(x)$ is dx^{n-m} for some $d \in \mathbb{Z}$. Thus $ab = a^2d$, so $a \mid b$, a contradiction.

Hence $f(x)g(x) \notin \langle f(x)^2, g(x)^2, h(x)^2, g(x)h(x)\rangle$,

so $\langle f(x)^2, g(x)^2, h(x)^2, g(x)h(x)\rangle \neq \langle f(x)^2, g(x)^2, h(x)^2, g(x)h(x), f(x)g(x)\rangle$, that is $\langle f(x), g(x), h(x)\rangle$ is not a cancellation ideal of $\mathbb{Z}[x]$. □

Example. Let $m \in \mathbb{N}$. Then $\langle 2, x^m\rangle = \langle 2, x^m, x^m\rangle$ is a cancellation ideal of $\mathbb{Z}[x]$ by Theorem 2.8.

Example. Let $h(x) \in \mathbb{Z}[x] \setminus \{0\}$ be such that its minimum degree monomial has degree at least 2. We have

$$\begin{aligned}
\langle 2+x, 2x+4x^2, h(x)\rangle &= \langle 2+x, (2x+4x^2) - (2+x)x, h(x)\rangle \\
&= \langle 2+x, 3x^2, h(x)\rangle.
\end{aligned}$$

Then $\langle 2+x, 2x+4x^2, h(x)\rangle$ is not a cancellation ideal of $\mathbb{Z}[x]$ by Theorem 2.8.

We consider ideals of $\mathbb{Z}[x, y]$ in Corollary 2.9-Theorem 2.16. Since $\mathbb{Z}$ is a unique factorization domain, $\mathbb{Z}[x, y]$ is a unique factorization domain.Corollary 2.9 follows from Theorem 2.6 directly.

Corollary 2.9. *Let $f(x,y)$, $g(x,y) \in \mathbb{Z}[x,y] \setminus \{0\}$ and $d(x,y)$ the greatest common divisor of $f(x,y)$ and $g(x,y)$. Then $\langle f(x,y), g(x,y)\rangle$ is a cancellation ideal of $\mathbb{Z}[x,y]$ if and only if $\langle f(x,y), g(x,y)\rangle = \langle d(x,y)\rangle$.*

Theorem 2.10. *Let $a, b, c \in \mathbb{Z} \setminus \{0\}$ and $m, n \in \mathbb{N}$.*
If $a \mid b$ and $a \mid c$, then $\langle a, bx^m, cy^n\rangle$ and $\langle a, cx^m, by^n\rangle$ are cancellation ideals of $\mathbb{Z}[x,y]$.
If $a \nmid b$ or $a \nmid c$, then $\langle a, bx^m, cy^n\rangle$ and $\langle a, cx^m, by^n\rangle$ are not cancellation ideals of $\mathbb{Z}[x,y]$.

Proof. Clearly , if $a \mid b$ and $a \mid c$, then $\langle a, bx^m, cy^n\rangle = \langle a\rangle = \langle a, cx^m, by^n\rangle$ is a cancellation ideal of $\mathbb{Z}[x,y]$ by Lemma 1.2.
Consider the cases $a \nmid b$ and $a \nmid c$.
Case 1: $a \nmid b$.
We have

$$\begin{aligned}
&\langle a, bx^m, cy^n\rangle\langle a^2, b^2x^{2m}, acy^n, c^2y^{2n}\rangle \\
=\ &\langle a^3, ab^2x^{2m}, a^2bx^m, ac^2y^{2n}, b^3x^{3m}, c^2bx^my^{2n}, ca^2y^n, cb^2y^nx^{2m}, abcx^my^n, c^3y^{3n}\rangle \\
=\ &\langle a, bx^m, cy^n\rangle\langle a^2, abx^m, acy^n, b^2x^{2m}, bcx^my^n, c^2y^{2n}\rangle.
\end{aligned}$$

Suppose that $abx^m \in \langle a^2, b^2x^{2m}, acy^n, c^2y^{2n}\rangle$. Then there exist $f_1(x,y)$, $f_2(x,y), f_3(x,y), f_4(x,y) \in \mathbb{Z}[x,y]$ such that
$abx^m = a^2f_1(x,y) + b^2x^{2m}f_2(x,y) + acy^nf_3(x,y) + c^2y^{2n}f_4(x,y)$.
Since each term in $b^2x^{2m}f_2(x,y)$ has degree at least $2m$ and each term in $acy^nf_3(x,y) + c^2y^{2n}f_4(x,y)$ is a multiple of y , abx^m must be a term of $a^2f_1(x,y)$.
Let $f_1(x,y) = \sum_{j=0}^{l}\sum_{i=0}^{k} a_{ij}x^iy^j$.
Then $ab = a_{m,0}a^2$, so $b = a_{m,0}a$ which contradicts to the fact that $a \nmid b$. Thus

$$\langle a^2, b^2x^{2m}, acy^n, c^2y^{2n}\rangle \neq \langle a^2, abx^m, acy^n, b^2x^{2m}, bcx^my^n, c^2y^{2n}\rangle.$$

Hence $\langle a, bx^m, cy^n\rangle$ is not a cancellation ideal.

