CHAPTER III

THE CHURCH-ROSSER THEOREM

The main topic of this chapter is the Church-Rosser theorem of which we will state and prove three versions. In the first section we state and prove some general lemmas which are used in the proof of the theorem. In the second section we define residuals and minimal complete developments (MCD's), two concepts which are the keys to proving the theorem. The Church-Rosser theorem for $\beta\delta$-reduction is proved in the third section by first proving a Church-Rosser theorem for MCD's. In the last section, we define $\beta\delta$-equality and state and prove relevant results about it.

3.1 Preliminary Lemmas

Lemma 3.1.1. Let R be a potential redex and P be a pattern. If $[N_1/x_1,\dots, N_k/x_k]P \equiv R$ for some distinct variables $x_1,\dots, x_k$, $k \geq 1$, and some terms $N_1,\dots, N_k$, then $P \equiv x_t$ for some $1 \leq t \leq k$.

Proof. Assume $[N_1/x_1,\dots, N_k/x_k]P \equiv R$ for some distinct variables $x_1,\dots, x_k$, $k \geq 1$, and some terms $N_1,\dots, N_k$. Suppose $P \not\equiv x_i$ for all $1 \leq i \leq k$. If P is an atom, then $P \equiv [N_1/x_1,\dots, N_k/x_k]P \equiv R$, which is a contradiction, since R contains an abstraction. Hence $P \equiv P_1P_2$ for some patterns $P_1$and P_2, where P_1 is not a variable.

Since R is a potential redex, $R \equiv AN$ for some abstraction A and some term N. So we have $AN \equiv R \equiv [N_1/x_1,\dots, N_k/x_k]P \equiv [N_1/x_1,\dots, N_k/x_k]P_1[N_1/x_1,\dots, N_k/x_k]P_2$. By Note 2.1.3(b), $A \equiv [N_1/x_1,\dots, N_k/x_k]P_1$. Since P_1 is not a variable, by Lemma 2.1.10(b) $[N_1/x_1,\dots, N_k/x_k]P_1$ is of the same form as P_1. This implies P_1 is an abstraction, which is impossible. □

Lemma 3.1.2. Let $\lambda P.Q$ be a simple abstraction with $FV(P) = \{x_1,..., x_k\}$, $k \geq 1$, and N be a term such that $\lambda P.Q \equiv_\alpha N$. Then $N \equiv \lambda[y_1/x_1,\ldots, y_k/x_k]P.Q'$ for some distinct variables $y_1,\ldots, y_k$ and some term Q' such that $\{y_1,\ldots, y_k\} \cap FV(\lambda P.Q) = \varnothing$ and $Q' \equiv_\alpha [y_1/x_1,\ldots, y_k/x_k]Q$.

Proof. Since $\lambda P.Q \equiv_\alpha N$, there exists a sequence of terms $\lambda P.Q \equiv A_1, A_2,\ldots, A_n \equiv N$, $n \geq 1$, such that for each $1 \leq i < n$, A_{i+1} is obtained from A_i by a single change of bound variable. Induct on n.

If $n = 1$, then, by Corollary 2.1.12(c) $N \equiv \lambda P.Q \equiv \lambda[x_1/x_1,..., x_k/x_k]P.Q$, where $\{x_1,..., x_k\} \cap FV(\lambda P.Q) = \varnothing$ and $Q \equiv [x_1/x_1,..., x_k/x_k]Q$.

Now suppose $n > 1$. Since $\lambda P.Q \equiv_\alpha A_{n-1}$, by induction $A_{n-1} \equiv \lambda[y_1/x_1,\ldots, y_k/x_k]P.Q'$ for some distinct variables $y_1,\ldots, y_k$ and some term Q', where $\{y_1,\ldots, y_k\} \cap FV(\lambda P.Q) = \varnothing$ and $Q' \equiv_\alpha [y_1/x_1,\ldots, y_k/x_k]Q$.

Since N is obtained from A_{n-1} by a single change of bound variable, there are two cases as follows.

Case 1. No variable in P has been changed.

By Lemma 2.2.4, $N \equiv \lambda[y_1/x_1,\ldots, y_k/x_k]P.Q''$ for some term Q'', where $Q'' \equiv_\alpha Q'$. Then we are finished since $Q'' \equiv_\alpha [y_1/x_1,\ldots, y_k/x_k]Q$.

Case 2. Some variable in P has been changed.

Then $N \equiv \lambda[w/y_t][y_1/x_1,\ldots, y_k/x_k]P.[w/y_t]Q'$ for some $1 \leq t \leq k$, and some variable w, where $w \notin FV(([y_1/x_1,\ldots, y_k/x_k]P)Q')$. Without loss of generality, assume $t = 1$. By Corollary 2.1.17(a), $N \equiv \lambda[w/x_1, y_2/x_2,\ldots, y_k/x_k]P.[w/y_1]Q'$, so it only remains to show that $[w/y_1]Q' \equiv_\alpha [w/x_1, y_2/x_2,..., y_k/x_k]Q$ and $\{w, y_2,..., y_k\} \cap FV(\lambda P.Q) = \varnothing$.

Suppose $w \in FV(\lambda P.Q)$. Since $Q' \equiv_\alpha [y_1/x_1,\ldots, y_k/x_k]Q$, by Lemma 2.2.5 $FV(Q') = FV([y_1/x_1,\ldots, y_k/x_k]Q)$. So we have

$w \in FV(Q) - \{x_1,..., x_k\} \subseteq FV([y_1/x_1,..., y_k/x_k]Q) = FV(Q')$, a contradiction. Hence $\{w, y_2,\ldots, y_k\} \cap FV(\lambda P.Q) = \varnothing$. Finally, by Lemma 2.2.7 and Corollary 2.2.8 we have $[w/y_1]Q' \equiv_\alpha [w/y_1][y_1/x_1,..., y_k/x_k]Q \equiv_\alpha [w/x_1, y_2/x_2,..., y_k/x_k]Q$. □

Lemma 3.1.3. Let P be a pattern with $FV(P) = \{x_1,..., x_k\}$, $k \geq 1$, $y_1,\ldots, y_k$ be distinct variables, and Q be a term. If $\{y_1,\ldots, y_k\} \cap FV(\lambda P.Q) = \varnothing$, then
$\lambda P.Q \equiv_\alpha \lambda[y_1/x_1,..., y_k/x_k]P.[y_1/x_1,..., y_k/x_k]Q$.

Proof. Assume $\{y_1,\ldots, y_k\} \cap FV(\lambda P.Q) = \varnothing$. Let $S = \{i \mid 1 \leq i \leq k \text{ and } y_i \not\equiv x_i\}$ and $|S| = m$ and induct on m.

Suppose m = 0, so that $y_i \equiv x_i$ for all $1 \leq i \leq k$. Hence
$\lambda P.Q \equiv \lambda[y_1/x_1,..., y_k/x_k]P.[y_1/x_1,..., y_k/x_k]Q$.

Now assume m > 0. Without loss of generality, assume $y_1 \not\equiv x_1$.

Case 1. $x_1 \not\equiv y_i$ for all $1 < i \leq k$.

Then $x_1 \notin FV([y_1/x_1,..., y_k/x_k](PQ))$. By the above assumption,

$$\begin{aligned} FV(Q) \cap (\{y_1,..., y_k\} - \{x_1,..., x_k\}) &= FV(Q) \cap (\{y_1,\ldots, y_k\} - FV(P)) \\ &= \{y_1,\ldots, y_k\} \cap FV(\lambda P.Q) = \varnothing. \end{aligned}$$

Hence $\lambda[y_1/x_1,..., y_k/x_k]P.[y_1/x_1,..., y_k/x_k]Q$

$\equiv_\alpha \lambda[x_1/y_1][y_1/x_1,..., y_k/x_k]P.[x_1/y_1][y_1/x_1,\ldots, y_k/x_k]Q$

$\equiv \lambda[x_1/x_1, y_2/x_2,\ldots, y_k/x_k]P.[x_1/y_1][y_1/x_1,\ldots, y_k/x_k]Q$ (by Corollary 2.1.17)

$\equiv_\alpha \lambda[x_1/x_1, y_2/x_2,\ldots, y_k/x_k]P.[x_1/x_1, y_2/x_2,\ldots, y_k/x_k]Q$ (by Corollary 2.2.8)

$\equiv_\alpha \lambda P.Q.$ (by induction)

Case 2. $x_1 \equiv y_t$ for some $1 < t \leq k$.

Without loss of generality, assume t = 2. Note that $y_2 \not\equiv x_2$ since $x_1 \not\equiv x_2$.

Choose a variable $w \notin FV(x_1\ldots x_k y_1\ldots y_k PQ)$. Then

$\lambda[y_1/x_1,..., y_k/x_k]P.[y_1/x_1,..., y_k/x_k]Q$

$\equiv \lambda[y_1/x_1, x_1/x_2, y_3/x_3,\ldots, y_k/x_k]P.[y_1/x_1, x_1/x_2, y_3/x_3,\ldots, y_k/x_k]Q$

$\equiv_\alpha \lambda[w/x_1][y_1/x_1, x_1/x_2, y_3/x_3,\ldots, y_k/x_k]P.[w/x_1][y_1/x_1, x_1/x_2, y_3/x_3,\ldots, y_k/x_k]Q$

$\equiv_\alpha \lambda[y_1/x_1, w/x_2, y_3/x_3,\ldots, y_k/x_k]P.[y_1/x_1, w/x_2, y_3/x_3,\ldots, y_k/x_k]Q$

$\equiv_\alpha \lambda[x_1/y_1][y_1/x_1, w/x_2, y_3/x_3, \ldots, y_k/x_k]P.[x_1/y_1][y_1/x_1, w/x_2, y_3/x_3, \ldots, y_k/x_k]Q$

$\equiv_\alpha \lambda[x_1/x_1, w/x_2, y_3/x_3, \ldots, y_k/x_k]P.[x_1/x_1, w/x_2, y_3/x_3, \ldots, y_k/x_k]Q$

$\equiv_\alpha \lambda P.Q.$ (by induction) □

Lemma 3.1.4. Let P and P' be patterns with $FV(P) \subseteq \{x_1, \ldots, x_k\}$, $k \geq 1$, and $P' \equiv [y_1/x_1, \ldots, y_k/x_k]P$ for some distinct variables $y_1, \ldots, y_k$ and let Q and N be terms. If $(\lambda P.Q)N$ is a β-redex, then $(\lambda P'.Q')[U_1/u_1, \ldots, U_m/u_m]N$ is also a β-redex for any distinct variables $u_1, \ldots, u_m$, $m \geq 1$, and any terms Q', $U_1, \ldots, U_m$.

Proof. Assume $(\lambda P.Q)N$ is a β-redex. Then there exist terms $N_1, \ldots, N_k$ such that $[N_1/x_1, \ldots, N_k/x_k]P \equiv N$.

Let $u_1, \ldots, u_m$, $m \geq 1$, be distinct variables and $U_1, \ldots, U_m$ be terms. Then

$[U_1/u_1, \ldots, U_m/u_m]N \equiv [U_1/u_1, \ldots, U_m/u_m][N_1/x_1, \ldots, N_k/x_k]P$

$\equiv [U_1/u_1, \ldots, U_m/u_m][N_1/y_1, \ldots, N_k/y_k][y_1/x_1, \ldots, y_k/x_k] P$

(by Corollary 2.1.17(a))

$\equiv [U_1/u_1, \ldots, U_m/u_m][N_1/y_1, \ldots, N_k/y_k]P'$

$\equiv [[U_1/u_1, \ldots, U_m/u_m]N_1/y_1, \ldots, [U_1/u_1, \ldots, U_m/u_m]N_k/y_k]P'.$

(by Corollary 2.1.17(a))

Hence $(\lambda P'.Q')[U_1/u_1, \ldots, U_m/u_m]N$ is a β-redex for any term Q'. □

Corollary 3.1.5. Let $(\lambda P.Q)N$ be a β-redex.

a. For any simple abstraction A such that $A \equiv_\alpha \lambda P.Q$, AN is a β-redex.

b. For any distinct variables $x_1, \ldots, x_k$, $k \geq 1$, and any terms $U_1, \ldots, U_k$, $[U_1/x_1, \ldots, U_k/x_k]((\lambda P.Q)N)$ is a β-redex.

