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นางสาวแสงแข ยินดีถิ่น

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DIVISIBLE COMMUTATIVE SEMIGROUPS

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ให้ $\mathbb{N}$, $\mathbb{R}^+$ และ $\mathbb{R}$ แทนเซตของจำนวนเต็มบวกทั้งหมด เซตของจำนวนจริงบวกทั้งหมดและเซตของจำนวนจริงทั้งหมด ตามลำดับ

ให้ $(S,+)$ แทนเซมิกรุป ถ้าสำหรับแต่ละ $x \in S$ และสำหรับแต่ละจำนวนเต็มบวก n มี $y \in S$ ซึ่ง $x = ny = y+\cdots+y$ (n ครั้ง) แล้ว จะกล่าวว่า S *หารลงตัวได้* เราจะกล่าวว่า เซมิกรุป S *พาวเวอร์แคนเซลเลทีฟ* ก็ต่อเมื่อสำหรับแต่ละ $x,y \in S$ และ $n \in \mathbb{N}$ ถ้า $nx = ny$ แล้ว $x = y$

ในการวิจัยนี้ เราหาเงื่อนไขจำเป็นและเพียงพอที่ทำให้เซมิกรุปย่อยของ $\mathbb{R}^+$ ภายใต้การบวกปกติและเซมิกรุปย่อยของ $\mathbb{R}^+$ ภายใต้การคูณปกติหารลงตัวได้ เราได้พิสูจน์ทฤษฎีบทเกี่ยวกับเซมิกรุปสลับที่ซึ่งพาวเวอร์แคนเซลเลทีฟและหารลงตัวได้

นอกจากนี้ เราให้ตัวอย่างเซมิกรุปย่อยไม่สลับที่บางชนิดซึ่งหารลงตัวได้ ของ $M_2(\mathbb{R})$ ภายใต้การคูณปกติ

ภาควิชา **คณิตศาสตร์** ลายมือชื่อนิสิต..

สาขาวิชา **คณิตศาสตร์** ลายมือชื่ออาจารย์ที่ปรึกษา.......................................

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Let $\mathbb{N}$, $\mathbb{R}^+$ and $\mathbb{R}$ denote the set of all positive integers, the set of all positive real numbers and the set of all real numbers, respectively.

Let $(S,+)$ be a semigroup. If for any element x of S and for any positive integer n, there is an element y of S such that $x = ny = y + \cdots + y$ (n times), then S is said to be *divisible*. A semigroup S is called *power cancellative* if and only if for $x,y \in S$ and $n \in \mathbb{N}$, $nx = ny$ implies that $x = y$.

In this research, we find necessary and sufficient conditions for subsemigroups of $\mathbb{R}^+$ under usual addition and $\mathbb{R}^+$ under usual multiplication to be divisible. We also prove a theorem on commutative power cancellative divisible semigroups .

Moreover, we give examples of some noncommutative divisible subsemigroups of $M_2(\mathbb{R})$ under usual multiplication.

Department **Mathematics** Student's signature.......................................

Field of study **Mathematics** Advisor's signature.......................................

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CHAPTER I

INTRODUCTION AND PRELIMINARIES

Throughout, let $\mathbb{N}, \mathbb{Q}^+, \mathbb{Q}, \mathbb{R}^+$ and $\mathbb{R}$, respectively, denote the set of positive integers, the set of positive rational numbers, the set of rational numbers, the set of positive real numbers and the set of real numbers. Also, let $\mathbb{Q}_0^+ = \mathbb{Q}^+ \cup \{0\}$.

Let S be a semigroup where the binary operation is denoted by $+$. If for any element x of S and for any positive integer n, there is an element y of S such that $x = ny = y + \cdots + y$ (n times), then S is said to be *divisible*. For example, the additive semigroup of all positive rational numbers and the additive semigroup of all positive real numbers are divisible semigroups. An element e of S is called an *idempotent* if $2e = e$. If every element of S is an idempotent, then S is a *band*. A semigroup S is called *power cancellative* if for $x, y \in S$ and $n \in \mathbb{N}, nx = ny$ implies that $x = y$.

If a is an element of S, then $< a > = \{a, 2a, 3a, \dots\}$ is the *monogenic subsemigroup of S generated by a*. The *order of a* is defined to be the order of $< a >$. If S has the property that $S = < a >$ for some $a \in S$, then we say that S is *monogenic*.

Theorem 1.1. ([6],J.M.Howie) *Let a be an element of a semigroup S. Then either: (i) all sums of a are distinct and the monogenic subsemigroup $< a >$ of S is isomorphic to the semigroup $(\mathbb{N}, +)$ or (ii) there exist positive integers r (the index of a) and m (the period of a) with the following properties :*

(1) $ra = (m + r)a$,

(2) *for all* $s, t \in \mathbb{N}, (r+s)a = (r+t)a$ *if and only if* $r+s \equiv r+t \pmod{m}$,

(3) $<a> = \{a, 2a, \ldots, (m+r-1)a\}$ *and the order of* $<a>$ *is* $m+r-1$,

(4) $K_a = \{ra, (r+1)a, \ldots, (m+r-1)a\}$ *is a cyclic subgroup of* $<a>$ *and the order of* K_a *is* m.

Let ρ be an equivalence relation on S. If $x \in S$, the *equivalence class of* ρ containing x is the class of all those elements of S that are equivalent to x. Let $\overline{x} = \{y \in S \mid y\rho x\}$ denote the equivalence class of ρ containing x. The set of all equivalence classes in S is denoted by S/ρ and called the *quotient semigroup of* S *by* ρ.

Throughout this thesis except the last chapter any semigroup is assumed to be commutative.

Let R and T be semigroups. A mapping $\varphi : R \to T$ is a *homomorphism* if for all $x, y \in R, \varphi(xy) = \varphi(x)\varphi(y)$. If φ maps R onto T, it is an *epimorphism* and T is a *homomorphic image* of R. A homomorphism φ which is a bijection of R onto T is *an isomorphism* and we write $R \cong T$.

The following basic theorems are used in this thesis.

Theorem 1.2. ([11],T.Tamura) *Any homomorphic image of a divisible semigroup is divisible.*

For each $\alpha \in \Gamma$, let S_α is semigroup and S_α^0 denote the semigroup S_α with two-sided identity 0 adjoined.