By interchanging x and y we can also that $\langle a, cx^m, by^n\rangle$ is not a cancellation ideal of $\mathbb{Z}[x,y]$.

<u>Case 2</u>: $a \nmid c$.

By Case 1, we have immediately that $\langle a, cx^m, by^n\rangle$ and $\langle a, bx^m, cy^n\rangle$ are not cancellation ideal of $\mathbb{Z}[x,y]$. $\square$

Theorem 2.11. *Let $l = ni$ where $i, n \in \mathbb{N}$ and $n \geq 2$. Then $\langle x^i - y^i, x^l, y^l\rangle$ is not a cancellation ideal of* $\mathbb{Z}[x,y]$.

Proof. We have

$$\begin{aligned}
&\langle x^i - y^i, x^l, y^l\rangle\langle (x^i - y^i)^2, x^{2l}, y^{2l}, x^l y^l\rangle \\
&\quad = \langle (x^i-y^i)^3, (x^i-y^i)x^{2l}, (x^i-y^i)y^{2l}, (x^i-y^i)^2x^l, x^{3l}, x^ly^{2l}, (x^i-y^i)^2y^l, x^{2l}y^l, y^{3l}, \\
&\qquad (x^i - y^i)x^l y^l\rangle \\
&\quad = \langle x^i - y^i, x^l, y^l\rangle\langle (x^i - y^i)^2, x^{2l}, y^{2l}, (x^i - y^i)x^l, (x^i - y^i)y^l, x^l y^l\rangle.
\end{aligned}$$

Suppose that $(x^i - y^i)x^l \in \langle (x^i - y^i)^2, x^{2l}, y^{2l}, x^l y^l\rangle$. Then there exist $f_1(x,y)$, $f_2(x,y), f_3(x,y), f_4(x,y) \in \mathbb{Z}[x,y]$ such that

$(x^i - y^i)x^l = f_1(x,y)(x^i - y^i)^2 + f_2(x,y)x^{2l} + f_3(x,y)y^{2l} + f_4(x,y)x^l y^l$.

Since each term in $f_2(x,y)x^{2l}+f_3(x,y)y^{2l}+f_4(x,y)x^ly^l$ has degree at least $2l$, $x^{l+i} - x^l y^i$ must be a term in $f_1(x,y)(x^{2i} - 2x^i y^i + y^{2i})$.

Let $f_1(x,y) \;=\; \displaystyle\sum_{j=0}^{k}\sum_{m=0}^{p} a_{mj}x^m y^j$.

We may assume that $k \geq l$

Note that for all $0 \leq j \leq k$,

$$a_{0,j} \;=\; 0. \tag{2.5}$$

By comparing the coefficients of x^{l+i} and $x^l y^i$, we get

$$a_{(l-i),0} \;=\; a_{(n-1)i,0} \;=\; 1 \tag{2.6}$$

$$\text{and } a_{(l-2i),i} - 2a_{(l-i),0} = a_{(n-2)i,i} - 2a_{(n-1)i,0} = -1. \tag{2.7}$$

From (2.6) and (2.7),we have

$$a_{(n-2)i,i} = 1. \tag{2.8}$$

Let $r \neq n$ and $2 \leq r < n$.

By comparing the coefficients of $x^{l-(r-1)i}y^{ri} = x^{(n-(r-1))i}y^{ri}$, we get

$$a_{(n-r-1)i,ri} - 2a_{(n-r)i,(r-1)i} + a_{(n-r+1)i,(r-2)i} = 0,$$

$$a_{(n-r-1)i,ri} = 2a_{(n-r)i,(r-1)i} - a_{(n-r+1)i,(r-2)i}.$$

If $r = 2$, then

$$a_{(n-3)i,2i} = 2a_{(n-2)i,i} - a_{(n-1)i,0} = 1, \tag{2.9}$$

from (2.6) and (2.8).

If $r = 3$, then

$$a_{(n-4)i,3i} = 2a_{(n-3)i,2i} - a_{(n-2)i,i} = 1, \tag{2.10}$$

from (2.8) and (2.9).

Continue this process, if $r = n-1$, then $a_{0,(n-1)i} = 2a_{i,(n-2)i} - a_{2i,(n-3)i} = 1$, contradict to (2.5). Hence $(x^i - y^i)x^l \notin \langle (x^i - y^i)^2, x^{2l}, y^{2l}, x^l y^l \rangle$, so $\langle (x^i - y^i)^2, x^{2l}, y^{2l}, x^l y^l \rangle \neq \langle (x^i - y^i)^2, x^{2l}, y^{2l}, (x^i - y^i)x^l, (x^i - y^i)y^l, x^l y^l \rangle$. Therefore ,$\langle x^i - y^i, x^l, y^l \rangle$ is not a cancellation ideal of $\mathbb{Z}[x, y]$. □

Theorem 2.12. *Let $i, l \in \mathbb{N}$ be such that $i < l$. If $a \in \mathbb{Z} \setminus \{1, -1\}$, then $\langle x^i + a, x^l, y^l \rangle$ and $\langle y^i + a, x^l, y^l \rangle$ are not cancellation ideals of $\mathbb{Z}[x, y]$.*

Proof. Assume that $a \in \mathbb{Z} \setminus \{1, -1\}$. We prove only the case $\langle x^i + a, x^l, y^l \rangle$ is not a cancellation ideal of $\mathbb{Z}[x, y]$.