Proof. Part (a) follows from Lemmas 2.2.4(b), 3.1.2 and 3.1.4, while Part (b) follows from Lemmas 2.1.10(c), 2.1.16 and 3.1.4. □

Lemma 3.1.6. Let $R \equiv (\lambda P.Q)N$ be a β-redex, $x_1,\ldots, x_k$, $k \geq 1$, be distinct variables, and $S, U_1,\ldots, U_k$ be terms. If $R \rhd_{1\beta} S$, then $[U_1/x_1,\ldots, U_k/x_k]R \rhd_{\beta} [U_1/x_1,\ldots, U_k/x_k]S$. To be precise, if $R \rhd_{1\beta} S$, then $[U_1/x_1,\ldots, U_k/x_k]R \rhd_{1\beta} S^*$ for some term S^*, where $S^* \equiv_{\alpha} [U_1/x_1,\ldots, U_k/x_k]S$.

Proof. Assume $R \rhd_{1\beta} S$.

Case 1. $FV(P) = \varnothing$.

Then $P \equiv N$ and $S \equiv Q$. Since $FV(N) = FV(P) = \varnothing$,

$$[U_1/x_1,\ldots, U_k/x_k]R \equiv [U_1/x_1,\ldots, U_k/x_k](\lambda P.Q)\, [U_1/x_1,\ldots, U_k/x_k]N$$
$$\equiv (\lambda P.[U_1/x_1,\ldots, U_k/x_k]Q)N$$
$$\rhd_{1\beta}[U_1/x_1,\ldots, U_k/x_k]Q.$$

Case 2. $FV(P) = \{y_1,\ldots, y_m\}$.

Then there exist terms $N_1,\ldots, N_m$ such that $[N_1/y_1,\ldots, N_m/y_m]P \equiv N$ and $S \equiv [N_1/y_1,\ldots, N_m/y_m]Q$. So we have

$$[U_1/x_1,\ldots, U_k/x_k]N \equiv [U_1/x_1,\ldots, U_k/x_k][N_1/y_1,\ldots, N_m/y_m]P$$
$$\equiv [[U_1/x_1,\ldots, U_k/x_k]N_1/y_1,\ldots,[U_1/x_1,\ldots, U_k/x_k]N_m/y_m]P$$

(by Corollary 2.1.17(a), since $FV(P) = \{y_1,\ldots, y_m\}$)

and $[U_1/x_1,\ldots, U_k/x_k]S \equiv [U_1/x_1,\ldots, U_k/x_k][N_1/y_1,\ldots, N_m/y_m]Q$.

There are cases and subcases as follows. (Note that

$$FV(Q) \cap (\{x_1,\ldots, x_k\} - \{y_1,\ldots, y_m\}) = FV(Q) \cap (\{x_1,\ldots, x_k\} - FV(P))$$
$$= \{x_1,\ldots, x_k\} \cap FV(\lambda P.Q).)$$

(2.1) $\{x_1,\ldots, x_k\} \cap FV(\lambda P.Q) = \varnothing$.

Then $[U_1/x_1,\ldots, U_k/x_k]R \equiv (\lambda P.Q)[U_1/x_1,\ldots, U_k/x_k]N$. By the note above $FV(Q) \cap (\{x_1,\ldots, x_k\} - \{y_1,\ldots, y_m\}) = \varnothing$, so we have

$$[U_1/x_1,\ldots, U_k/x_k]R \rhd_{1\beta} [[U_1/x_1,\ldots, U_k/x_k]N_1/y_1,\ldots,[U_1/x_1,\ldots, U_k/x_k]N_m/y_m]Q$$
$$\equiv_{\alpha}[U_1/x_1,\ldots, U_k/x_k][N_1/y_1,\ldots, N_k/y_k]Q. \qquad \text{(by Corollary 2.2.8)}$$

(2.2) $\{x_1,\ldots, x_k\} \cap FV(\lambda P.Q) = \{x_{i_1},\ldots, x_{i_n}\}$.

Then $FV(Q) \cap (\{x_1,\ldots, x_k\} - \{y_1,\ldots, y_m\}) = \{x_{i_1},\ldots, x_{i_n}\}$.

(2.2.1) $FV(P) \cap FV(U_{i_1}\ldots U_{i_n}) = \varnothing$.

Then $[U_1/x_1,\ldots, U_k/x_k]R \equiv (\lambda P.[U_{i_1}/x_{i_1},\ldots, U_{i_n}/x_{i_n}]Q)\,[U_1/x_1,\ldots, U_k/x_k]N$.

Hence $[U_1/x_1,\ldots, U_k/x_k]R$

$\triangleright_{1\beta}$ $[[U_1/x_1,\ldots, U_k/x_k]N_1/y_1,\ldots,[U_1/x_1,\ldots, U_k/x_k]N_m/y_m][U_{i_1}/x_{i_1},\ldots, U_{i_n}/x_{i_n}]Q$

$\equiv_\alpha$ $[[U_1/x_1,\ldots, U_k/x_k]N_1/y_1,\ldots,[U_1/x_1,\ldots, U_k/x_k]N_m/y_m, U_{i_1}/x_{i_1},\ldots, U_{i_n}/x_{i_n}]Q$

(by Corollary 2.2.8, since $\{y_1,\ldots, y_m\} \cap FV(x_{i_1}\ldots x_{i_n}U_{i_1}\ldots U_{i_n}) = \varnothing$)

$\equiv_\alpha$ $[U_1/x_1,\ldots, U_k/x_k][N_1/y_1,\ldots, N_m/y_m]Q$.

(by Corollary 2.2.8, since $FV(Q) \cap (\{x_1,\ldots, x_k\} - \{y_1,\ldots, y_m\}) = \{x_{i_1},\ldots, x_{i_n}\}$)

(2.2.2) $FV(P) \cap FV(U_{i_1}\ldots U_{i_n}) = \{y_{j_1},\ldots, y_{j_t}\}$, where for each $1\leq r \leq t$, y_{j_r} is the r^{th} variable in $FV(P) \cap FV(U_{i_1}\ldots U_{i_n})$.

By Lemma 2.1.10(c), there exist variables $z_{j_1},\ldots, z_{j_t}$ (as in the lemma) such that

$[U_1/x_1,\ldots, U_k/x_k](\lambda P.Q)$

$\equiv \lambda[z_{j_t}/y_{j_t}]\ldots [z_{j_1}/y_{j_1}]P.[U_{i_1}/x_{i_1},\ldots, U_{i_n}/x_{i_n}][z_{j_t}/y_{j_t}]\ldots[z_{j_1}/y_{j_1}]Q$

$\equiv \lambda[z_{j_1}/y_{j_1},\ldots, z_{j_t}/y_{j_t}]P.[U_{i_1}/x_{i_1},\ldots, U_{i_n}/x_{i_n}][z_{j_t}/y_{j_t}]\ldots[z_{j_1}/y_{j_1}]Q$

(by Corollary 1.1.17(c))

$\equiv \lambda[z_1/y_1,\ldots, z_m/y_m]P.[U_{i_1}/x_{i_1},\ldots, U_{i_n}/x_{i_n}][z_{j_t}/y_{j_t}]\ldots[z_{j_1}/y_{j_1}]Q$,

where $z_r \equiv y_r$ if $r \notin \{j_1,\ldots, j_t\}$. (by Lemma 2.1.11(b))

From the above, we have

$[U_1/x_1,\ldots, U_k/x_k]N$

$\equiv [[U_1/x_1,\ldots, U_k/x_k]N_1/y_1,\ldots,[U_1/x_1,\ldots, U_k/x_k]N_m/y_m]P$

$\equiv [[U_1/x_1,\ldots, U_k/x_k]N_1/z_1,\ldots,[U_1/x_1,\ldots, U_k/x_k]N_m/z_m][z_1/y_1,\ldots, z_m/y_m]P$.

(by Corollary 2.1.17(a))

Hence $[U_1/x_1,\ldots, U_k/x_k]R$

$\triangleright_{1\beta}$ $[[U_1/x_1,\ldots, U_k/x_k]N_1/z_1,\ldots,[U_1/x_1,\ldots, U_k/x_k]N_m/z_m][U_{i_1}/x_{i_1},\ldots, U_{i_n}/x_{i_n}]$
$[z_{j_t}/y_{j_t}]\ldots[z_{j_1}/y_{j_1}]Q$

$\equiv_\alpha$ $[[U_1/x_1,\ldots, U_k/x_k]N_1/z_1,\ldots,[U_1/x_1,\ldots, U_k/x_k]N_m/z_m][U_{i_1}/x_{i_1},\ldots, U_{i_n}/x_{i_n}]$
$[z_1/y_1,\ldots, z_m/y_m]Q$ (by Corollary 2.2.8 and Lemmas 2.1.11(b), 2.2.7)

$\equiv_\alpha$ $[[U_1/x_1,\ldots, U_k/x_k]N_1/z_1,\ldots,[U_1/x_1,\ldots, U_k/x_k]N_m/z_m, U_{i_1}/x_{i_1},\ldots, U_{i_n}/x_{i_n}]$
$[z_1/y_1,\ldots, z_m/y_m]Q$

(by Corollary 2.2.8, since $\{z_1,\ldots, z_m\} \cap FV(x_{i_1}\ldots x_{i_n}U_{i_1}\ldots U_{i_n}) = \varnothing$)

$\equiv_\alpha [U_1/x_1,\ldots, U_k/x_k][N_1/z_1,\ldots, N_m/z_m][z_1/y_1,\ldots, z_m/y_m]Q$

(by Corollary 2.2.8, since $FV([z_1/y_1,\ldots, z_m/y_m]Q) \cap (\{x_1,\ldots, x_k\} - \{z_1,\ldots, z_m\})$

$= FV(\lambda P.Q) \cap \{x_1,\ldots, x_k\} = \{x_{i_1},\ldots, x_{i_n}\}$)

$\equiv_\alpha [U_1/x_1,\ldots, U_k/x_k][N_1/y_1,\ldots, N_m/y_m]Q$.

(by Corollary 2.2.8 and Lemma 2.2.7, since $z_r \notin \{y_1,\ldots, y_m\}$ implies $z_r \notin FV(Q)$) □

Lemma 3.1.7. Let $R \equiv (\lambda P.Q \mid A)N$ be a δ-redex, $x_1,\ldots, x_k$, $k \geq 1$, be distinct variables, and $U_1,\ldots, U_k$ be terms. If $R \rhd_{1\delta} S$, then $[U_1/x_1,\ldots, U_k/x_k]R \rhd_{1\delta} [U_1/x_1,\ldots, U_k/x_k]S$.

Proof. Assume $R \rhd_{1\delta} S$.

Case 1. $S \equiv (\lambda P.Q)N$.