Let $\sum\limits_{\alpha \in \Gamma} S_\alpha^0 = \{f : \Gamma \to \bigcup\limits_{\alpha \in \Gamma} S_\alpha^0 \mid f(\alpha) \in S_\alpha^0$ for all $\alpha \in \Gamma$ and $f(\alpha) = 0$ for all but finitely many components$\}$. The semigroup obtained as the direct sum $\sum\limits_{\alpha \in \Gamma} S_\alpha^0$ excluding the identity is called the *annexed sum* of S_α, and it is denoted by $\tilde{\sum\limits_{\alpha \in \Gamma}} S_\alpha$.

Theorem 1.3. ([11],T.Tamura) *If* S_α *is a divisible semigroup for all* $\alpha \in \Gamma$, *then* $\tilde{\sum\limits_{\alpha \in \Gamma}} S_\alpha$ *is also a divisible semigroup.*

There are exactly 15 types of multiplicative interval semigroups on $\mathbb{R}$. This was proved by S.Ritkeao in [9].

Theorem 1.4. ([9],S.Ritkeao) *A subset S of $\mathbb{R}$ is a multiplicative interval semigroup on $\mathbb{R}$ if and only if S is one of the following types :*

(1) $\mathbb{R}$, *(2)* $\{0\}$, *(3)* $\{1\}$, *(4)* $(0,\infty)$, *(5)* $[0,\infty)$,

(6) (a,∞) *where* $a \geqslant 1$,

(7) $[a,\infty)$ *where* $a \geqslant 1$,

(8) $(0,b)$ *where* $0 < b \leqslant 1$,

(9) $(0,b]$ *where* $0 < b \leqslant 1$,

(10) $[0,b)$ *where* $0 < b \leqslant 1$,

(11) $[0,b]$ *where* $0 < b \leqslant 1$,

(12) (a,b) *where* $-1 \leqslant a < 0 < a^2 \leqslant b \leqslant 1$,

(13) $(a,b]$ *where* $-1 \leqslant a < 0 < a^2 \leqslant b \leqslant 1$,

(14) $[a,b)$ *where* $-1 < a < 0 < a^2 < b < 1$,

(15) $[a,b]$ *where* $-1 \leqslant a < 0 < a^2 \leqslant b \leqslant 1$.

There are exactly 6 types of additive interval semigroups on $\mathbb{R}$ and a proof was given by K.Palasri in [10].

Theorem 1.5. ([10],K.Palasri) *A subset S of $\mathbb{R}$ is an additive interval semigroup on $\mathbb{R}$ if and only if S is one of the following types :*

(1) $\{0\}$, *(2)* $\mathbb{R}$,

(3) (a,∞) *where* $a \geqslant 0$, *(4)* $[a,\infty)$ *where* $a \geqslant 0$,

(5) $(-\infty,b)$ *where* $b \leqslant 0$, *(6)* $(-\infty,b]$ *where* $b \leqslant 0$.

The notions of divisible commutative groups and divisible commutative semigroups have long been studied. See in [1],[2],[3] and [5] for examples. In [8], we see that a commutative group is divisible if and only if it is injective. This statement was given by Baer. The notions of theme are still interesting in the last two decades. We can see in [4] and [5] that divisible semigroups are linked to Lie groups. The authors are interested in the structure of groups which contain a nontrivial divisible subsemigroups and they require that the enclosing group is 'as small as possible'. Every divisible group is the n^{th} root group for all natural numbers n, and we may glance the n^{th} root group in [7].

The study of divisible semigroups which are not related to other subjects is quite interesting in its own, so we study properties of special divisible commutative semigroup in this research. If we look at the statement given by Baer, mentioned above, one can see that the 'if part' is still true by changing the word 'group' to the word 'semigroup'. We study some commutative semigroups of which the 'only if part' still holds. However, our characterizations may not be related to injectivity. Moreover, general properties of divisible commutative semigroups are investigated.

Interval semigroups of real numbers under both multiplication and addition seem to be interesting. There are exactly 15 types of multiplicative interval semigroups of real numbers which were introduced by S.Ritkeao in [9] and there are exactly 6 types of additive interval semigroups of real numbers which were given by K.Palasri in [10]. We characterize such multiplicative interval semigroups and additive interval semigroups which are divisible semigroups in Chapter II.

In Chapter III we have to search the conditions that additive subsemigroups of $\mathbb{R}^+$ and multiplicative subsemigroups of $\mathbb{R}^+$ are divisible. Moreover, we prove a theorem on commutative power cancellative divisible semigroups.

We provide some noncommutative divisible subsemigroups of $M_2(\mathbb{R})$ under usual multiplication. This is the purpose of Chapter IV.

CHAPTER II

DIVISIBLE INTERVAL SUBSEMIGROUPS OF $\mathbb{R}$

From Theorem 1.4, we know that there are exactly 15 types of multiplicative interval semigroups on $\mathbb{R}$ and, from Theorem 1.5, there are exactly 6 types of additive interval semigroups on $\mathbb{R}$.

The purpose of this chapter is to show that there are 10 multiplicative divisible interval semigroups on $\mathbb{R}$ and 6 additive divisible interval semigroups on $\mathbb{R}$.

Theorem 2.1. *For a multiplicative interval semigroup S on $\mathbb{R}$, S is divisible if and only if S is* $\{0\}, \{1\}, (0,\infty), [0,\infty), (1,\infty), [1,\infty), (0,1), (0,1], [0,1)$ *or* $[0,1]$.

Proof. Assume that S is a multiplicative interval semigroup on $\mathbb{R}$. Since S is a multiplicative interval semigroup on $\mathbb{R}$, by Theorem 1.4, S belongs to one of the following types :

(1) $\mathbb{R}$, (2) $\{0\}$, (3) $\{1\}$, (4) $(0,\infty)$, (5) $[0,\infty)$,

(6) (a,∞) where $a \geqslant 1$,

(7) $[a,\infty)$ where $a \geqslant 1$,

(8) $(0,b)$ where $0 < b \leqslant 1$,

(9) $(0,b]$ where $0 < b \leqslant 1$,

(10) $[0,b)$ where $0 < b \leqslant 1$,

(11) $[0,b]$ where $0 < b \leqslant 1$,

(12) (a,b) where $-1 \leqslant a < 0 < a^2 \leqslant b \leqslant 1$,

(13) $(a,b]$ where $-1 \leqslant a < 0 < a^2 \leqslant b \leqslant 1$,

(14) $[a,b)$ where $-1 < a < 0 < a^2 < b < 1$,

(15) $[a,b]$ where $-1 \leqslant a < 0 < a^2 \leqslant b \leqslant 1$.