If $a = 0$, then $\langle x^i + a, x^l, y^l \rangle = \langle x^i, y^l \rangle$. By Corollary 2.9,$\langle x^i + a, x^l, y^l \rangle$ is not a cancellation ideal of $\mathbb{Z}[x,y]$.

Consider the case $a \in \mathbb{Z} \setminus \{1,-1,0\}$. We have

$$\begin{aligned}
&\langle x^i+a, x^l, y^l\rangle\langle (x^i+a)^2, x^{2l}, y^{2l}, x^ly^l\rangle \\
&= \langle (x^i+a)^3, (x^i+a)x^{2l}, (x^i+a)y^{2l}, (x^i+a)x^ly^l, (x^i+a)^2x^l, x^{3l}, x^ly^{2l}, x^{2l}y^l, \\
&\quad (x^i+a)^2y^l, y^{3l}\rangle \\
&= \langle x^i+a, x^l, y^l\rangle\langle (x^i+a)^2, x^{2l}, y^{2l}, (x^i+a)x^l, (x^i+a)y^l, x^ly^l\rangle.
\end{aligned}$$

Suppose that $(x^i+a)y^l \in \langle (x^i+a)^2, x^{2l}, y^{2l}, x^ly^l\rangle$. Then there exist $f_1(x,y), f_2(x,y)$, $f_3(x,y), f_4(x,y) \in \mathbb{Z}[x,y]$ such that

$$\begin{aligned}
(x^i+a)y^l &= f_1(x,y)(x^i+a)^2 + f_2(x,y)x^{2l} + f_3(x,y)y^{2l} + f_4(x,y)x^ly^l, \\
x^iy^l + ay^l &= f_1(x,y)(x^{2i}+2ax^i+a^2) + f_2(x,y)x^{2l} + f_3(x,y)y^{2l} + f_4(x,y)x^ly^l.
\end{aligned}$$

Since each term in $f_2(x,y)x^{2l} + f_3(x,y)y^{2l} + f_4(x,y)x^ly^l$ has degree at least $2l$, $x^iy^l + ay^l$ must be terms in $f_1(x,y)(x^{2i}+2ax^i+a^2)$.

Let $f_1(x,y) = \sum_{j=0}^{k}\sum_{m=0}^{p} a_{mj}x^my^j$.

By comparing the coefficients of y^l, we get $a = a^2a_{0,l}$. Since $a \neq 0, 1 = aa_{0,l}$. This implies that $a=1$ or $a=-1$, a contradiction.

Thus $\langle (x^i+a)^2, x^{2l}, y^{2l}, x^ly^l\rangle \neq \langle (x^i+a)^2, x^{2l}, y^{2l}, (x^i+a)x^l, (x^i+a)y^l, x^ly^l\rangle$.

Hence $\langle x^i+a, x^l, y^l\rangle$ is not a cancellation ideal of $\mathbb{Z}[x,y]$. □

Corollary 2.13. *Let $a \in \mathbb{Z}, i \in \mathbb{N}$ and $l = 2i$. Then $\langle x^i+a, x^l, y^l\rangle$ is a cancellation ideal of $\mathbb{Z}[x,y]$ if and only if $a \in \{1,-1\}$.*

Proof. ($\rightarrow$)Assume that $a \in \mathbb{Z} \setminus \{1,-1\}$. By Theorem 2.12 and $l = 2i > i$, $\langle x^i+a, x^l, y^l\rangle$ is not a cancellation ideal of $\mathbb{Z}[x,y]$.

($\leftarrow$) Assume that $a \in \{1,-1\}$. Then $x^l - 1 = x^{2i} - 1 = (x^i-1)(x^i+1)$ $\in \langle x^i+a, x^l, y^l\rangle$, so $1 = x^l - (x^l-1) \in \langle x^i+a, x^l, y^l\rangle$.

Thus $\langle x^i+a, x^l, y^l\rangle = \mathbb{Z}[x,y]$, which is a cancellation ideal of $\mathbb{Z}[x,y]$. □

Theorem 2.14. *$\langle x^k y^l\ ,\ x^m\ ,\ y^n \rangle$ is not a cancellation ideal of $\mathbb{Z}[x,y]$ for $1 \le k < m$ and $1 \le l < n$.*

Proof. We have

$$\begin{aligned}
&\langle x^k y^l, x^m, y^n \rangle \langle x^{2k} y^{2l}, x^{2m}, y^{2n}, x^{k+m} y^l \rangle \\
&= \langle x^{3k} y^{3l}, x^{2m+k} y^l, x^k y^{2n+l}, x^{2k+m} y^{2l}, x^{3m}, x^m y^{2n}, x^{2k} y^{2l+n}, x^{2m} y^n, \\
&\quad y^{3n}, x^{k+m} y^{l+n} \rangle \\
&= \langle x^k y^l, x^m, y^n \rangle \langle x^{2k} y^{2l}, x^{2m}, y^{2n}, x^{k+m} y^l, x^k y^{l+n}, x^m y^n \rangle.
\end{aligned}$$

Suppose that $x^k y^{l+n} \in \langle x^{2k} y^{2l}, x^{2m}, y^{2n}, x^{k+m} y^l \rangle$.