Then $(\lambda P.Q)N$ is a β-redex. So we have $[U_1/x_1,\ldots, U_k/x_k](\lambda P.Q)\,[U_1/x_1,\ldots, U_k/x_k]N \equiv [U_1/x_1,\ldots, U_k/x_k]((\lambda P.Q)N)$ which is a β-redex by Corollary 3.1.5(b). Hence

$[U_1/x_1,\ldots, U_k/x_k]R \equiv ([U_1/x_1,\ldots, U_k/x_k](\lambda P.Q) \mid [U_1/x_1,\ldots, U_k/x_k]A)[U_1/x_1,\ldots, U_k/x_k]N$

$\rhd_{1\delta} [U_1/x_1,\ldots, U_k/x_k](\lambda P.Q)\,[U_1/x_1,\ldots, U_k/x_k]N$

$\equiv [U_1/x_1,\ldots, U_k/x_k]((\lambda P.Q)N)$.

Case 2. $S \equiv AN$.

Suppose $[U_1/x_1,\ldots, U_k/x_k]R \not\rhd_{1\delta} [U_1/x_1,\ldots, U_k/x_k](AN)$.

Subcase 2.1. $FV(N) = \varnothing$, so $FV([U_1/x_1,\ldots, U_k/x_k]N) = \varnothing$.

Then $([U_1/x_1,\ldots, U_k/x_k](\lambda P.Q))N'$ is a β-redex for some term N' such that $[U_1/x_1,\ldots, U_k/x_k]N \rhd_{\beta\gamma} N'$ i.e. $N \rhd_{\beta\gamma} N'$.

By Lemmas 2.1.10(c) and 2.1.16, $[U_1/x_1,\ldots, U_k/x_k](\lambda P.Q) \equiv \lambda P'.Q'$ for some term Q' and some pattern P' such that $P' \equiv [z_1/u_1,\ldots, z_t/u_t]P$ for some distinct variables $z_1,\ldots, z_t$, where $FV(P) \subseteq \{u_1,\ldots, u_t\}$, $t \geq 1$. Note that $[u_1/z_1,\ldots, u_t/z_t]P' \equiv P$ (by Corollary 2.1.17).

Since $(\lambda P'.Q')N' \equiv ([U_1/x_1,\ldots, U_k/x_k](\lambda P.Q))N'$ which is a β-redex, by

Lemma 3.1.4 $(\lambda P.Q)N'$ is a β-redex. Since $N \rhd_{\beta\gamma} N'$, $R \not\rhd_{1\delta} AN$, a contradiction.

Subcase 2.2. $FV(N) = \{y_1,\ldots, y_m\}$.

Subcase 2.2.1. $FV([U_1/x_1,\ldots, U_k/x_k]N) = \varnothing$.

Then $([U_1/x_1,\ldots, U_k/x_k](\lambda P.Q))N'$ is a β-redex for some term N' such that $[U_1/x_1,\ldots, U_k/x_k]N \rhd_{\beta\gamma} N'$.

By Corollary 2.1.12(b), $[U_1/x_1,\ldots, U_k/x_k]N \equiv [V_1/y_1,\ldots, V_m/y_m]N$ for some terms $V_1,\ldots, V_m$. Hence $[V_1/y_1,\ldots, V_m/y_m]N \rhd_{\beta\gamma} N'$. As in case 2.1, $(\lambda P.Q)N'$ is a β-redex. Hence $R \not\rhd_{1\delta} AN$, a contradiction.

Subcase 2.2.2. $FV([U_1/x_1,\ldots, U_k/x_k]N) = \{u_1,\ldots, u_r\}$.

Then $([U_1/x_1,\ldots, U_k/x_k](\lambda P.Q))N'$ is a β-redex for some term N' such that $[V_1/u_1,\ldots, V_r/u_r][U_1/x_1,\ldots, U_k/x_k]N \rhd_{\beta\gamma} N'$ for some terms $V_1,\ldots, V_r$. By Corollaries 2.2.8 and 2.1.12(b),
$[V_1/u_1,\ldots, V_r/u_r][U_1/x_1,\ldots, U_k/x_k]N \equiv_\alpha [W_1/y_1,\ldots,W_m/y_m]N$ for some terms $W_1,\ldots,W_m$. So we have $[W_1/y_1,\ldots,W_m/y_m]N \rhd_{\beta\gamma} N'$. As above, this leads to a contradiction. □

Corollary 3.1.8. Let $x_1,\ldots, x_k$, $k \geq 1$, be distinct variables and $M, M', U_1,\ldots, U_k$ be terms.

a. If $M \rhd_{1\beta} M'$, then $[U_1/x_1,\ldots, U_k/x_k]M \rhd_\beta [U_1/x_1,\ldots, U_k/x_k]M'$.

b. If $M \rhd_{1\delta} M'$, then $[U_1/x_1,\ldots, U_k/x_k]M \rhd_{1\delta} [U_1/x_1,\ldots, U_k/x_k]M'$.

c. If $M \rhd_{\beta\delta} M'$, then $[U_1/x_1,\ldots, U_k/x_k]M \rhd_{\beta\delta} [U_1/x_1,\ldots, U_k/x_k]M'$.

d. If R is a contractible redex, then so is $[U_1/x_1,\ldots, U_k/x_k]R$.

Proof. Parts (a) and (b) follow from Lemmas 3.1.6 and 3.1.7 respectively. Part (c) follows from Parts (a) and (b), and Lemma 2.2.7. Part (d) follows from Lemmas 3.1.6 and 3.1.7. □

Lemma 3.1.9. Let A be an abstraction, and A' and N be terms such that $A \rhd_{1\beta,1\delta} A'$. If AN is a contractible redex, then so is $A'N$.

Proof. Assume AN is a contractible redex and let R be the occurrence of a potential redex in A which is contracted when $A \rhd_{1\beta,1\delta} A'$.

Case 1. $A \equiv \lambda P.Q$.

Since $A \rhd_{1\beta,1\delta} A'$, by Note 2.3.14 $A' \equiv (\lambda P.Q')$ for some term Q'. By Lemma 3.1.4, $A'N$ is a β-redex.

Case 2. $A \equiv (\lambda P.Q \,|\, B)$.

Since $A \rhd_{1\beta,1\delta} A'$, by Note 2.3.14 $A' \equiv (\lambda P.Q' \,|\, B')$ for some term Q' and some abstraction B' such that either $Q \rhd_{1\beta,1\delta} Q'$ and $B \equiv B'$ or $Q \equiv Q'$ and $B \rhd_{1\beta,1\delta} B'$.

Suppose $A'N$ is not contractible. Then there exists a term N' such that $(\lambda P.Q')N'$ is a β-redex and $[U_1/x_1,..., U_k/x_k]N \rhd_{\beta\gamma} N'$ for some distinct variables $x_1,..., x_k$, $k \geq 1$, and some terms $U_1,\ldots, U_k$. By Lemma 3.1.4, $(\lambda P.Q)N'$ is also a β-redex. Since AN is contractible, we must have that $(\lambda P.Q)N$ is a β-redex. But then $(\lambda P.Q')N$ is a β-redex. Hence $A'N \rhd_{1\delta} (\lambda P.Q')N$, a contradiction. Therefore $A'N$ is contractible. □

Lemma 3.1.10. Let P be a pattern with $FV(P) = \{x_1,..., x_k\}$, $k \geq 1$, and $N, U_1,\ldots, U_k$ be terms. If $[U_1/x_1,..., U_k/x_k]P \rhd_{\beta\delta} N$, then $N \equiv [V_1/x_1,...,V_k/x_k]P$ for some terms $V_1,\ldots,V_k$ such that $U_i \rhd_{\beta\delta} V_i$ for all $1 \leq i \leq k$.

Proof. Assume $[U_1/x_1,..., U_k/x_k]P \rhd_{\beta\delta} N$. Induct on P.

i. $P \equiv x_1$.

Let $V_1 \equiv N$, and observe that $N \equiv V_1 \equiv [V_1/x_1]P$ and $U_1 \equiv [U_1/x_1]P \rhd_{\beta\delta} N \equiv V_1$.

ii. $P \equiv P_1P_2$.

By Lemma 3.1.1, substituting into P cannot produce a potential redex. Since $[U_1/x_1,..., U_k/x_k]P_1[U_1/x_1,..., U_k/x_k]P_2 \equiv [U_1/x_1,..., U_k/x_k]P \rhd_{\beta\delta} N$, by Corollary 2.3.15(a) $N \equiv N_1N_2$ for some terms N_1 and N_2, where $[U_1/x_1,..., U_k/x_k]P_i \rhd_{\beta\delta} N_i$, $i = 1, 2$.

Since $FV(P) = \{x_1,\ldots, x_k\}$, $FV(P_1) \neq \emptyset$ or $FV(P_2) \neq \emptyset$. Without loss of generality, assume $FV(P_1) \neq \emptyset$. The proof for the case $FV(P_2) \neq \emptyset$ is similar.

Case 1. $FV(P_2) = \emptyset$.

Then $FV(P_1) = \{x_1,\ldots, x_k\}$. Since $[U_1/x_1,\ldots, U_k/x_k]P_1 \rhd_{\beta\delta} N_1$, by induction $N_1 \equiv [V_1/x_1,\ldots,V_k/x_k]P_1$ for some terms $V_1,\ldots,V_k$, where $U_i \rhd_{\beta\delta} V_i$ for all $1 \leq i \leq k$. Since $[U_1/x_1,\ldots, U_k/x_k]P_2 \rhd_{\beta\delta} N_2$ and $FV(P_2) = \emptyset$, $P_2 \rhd_{\beta\delta} N_2$, so in fact $P_2 \equiv N_2$, since P_2 contains no bound variables. Hence

$$\begin{aligned} N \equiv N_1N_2 &\equiv ([V_1/x_1,\ldots,V_k/x_k]P_1)P_2 \\ &\equiv [V_1/x_1,\ldots,V_k/x_k]P_1[V_1/x_1,\ldots,V_k/x_k]P_2 \\ &\equiv [V_1/x_1,\ldots,V_k/x_k](P_1P_2) \\ &\equiv [V_1/x_1,\ldots,V_k/x_k]P. \end{aligned}$$

Case 2. $FV(P_2) = \{x_{j_1},\ldots, x_{j_n}\}$.

Since $FV(P) = \{x_1,\ldots, x_k\}$ and no variable occurs in both P_1 and P_2, $FV(P_1) = \{x_{i_1},\ldots, x_{i_m}\}$, where $\{i_1,\ldots, i_m\} \cup \{j_1,\ldots, j_n\} = \{1,\ldots, k\}$ and $\{i_1,\ldots, i_m\} \cap \{j_1,\ldots, j_n\} = \emptyset$.

By Corollary 2.1.10(b), $[U_1/x_1,\ldots, U_k/x_k]P_1 \equiv [U_{i_1}/x_{i_1},\ldots, U_{i_m}/x_{i_m}]P_1$ and $[U_1/x_1,\ldots, U_k/x_k]P_2 \equiv [U_{j_1}/x_{j_1},\ldots, U_{j_n}/x_{j_n}]P_2$. By induction, $N_1 \equiv [V_{i_1}/x_{i_1},\ldots,V_{i_m}/x_{i_m}]P_1$ and $N_2 \equiv [V_{j_1}/x_{j_1},\ldots,V_{j_n}/x_{j_n}]P_2$ for some terms $V_{i_1},\ldots,V_{i_m}$, $V_{j_1},\ldots,V_{j_n}$, where $U_r \rhd_{\beta\delta} V_r$ for all $1 \leq r \leq k$. Hence

$$\begin{aligned} N \equiv N_1N_2 &\equiv [V_{i_1}/x_{i_1},\ldots,V_{i_m}/x_{i_m}]P_1[V_{j_1}/x_{j_1},\ldots,V_{j_n}/x_{j_n}]P_2 \\ &\equiv [V_1/x_1,\ldots,V_k/x_k]P_1[V_1/x_1,\ldots,V_k/x_k]P_2 \\ &\equiv [V_1/x_1,\ldots,V_k/x_k]P. \end{aligned}$$

□

Lemma 3.1.11. If we replace $\rhd_{\beta\delta}$ in Lemma 3.1.10 by $\equiv_\alpha$, then the lemma remains true.