Case 1 : $S = \{0\}, \{1\}, (0,\infty), [0,\infty), (1,\infty), [1,\infty), (0,1), (0,1], [0,1)$ or $[0,1]$. Let $s \in S$ and $n \in \mathbb{N}$. Then $\sqrt[n]{s} \in S$ and $(\sqrt[n]{s})^n = s$. So S is divisible.

Case 2 : $S = (a,\infty), [a,\infty), (0,b), [0,b), (0,b]$ or $[0,b]$ for some $a > 1$ and $0 < b < 1$. For $c \in \{a,b\}, c^2 \in S$ (depends on S). Since the only positive real number x such that $x^4 = c^2$ is $x = \sqrt{c}$ which $\sqrt{c} < a$ if $c = a$ or $\sqrt{c} > b$ if $c = b$, there is no $x \in S$ such that $x^4 = c^2$. So S is not divisible.

Case 3 : $S = \mathbb{R}, (a,b), (a,b], [a,b]$ or $[c,d)$ where $-1 \leqslant a < 0 < a^2 \leqslant b \leqslant 1$ and $-1 < c < 0 < c^2 < d < 1$. There exists $x \in S$ such that $x < 0$. So that S is not divisible.

Hence S is $\{0\}, \{1\}, (0,\infty), [0,\infty), (1,\infty), [1,\infty), (0,1), (0,1], [0,1)$ or $[0,1]$ if and only if S is divisible. □

Theorem 2.2. *For an additive interval semigroup S on $\mathbb{R}$, S is divisible if and only if S is $\{0\}, \mathbb{R}, (0,\infty), [0,\infty), (-\infty,0)$ or $(-\infty,0]$.*

Proof. Suppose that S is an additive interval semigroup on $\mathbb{R}$. Since S is an additive interval semigroup on $\mathbb{R}$, by Theorem 1.5, the type of S is one of the followings :

(1) $\{0\}$, (2) $\mathbb{R}$,

(3) (a,∞) where $a \geqslant 0$, (4) $[a,\infty)$ where $a \geqslant 0$,

(5) $(-\infty,b)$ where $b \leqslant 0$, (6) $(-\infty,b]$ where $b \leqslant 0$.

Case 1 : $S = \{0\}, \mathbb{R}, (0,\infty), [0,\infty), (-\infty,0)$ or $(-\infty,0]$. Let $a \in S$ and $n \in \mathbb{N}$. Then $\frac{a}{n} \in S$ and $n(\frac{a}{n}) = a$. So S is divisible.

Case 2 : $S = (a,\infty), [a,\infty), (-\infty,b)$ or $(-\infty,b]$ for some $a > 0$ and $b < 0$. For $c \in \{a,b\}$, $2c \in S$ (depends on S). Since the only positive real number x such that $4x = 2c$ is $x = \frac{c}{2}$ which $\frac{c}{2} < a$ if $c = a$ or $\frac{c}{2} > b$ if $c = b$, there is no $x \in S$ such that $4x = 2c$.

Therefore S is $\{0\}, \mathbb{R}, (0,\infty), [0,\infty), (-\infty,0)$ or $(-\infty,0]$ if and only if S is divisible. $\square$

The following corollaries are immediate consequences of Theorem 2.1 and Theorem 2.2, respectively.

Corollary 2.3. *For a multiplicative interval semigroup S on $\mathbb{R}^+$, S is divisible if and only if S is $\{1\}, (0,\infty), (1,\infty), [1,\infty), (0,1)$ or $(0,1]$.*

Corollary 2.4. *For an additive interval semigroup S on $\mathbb{R}^+$, S is divisible if and only if $S = \mathbb{R}^+$.*

CHAPTER III

DIVISIBLE SUBSEMIGROUPS OF $\mathbb{R}^+$

It is known that $\mathbb{R}^+$ under usual addition and $\mathbb{R}^+$ under usual multiplication are divisible semigroups. The first purpose of this chapter is to find conditions when an additive subsemigroup of $\mathbb{R}^+$ is divisible and a multiplicative subsemigroup of $\mathbb{R}^+$ is divisible.

The second purpose of this chapter is to prove when a commutative power cancellative semigroup is divisible. Note that this theorem was stated in [11] without proof.

Theorem 3.1. *Let T be a subsemigroup of $\mathbb{R}^+$ under addition. Then T is a divisible subsemigroup of $\mathbb{R}^+$ if and only if there exists a basis B of $\mathbb{R}$ over $\mathbb{Q}$ such that T is a divisible subsemigroup of the semigroup T_C for some $\varnothing \neq C \subseteq B$, where $T_C = \{x \in \mathbb{R}^+ \mid x$ is a $\mathbb{Q}$-linear combination of elements in $C\}$.*

Proof. Assume that T is a divisible subsemigroup of $\mathbb{R}^+$ under addition.

Let $\mathcal{A} = \{D \mid \varnothing \neq D \subseteq T$ and D is a $\mathbb{Q}$-linearly independent subset of $\mathbb{R}\}$. Since $\varnothing \neq T \subseteq \mathbb{R}^+$, there exists $a \in T$ such that $\{a\} \subseteq T$ so that $\{a\}$ is a $\mathbb{Q}$-linearly independent subset of $\mathbb{R}$. As a result, $\{a\} \in \mathcal{A}$ and $\mathcal{A} \neq \varnothing$. We know that $\mathcal{A}$ is a partially order set under inclusion. Let $\mathcal{C}$ be a chain in $\mathcal{A}$. Let $A = \bigcup_{D\in\mathcal{C}} D$. Obviously, $D \subseteq A$ for every $D \in \mathcal{A}$.

First, we show that $A \in \mathcal{A}$. Since $D \subseteq T$ for all $D \in \mathcal{C}$, $A = \bigcup_{D\in\mathcal{C}} D \subseteq T$. Suppose that $\sum_{i=1}^{n} \alpha_i v_i = 0$ where $v_1, v_2, \ldots, v_n \in A$ are all distinct and $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{Q}$. So, for each $i \in \{1, 2, \ldots, n\}$, there exists $D_i \in \mathcal{C}$ such that $v_i \in D_i$.

Since $D_1, D_2, \ldots, D_n \in \mathcal{C}$ and $\mathcal{C}$ is a chain in $\mathcal{A}$, there exists $j \in \{1, 2, \ldots, n\}$ such that $D_1, D_2, \ldots, D_n \subseteq D_j$. Now we have $v_1, v_2, \ldots, v_n \in D_j$ which is a $\mathbb{Q}$-linearly independent subset of $\mathbb{R}$. Thus $\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$. This shows that A is a $\mathbb{Q}$-linearly independent subset of $\mathbb{R}$, so that $A \in \mathcal{A}$. By Zorn's lemma, $\mathcal{A}$ has a maximal element C. Since $C \subseteq \mathbb{R}^+$ and C is a $\mathbb{Q}$-linearly independent subset of $\mathbb{R}$, there exists a basis B of $\mathbb{R}$ over $\mathbb{Q}$ such that $C \subseteq B$.