Then there exist $f_1(x,y), f_2(x,y), f_3(x,y), f_4(x,y) \in \mathbb{Z}[x,y]$ such that

$$x^k y^{l+n} = f_1(x,y) x^{2k} y^{2l} + f_2(x,y) x^{2m} + f_3(x,y) y^{2n} + f_4(x,y) x^{k+m} y^l. \quad (2.11)$$

Since each term in $f_1(x,y)x^{2k}y^{2l} + f_2(x,y)x^{2m} + f_4(x,y)x^{k+m}y^l$ has degree of x greater than k and each term in $f_3(x,y)y^{2n}$ has degree of y greater than $l+n$, it is impossible to write $x^k y^{l+n}$ as the sum in (2.11), a contradiction.

Thus $\langle x^{2k} y^{2l}, x^{2m}, y^{2n}, x^{k+m} y^l \rangle \neq \langle x^{2k} y^{2l}, x^{2m}, y^{2n}, x^{k+m} y^l, x^k y^{l+n}, x^m y^n \rangle$, so $\langle x^k y^l, x^m, y^n \rangle$ is not a cancellation ideal. □

Theorem 2.15. *Let $f(x,y), g(x,y), h(x,y) \in \mathbb{Z}[x,y] \setminus \{0\}$ be such that $ax^{m_1}y^{m_2}, bx^{n_1}y^{n_2}$ and $cx^{l_1}y^{l_2}$ are the minimum degree monomials in $f(x,y), g(x,y), h(x,y)$, respectively, where $a \neq 0, a \nmid b$ and $0 \le m_1 + m_2 < n_1 + n_2 \le l_1 + l_2$.*
Then $\langle f(x,y), g(x,y), h(x,y) \rangle$ is not a cancellation ideal of $\mathbb{Z}[x,y]$.

Proof. We have

$$\begin{aligned}
&\langle f(x,y), g(x,y), h(x,y) \rangle \langle f(x,y)^2, g(x,y)^2, h(x,y)^2, g(x,y)h(x,y) \rangle \\
&= \langle f(x,y)^3, f(x,y)g(x,y)^2, f(x,y)h(x,y)^2, f(x,y)g(x,y)h(x,y), g(x,y)f(x,y)^2, \\
&\quad g(x,y)^3, g(x,y)h(x,y)^2, g(x,y)^2h(x,y), f(x,y)^2h(x,y), h(x,y)^3 \rangle
\end{aligned}$$

$= \langle f(x,y), g(x,y), h(x,y)\rangle\langle f(x,y)^2, g(x,y)^2, h(x,y)^2, g(x,y)h(x,y),$
$f(x,y)g(x,y)\rangle$

Suppose that $f(x,y)g(x,y) \in \langle f(x,y)^2, g(x,y)^2, h(x,y)^2, g(x,y)h(x,y)\rangle$. Then there exist $f_1(x,y), f_2(x,y), f_3(x,y), f_4(x,y) \in \mathbb{Z}[x,y]$ such that
$f(x,y)g(x,y) = f_1(x,y)f(x,y)^2+f_2(x,y)g(x,y)^2+f_3(x,y)h(x,y)^2+f_4(x,y)g(x,y)h(x,y)$
Consider the term $abx^{m_1+n_1}y^{m_2+n_2}$ in $f(x,y)g(x,y)$.
Since each nonzero term in $f_2(x,y)g(x,y)^2+f_3(x,y)h(x,y)^2+f_4(x,y)g(x,y)h(x,y)$, if exist, has degree at least $2n_1 + 2n_2$,we have $abx^{m_1+n_1}y^{m_2+n_2}$ is the term in $f_1(x,y)f(x,y)^2$. Since the minimum degree monomial in $f(x,y)^2$ is $a^2x^{2m_1}y^{2m_2}$ and $a \neq 0$, the minimum degree monomial in $f_1(x,y)$ is $dx^{n_1-m_1}y^{n_2-m_2}$ for some $d \in \mathbb{Z}$. Thus $ab = a^2d$, so $a \mid b$, a contradiction.
Hence $f(x,y)g(x,y) \notin \langle f(x,y)^2, g(x,y)^2, h(x,y)^2, g(x,y)h(x,y)\rangle$, so
$\langle f(x,y)^2, g(x,y)^2, h(x,y)^2, g(x,y)h(x,y)\rangle \neq$
$\langle f(x,y)^2, g(x,y)^2, h(x,y)^2, g(x,y)h(x,y), f(x,y)g(x,y)\rangle$,
that is $\langle f(x,y), g(x,y), h(x,y)\rangle$ is not a cancellation ideal. $\square$

Theorem 2.16. *Let $f(x), g(x), h(y) \in \mathbb{Z}[x,y] \smallsetminus \{0\}$ be such that $h(y)$ is a polynomial which has no the constant term.*
Assume that $\langle f(x), g(x), h(y)\rangle$ is not an ideal generated by one or two polynomials in $\mathbb{Z}[x,y]$. Then $\langle f(x), g(x), h(y)\rangle$ is not a cancellation ideal of $\mathbb{Z}[x,y]$.