Proof. This can be proved in the same way as Lemma 3.1.10. □

Lemma 3.1.12. Let A be an abstraction, and N and N' be terms such that $N \rhd_{\beta\delta} N'$. If AN is a contractible redex, then so is AN'.

Proof. Assume AN is a contractible redex.

Case 1. $A \equiv \lambda P.Q$.

Subcase 1.1. $FV(P) = \varnothing$.

Since AN is a β-redex, $P \equiv N$, so $P \equiv N \rhd_{\beta\delta} N'$. This implies $P \equiv N'$ since P contains no bound variables. Thus $(\lambda P.Q)N'$ is a β-redex. That is, AN' is a contractible redex.

Subcase 1.2. $FV(P) = \{x_1,..., x_k\}$.

Then $[N_1/x_1,..., N_k/x_k]P \equiv N$ for some terms $N_1,\ldots, N_k$. Since $N \rhd_{\beta\delta} N'$, by Lemma 3.1.10 $N' \equiv [N_1'/x_1,..., N_k'/x_k]P$ for some terms $N_1',\ldots, N_k'$. Hence $(\lambda P.Q)N'$is a β-redex, so $AN'$is contractible.

Case 2. $A \equiv (\lambda P.Q \mid B)$.

Suppose AN' is not contractible. Then there exists a term N^* such that $(\lambda P.Q)N^*$ is a β-redex and $[U_1/y_1,..., U_m/y_m]N' \rhd_{\beta\gamma} N^*$ for some distinct variables $y_1,\ldots y_m$, $m \geq 1$, and some terms $U_1,\ldots, U_m$. Since $N \rhd_{\beta\delta} N'$, by Corollary 3.1.8(c) $[U_1/y_1,..., U_m/y_m]N \rhd_{\beta\delta} [U_1/y_1,..., U_m/y_m]N'$. By Note 2.3.8(b), $[U_1/y_1,..., U_m/y_m]N \rhd_{\beta\gamma} [U_1/y_1,..., U_m/y_m]N'$. By the transitivity of the relation $\rhd_{\beta\gamma}$, $[U_1/y_1,..., U_m/y_m]N \rhd_{\beta\gamma} N^*$. Since AN is contractible and $(\lambda P.Q)N^*$ is a β-redex, this implies $(\lambda P.Q)N$ is a β-redex. By Case 1, $(\lambda P.Q)N'$ is a β-redex. Hence $AN' \rhd_{1\delta} (\lambda P.Q)N'$, a contradiction. Thus AN' is contractible. □

Lemma 3.1.13. Let R be a contractible redex, and $R'$and S be terms such that $R \equiv_\alpha R'$. If $R \rhd_{1\beta} S$ (respectively $R \rhd_{1\delta} S$), then $R' \rhd_{1\beta} S'$ (respectively $R' \rhd_{1\delta} S'$) for some term S', where $S' \equiv_\alpha S$.

Proof. First, assume $R \equiv (\lambda P.Q)N \rhd_{1\beta} S$.

Case 1. $FV(P) = \varnothing$.

Since R is a β-redex, $P \equiv N$ and $S \equiv Q$. Since $R \equiv_\alpha R'$, $R' \equiv (\lambda P.\ Q')N'$ for some terms Q' and N', where $Q' \equiv_\alpha Q$ and $N' \equiv_\alpha N$. Since $N \equiv P$, N contains no bound variables. This implies $N' \equiv N \equiv P$. Hence $R' \equiv (\lambda P.Q')N' \rhd_{1\beta} Q'$ and $Q' \equiv_\alpha Q$.

Case 2. $FV(P) = \{x_1,\dots, x_k\}$.

Then $[N_1/x_1,\dots, N_k/x_k]P \equiv N$ and $S \equiv [N_1/x_1,\dots, N_k/x_k]Q$ for some terms $N_1,\dots, N_k$. Since $R \equiv_\alpha R'$, by Lemmas 2.2.4 and 3.1.2 $R' \equiv (\lambda P'.Q')N'$ for some pattern P' and some terms Q' and N' such that $N' \equiv_\alpha N$, $P' \equiv [y_1/x_1,\dots, y_k/x_k]P$ and $Q' \equiv_\alpha [y_1/x_1,\dots, y_k/x_k]Q$, for some distinct variables $y_1,\dots, y_k$, where $\{y_1,\dots, y_k\} \cap FV(\lambda P.Q) = \varnothing$. Since $N' \equiv_\alpha N \equiv [N_1/x_1,\dots, N_k/x_k]P$, by Lemma 3.1.11 $N' \equiv [N_1'/x_1,\dots, N_k'/x_k]P$ for some terms $N_1',\dots, N_k'$ such that $N_i' \equiv_\alpha N_i$ for all $1 \le i \le k$.

So we have $N' \equiv [N_1'/x_1,\dots, N_k'/x_k]P$

$\equiv [N_1'/y_1,\dots, N_k'/y_k][y_1/x_1,\dots, y_k/x_k]P$ (by Corollary 2.1.17)

$\equiv [N_1'/y_1,\dots, N_k'/y_k]P'$.

Hence $R' \rhd_{1\beta} [N_1'/y_1,\dots, N_k'/y_k]Q'$, and we have

$[N_1'/y_1,\dots, N_k'/y_k]Q' \equiv_\alpha [N_1/y_1,\dots, N_k/y_k][y_1/x_1,\dots, y_k/x_k]Q$ (by Lemma 2.2.7)

$\equiv_\alpha [N_1/x_1,\dots, N_k/x_k]Q.$ (by Corollary 2.2.8)

Now, assume $R \equiv (\lambda P.Q \mid A)N \rhd_{1\delta} S$.

Since $R \equiv_\alpha R'$, by Lemma 2.2.4 $R' \equiv (\lambda P'.Q' \mid A')N'$ for some abstractions $\lambda P'.Q'$ and A' and some term N', where $\lambda P'.Q' \equiv_\alpha \lambda P.Q$, $A' \equiv_\alpha A$, and $N' \equiv_\alpha N$.

Case 1. $S \equiv (\lambda P.Q)N$.

Then $(\lambda P.Q)N$ is a β-redex. Since $\lambda P'.Q' \equiv_\alpha \lambda P.Q$, by Corollary 3.1.5(a) $(\lambda P'.Q')N$ is a β-redex. Since $N' \equiv_\alpha N$, by Lemma 3.1.12 $(\lambda P'.Q')N'$ is a β-redex. Hence $R' \rhd_{1\delta} (\lambda P'.Q')N'$, where $(\lambda P'.Q')N' \equiv_\alpha (\lambda P.Q)N$.

Case 2. $S \equiv AN$.

Suppose $R' \not\rhd_{1\delta} A'N'$. Then there exists a term N^* such that $(\lambda P'.Q')N^*$ is a β-redex and $[U_1/x_1,..., U_k/x_k]N' \triangleright_{\beta\gamma} N^*$ for some distinct variables $x_1,..., x_k$, $k \geq 1$, and some terms $U_1,\dots, U_k$. Since $N \equiv_\alpha N'$, by Lemma 2.2.7 $[U_1/x_1,..., U_k/x_k]N \equiv_\alpha [U_1/x_1,..., U_k/x_k]N'$. Hence $[U_1/x_1,..., U_k/x_k]N \triangleright_{\beta\gamma} N^*$. Since $\lambda P.Q \equiv_\alpha \lambda P'.Q'$ and $(\lambda P'.Q')N^*$ is a β-redex, by Corollary 3.1.5(a) $(\lambda P.Q)N^*$ is also a β-redex. Hence $R \not\triangleright_{1\delta} AN$, a contradiction. Thus $R' \triangleright_{1\delta} A'N'$, where $A'N' \equiv_\alpha AN$. □

Corollary 3.1.14.

a. Let M, M′, and N be terms such that $M \equiv_\alpha M'$. If $M \triangleright_{1\beta} N$ (respectively $M \triangleright_{1\delta} N$), then $M' \triangleright_{1\beta} N'$(respectively $M' \triangleright_{1\delta} N'$) for some term N′, where $N' \equiv_\alpha N$.

b. If R is a contractible redex and R′is a term such that $R \equiv_\alpha R'$, then R′ is also a contractible redex.

c. Let R be a potential redex and S be a term such that $R \triangleright_{\beta\delta} S$ by a sequence of terms $R \equiv R_1, R_2,\dots, R_n \equiv S$, $n \geq 1$, where for each $1 \leq i < n$, R_i is not the potential redex which is contracted. If R is a contractible redex, then so is S.

Proof. Parts (a) and (b) follow from Lemma 3.1.13, while Part (c) follows from Lemmas 3.1.9, 3.1.12 and Part (b). □

Lemma 3.1.15. For any βδ-normal form M and any term N, if $M \triangleright_{\beta\delta} N$, then $M \equiv_\alpha N$.

Proof. Let M be a βδ-normal form and N be a term such that $M \triangleright_{\beta\delta} N$. Then there exists a sequence of terms $M \equiv M_1,\dots, M_n \equiv N$, $n \geq 1$, such that for each $1 \leq i < n$, $M_i \equiv_\alpha M_{i+1}$ or $M_i \triangleright_{1\beta,1\delta} M_{i+1}$. Induct on n.

If $n = 1$, then $M \equiv N$.

Now, suppose $n > 1$. By induction, $M \equiv_\alpha M_{n-1}$. Suppose M_{n-1} contains a contractible redex R. Since $M \equiv_\alpha M_{n-1}$, M contains a potential redex R_0 such that $R_0 \equiv_\alpha R$. By Corollary 3.1.14, R_0 is also a contractible redex, so M contains a contractible redex, which is a contradiction. Hence M_{n-1} contains no contractible redexes and so $M_{n-1} \not\triangleright_{1\beta,1\delta} N$. Therefore $M_{n-1} \equiv_\alpha N$. Thus $M \equiv_\alpha N$. □

3.2 Residuals and Minimal Complete Developments

To prove the Church-Rosser theorem, we need to look at a restricted set of $\beta\delta$-reductions, called minimal complete developments (MCD's). In this section we will define this type of reduction and prove those basic properties concerning it which are needed for proving the Church-Rosser theorem.

Definition 3.2.1. Let R and S be occurrences of contractible redexes in a term M. When R is contracted, let M change to M'.

The **residuals** of S with respect to R are occurrences of potential redexes in M', defined as follows.

Case 1. R and S are non-overlapping parts of M.

Then contracting R leaves S unchanged. This unchanged S in M' is the residual of S.

Case 2. $R \equiv S$.

Then contracting R is the same as contracting S. We say S has no residuals in M'.

Case 3. R is part of S and $R \not\equiv S$.

Since S is a potential redex, $S \equiv AN$ for some abstraction A, and some term N. So R is either in A or in N. Then contracting R changes S to S', where $S' \equiv A'N'$ for some abstraction A' and some term N' such that either $A \triangleright_{1\beta,1\delta} A'$ and $N \equiv N'$ or $A \equiv A'$ and $N \triangleright_{1\beta,1\delta} N'$. This S' is the residual of S.

Case 4. S is part of R and $S \not\equiv R$.

There are cases and subcases as follows.

(4.1) $R \equiv (\lambda P.Q)N$.

(4.1.1) $FV(P) = \varnothing$.