Next, we have to show that $T \subseteq T_C$. Let $x \in T$. If $x \in C$, then $x = 1 \cdot x \in T_C$. Assume that $x \notin C$. So $C \cup \{x\}$ is a $\mathbb{Q}$-linearly dependent subset of $\mathbb{R}$. Thus there are distinct elements $c_1, c_2, \ldots, c_n \in C$ and $\alpha, \alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{Q}$, not all of them 0, such that $\alpha x + \alpha_1 c_1 + \cdots + \alpha_n c_n = 0$. Suppose that $\alpha = 0$. Then $\alpha_1 c_1 + \cdots + \alpha_n c_n = 0$. Since $c_1, c_2, \ldots, c_n \in C$ and C is linearly independent, $\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$, which is a contradiction. So $\alpha \neq 0$. Thus $x = (\frac{-\alpha_1}{\alpha})c_1 + \cdots + (\frac{-\alpha_n}{\alpha})c_n \in T_C$. Hence $T \subseteq T_C$.

The converse follows directly from the assumption. □

Theorem 3.2. *Let T be a subsemigroup of $\mathbb{R}^+$ under multiplication. Then T is a divisible subsemigroup of $\mathbb{R}^+$ if and only if there exists a basis B of $\mathbb{R}^+$ over $\mathbb{Q}$ such that T is a divisible subsemigroup of the semigroup T_C for some $\varnothing \neq C \subseteq B$, where $T_C = \{x \in \mathbb{R}^+ \mid x$ is a $\mathbb{Q}$- linear combination of elements in $C\}$.*

Proof. Suppose that T is a divisible subsemigroup of $\mathbb{R}^+$ under multiplication.By Zorn's lemma, there exists $\varnothing \neq C \subseteq T$ such that C is a maximal $\mathbb{Q}$-linearly independent subset of $(\mathbb{R}^+, \cdot)$ and can be extended to a basis B of $(\mathbb{R}^+, \cdot)$ over $\mathbb{Q}$ where scalar multiplication αr is r^α where $r \in \mathbb{R}^+$ and $\alpha \in \mathbb{Q}$.

To show that $T \subseteq T_C$, let $x \in T$. If $x \in C$, then $x = x^1 \in T_C$. Assume that $x \notin C$. Thus $C \cup \{x\}$ is a $\mathbb{Q}$-linearly dependent subset of $\mathbb{R}^+$. Hence there are distinct elements $c_1, c_2, \ldots, c_n \in C$ and $\alpha, \alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{Q}$, not all 0, such that $x^\alpha {c_1}^{\alpha_1} {c_2}^{\alpha_2} \ldots {c_n}^{\alpha_n} = 1$. Suppose that $\alpha = 0$. Then ${c_1}^{\alpha_1} {c_2}^{\alpha_2} \ldots {c_n}^{\alpha_n} = 1$. Since

$c_1, c_2, \ldots, c_n \in C$ and C is linearly independent, $\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$, which is a contradiction. So $\alpha \neq 0$. Thus $x = {c_1}^{(\frac{-\alpha_1}{\alpha})}{c_2}^{(\frac{-\alpha_2}{\alpha})} \cdots {c_n}^{(\frac{-\alpha_n}{\alpha})} \in T_C$. Hence $T \subseteq T_C$.

The converse follows immediately form the assumption. $\square$

In Lemmas 3.3-3.7, Theorem 3.8 and Corollary 3.9, α and β are rational numbers.

Lemma 3.3. *Given $\delta \geqslant \sqrt{2}$. Then $S_2 = \{\alpha - \beta\sqrt{2} \mid \alpha \geqslant 0$ and ($\beta \leqslant 0$ or $\frac{\alpha}{\beta} \geqslant \delta)\} \setminus \{0\}$ is a semigroup and is divisible.*

Proof. Let $\delta \geqslant \sqrt{2}$ be fixed. Note that $S_2 \subseteq \mathbb{R}^+$. Let $a, b \in S_2$. So $a = \alpha_1 - \beta_1\sqrt{2}$ for some $\alpha_1 \geqslant 0$ and ($\beta_1 \leqslant 0$ or $\frac{\alpha_1}{\beta_1} \geqslant \delta$) and $b = \alpha_2 - \beta_2\sqrt{2}$ for some $\alpha_2 \geqslant 0$ and ($\beta_2 \leqslant 0$ or $\frac{\alpha_2}{\beta_2} \geqslant \delta$). Thus $a + b = (\alpha_1 + \alpha_2) - (\beta_1 + \beta_2)\sqrt{2}$. Since $a, b > 0$ and $\alpha_1, \alpha_2 \geqslant 0$, it follows that $a + b > 0$ and $\alpha_1 + \alpha_2 \geqslant 0$, respectively.

Case 1 : $\beta_1, \beta_2 \leqslant 0$. Then $\beta_1 + \beta_2 \leqslant 0$. So $a + b \in S_2$.

Case 2 : $\beta_1 \leqslant 0$ and $\frac{\alpha_2}{\beta_2} \geqslant \delta$. If $\beta_1 + \beta_2 \leqslant 0$, then $a + b \in S_2$. If $\beta_1 + \beta_2 > 0$, then $\delta(\beta_1 + \beta_2) > 0$. Thus $\alpha_1 + \alpha_2 \geqslant 0 + \delta\beta_2 \geqslant \delta\beta_1 + \delta\beta_2 = \delta(\beta_1 + \beta_2)$. So $\frac{\alpha_1+\alpha_2}{\beta_1+\beta_2} \geqslant \delta$. Hence $a + b \in S_2$.

Case 3 : $\frac{\alpha_1}{\beta_1} \geqslant \delta$ and $\frac{\alpha_2}{\beta_2} \geqslant \delta$. Since $\beta_1, \beta_2 > 0$, $\beta_1 + \beta_2 > 0$. Thus $\alpha_1 + \alpha_2 \geqslant \delta\beta_1 + \delta\beta_2 = \delta(\beta_1 + \beta_2)$. So $\frac{\alpha_1+\alpha_2}{\beta_1+\beta_2} \geqslant \delta$. Hence $a + b \in S_2$.

Therefore S_2 is a semigroup.