Proof. We have

$$\begin{aligned}
&\langle f(x), g(x), h(y)\rangle\langle f(x)^2, g(x)^2, h(y)^2, g(x)h(y)\rangle\\
&\quad= \langle f(x)^3, f(x)g(x)^2, f(x)h(y)^2, f(x)g(x)h(y), g(x)f(x)^2,\\
&\quad\ g(x)^3, g(x)h(y)^2, g(x)^2h(y), f(x)^2h(y), h(y)^3\rangle\\
&\quad= \langle f(x), g(x), h(y)\rangle\langle f(x)^2, g(x)^2, h(y)^2, g(x)h(y), f(x)g(x)\rangle.
\end{aligned}$$

Suppose that $f(x)g(x) \in \langle f(x)^2, g(x)^2, h(y)^2, g(x)h(y)\rangle$. Then there exist $f_1(x,y)$,

$f_2(x,y), f_3(x,y), f_4(x,y) \in \mathbb{Z}[x,y]$ such that

$$\begin{aligned} f(x)g(x) &= f_1(x,y)f(x)^2 + f_2(x,y)g(x)^2 + f_3(x,y)h(y)^2 + f_4(x,y)g(x)h(y) \\ &= (f_{1x}(x,y) + f_{1y}(x,y))f(x)^2 + (f_{2x}(x,y) + f_{2y}(x,y))g(x)^2 + \\ &\quad f_3(x,y)h(y)^2 + f_4(x,y)g(x)h(y), \end{aligned}$$

where $f_{ix}(x,y)$ is the partial polynomial of $f_i(x,y)$ which has no terms in y and $f_{iy}(x,y) = f_i(x,y) - f_{ix}(x,y)$ for all i =1,2.

Then $f(x)g(x) = (f_{1x}(x,y)f(x)^2 + f_{2x}(x,y)g(x)^2) + (f_{1y}(x,y)f(x)^2$

$+f_{2y}(x,y)g(x)^2 + f_3(x,y)h(y)^2 + f_4(x,y)g(x)h(y))$.

This implies $f_{1y}(x,y)f(x)^2 + f_{2y}(x,y)g(x)^2 + f_3(x,y)h(y)^2 + f_4(x,y)g(x)h(y) = 0$, so

$$f(x)g(x) = f_{1x}(x,y)f(x)^2 + f_{2x}(x,y)g(x)^2. \tag{2.12}$$

We can write that $f(x) = d(x)\alpha(x)$ and $g(x) = d(x)\beta(x)$ for some $d(x), \alpha(x), \beta(x)$ $\in \mathbb{Z}[x,y]$, and $\alpha(x)$ and $\beta(x)$ have no nonunit common factor in $\mathbb{Z}[x,y]$.

By (2.12),

$$\begin{aligned} d(x)\alpha(x)d(x)\beta(x) &= f_{1x}(x,y)d(x)^2\alpha(x)^2 + f_{2x}(x,y)d(x)^2\beta(x)^2, \\ \alpha(x)\beta(x) &= f_{1x}(x,y)\alpha(x)^2 + f_{2x}(x,y)\beta(x)^2, \\ \alpha(x)(\beta(x) - f_{1x}(x,y)\alpha(x)) &= f_{2x}(x,y)\beta(x)^2. \end{aligned} \tag{2.13}$$

Since $\alpha(x)$ and $\beta(x)$ have no nonunit common factor , $\alpha(x) \mid f_{2x}(x,y)$. Then $f_{2x}(x,y) = \alpha(x)h_1(x,y)$ for some $h_1(x,y) \in \mathbb{Z}[x,y]$.

By (2.13),

$$\begin{aligned} \alpha(x)(\beta(x) - f_{1x}(x,y)\alpha(x)) &= \alpha(x)h_1(x,y)\beta(x)^2, \\ \beta(x) &= f_{1x}(x,y)\alpha(x) + h_1(x,y)\beta(x)^2, \text{since } \alpha(x) \neq 0, \\ \beta(x)(1 - h_1(x,y)\beta(x)) &= f_{1x}(x,y)\alpha(x). \end{aligned} \tag{2.14}$$

Since $\alpha(x)$ and $\beta(x)$ have no nonunit common factor , $\beta(x) \mid f_{1x}(x,y)$.Thus $f_{1x}(x,y) = \beta(x)h_2(x,y)$ for some $h_2(x,y) \in \mathbb{Z}[x,y]$.

By (2.14),

$$\beta(x)(1-h_1(x,y)\beta(x)) = \beta(x)h_2(x,y)\alpha(x),$$
$$1 = h_1(x,y)\beta(x) + h_2(x,y)\alpha(x), \text{since } \beta(x) \neq 0,$$
$$d(x) = h_1(x,y)d(x)\beta(x) + h_2(x,y)d(x)\alpha(x),$$
$$d(x) = h_1(x,y)g(x) + h_2(x,y)f(x).$$

That is $\langle d(x)\rangle = \langle f(x), g(x)\rangle$, so $\langle f(x), g(x), h(y)\rangle = \langle f(x), g(x)\rangle + \langle h(y)\rangle$
$= \langle d(x)\rangle + \langle h(y)\rangle$
$= \langle d(x), h(y)\rangle$, a contradiction.