Since R is a β-redex, $P \equiv N$ and $R \triangleright_{1\beta} Q$. Since S is a potential redex in R, S is in Q. Since $R \triangleright_{1\beta} Q$, contracting R leaves S unchanged in M'; this is the residual of S.

(4.1.2) $FV(P) = \{x_1, \dots, x_k\}$, $k \geq 1$.

Then $[N_1/x_1,\ldots, N_k/x_k]P \equiv N$ for some terms $N_1,\ldots, N_k$ and $R \triangleright_{1\beta} [N_1/x_1,\ldots, N_k/x_k]Q$.

(4.1.2.1) S is in Q.

Then S changes to S', where S' is either S or some substitution of S. This S' is the residual of S.

(4.1.2.2) S is in N.

Then S is in $[N_1/x_1,\ldots, N_k/x_k]P$. By Lemma 3.1.1, S is in N_t for some $1 \le t \le k$. Hence there is an occurrence of S in each N_t substituted for an occurrence of x_t in Q. These are the residuals of S. (Note that S may have many or no residuals.)

(4.2) $R \equiv (\lambda P.Q \mid A)N$.

(4.2.1) $R \triangleright_{1\delta} (\lambda P.Q)N$.

If S is in Q or N, then contracting R leaves S unchanged, and this is the residual of S in M'. If S is in A, then S has no residuals in M'.

(4.2.2) $R \triangleright_{1\delta} AN$.

If S is in A or N, then this unchanged S in A or N is the residual of S in M'. If S is in Q, then S has no residuals in M'.

Notes 3.2.2.

a. Except in case 4.1.2.2, S has at most one residual.

b. Each residual is a contractible redex. (The residual in Case 3 is contractible by Lemmas 3.1.9 and 3.1.12, and the residual in (4.1.2.1) is contractible by Corollary 3.1.8(d)).

Definition 3.2.3. If $\mathscr{R} = \{R_i \mid 1 \le i \le n\}$, $n \ge 0$, is a set of occurrences of potential redexes in a term M, then an R_i is called **minimal** (with respect to $\mathscr{R}$) if it properly contains no other $R_j \in \mathscr{R}$.

Let $\mathscr{R} = \{R_i \mid 1 \le i \le n\}$, $n \ge 0$, be a set of occurrences of contractible redexes in a term M. For any term M', we say M' is obtained from M by a **minimal complete**

development (MCD) of $\mathscr{R}$, denoted by M $\triangleright_{mcd}$ M′ (of $\mathscr{R}$), if M′ is obtained from M by the following process.

First contract any minimal R_i; without loss of generality let i = 1. By Definition 3.2.1, this leaves n – 1 residuals R_2', R_3',…, R_n'. Contract any minimal R_t'. This leaves n – 2 residuals. Repeat this process until no residuals are left. Then make as many α-steps as you like.

Notes 3.2.4.

a. In any non-empty set of potential redexes, there is always a minimal member.

b. If n = 0, an MCD is just a finite sequence of α-steps.

c. A single β-contraction or a single δ-contraction is an MCD of a one member set.

d. There exist reductions which are not MCD's, for example

$$(\lambda x.xy)(\lambda z.z) \triangleright_{1\beta} (\lambda z.z)y \triangleright_{1\beta} y.$$

e. The relation $\triangleright_{mcd}$ is not transitive. For example, in (d) there is clearly no MCD from (λx.xy)(λz.z) to y.

f. If M $\triangleright_{mcd}$ M′ and N $\triangleright_{mcd}$ N′, then MN $\triangleright_{mcd}$ M′N′ and λP.M $\triangleright_{mcd}$ λP.M′.

g. Each MCD is a βδ-reduction.

h. For any contractible redex L, if L $\triangleright_{mcd}$ M of $\mathscr{R}$, without α-steps, where L $\notin \mathscr{R}$, and M $\triangleright_{1\beta,1\delta}$ N, with M being the potential redex contracted, then L $\triangleright_{mcd}$ N of $\mathscr{R} \cup \{L\}$, without α-steps.

Lemma 3.2.5. If we replace $\triangleright_{\beta\delta}$ in Corollary 2.3.15 by $\triangleright_{mcd}$, then the corollary remains true.

Proof. This is obvious for Part (b), since all potential redexes in λP.Q are in Q. Part (a) follows from the fact that the sets of potential redexes in M_1 and M_2 are disjoint when M_1 and M_2 are non-overlapping. The argument for Part (a) also applies to Part (c). □

Lemma 3.2.6. If we replace $\rhd_{\beta\delta}$ in Lemma 3.1.10 by $\rhd_{mcd}$, then the lemma remains true.

Proof. This can be proved in the same way as Lemma 3.1.10. □

Lemma 3.2.7. For any terms M, N and M′, if $M \rhd_{mcd} N$ and $M \equiv_\alpha M'$, then $M' \rhd_{mcd} N$.

Lemma 3.2.8. For any distinct variables $x_1,\ldots, x_k$, $k \geq 1$, and any terms $M, N, U_1,\ldots, U_k, V_1,\ldots,V_k$, if $M \rhd_{mcd} N$ and $U_i \rhd_{mcd} V_i$ for all $1 \leq i \leq k$, then $[U_1/x_1,\ldots, U_k/x_k]M \rhd_{mcd} [V_1/x_1,\ldots,V_k/x_k]N$.

Proof of Lemmas 3.2.7 and 3.2.8.

Let $x_1,\ldots, x_k$, $k \geq 1$, be distinct variables and $M, N, M', U_1,\ldots, U_k, V_1,\ldots, V_k$ be terms such that $M \rhd_{mcd} N$, $M \equiv_\alpha M'$ and $U_i \rhd_{mcd} V_i$ for all $1 \leq i \leq k$. Then N is obtained from M by the given MCD of a set $\mathscr{R}$. By Definition 3.2.3 (for Lemma 3.2.7) and Lemma 2.2.7 (for Lemma 3.2.8), we may assume that the MCD $M \rhd_{mcd} N$ has no α-steps.

For Lemma 3.2.8, first suppose $\{x_1,\ldots, x_k\} \cap FV(M) = \varnothing$. Then by Corollary 2.1.12(a) $[U_1/x_1,\ldots, U_k/x_k]M \equiv M$. Since $M \rhd_{mcd} N$, we have $M \rhd_{\beta\delta} N$. By Lemma 2.3.16(a), $FV(N) \subseteq FV(M)$. Hence $\{x_1,\ldots, x_k\} \cap FV(N) = \varnothing$. Thus $[V_1/x_1,\ldots,V_k/x_k]N \equiv N$. So we have

$[U_1/x_1,\ldots, U_k/x_k]M \equiv M \rhd_{mcd} N \equiv [V_1/x_1,\ldots,V_k/x_k]N$, and we are finished.

Thus for Lemma 3.2.8 we may assume $\{x_1,\ldots, x_k\} \cap FV(M) \neq \varnothing$, and in fact, by Corollary 2.1.12(b) we may assume that $\{x_1,\ldots, x_k\} \subseteq FV(M)$.

Now we will prove both lemmas simultaneously by induction on M.

i. M is an atom.

Proof of Lemma 3.2.7.

Since $M \equiv_\alpha M'$, in fact $M \equiv M'$. Since $M \rhd_{mcd} N$, this implies $M' \rhd_{mcd} N$.

Proof of Lemma 3.2.8.

By our assumption, $M \equiv x_1$ and $k = 1$. Since $M \rhd_{mcd} N$, it must be that $N \equiv M$. Hence $[U_1/x_1]M \equiv U_1 \rhd_{mcd} V_1 \equiv [V_1/x_1]N$.

ii. $M \equiv \lambda P.Q$.

Since $M \rhd_{mcd} N$, without α-steps, $N \equiv \lambda P.Q_0$ for some term Q_0 such that $Q \rhd_{mcd} Q_0$. Note that $FV(Q_0) \subseteq FV(Q)$ (by Lemma 2.3.16(a)).

Proof of Lemma 3.2.7.

Since $M \equiv_\alpha M'$, by Lemmas 2.2.4 and 3.1.2 M' is one of the following forms.

1. $M' \equiv \lambda P.Q'$, where $Q' \equiv_\alpha Q$.

By induction $Q' \rhd_{mcd} Q_0$. Hence $M' \equiv \lambda P.Q' \rhd_{mcd} \lambda P.Q_0 \equiv N$.

2. $M' \equiv \lambda[z_1/y_1,\ldots, z_m/y_m]P.Q'$, where $FV(P) = \{y_1,\ldots, y_m\}$, $m \geq 1$, $z_1,\ldots, z_m$ are distinct variables and Q' is a term such that $\{z_1,\ldots, z_m\} \cap FV(\lambda P.Q) = \varnothing$ and $Q' \equiv_\alpha [z_1/y_1,\ldots, z_m/y_m]Q$.

Since $Q \rhd_{mcd} Q_0$, by induction (3.2.8) $[z_1/y_1,\ldots, z_m/y_m]Q \rhd_{mcd} [z_1/y_1,\ldots, z_m/y_m]Q_0$. Hence, by induction (3.2.7) $Q' \rhd_{mcd} [z_1/y_1,\ldots, z_m/y_m]Q_0$. Hence $M' \equiv \lambda[z_1/y_1,\ldots, z_m/y_m]P.Q'$

$$\rhd_{mcd} \lambda[z_1/y_1,\ldots, z_m/y_m]P.[z_1/y_1,\ldots, z_m/y_m]Q_0$$

$$\equiv_\alpha \lambda P.Q_0 \equiv N. \qquad \text{(by Lemma 3.1.3)}$$

Proof of Lemma 3.2.8.

By Lemma 2.1.10(b), $[U_1/x_1,\ldots, U_k/x_k]M$ is also a simple abstraction. Hence by Lemmas 2.2.5(b), 2.2.7 and 3.2.7 we may assume that no variable bound in M is free in $x_1\ldots x_kU_1\ldots U_k$. So $FV(P) \cap FV(x_1\ldots x_kU_1\ldots U_k) = \varnothing$. By Lemma 2.3.16(a), $FV(V_i) \subseteq FV(U_i)$ for all $1 \leq i \leq k$. Hence $FV(P) \cap FV(x_1\ldots x_kV_1\ldots V_k) = \varnothing$. Thus

$$[U_1/x_1,\ldots, U_k/x_k]M \equiv \lambda P.[U_1/x_1,\ldots, U_k/x_k]Q \qquad \text{(by Corollary 2.1.12(d))}$$

$$\rhd_{mcd} \lambda P.[V_1/x_1,\ldots,V_k/x_k]Q_0 \qquad \text{(by induction)}$$

$$\equiv [V_1/x_1,\ldots,V_k/x_k](\lambda P.Q_0)$$

$$\equiv [V_1/x_1,\ldots,V_k/x_k]N.$$

iii. $M \equiv (\lambda P.Q \mid A)$.

Since $M \triangleright_{mcd} N$, $N \equiv (\lambda P_0.Q_0 \mid A_0)$ for some abstractions $\lambda P_0.Q_0$ and A_0 such that $\lambda P.Q \triangleright_{mcd} \lambda P_0.Q_0$ and $A \triangleright_{mcd} A_0$.

Proof of Lemma 3.2.7.

Since $M \equiv_\alpha M'$, we must have that $M' \equiv (\lambda P'.Q' \mid A')$ for some abstractions $\lambda P'.Q'$ and A' such that $\lambda P'.Q' \equiv_\alpha \lambda P.Q$ and $A' \equiv_\alpha A$. By induction, $\lambda P'.Q' \triangleright_{mcd} \lambda P_0.Q_0$ and $A' \triangleright_{mcd} A_0$. Hence $M' \equiv (\lambda P'.Q' \mid A') \triangleright_{mcd} (\lambda P_0.Q_0 \mid A_0) \equiv N$.