To show that S_2 is divisible, let $\alpha - \beta\sqrt{2} \in S_2$ where $\alpha \geqslant 0$ and ($\beta \leqslant 0$ or $\frac{\alpha}{\beta} \geqslant \delta$) and $n \in \mathbb{N}$. Since $\alpha \geqslant 0$, $\frac{\alpha}{n} \geqslant 0$.

Case 1 : $\beta \leqslant 0$. Then $\frac{\beta}{n} \leqslant 0$. So $\frac{\alpha}{n} - \frac{\beta}{n}\sqrt{2} \in S_2$.

Case 2 : $\frac{\alpha}{\beta} \geqslant \delta$. Thus $\frac{\alpha}{n} - \frac{\beta}{n}\sqrt{2} \in S_2$.

Hence $\frac{\alpha}{n} - \frac{\beta}{n}\sqrt{2} \in S_2$. Thus $n(\frac{\alpha}{n} - \frac{\beta}{n}\sqrt{2}) = n(\frac{\alpha-\beta\sqrt{2}}{n}) = \alpha - \beta\sqrt{2}$.

Therefore S_2 is divisible. $\square$

Lemma 3.4. *Given* $0 < \gamma \leqslant \sqrt{2}$. *Then* $S_3{}' = \{-\alpha + \beta\sqrt{2} \mid \beta \geqslant 0$ *and* $(\alpha \leqslant 0$ *or* $0 < \frac{\alpha}{\beta} < \gamma)\} \setminus \{0\}$ *is a semigroup and is divisible.*

Proof. Let $0 < \gamma \leqslant \sqrt{2}$ be fixed. Let $a, b \in S_3'$. So $a = -\alpha_1 + \beta_1\sqrt{2}$ for some $\beta_1 \geqslant 0$ and ($\alpha_1 \leqslant 0$ or $0 < \frac{\alpha_1}{\beta_1} < \gamma$) and $b = -\alpha_2 + \beta_2\sqrt{2}$ for some $\beta_2 \geqslant 0$ and ($\alpha_2 \leqslant 0$ or $0 < \frac{\alpha_2}{\beta_2} < \gamma$). Thus $a + b = -(\alpha_1 + \alpha_2) + (\beta_1 + \beta_2)\sqrt{2}$. Since $a, b > 0$ and $\beta_1, \beta_2 \geqslant 0$, it follows that $a + b > 0$ and $\beta_1 + \beta_2 \geqslant 0$, respectively.

Case 1 : $\alpha_1, \alpha_2 \leqslant 0$. Then $\alpha_1 + \alpha_2 \leqslant 0$. So $a + b \in S_3'$.

Case 2 : $\alpha_1 \leqslant 0$ and $0 < \frac{\alpha_2}{\beta_2} < \gamma$. If $\alpha_1 + \alpha_2 \leqslant 0$, then $a + b \in S_3'$. If $\alpha_1 + \alpha_2 > 0$, then $0 < \alpha_1 + \alpha_2 < 0 + \gamma\beta_2 \leqslant \gamma\beta_1 + \gamma\beta_2 = \gamma(\beta_1 + \beta_2)$. Since $\beta_1 \geqslant 0$ and $\beta_2 > 0, \beta_1 + \beta_2 > 0$. So $0 < \frac{\alpha_1 + \alpha_2}{\beta_1 + \beta_2} < \gamma$. Thus $a + b \in S_3'$.

Case 3 : $0 < \frac{\alpha_1}{\beta_1} < \gamma$ and $0 < \frac{\alpha_2}{\beta_2} < \gamma$. If $\alpha_1 + \alpha_2 \leqslant 0$, then $a + b \in S_3'$. If $\alpha_1 + \alpha_2 > 0$, then $0 < \alpha_1 + \alpha_2 < \gamma\beta_1 + \gamma\beta_2 = \gamma(\beta_1 + \beta_2)$. Thus $\frac{\alpha_1 + \alpha_2}{\beta_1 + \beta_2} < \gamma$. So $a + b \in S_3'$.

Hence S_3' is a semigroup.

To show that S_3' is divisible, let $-\alpha + \beta\sqrt{2} \in S_3'$ where $\beta \geqslant 0$ and ($\alpha \leqslant 0$ or $\frac{\alpha}{\beta} < \gamma$) and $n \in \mathbb{N}$. Since $\beta \geqslant 0, \frac{\beta}{n} \geqslant 0$.

Case 1 : $\alpha \leqslant 0$. Then $\frac{\alpha}{n} \leqslant 0$. So $\frac{-\alpha}{n} + \frac{\beta}{n}\sqrt{2} \in S_3'$.

Case 2 : $0 < \frac{\alpha}{\beta} < \gamma$. So $\frac{-\alpha}{n} + \frac{\beta}{n}\sqrt{2} \in S_3'$.

Thus $\frac{-\alpha}{n} + \frac{\beta}{n}\sqrt{2} \in S_3'$. So $n(\frac{-\alpha}{n} + \frac{\beta}{n}\sqrt{2}) = n(\frac{-\alpha + \beta\sqrt{2}}{n}) = -\alpha + \beta\sqrt{2}$.

Hence S_3' is divisible. □

Lemma 3.5. *Let* S_2 *and* S_3' *be defined as in Lemma 3.3 and Lemma 3.4, respectively. Then* $S_2 \cup S_3{}'$ *is a semigroup and is divisible.*

Proof. Let $0 < \gamma \leqslant \sqrt{2} \leqslant \delta$ be fixed. Let $a, b \in S_2 \cup S_3'$. If $a, b \in S_2$ or $a, b \in S_3'$, then $a + b \in S_2$ or $a + b \in S_3'$, so that $a + b \in S_2 \cup S_3'$.

Assume that $a \in S_2$ and $b \in S_3'$. So $a = \alpha_1 - \beta_1\sqrt{2}$ for $\alpha_1 \geqslant 0$ and ($\beta_1 \leqslant 0$ or $\frac{\alpha_1}{\beta_1} \geqslant \delta$) and $b = -\alpha_2 + \beta_2\sqrt{2}$ for some $\beta_2 \geqslant 0$ and ($\alpha_2 \leqslant 0$ or

$0 < \frac{\alpha_2}{\beta_2} < \gamma$). Thus $a+b = (\alpha_1-\beta_1\sqrt{2})+(-\alpha_2+\beta_2\sqrt{2}) = (\alpha_1-\alpha_2)-(\beta_1-\beta_2)\sqrt{2} = -(\alpha_2-\alpha_1)+(\beta_2-\beta_1)\sqrt{2}$.