Then $f(x)g(x) \notin \langle f(x)^2, g(x)^2, h(y)^2, g(x)h(y)\rangle$,
so $\langle f(x)^2, g(x)^2, h(y)^2, g(x)h(y)\rangle \neq \langle f(x)^2, g(x)^2, h(y)^2, g(x)h(y), f(x)g(x)\rangle$.
Hence $\langle f(x), g(x), h(y)\rangle$ is not a cancellation ideal of $\mathbb{Z}[x,y]$. □

Theorem 2.17. *Every proper ideal I in any Boolean ring R with 1 is not a cancellation ideal of R.*

Proof. Let I be a proper ideal in a Boolean ring R with 1. Then $I^2 = I = IR$. But $I \neq R$, so I is not a cancellation ideal of R. □

Theorem 2.18. *Let R be a commutative Artinian ring with 1 , and suppose that R is not a field.Then the following statements hold.*
(i) For all $a \in R$ such that $\langle a\rangle \neq R$, $\langle a\rangle$ is not a cancellation ideal of R.
(ii)For all $b, c \in R$ such that $\langle b, c\rangle \neq R$, $\langle b, c\rangle$ is not a cancellation ideal of R.

Proof. (i) Let $a \in R$. We have $\langle a\rangle \supseteq \langle a^2\rangle \supseteq \langle a^3\rangle \supseteq \cdots$. Since R satisfies the descending chain condition, choose the smallest positive integer n such that $\langle a^n\rangle = \langle a^{n+1}\rangle$.
Case 1: $n = 1$.

Since $\langle a\rangle R = \langle a\rangle = \langle a^2\rangle = \langle a\rangle\langle a\rangle$ and $\langle a\rangle \neq R$, $\langle a\rangle$ is not a cancellation ideal of R.

Case 2: $n > 1$.

Since $\langle a\rangle\langle a^{n-1}\rangle = \langle a^n\rangle = \langle a^{n+1}\rangle = \langle a\rangle\langle a^n\rangle$ and

$\langle a^{n-1}\rangle \neq \langle a^n\rangle$, $\langle a\rangle$ is not a cancellation ideal of R.

(ii) We have $\langle b, c\rangle \supseteq \langle b^2, bc, c^2\rangle \supseteq \langle b^3, b^2c, bc^2, c^3\rangle \supseteq \cdots$. Since R satisfies the descending chain condition , choose the smallest positive integer n such that $\langle b^n, b^{n-1}c, ..., bc^{n-1}, c^n\rangle = \langle b^{n+1}, b^n c, ..., bc^n, c^{n+1}\rangle$.

Case 1: $n = 1$.

Since $\langle b, c\rangle R = \langle b, c\rangle = \langle b^2, bc, c^2\rangle = \langle b, c\rangle\langle b, c\rangle$ and $\langle b, c\rangle \neq R$, $\langle b, c\rangle$ is not a cancellation ideal of R.

Case 2: $n > 1$.

Since $\langle b, c\rangle\langle b^{n-1}, b^{n-2}c, ..., bc^{n-2}, c^{n-1}\rangle = \langle b^n, b^{n-1}c, ..., bc^{n-1}, c^n\rangle$

$= \langle b^{n+1}, b^n c, ..., bc^n, c^{n+1}\rangle = \langle b, c\rangle\langle b^n, b^{n-1}c, ..., bc^{n-1}, c^n\rangle$ and

$\langle b^{n-1}, b^{n-2}c, ..., bc^{n-2}, c^{n-1}\rangle \neq \langle b^n, b^{n-1}c, ..., bc^{n-1}, c^n\rangle$, $\langle b, c\rangle$ is not a cancellation ideal of R. $\square$

CHAPTER III

CANCELLATION IDEALS BELONGING TO IDEALS

In this chapter, we consider cancellation ideals of $\mathbb{Z}[x]$ belonging to ideals $\langle 2, x^m \rangle$, which we have already showed in the example follow from Theorem 2.8 that it is not a cancellation ideal of $\mathbb{Z}[x]$ for all $m \in \mathbb{N}$.

Theorem 3.1. *Let $a \in \mathbb{Z} \setminus \{0\}$ and J a cancellation ideal belonging to ideal $\langle x+a, x^2 \rangle$. Suppose that $J \cap \mathbb{Z}$ is a prime ideal of $\mathbb{Z}$. Then $J = \mathbb{Z}[x]$.*

Proof. Since $x^2 - a^2 = (x+a)(x-a)$ and $x^2 \in J,\ a^2 = x^2 - (x^2 - a^2) \in J$.
By the assumption, $a \in J$, so $x = (x+a) - a \in J$.
If $a = 1$ or $a = -1$, then $J = \mathbb{Z}[x]$.
Next, consider the case $a \in \mathbb{Z} \setminus \{0, 1, -1\}$.
Clearly, $|a| \in J$. Assume that $|a| = p_1^{r_1} p_2^{r_2} \cdots p_n^{r_n}$, where p_i is a prime divisor of a, and $r_i \in \mathbb{N}$ for all $i \in \{1, 2, ..., n\}$.
Since $p_1^{r_1} p_2^{r_2} \cdots p_n^{r_n} = |a| \in J,\ p_i \in J$ for some $i \in \{1, 2, ..., n\}$, by the assumption.
Thus $\langle p_i, x \rangle \subseteq J \subseteq \mathbb{Z}[x]$. By Theorem 2.5, $\langle p_i, x \rangle$ is not a cancellation ideal of $\mathbb{Z}[x]$. Since J is a cancellation ideal of $\mathbb{Z}[x]$ and $\langle p_i, x \rangle$ is a maximal ideal in $\mathbb{Z}[x],\ J = \mathbb{Z}[x]$. □

The following two examples give us for the motivation of Theorem 3.2.