Proof of Lemma 3.2.8.

By induction,

$$
\begin{aligned}
[U_1/x_1,\dots,U_k/x_k]M &\equiv ([U_1/x_1,\dots,U_k/x_k](\lambda P.Q) \mid [U_1/x_1,\dots,U_k/x_k]A) \\
&\triangleright_{mcd} ([V_1/x_1,\dots,V_k/x_k](\lambda P_0.Q_0) \mid [V_1/x_1,\dots,V_k/x_k]A_0) \\
&\equiv [V_1/x_1,\dots,V_k/x_k](\lambda P_0.Q_0 \mid A_0) \\
&\equiv [V_1/x_1,\dots,V_k/x_k]N.
\end{aligned}
$$

iv. $M \equiv M_1M_2$.

Case 1. $M \notin \mathscr{R}$.

This case can be proved in the same way as (iii).

Case 2. $M \in \mathscr{R}$.

Since $M \in \mathscr{R}$ and $M \triangleright_{mcd} N$, without α-steps, by Definition 3.2.3 $M \triangleright_{mcd} M_1^0M_2^0$ for some terms M_1^0 and M_2^0 such that $M_1 \triangleright_{mcd} M_1^0$ and $M_2 \triangleright_{mcd} M_2^0$, both without α-steps, and $M_1^0M_2^0 \triangleright_{1\beta,1\delta} N$, with $M_1^0M_2^0$ being the potential redex contracted.

Proof of Lemma 3.2.7.

Since $M \equiv_\alpha M'$, we have that $M' \equiv M_1'M_2'$ for some terms M_1' and M_2' such that $M_i' \equiv_\alpha M_i$, $i = 1, 2$. By induction, $M_1' \triangleright_{mcd} M_1^0$ and $M_2' \triangleright_{mcd} M_2^0$. Hence $M_1' \triangleright_{mcd} M_1^*$ and $M_2' \triangleright_{mcd} M_2^*$, both without α-steps, for some terms M_1^* and M_2^*, where $M_i^* \equiv_\alpha M_i^0$, $i = 1, 2$. Since $M_1^*M_2^* \equiv_\alpha M_1^0M_2^0$ and $M_1^0M_2^0 \triangleright_{1\beta,1\delta} N$, by Lemma 3.1.13 $M_1^*M_2^* \triangleright_{1\beta,1\delta} M^*$ for some term M^*, where $M^* \equiv_\alpha N$. Hence $M' \equiv M_1'M_2' \triangleright_{mcd} M_1^*M_2^* \triangleright_{1\beta,1\delta} M^* \equiv_\alpha N$. Since $M \equiv_\alpha M'$, by Corollary 3.1.14(b) M' is contractible. By Note 3.2.4(h), $M' \rhd_{mcd} N$.

Proof of Lemma 3.2.8.

Since $M_1 \rhd_{mcd} M_1^0$ and $M_2 \rhd_{mcd} M_2^0$, by induction $[U_1/x_1,\ldots, U_k/x_k]M_i \rhd_{mcd} [V_1/x_1,\ldots,V_k/x_k]M_i^0$, i = 1, 2. Hence $[U_1/x_1,\ldots, U_k/x_k]M_i \rhd_{mcd} M_i^*$, without α-steps, for some term M_i^* such that $M_i^* \equiv_\alpha [V_1/x_1,\ldots,V_k/x_k]M_i^0$, i = 1, 2. Since $M_1^0M_2^0 \rhd_{1\beta,1\delta} N$, by Lemmas 3.1.6 and 3.1.7 $[V_1/x_1,\ldots,V_k/x_k](M_1^0M_2^0) \rhd_{1\beta,1\delta} N^*$ for some term N^*, where $N^* \equiv_\alpha [V_1/x_1,\ldots,V_k/x_k]N$. Since $M_1^*M_2^* \equiv_\alpha [V_1/x_1,\ldots,V_k/x_k](M_1^0M_2^0)$, by Lemma 3.1.13 $M_1^*M_2^* \rhd_{1\beta,1\delta} M^*$ for some term M^* such that $M^* \equiv_\alpha N^*$.

Hence

$$
\begin{aligned}
[U_1/x_1,\ldots, U_k/x_k]M &\equiv [U_1/x_1,\ldots, U_k/x_k]M_1[U_1/x_1,\ldots, U_k/x_k]M_2 \\
&\rhd_{mcd} M_1^*M_2^* \\
&\rhd_{1\beta,1\delta} M^* \\
&\equiv_\alpha N^* \\
&\equiv_\alpha [V_1/x_1,\ldots,V_k/x_k]N.
\end{aligned}
$$

Since $M \in \mathscr{R}$, M is contractible. Hence, by Corollary 3.1.8(d) $[U_1/x_1,\ldots, U_k/x_k]M$ is contractible. Thus, by Note 3.2.4(h) $[U_1/x_1,\ldots, U_k/x_k]M \rhd_{mcd} [V_1/x_1,\ldots,V_k/x_k]N$. □

3.3 The Church-Rosser Theorem for βδ-Reduction

Our goal in this section is to prove the Church-Rosser theorem for βδ-reduction. To make the proof easier to follow, we split it into two steps. The conclusion of the first step is important enough to be called a theorem in its own right.

Theorem 3.3.1 (The Church-Rosser theorem for MCD's). For any terms L, M and N, if $L \rhd_{mcd} M$ and $L \rhd_{mcd} N$, then there exists a term T such that $M \rhd_{mcd} T$ and $N \rhd_{mcd} T$.

Proof. Let L, M and N be terms such that $L \rhd_{mcd} M$ and $L \rhd_{mcd} N$.

Then M (respectively N) is obtained from L by the given MCD of a set $\mathscr{R}_M$ (respectively $\mathscr{R}_N$). By Lemma 3.2.7, it is sufficient to consider the case in which the given MCD's have no α-steps. Induct on L.

i. L is an atom.

Since $L \rhd_{mcd} M$ and $L \rhd_{mcd} N$, it must be that $M \equiv L \equiv N$ and we are finished.

ii. $L \equiv \lambda P.Q$.

Since $L \rhd_{mcd} M$ and $L \rhd_{mcd} N$, both without α-steps, $M \equiv \lambda P.Q^M$ and $N \equiv \lambda P.Q^N$ for some terms Q^M and Q^N such that $Q \rhd_{mcd} Q^M$ and $Q \rhd_{mcd} Q^N$. By induction, there exists a term Q^* such that $Q^M \rhd_{mcd} Q^*$ and $Q^N \rhd_{mcd} Q^*$. Let $T \equiv \lambda P.Q^*$. Then $M \equiv \lambda P.Q^M \rhd_{mcd} \lambda P.Q^* \equiv T$ and $N \equiv \lambda P.Q^N \rhd_{mcd} \lambda P.Q^* \equiv T$.

iii. $L \equiv (\lambda P.Q \mid A)$.

Since $L \rhd_{mcd} M$ and $L \rhd_{mcd} N$, both without α-steps, $M \equiv (\lambda P.Q^M \mid A^M)$ and $N \equiv (\lambda P.Q^N \mid A^N)$ for some terms Q^M and Q^N and some abstractions A^M and A^N such that $Q \rhd_{mcd} Q^M$, $Q \rhd_{mcd} Q^N$, $A \rhd_{mcd} A^M$ and $A \rhd_{mcd} A^N$. By induction, there exist terms Q^* and A^* such that $Q^M \rhd_{mcd} Q^*$, $Q^N \rhd_{mcd} Q^*$, $A^M \rhd_{mcd} A^*$, and $A^N \rhd_{mcd} A^*$. By Lemmas 3.2.5 and 3.1.2, A^* is also an abstraction. Let $T \equiv (\lambda P.Q^* \mid A^*)$. Then $M \equiv (\lambda P.Q^M \mid A^M) \rhd_{mcd} (\lambda P.Q^* \mid A^*) \equiv T$ and, similarly, $N \rhd_{mcd} T$.

iv. $L \equiv L_1L_2$.

Case 1. $L \notin \mathscr{R}_M$ and $L \notin \mathscr{R}_N$.

This case can be proved in the same way as (iii).

Case 2. $L \in \mathscr{R}_M$ or $L \in \mathscr{R}_N$.

Without loss of generality, assume that $L \in \mathscr{R}_M$. There are cases and subcases as follows.

(2.1) $L_1 \equiv \lambda P.Q$.

Since $L \in \mathscr{R}_M$ and $(\lambda P.Q)L_2 \equiv L \rhd_{mcd} M$, without α-steps, by Definition 3.2.3 $L \rhd_{mcd} (\lambda P.Q^M)L_2^M$ for some terms Q^M and L_2^M such that $Q \rhd_{mcd} Q^M$ and $L_2 \rhd_{mcd} L_2^M$, and $(\lambda P.Q^M)L_2^M \rhd_{1\beta} M$, with $(\lambda P.Q^M)L_2^M$ being the β-redex contracted.

(2.1.1) $L \in \mathscr{R}_N$.

Similar to the above, $L \triangleright_{mcd} (\lambda P.Q^N)L_2^N$ for some terms Q^N and L_2^N such that $Q \triangleright_{mcd} Q^N$ and $L_2 \triangleright_{mcd} L_2^N$, and $(\lambda P.Q^N)L_2^N \triangleright_{1\beta} N$, with $(\lambda P.Q^N)L_2^N$ being the β-redex contracted. By induction, there exist terms Q^* and L_2^* such that $Q^M \triangleright_{mcd} Q^*$, $Q^N \triangleright_{mcd} Q^*$, $L_2^M \triangleright_{mcd} L_2^*$, and $L_2^N \triangleright_{mcd} L_2^*$.

(2.1.1.1) $FV(P) = \varnothing$.

Since $(\lambda P.Q^M)L_2^M \triangleright_{1\beta} M$ and $(\lambda P.Q^N)L_2^N \triangleright_{1\beta} N$, $M \equiv Q^M$ and $N \equiv Q^N$. Hence $M \equiv Q^M \triangleright_{mcd} Q^*$ and $N \equiv Q^N \triangleright_{mcd} Q^*$ so we are finished with $T \equiv Q^*$.

(2.1.1.2) $FV(P) = \{x_1,\ldots, x_k\}$.

Since $(\lambda P.Q^M)L_2^M \triangleright_{1\beta} M$ and $(\lambda P.Q^N)L_2^N \triangleright_{1\beta} N$, there exist terms $U_1,\ldots, U_k,V_1,\ldots,V_k$ such that $[U_1/x_1,\ldots, U_k/x_k]P \equiv L_2^M$, $[V_1/x_1,\ldots,V_k/x_k]P \equiv L_2^N$, $M \equiv [U_1/x_1,\ldots, U_k/x_k]Q^M$, and $N \equiv [V_1/x_1,\ldots,V_k/x_k]Q^N$.