Case 1 : $\beta_1 \leqslant 0 \leqslant \beta_2$ and $\alpha_2 \leqslant 0 \leqslant \alpha_1$. Then $\alpha_1 - \alpha_2 \geqslant 0$ and $\beta_1 - \beta_2 \leqslant 0$. So $a+b \in S_2$. Thus $a+b \in S_2 \cup S_3'$.

Case 2 : $\beta_1 \leqslant 0$ and $\frac{\alpha_2}{\beta_2} < \gamma$. Since $\alpha_1 \geqslant 0$ and $\beta_1 \leqslant 0, a \in S_3'$. Since $b \in S_3'$ and S_3' is a semigroup, $a+b \in S_3'$. Thus $a+b \in S_2 \cup S_3'$.

Case 3 : $\frac{\alpha_1}{\beta_1} \geqslant \delta$ and $\alpha_2 \leqslant 0$. Since $\beta_2 \geqslant 0$ and $\alpha_2 \leqslant 0, b \in S_2$. Since $a \in S_2$ and S_2 is a semigroup, $a+b \in S_2$. Thus $a+b \in S_2 \cup S_3'$.

Case 4 : $\frac{\alpha_1}{\beta_1} \geqslant \delta$ and $\frac{\alpha_2}{\beta_2} < \gamma$. By assumption, $\alpha_1 - \alpha_2 \geqslant \delta\beta_1 - \delta\beta_2 = \delta(\beta_1 - \beta_2)$ and $\alpha_2 - \alpha_1 < \gamma\beta_2 - \gamma\beta_1 = \gamma(\beta_2 - \beta_1)$.

Subcase 4.1 : $\beta_1 - \beta_2 = 0$. Then $\alpha_1 - \alpha_2 \geqslant 0$. Thus $a+b \in S_2$. So $a+b \in S_2 \cup S_3'$.

Subcase 4.2 : $\beta_1 - \beta_2 > 0$. So $\frac{\alpha_1-\alpha_2}{\beta_1-\beta_2} \geqslant \delta$. Thus $a+b \in S_2$. So $a+b \in S_2 \cup S_3'$.

Subcase 4.3 : $\beta_1 - \beta_2 < 0$. If $\alpha_2 - \alpha_1 \leqslant 0$, then $a+b \in S_3'$. So $a+b \in S_2 \cup S_3'$. If $\alpha_2 - \alpha_1 > 0$, then $\frac{\alpha_2-\alpha_1}{\beta_2-\beta_1} < \gamma$. Thus $a+b \in S_3'$. So $a+b \in S_2 \cup S_3'$.

Hence $S_2 \cup S_3'$ is a semigroup.

Since S_2 and S_3' are divisible, it is obvious that $S_2 \cup S_3'$ is divisible. □

Lemma 3.6. *Let S be an additive subsemigroup of $T_{\{1,\sqrt{2}\}}$ containing 1 and $\sqrt{2}$. Suppose that S is divisible. Let*

$$A_1 = \{\alpha - \beta\sqrt{2} \mid \alpha, \beta > 0 \text{ and } \frac{\alpha}{\beta} > \sqrt{2}\} \text{ and}$$

$$A_2 = \{-\alpha + \beta\sqrt{2} \mid \alpha, \beta > 0 \text{ and } \frac{\alpha}{\beta} < \sqrt{2}\}.$$

If $A_1 \cap S \neq \varnothing$ and $A_2 \cap S = \varnothing$, then there exists $\delta \geqslant \sqrt{2}$ such that $S = S_2$ or $S = S_2'$ where

$$S_2 = \{\alpha - \beta\sqrt{2} \mid \alpha \geqslant 0 \textit{ and } (\beta \leqslant 0 \textit{ or } \frac{\alpha}{\beta} \geqslant \delta)\} \setminus \{0\} \textit{ and}$$

$$S_2' = \{\alpha - \beta\sqrt{2} \mid \alpha \geqslant 0 \textit{ and } (\beta \leqslant 0 \textit{ or } \frac{\alpha}{\beta} > \delta)\} \setminus \{0\}.$$

Proof. Assume that $A_1 \cap S \neq \varnothing$ and $A_2 \cap S = \varnothing$. Then there exists $a \in A_1 \cap S$ and $a = \alpha_1 - \alpha_2\sqrt{2}$ for some $\alpha_1, \alpha_2 > 0$ and $\frac{\alpha_1}{\alpha_2} > \sqrt{2}$. Let $B_1 = \{\frac{\alpha}{\beta} \mid \alpha - \beta\sqrt{2} \in A_1 \cap S\}$. Since $B_1 \neq \varnothing$ and B_1 is bounded below, $\inf B_1$ exists. Let $\delta = \inf B_1$. So $\delta \geqslant \sqrt{2}$.

Case 1 : There exist $\alpha, \beta > 0$, $\alpha - \beta\sqrt{2} \in S$ and $\frac{\alpha}{\beta} = \delta$. We need to show that $S = S_2$. Let $u - v\sqrt{2} \in S$ where $u, v \in \mathbb{Q}$. If $u \geqslant 0$ and $v \leqslant 0$, then $u - v\sqrt{2} \in S_2$. Assume $u > 0$ and $v > 0$. Since $\frac{u}{v} > \sqrt{2}$, $u - v\sqrt{2} \in A_1 \cap S$. So $\frac{u}{v} \in B_1$. Since δ is a lower bound of B_1, $\frac{u}{v} \geqslant \delta$. Hence $u - v\sqrt{2} \in S_2$. Therefore $S \subseteq S_2$.

Let $u - v\sqrt{2} \in S_2$ where $u \geqslant 0$ and ($v \leqslant 0$ or $\frac{u}{v} \geqslant \delta$). If $v \leqslant 0$, then $u - v\sqrt{2} \in S$. Assume $\frac{u}{v} \geqslant \delta$. So $u, v > 0$. Since $\alpha - \beta\sqrt{2} \in S$ and S is divisible, for all $n \in \mathbb{N}$ there exists $b \in S$ such that $\alpha - \beta\sqrt{2} = nb$. Thus for all $n \in \mathbb{N}$, $b = \frac{\alpha}{n} - \frac{\beta}{n}\sqrt{2} \in S$. Since S is a semigroup, $m(\frac{\alpha}{n} - \frac{\beta}{n}\sqrt{2}) \in S$ for all $m, n \in \mathbb{N}$. So $\frac{m}{n}\alpha - \frac{m}{n}\beta\sqrt{2} \in S$ for all $m, n \in \mathbb{N}$. Since $v, \beta \in \mathbb{Q}^+$, $v = \frac{p}{q}$ and $\beta = \frac{r}{s}$ for some $p, q, r, s \in \mathbb{N}$.