Example. Let J be a cancellation ideal belonging to ideal $\langle 2, x^2 \rangle$. We claim that $J = \mathbb{Z}[x]$.

Let $I = \langle 2, x^2\rangle$. Since I is not a cancellation ideal of $\mathbb{Z}[x]$, $I \subsetneq J$. Since $x^2 \in I$, there exists $g(x) = a + bx \in J \setminus I$ for some $a, b \in \mathbb{Z}$.

Case 1: b is even.

Then $a = a + bx - 2mx \in J \setminus I$ where $m = \frac{b}{2}$, so a must be odd and $a \in J \setminus I$. Now J is an ideal of $\mathbb{Z}[x]$ contains 2 and the odd number a, so $1 \in J$. Thus $J = \mathbb{Z}[x]$ as required.

Case 2: b is odd.

Subcase 2.1 : a is even.

Then $bx \in J$ and $x = bx - (b-1)x \in J \setminus I$ since $b - 1 \in \langle 2\rangle \subseteq J$. This implies that the maximal ideal $\langle 2, x\rangle$ is contained in J. Since $\langle 2, x\rangle$ is not a cancellation ideal of $\mathbb{Z}[x]$, $J = \mathbb{Z}[x]$.

Subcase 2.2: a is odd.

Then $a - 1$, $b - 1 \in \langle 2\rangle$ and $1 + x = a + bx - [(a-1) + (b-1)x] \in J \setminus I$. Now $1+x$, $x^2 \in J$ and $x = (1+x)x - x^2 \in J$, we have that $\langle 2, x\rangle \subseteq J$. Hence $J = \mathbb{Z}[x]$. Therefore, $\mathbb{Z}[x]$ is the only cancellation ideal of $\mathbb{Z}[x]$ belonging to ideal I.

Example. Let J be a cancellation ideal belonging to ideal $\langle 2, x^3\rangle$. We claim that $J = \mathbb{Z}[x]$.

As in the previous example, there exists a polynomial $a + bx + cx^2 \in J \setminus I$ where $a, b, c \in \mathbb{Z}$. Since $a + bx + cx^2 \notin I$, at least one of integers a, b, c must be odd.

Case 1: a is even.

Then $bx + cx^2 \in J \setminus I$ and one of b or c must be odd. Since $2 \in J$, we may assume that b, $c \in \{0, 1\}$

Subcase 1.1: $b = 1$ and $c = 0$.

Then $x \in J$. Since $\langle 2, x\rangle$ is a maximal ideal which is not a cancellation ideal, $\langle 2, x\rangle \subsetneq J$. Thus $J = \mathbb{Z}[x]$.

Subcase 1.2: $b = 0$ and $c = 1$.

Then $x^2 \in J$, so J is a cancellation ideal belonging to the ideal $\langle 2, x^2 \rangle$. By the previous example, $J = \mathbb{Z}[x]$.

Subcase 1.3: $b = c = 1$

We have $x + x^2, x^3 \in J$. Since

$$\begin{aligned} \langle x + x^2, x^3 \rangle &= \langle x + x^2, x^3 - (x + x^2)x \rangle \\ &= \langle x + x^2, x^2 \rangle \\ &= \langle (x + x^2) - x^2, x^2 \rangle \\ &= \langle x, x^2 \rangle \\ &= \langle x \rangle, \end{aligned}$$

$x \in J$.

Thus $\langle 2, x \rangle \subsetneq J$, so $J = \mathbb{Z}[x]$.

Case 2: a is odd.

We may assume that $a = 1$ and $b, c \in \{0, 1\}$.

Subcase 2.1: $b = c = 0$.

Thus $1 = a + bx + cx^2 \in J$, so $J = \mathbb{Z}[x]$.

Subcase 2.2: $b = 0$ and $c = 1$.

We have $1 + x^2, x^3 \in J$. Since

$$\begin{aligned} \langle 1 + x^2, x^3 \rangle &= \langle 1 + x^2, x^3 - (1 + x^2)x \rangle \\ &= \langle 1 + x^2, x \rangle, \end{aligned}$$

$x \in J$. Then $\langle 2, x \rangle \subsetneq J$ and we get $J = \mathbb{Z}[x]$.

Subcase 2.3: $b = 1$ and $c = 0$.

We have $1+x \in J$. Since $\langle 1+x, x^3\rangle \subseteq J$ and

$$\begin{aligned}\langle 1+x, x^3\rangle &= \langle 1+x, x^3-(1+x)x^2\rangle \\ &= \langle 1+x, x^2\rangle,\end{aligned}$$

$x^2 \in J$. Thus $\langle 2, x^2\rangle \subseteq J$, and we have $J = \mathbb{Z}[x]$ by the previous example.