Since $L_2^M \triangleright_{mcd} L_2^*$ and $L_2^N \triangleright_{mcd} L_2^*$, by Lemma 3.2.6 $L_2^* \equiv [U_1'/x_1,\ldots, U_k'/x_k]P$ and $L_2^* \equiv [V_1'/x_1,\ldots,V_k'/x_k]P$ for some terms $U_1',\ldots, U_k'$, $V_1',\ldots,V_k'$ such that $U_i \triangleright_{mcd} U_i'$, and $V_i \triangleright_{mcd} V_i'$ for all $1 \le i \le k$. Since $[U_1'/x_1,\ldots, U_k'/x_k]P \equiv L_2^* \equiv [V_1'/x_1,\ldots,V_k'/x_k]P$, for each $1 \le i \le k$, $U_i' \equiv V_i'$, so let $W_i \equiv U_i \equiv V_i$. Then $U_i \triangleright_{mcd} W_i$ and $V_i \triangleright_{mcd} W_i$ for all $1 \le i \le k$. Thus, by Lemma 3.2.7

$M \equiv [U_1/x_1,\ldots, U_k/x_k]Q^M \triangleright_{mcd} [W_1/x_1,\ldots,W_k/x_k]Q^*$ and

$N \equiv [V_1/x_1,\ldots,V_k/x_k]Q^N \triangleright_{mcd} [W_1/x_1,\ldots,W_k/x_k]Q^*$ so we are finished with $T \equiv [W_1/x_1,\ldots,W_k/x_k]Q^*$.

(2.1.2) $L \notin \mathscr{R}_N$.

Since $(\lambda P.Q)L_2 \equiv L \triangleright_{mcd} N$, without α-steps, $N \equiv (\lambda P.Q^N)L_2^N$ for some terms Q^N and L_2^N such that $Q \triangleright_{mcd} Q^N$ and $L_2 \triangleright_{mcd} L_2^N$.

By induction, there exist terms Q^* and L_2^* such that $Q^N \triangleright_{mcd} Q^*$ and $L_2^N \triangleright_{mcd} L_2^*$, both without α-steps, and $Q^M \triangleright_{mcd} Q^*$ and $L_2^M \triangleright_{mcd} L_2^*$.

Since $(\lambda P.Q^M)L_2^M$ is a β-redex and $L_2^M \triangleright_{mcd} L_2^*$, then $L_2^M \triangleright_{\beta\delta} L_2^*$, and by Lemma 3.1.12 $(\lambda P.Q^M)L_2^*$ is a β-redex. Hence $(\lambda P.Q^*)L_2^*$ is a β-redex by Lemma 3.1.4. Note that, by Corollary 3.1.14(c) N is contractible since $L \triangleright_{mcd} N$ and

$L \in \mathscr{R}_M$ which implies L is contractible.

(2.1.2.1) $FV(P) = \varnothing$.

Since $(\lambda P.Q^M)L_2^M \triangleright_{1\beta} M$ and $(\lambda P.Q^*)L_2^*$ is a β-redex, $M \equiv Q^M$ and $(\lambda P.Q^*)L_2^* \triangleright_{1\beta} Q^*$. Hence $M \equiv Q^M \triangleright_{mcd} Q^*$ and $N \equiv (\lambda P.Q^N)L_2^N \triangleright_{mcd} (\lambda P.Q^*)L_2^* \triangleright_{1\beta} Q^*$. Thus we are finished with $T \equiv Q^*$.

(2.1.2.2) $FV(P) = \{x_1,\ldots, x_k\}$.

Since $(\lambda P.Q^M)L_2^M \triangleright_{1\beta} M$, there exist terms $U_1,\ldots, U_k$ such that $[U_1/x_1,\ldots, U_k/x_k]P \equiv L_2^M$ and $M \equiv [U_1/x_1,\ldots, U_k/x_k]Q^M$. Since $L_2^M \triangleright_{mcd} L_2^*$, by Lemma 3.2.6 $L_2^* \equiv [V_1/x_1,\ldots,V_k/x_k]P$ for some terms $V_1,\ldots,V_k$ such that $U_i \triangleright_{mcd} V_i$ for all $1 \le i \le k$. Thus

$M \equiv [U_1/x_1,\ldots, U_k/x_k]Q^M \triangleright_{mcd} [V_1/x_1,\ldots,V_k/x_k]Q^*$ (by Lemma 3.2.8), and

$N \equiv (\lambda P.Q^N)L_2^N \triangleright_{mcd} (\lambda P.Q^*)L_2^* \triangleright_{1\beta} [V_1/x_1,\ldots,V_k/x_k]Q^*$, so we are finished with $T \equiv [V_1/x_1,\ldots,V_k/x_k]Q^*$.

(2.2) $L_1 \equiv (\lambda P.Q \mid A)$.

Since $L \in \mathscr{R}_M$ and $(\lambda P.Q \mid A)L_2 \equiv L \triangleright_{mcd} M$, without α-steps, by Definition 3.2.3 $L \triangleright_{mcd} (\lambda P.Q^M \mid A^M)L_2^M$ for some terms Q^M and L_2^M, and some abstraction A^M such that $Q \triangleright_{mcd} Q^M$, $A \triangleright_{mcd} A^M$ and $L_2 \triangleright_{mcd} L_2^M$, and $(\lambda P.Q^M \mid A^M)L_2^M \triangleright_{1\delta} M$, with $(\lambda P.Q^M \mid A^M)L_2^M$ being the δ-redex contracted.

(2.2.1) $L \in \mathscr{R}_N$.

Similar to the above, $L \triangleright_{mcd} (\lambda P.Q^N \mid A^N)L_2^N$ for some terms Q^N and L_2^N, and some abstraction A^N such that $Q \triangleright_{mcd} Q^N$, $A \triangleright_{mcd} A^N$ and $L_2 \triangleright_{mcd} L_2^N$, and $(\lambda P.Q^N \mid A^N)L_2^N \triangleright_{1\delta} N$, with $(\lambda P.Q^N \mid A^N)L_2^N$ being the δ-redex contracted. By induction, there exist terms Q^*, A^* and L_2^* such that $Q^M \triangleright_{mcd} Q^*$, $Q^N \triangleright_{mcd} Q^*$, $A^M \triangleright_{mcd} A^*$, $A^N \triangleright_{mcd} A^*$, $L_2^M \triangleright_{mcd} L_2^*$, and $L_2^N \triangleright_{mcd} L_2^*$.

(2.2.1.1) $(\lambda P.Q^M \mid A^M)L_2^M \triangleright_{1\delta} (\lambda P.Q^M)L_2^M$.

Then $(\lambda P.Q^M)L_2^M$ is a β-redex and $M \equiv (\lambda P.Q^M)L_2^M$. Since $L_2^M \triangleright_{mcd} L_2^*$, by Note 3.2.4(g), Lemmas 3.1.12 and 3.1.4 $(\lambda P.Q^N)L_2^*$ is a β-redex. Since

$L_2^N \rhd_{mcd} L_2^*$, we have that $L_2^N \rhd_{\beta\delta} L_2^*$, and so $L_2^N \rhd_{\beta\gamma} L_2^*$. Hence $(\lambda P.Q^N \mid A^N)L_2^N \not\rhd_{1\delta} A^N L_2^N$. Since $(\lambda P.Q^N \mid A^N)L_2^N \rhd_{1\delta} N$, it must be that $N \equiv (\lambda P.Q^N)L_2^N$. Thus $M \equiv (\lambda P.Q^M)L_2^M \rhd_{mcd} (\lambda P.Q^*)L_2^*$ and $N \equiv (\lambda P.Q^N)L_2^N \rhd_{mcd} (\lambda P.Q^*)L_2^*$ so we are finished with $T \equiv (\lambda P.Q^*)L_2^*$.

(2.2.1.2) $(\lambda P.Q^M \mid A^M)L_2^M \rhd_{1\delta} A^M L_2^M$.

Suppose $(\lambda P.Q^N)L_2^N$ is a β-redex. Since $L_2^N \rhd_{mcd} L_2^*$, an argument similar to the one above shows that $(\lambda P.Q^M)L_2^*$ is a β-redex. Since $L_2^M \rhd_{mcd} L_2^*$, $L_2^M \rhd_{\beta\gamma} L_2^*$. Hence $(\lambda P.Q^M \mid A^M)L_2^M \not\rhd_{1\delta} A^M L_2^M$, a contradiction. Hence $(\lambda P.Q^N)L_2^N$ is not a β-redex. Since $(\lambda P.Q^N \mid A^N)L_2^N \rhd_{1\delta} N$, $N \equiv A^N L_2^N$. Thus $M \equiv A^M L_2^M \rhd_{mcd} A^* L_2^*$ and $N \equiv A^N L_2^N \rhd_{mcd} A^* L_2^*$ so we are finished with $T \equiv A^* L_2^*$.

(2.2.2) $L \notin \mathscr{R}_N$.

Since $(\lambda P.Q \mid A)L_2 \equiv L \rhd_{mcd} N$, without α-steps, $N \equiv (\lambda P.Q^N \mid A^N)L_2^N$ for some terms Q^N and L_2^N and some abstraction A^N such that $Q \rhd_{mcd} Q^N$, $A \rhd_{mcd} A^N$, and $L_2 \rhd_{mcd} L_2^N$. By induction, there exist terms Q^*, A^*, and L_2^* such that $Q^N \rhd_{mcd} Q^*$, $A^N \rhd_{mcd} A^*$, and $L_2^N \rhd_{mcd} L_2^*$, all without α-steps, and $Q^M \rhd_{mcd} Q^*$, $A^M \rhd_{mcd} A^*$, and $L_2^M \rhd_{mcd} L_2^*$. Note that A^* is an abstraction by Lemmas 3.2.5 and 3.1.2.

(2.2.2.1) $(\lambda P.Q^M \mid A^M)L_2^M \rhd_{1\delta} (\lambda P.Q^M)L_2^M$.

Then $(\lambda P.Q^M)L_2^M$ is a β-redex and $M \equiv (\lambda P.Q^M)L_2^M$. Since $L_2^M \rhd_{mcd} L_2^*$, we have that $(\lambda P.Q^*)L_2^*$ is a β-redex. So we have $M \equiv (\lambda P.Q^M)L_2^M \rhd_{mcd} (\lambda P.Q^*)L_2^*$ and $N \equiv (\lambda P.Q^N \mid A^N)L_2^N \rhd_{mcd} (\lambda P.Q^* \mid A^*)L_2^* \rhd_{1\delta} (\lambda P.Q^*)L_2^*$ and we are finished with $T \equiv (\lambda P.Q^*)L_2^*$.

(2.2.2.2) $(\lambda P.Q^M \mid A^M)L_2^M \rhd_{1\delta} A^M L_2^M$.

Suppose $(\lambda P.Q^* \mid A^*)L_2^* \not\rhd_{1\delta} A^* L_2^*$. Then $(\lambda P.Q^*)L_2^+$ is a β-redex for some term L_2^+ such that $[U_1/x_1,\ldots, U_k/x_k]L_2^* \rhd_{\beta\gamma} L_2^+$ for some distinct variables $x_1,\ldots, x_k$, $k \geq 1$, and some terms $U_1,\ldots, U_k$. Since $L_2^M \rhd_{mcd} L_2^*$, we have that $L_2^M \rhd_{\beta\delta} L_2^*$. By Corollary 3.1.8(c), $[U_1/x_1,\ldots, U_k/x_k]L_2^M \rhd_{\beta\delta} [U_1/x_1,\ldots, U_k/x_k]L_2^*$, so $[U_1/x_1,\ldots, U_k/x_k]L_2^M \rhd_{\beta\gamma} [U_1/x_1,\ldots, U_k/x_k]L_2^*$. Since the relation $\rhd_{\beta\gamma}$ is transitive, this shows that $[U_1/x_1,\ldots, U_k/x_k]L_2^M \rhd_{\beta\gamma} L_2^+$. Since $(\lambda P.Q^*)L_2^+$ is a β-redex, $(\lambda P.Q^M)L_2^+$ is also a β-redex. Hence $(\lambda P.Q^M \mid A^M)L_2^M \not\rhd_{1\delta} A^M L_2^M$, a contradiction. Thus $(\lambda P.Q^* \mid A^*)L_2^* \rhd_{1\delta} A^* L_2^*$. Hence $M \equiv A^M L_2^M \rhd_{mcd} A^* L_2^*$ and $N \equiv (\lambda P.Q^N \mid A^N)L_2^N \rhd_{mcd} (\lambda P.Q^* \mid A^*)L_2^* \rhd_{1\delta} A^* L_2^*$ so we are finished with $T \equiv A^* L_2^*$. □

Theorem 3.3.2 (The Church-Rosser theorem for βδ-reduction). For any terms L, M and N, if $L \rhd_{\beta\delta} M$ and $L \rhd_{\beta\delta} N$, then there exists a term T such that $M \rhd_{\beta\delta} T$ and $N \rhd_{\beta\delta} T$.