Subcase 1.1 : $\frac{u}{v} = \delta$. So $u - v\sqrt{2} = \delta v - v\sqrt{2} = \delta v\frac{\alpha}{\alpha} - \frac{v\beta}{\beta}\sqrt{2} = \frac{\delta v\alpha}{\delta\beta} - \frac{v\beta}{\beta}\sqrt{2} = \frac{ps}{qr}\alpha - \frac{ps}{qr}\beta\sqrt{2}$. Since ps and $qr \in \mathbb{N}$, $u - v\sqrt{2} \in S$.

Subcase 1.2 : $\frac{u}{v} > \delta$. Let $u' - v'\sqrt{2} \in S$ where $u', v' > 0$ and $\frac{u}{v} > \frac{u'}{v'} > \delta$. Let $q = \frac{v}{v'}$. Then $qu' - v\sqrt{2} = qu' - qv'\sqrt{2} = q(u' - v'\sqrt{2}) \in S$. Since $\frac{u}{v} > \frac{u'}{v'} = \frac{qu'}{qv'} = \frac{qu'}{v}$ and $v > 0, u > qu'$. So $u - qu' > 0$. Thus $u - v\sqrt{2} = (u - qu') + (qu' - v\sqrt{2}) \in S$.

Hence $S_2 \subseteq S$. Therefore $S = S_2$.

Case 2 : For all $\alpha, \beta > 0$, $\alpha - \beta\sqrt{2} \in S$ implies $\frac{\alpha}{\beta} > \delta$. We want to show that $S = S_2'$. By assumption, $S \subseteq S_2'$. Let $u - v\sqrt{2} \in S_2'$ where $u \geqslant 0$ and ($v \leqslant 0$ or $\frac{u}{v} > \delta$). If $v \leqslant 0$, then $u - v\sqrt{2} \in S$. Assume that $\frac{u}{v} > \delta$. It can be proved similarly to Subcase 1.2 that $S_2' \subseteq S$.

Hence $S = S_2'$. $\square$

Lemma 3.7. *Let S be an additive subsemigroup of $T_{\{1,\sqrt{2}\}}$ containing 1 and $\sqrt{2}$. Suppose that S is divisible. Let A_1 and A_2 be defined as in Lemma 3.6. If $A_1 \cap S = \varnothing$ and $A_2 \cap S \neq \varnothing$, then there exists $0 < \gamma \leqslant \sqrt{2}$ such that $S = S_3$ or $S = S_3'$ where*

$$S_3 = \{-\alpha + \beta\sqrt{2} \mid \beta \geqslant 0 \text{ and } (\alpha \leqslant 0 \text{ or } 0 < \tfrac{\alpha}{\beta} \leqslant \gamma)\} \setminus \{0\} \text{ and}$$

$$S_3{}' = \{-\alpha + \beta\sqrt{2} \mid \beta \geqslant 0 \text{ and } (\alpha \leqslant 0 \text{ or } 0 < \tfrac{\alpha}{\beta} < \gamma)\} \setminus \{0\}.$$

Proof. Suppose that $A_1 \cap S = \varnothing$ and $A_2 \cap S \neq \varnothing$. Thus there exists $a \in A_2 \cap S$ and $a = -\alpha_1 + \alpha_2\sqrt{2}$ for some $\alpha_1, \alpha_2 > 0$ and $\frac{\alpha_1}{\alpha_2} < \sqrt{2}$. Let $B_2 = \{\frac{\alpha}{\beta} \mid -\alpha + \beta\sqrt{2} \in A_2 \cap S\}$. Then we can see that $B_2 \neq \varnothing$ and B_2 is bounded above, so that sup B_2 exists. Let $\gamma = \sup B_2$. So $0 < \gamma \leqslant \sqrt{2}$.

Case 1 : There exists $\alpha, \beta > 0$, $-\alpha + \beta\sqrt{2} \in S$ and $\frac{\alpha}{\beta} = \gamma$. We need to show that $S = S_3$. Let $-u + v\sqrt{2} \in S$ where $u, v \in \mathbb{Q}$. If $v \geqslant 0$ and $u \leqslant 0$, then $-u + v\sqrt{2} \in S_3$. Assume that $v > 0$ and $u > 0$. Since $\frac{u}{v} < \sqrt{2}$, $-u + v\sqrt{2} \in A_2 \cap S$. So $\frac{u}{v} \in B_2$. Since γ is an upper bound of B_2, $\frac{u}{v} \leqslant \gamma$. Hence $-u + v\sqrt{2} \in S_3$. Therefor $S \subseteq S_3$.

Let $-u + v\sqrt{2} \in S_3$ where $v \geqslant 0$ and ($u \leqslant 0$ or $0 < \frac{u}{v} \leqslant \gamma$). If $u \leqslant 0$, then $-u + v\sqrt{2} \in S$. Assume that $0 < \frac{u}{v} \leqslant \gamma$. So $u, v > 0$. Since $-\alpha + \beta\sqrt{2} \in S$ and S is divisible, for all $n \in \mathbb{N}$ there exists $b \in S$ such that $-\alpha + \beta\sqrt{2} = nb$. Thus for all $n \in \mathbb{N}$, $b = \frac{-\alpha}{n} + \frac{\beta}{n}\sqrt{2} \in S$. Since S is a semigroup, $m(\frac{-\alpha}{n} + \frac{\beta}{n}\sqrt{2})$ for all $m, n \in \mathbb{N}$. So $\frac{-m}{n}\alpha + \frac{m}{n}\beta\sqrt{2} \in S$ for all $m, n \in \mathbb{N}$. Since $v, \beta \in \mathbb{Q}^+, v = \frac{p}{q}$ and $\beta = \frac{r}{s}$ for some $p, q, r, s \in \mathbb{N}$.

Subcase 1.1 : $\frac{u}{v} = \gamma$. So $-u + v\sqrt{2} = -\gamma v + v\sqrt{2} = -\gamma v\frac{\alpha}{\alpha} + \frac{v\beta}{\beta}\sqrt{2} = \frac{-\gamma v\alpha}{\gamma\beta} + \frac{v\beta}{\beta}\sqrt{2} = \frac{-ps}{qr}\alpha + \frac{ps}{qr}\beta\sqrt{2}$. Since ps and $qr \in \mathbb{N}, -u + v\sqrt{2} \in S$.