Subcase 2.4: $b=c=1$.

We have $1+x+x^2 \in J$. Since

$$\begin{aligned}\langle 1+x+x^2, x^3\rangle &= \langle 1+x+x^2, x^3-(1+x+x^2)x\rangle \\ &= \langle 1+x+x^2, x+x^2\rangle, \\ &= \langle (1+x+x^2)-(x+x^2), x+x^2\rangle \\ &= \langle 1, x+x^2\rangle \\ &= \mathbb{Z}[x],\end{aligned}$$

$J = \mathbb{Z}[x]$.

Therefore, $\mathbb{Z}[x]$ is the only cancellation ideal of $\mathbb{Z}[x]$ belonging to I.

Theorem 3.2. *Let $m \in \mathbb{N}$ and J a cancellation ideal belonging to the ideal $\langle 2, x^m\rangle$. Then $J = \mathbb{Z}[x]$.*

Proof. We will prove the theorem by induction on m. Assume that J is a cancellation ideal belonging to ideal $\langle 2, x^m\rangle$.

The case of $m=1$ is obtained from the fact that $\langle 2, x\rangle$ is a maximal ideal which is not a cancellation ideal of $\mathbb{Z}[x]$. Next, let $m \geq 2$. Suppose that the statement is true for $\langle 2, x^l\rangle$ for all $l \in \{1, 2, ..., m-1\}$.

Let $I = \langle 2, x^m\rangle$. Since I is not a cancellation ideal of $\mathbb{Z}[x]$, $I \subsetneq J$. Then there exists $f(x) \in J \setminus I$, say $f(x) = \sum_{i=0}^{n} a_i x^i$, where $a_i \in \mathbb{Z}$ for all $i \in \{0, 1, \ldots, n\}$.

If $n \geq m$, then let $g(x) = a_m + a_{m+1}x + ... + a_n x^{n-m}$.

Thus $h(x) := f(x) - x^m g(x) \in J \setminus I$ and $\deg h(x) < m$.

If $n < m$, then let $h(x) := f(x)$.

Assume that $h(x) = b_{m-1}x^{m-1} + b_{m-2}x^{m-2} + ... + b_1x + b_0$, $b_j \in \mathbb{Z}$ for all $j \in \{0, 1, ..., m-1\}$.

Since $2 \in J$, we may assume that $b_j \in \{0, 1\}$ for all $j \in \{0, 1, ..., m-1\}$. Clearly, there exists $p \in \{0, 1, ..., m-1\}$ such that $b_p \neq 0$.

Let $s =$ the number of nonzero coefficients of $h(x)$.

Case 1: $s = m$.

Then $x^m - 1 = (x-1)(x^{m-1} + x^{m-2} + ... + x + 1) = (x-1)h(x) \in J$.

Since $x^m - 1 \in J$ and $x^m \in J$, $1 = x^m - (x^m - 1) \in J$. Then $J = \mathbb{Z}[x]$.

Case 2: $s = 1$.

That is, there exists $k \in \{0, 1, ..., m-1\}$ such that $b_k = 1$ and $b_j = 0$ for all $j \in \{0, 1, ..., m-1\} \setminus \{k\}$.

If $k = 0$, then $1 = h(x) \in J$, so $J = \mathbb{Z}[x]$.

If $1 \leq k \leq m-1$, $h(x) = b_kx^k = x^k$, so $\langle 2, x^k \rangle \subseteq J$. By induction hypothesis, $J = \mathbb{Z}[x]$.

Case 3: $1 < s < m$.

Let $r = deg\ h(x)$. Since $x^m \in J$ and $x^{m-r}h(x) \in J$, $h_1(x) := x^{m-r}h(x) - x^m \in J$.

Let d_1 be the number of nonzero terms of $h_1(x)$.

If $d_1 = 1$, then $h_1(x) = x^{i_1}$ for some $i_1 \in \{1, ..., m-1\}$. Thus $\langle 2, x^{i_1} \rangle \subseteq J$, so $J = \mathbb{Z}[x]$, by induction hypothesis.

If $d_1 > 1$, let $n_1 = \deg\ h_1(x)$. Since $x^m \in J$ and $x^{m-n_1}h_1(x) \in J$, $h_2(x) := x^{m-n_1}h_1(x) - x^m \in J$. Thus the number of nonzero terms of $h_2(x)$ is less than the number of nonzero terms of $h_1(x)$.

Let d_2 be the number of nonzero terms of $h_2(x)$.

If $d_2 = 1$, then $h_2(x) = x^{i_2}$ for some $i_2 \in \{1, ..., m-1\}$. Thus $\langle 2, x^{i_2} \rangle \subseteq J$, so $J = \mathbb{Z}[x]$, by induction hypothesis.

Continue this process, there exists $t \in \mathbb{N}$ such that $h_t(x) = x^{i_t} \in J$ for some $i_t \in \{1, ..., m-1\}$. Thus $\langle 2, x^{i_t} \rangle \subseteq J$, so $J = \mathbb{Z}[x]$, by induction hypothesis.

Therefore, $J = \mathbb{Z}[x]$ for all $m \in \mathbb{N}$. $\square$

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