Proof. Let L, M, and N be terms.

Claim. If $L \rhd_{mcd} M$ and $L \rhd_{\beta\delta} N$, then there exists a term T such that $M \rhd_{\beta\delta} T$ and $N \rhd_{mcd} T$.

Proof of Claim. Assume $L \rhd_{mcd} M$ and $L \rhd_{\beta\delta} N$.

Then there exists a sequence of terms $L \equiv N_1, N_2,\ldots, N_n \equiv N$, $n \geq 1$, as in Definition 2.3.7. Induct on n.

If $n = 1$, then $L \equiv N$, so $M \rhd_{\beta\delta} M$ and $N \equiv L \rhd_{mcd} M$ and we are finished.

Now, assume $n > 1$. By induction, there exists a term T_0 such that $N_{n-1} \rhd_{mcd} T_0$ and $M \rhd_{\beta\delta} T_0$. Since $N_{n-1} \equiv_\alpha N$ or $N_{n-1} \rhd_{1\beta,1\delta} N$, we have that $N_{n-1} \rhd_{mcd} N$. Hence, by Theorem 3.3.1 there exists a term T such that $N \rhd_{mcd} T$ and $T_0 \rhd_{mcd} T$, so that $T_0 \rhd_{\beta\delta} T$. Since $M \rhd_{\beta\delta} T_0$ and the relation $\rhd_{\beta\delta}$ is transitive, $M \rhd_{\beta\delta} T$. Thus we have $M \rhd_{\beta\delta} T$ and $N \rhd_{mcd} T$. So we have the claim.

Now we can prove the theorem.

Assume $L \rhd_{\beta\delta} M$ and $L \rhd_{\beta\delta} N$. Then there exists a sequence of terms $L \equiv M_1, M_2,\ldots, M_m \equiv M$ as in Definition 2.3.7. Induct on m.

If $m = 1$, then $L \equiv M$, so $M \equiv L \rhd_{\beta\delta} N$ and $N \rhd_{\beta\delta} N$ and we are finished.

Now, suppose $m > 1$. By induction, there exists a term T_0 such that $M_{m-1} \rhd_{\beta\delta} T_0$ and $N \rhd_{\beta\delta} T_0$. Since $M_{m-1} \equiv_\alpha M$ or $M_{m-1} \rhd_{1\beta,1\delta} M$, we have that $M_{m-1} \rhd_{mcd} M$. By the claim, there exists a term T such that $M \rhd_{\beta\delta} T$ and $T_0 \rhd_{mcd} T$, so that $T_0 \rhd_{\beta\delta} T$. Since $N \rhd_{\beta\delta} T_0$ and the relation $\rhd_{\beta\delta}$ is transitive, $N \rhd_{\beta\delta} T$. Thus $M \rhd_{\beta\delta} T$ and $N \rhd_{\beta\delta} T$. □

Corollary 3.3.3. For any term L, if L has βδ-normal forms M and N, then $M \equiv_\alpha N$.

Proof. Let L, M and N be terms such that $L \rhd_{\beta\delta} M$ and $L \rhd_{\beta\delta} N$, and M and N are βδ-normal forms.

By Theorem 3.3.2, there exists a term T such that $M \rhd_{\beta\delta} T$ and $N \rhd_{\beta\delta} T$. By Lemma 3.1.15, $M \equiv_\alpha T$ and $N \equiv_\alpha T$. Hence $M \equiv_\alpha N$. □

3.4 βδ-Equality

We know that the relation βδ-reduction, which is defined in Chapter II, is transitive and reflexive but is not symmetric. In this section, we will define an equivalence relation which is closely connected to βδ-reduction, called βδ-equality.

Definition 3.4.1 For any terms M and M', we say M is **βδ-equal** or **βδ-convertible** to M', denoted by $M =_{\beta\delta} M'$, if there exists a sequence of terms $M \equiv M_1, M_2, \ldots, M_n \equiv M'$, $n \geq 1$, such that for each $1 \leq i < n$, $M_i \rhd_{1\beta,1\delta} M_{i+1}$, $M_{i+1} \rhd_{1\beta,1\delta} M_i$, or $M_i \equiv_\alpha M_{i+1}$.

Note 3.4.2. βδ-equality is reflexive, transitive and symmetric.

Theorem 3.4.3 (The Church-Rosser theorem for βδ-equality). For any terms M and N, if $M =_{\beta\delta} N$, then there exists a term T such that $M \rhd_{\beta\delta} T$ and $N \rhd_{\beta\delta} T$.

Proof. Let M and N be terms such that $M =_{\beta\delta} N$.

Then there exists a sequence of terms $M \equiv M_1, M_2, \ldots, M_n \equiv N$, $n \geq 1$, as in

Definition 3.4.1. Induct on n.

If n =1, then $M \equiv N$ and we are finished.

Now, suppose n > 1. Since $M =_{\beta\delta} M_{n-1}$, by induction there exists a term T_0 such that $M \rhd_{\beta\delta} T_0$ and $M_{n-1} \rhd_{\beta\delta} T_0$. Since $M_{n-1} \rhd_{1\beta,1\delta} M_n$, $M_n \rhd_{1\beta,1\delta} M_{n-1}$, or $M_{n-1} \equiv_\alpha M_n$, we have that $M_{n-1} \rhd_{\beta\delta} N$ or $N \rhd_{\beta\delta} M_{n-1}$.

Case 1. $M_{n-1} \rhd_{\beta\delta} N$.

Since $M_{n-1} \rhd_{\beta\delta} T_0$, by Theorem 3.3.2 there exists a term T such that $N \rhd_{\beta\delta} T$ and $T_0 \rhd_{\beta\delta} T$. Since $M \rhd_{\beta\delta} T_0$ and the relation $\rhd_{\beta\delta}$ is transitive, $M \rhd_{\beta\delta} T$.

Case 2. $N \rhd_{\beta\delta} M_{n-1}$.

Since $M_{n-1} \rhd_{\beta\delta} T_0$, we have that $N \rhd_{\beta\delta} T_0$. So we have $M \rhd_{\beta\delta} T_0$ and $N \rhd_{\beta\delta} T_0$.

□

Corollary 3.4.4. For any terms M and N, if $M =_{\beta\delta} N$ and N is in βδ-normal form, then $M \rhd_{\beta\delta} N$.

Proof. Let M and N be terms such that $M =_{\beta\delta} N$ and N is in βδ-normal form.

By Theorem 3.4.3, there exists a term T such that $M \rhd_{\beta\delta} T$ and $N \rhd_{\beta\delta} T$. Since N is in βδ-normal form, by Lemma 3.1.14 $N \equiv_\alpha T$. Since $M \rhd_{\beta\delta} T$, we have that $M \rhd_{\beta\delta} N$. □

Corollary 3.4.5. For any terms M and N, if $M =_{\beta\delta} N$, then either M and N do not have βδ-normal forms, or M and N both have the same βδ-normal forms.

Proof. Let M and N be terms such that $M =_{\beta\delta} N$.

Suppose M or N has a βδ-normal form. We want to show that M and N both have the same βδ-normal forms. Without loss of generality, assume that M has a βδ-normal form, as the case that N has a βδ-normal form can be proved similarly.

Let M' be a βδ-normal form of M. Then $M \rhd_{\beta\delta} M'$ and M' is a βδ-normal form. Since $M =_{\beta\delta} N$ and $M \rhd_{\beta\delta} M'$, by Definitions 2.3.7 and 3.4.1 $N =_{\beta\delta} M'$. Since

M' is a $\beta\delta$-normal form, by Corollary 3.4.4 $N \rhd_{\beta\delta} M'$. Hence M' is also a $\beta\delta$-normal form of N. This implies that N has a $\beta\delta$-normal form and every $\beta\delta$-normal form of M is a $\beta\delta$-normal form of N. Similarly, we can prove that every $\beta\delta$-normal form of N is also a $\beta\delta$-normal form of M. Thus M and N both have the same $\beta\delta$-normal forms. □

Corollary 3.4.6. Two $\beta\delta$-equal terms in $\beta\delta$-normal form must be congruent.

Proof. Let M and N be terms in $\beta\delta$-normal form such that $M =_{\beta\delta} N$. Since N is a $\beta\delta$-normal form, by Corollary 3.4.4 $M \rhd_{\beta\delta} N$. Since M is a $\beta\delta$-normal form, by Lemma 3.1.15 $M \equiv_\alpha N$. □

Corollary 3.4.7. For any term M, if N_1 and N_2 are $\beta\delta$-normal forms such that $M =_{\beta\delta} N_1$ and $M =_{\beta\delta} N_2$, then $N_1 \equiv_\alpha N_2$.

Proof. Let M be a term, and N_1 and N_2 be $\beta\delta$-normal forms such that $M =_{\beta\delta} N_1$ and $M =_{\beta\delta} N_2$. By Note 3.4.2, $N_1 =_{\beta\delta} N_2$. Since $N_1$and N_2 are $\beta\delta$-normal forms, by Corollary 3.4.6 $N_1 \equiv_\alpha N_2$. □

Corollary 3.4.8. Let $M_0, M_1,\ldots, M_m, N_0, N_1,\ldots, N_n$, $m \geq 1$, $n \geq 1$, be terms. If $M_0M_1\ldots M_m =_{\beta\delta} N_0N_1\ldots N_n$, and M_0M_1 and N_0N_1 are not contractible redexes, then $m = n$ and $M_i =_{\beta\delta} N_i$ for all $1 \leq i \leq m$.

Proof. Assume $M_0M_1\ldots M_m =_{\beta\delta} N_0N_1\ldots N_n$, and M_0M_1 and N_0N_1 are not contractible redexes.

By Theorem 3.4.3, there exists a term T such that $M_0M_1\ldots M_m \rhd_{\beta\delta} T$ and $N_0N_1\ldots N_n \rhd_{\beta\delta} T$. Since M_0M_1 is not a contractible redex, each contractible redex in $M_0M_1\ldots M_m$ must be in an M_i. Hence $T \equiv T_0T_1\ldots T_m$ for some terms $T_0, T_1,\ldots, T_m$ such that $M_i \rhd_{\beta\delta} T_i$ for all $1 \leq i \leq m$. Similarly, $T \equiv T_0'T_1'\ldots T_n'$for some terms $T_0', T_1',\ldots, T_n'$ such that $N_i \rhd_{\beta\delta} T_i'$ for all $1 \leq i \leq n$. So we have

$T_0T_1 \ldots T_m \equiv T_0'T_1' \ldots T_n'$. Thus, by Note 2.1.3(b) $m = n$ and $T_i \equiv T_i'$ for all $1 \leq i \leq m$. Since for each $1 \leq i \leq m$, $M_i \rhd_{\beta\delta} T_i$ and $N_i \rhd_{\beta\delta} T_i$, by Definitions 2.3.7 and 3.4.1 $M_i =_{\beta\delta} N_i$ for all $1 \leq i \leq m$. □