Subcase 1.2 : $\frac{u}{v} < \gamma$. Let $-u' + v'\sqrt{2} \in S$ where $u', v' > 0$ and

$\frac{u}{v} < \frac{u'}{v'} < \gamma$. Let $q = \frac{v}{v'}$. Then $-qu' + v\sqrt{2} = -qu' + qv'\sqrt{2} = q(-u' + v'\sqrt{2}) \in S$. Since $\frac{u}{v} < \frac{u'}{v'} = \frac{qu'}{qv'} = \frac{qu'}{v}$ and $v > 0, u < qu'$. So $qu' - u > 0$. Thus $-u + v\sqrt{2} = (qu' - u) + (-qu' + v\sqrt{2}) \in S$.

Hence $S_3 \subseteq S$. Therefore $S = S_3$.

Case 2 : For all $\alpha, \beta > 0$, $-\alpha + \beta\sqrt{2} \in S$ implies $\frac{\alpha}{\beta} < \gamma$. We want to show that $S = S_3'$. By assumption, $S \subseteq S_3'$. Let $-u + v\sqrt{2} \in S_3'$ where $v \geqslant 0$ and ($u \leqslant 0$ or $0 < \frac{u}{v} < \gamma$). If $u \leqslant 0$, then $-u + v\sqrt{2} \in S$. Assume that $0 < \frac{u}{v} < \gamma$. The proof is similar to Subcase 1.2. Thus $S_3' \subseteq S$.

Hence $S = S_3'$. □

Theorem 3.8. *Let S be an additive subsemigroup of $T_{\{1,\sqrt{2}\}}$ containing 1 and $\sqrt{2}$. Then S is divisible if and only if S is one of the following types :*

(1) $S_1 = \{\alpha + \beta\sqrt{2} \mid \alpha, \beta \geqslant 0\} \setminus \{0\}$,

(2) $S_2 = \{\alpha - \beta\sqrt{2} \mid \alpha \geqslant 0$ *and* $(\beta \leqslant 0$ *or* $\frac{\alpha}{\beta} \geqslant \delta)\} \setminus \{0\}$ *where* $\delta \geqslant \sqrt{2}$ *is fixed,*

$S_2{}' = \{\alpha - \beta\sqrt{2} \mid \alpha \geqslant 0$ *and* $(\beta \leqslant 0$ *or* $\frac{\alpha}{\beta} > \delta)\} \setminus \{0\}$ *where* $\delta \geqslant \sqrt{2}$ *is fixed,*

(3) $S_3 = \{-\alpha + \beta\sqrt{2} \mid \beta \geqslant 0$ *and* $(\alpha \leqslant 0$ *or* $0 < \frac{\alpha}{\beta} \leqslant \gamma)\} \setminus \{0\}$ *where* $0 < \gamma \leqslant \sqrt{2}$ *is fixed,*

$S_3{}' = \{-\alpha + \beta\sqrt{2} \mid \beta \geqslant 0$ *and* $(\alpha \leqslant 0$ *or* $0 < \frac{\alpha}{\beta} < \gamma)\} \setminus \{0\}$ *where* $0 < \gamma \leqslant \sqrt{2}$ *is fixed or*

(4) $S_2 \cup S_3$ *or* $S_2 \cup S_3{}'$ *or* $S_2{}' \cup S_3$ *or* $S_2{}' \cup S_3{}'$.

Proof. By Lemma 3.3, Lemma 3.4 and Lemma 3.5, S_2, S_3' and $S_2 \cup S_3'$ are divisible semigroups, respectively. It can be proved similarly to Lemma 3.3, Lemma 3.4 and Lemma 3.5, that the others are also divisible semigroups.

For the converse, assume that S is divisible. Let S_i be as the above sets for all $i \in \{1, 2, 3\}$ and S_j' be as the above sets for all $j \in \{2, 3\}$. Since $\alpha, \beta\sqrt{2} \in S$ for all $\alpha, \beta \in \mathbb{Q}^+$ and S is a semigroup, $\alpha + \beta\sqrt{2} \in S$ for all $\alpha, \beta \in \mathbb{Q}^+$. If either $\alpha = 0$ or $\beta = 0$, then $\alpha + \beta\sqrt{2} \in S$. Thus $S_1 \subseteq S$. Let A_1 and A_2 be defined as in Lemma 3.6. Then there are 4 cases to be considered as follows :

Case 1 : $A_1 \cap S = \varnothing$ and $A_2 \cap S = \varnothing$. To show that $S \subseteq S_1$, let $\alpha + \beta\sqrt{2} \in S$. If ($\alpha \geqslant 0$ and $\beta < 0$) or ($\alpha < 0$ and $\beta > 0$), then $\alpha + \beta\sqrt{2} \in A_1 \cap S$ or $\alpha + \beta\sqrt{2} \in A_2 \cap S$, which is a contradiction. So $\alpha, \beta \geqslant 0$. Thus $\alpha + \beta\sqrt{2} \in S_1$. Hence $S \subseteq S_1$. Since $S_1 \subseteq S$, $S = S_1$.

Case 2 : $A_1 \cap S \neq \varnothing$ and $A_2 \cap S = \varnothing$. By Lemma 3.6, there exists $\delta \geqslant \sqrt{2}$ such that $S = S_2$ or $S = S_2'$.

Case 3 : $A_1 \cap S = \varnothing$ and $A_2 \cap S \neq \varnothing$. By Lemma 3.7, there exists $0 < \gamma \leqslant \sqrt{2}$ such that $S = S_3$ or $S = S_3'$.

Case 4 : $A_1 \cap S \neq \varnothing$ and $A_2 \cap S \neq \varnothing$. Thus there exist $a \in A_1 \cap S$ and $b \in A_2 \cap S$. So $a = \alpha_1 - \beta_1\sqrt{2}$ for some $\alpha_1, \beta_1 > 0$ and $\frac{\alpha_1}{\beta_1} > \sqrt{2}$ and $b = -\alpha_2 + \beta_2\sqrt{2}$ for some $\alpha_2, \beta_2 > 0$ and $\frac{\alpha_2}{\beta_2} < \sqrt{2}$. Let

$$B_1 = \left\{ \frac{\alpha}{\beta} \mid \alpha - \beta\sqrt{2} \in A_1 \cap S \right\} \text{ and}$$
$$B_2 = \left\{ \frac{\alpha}{\beta} \mid -\alpha + \beta\sqrt{2} \in A_2 \cap S \right\}.$$

Since $B_1, B_2 \neq \varnothing, B_1$ is bounded below and B_2 is bounded above, let $\delta = \inf B_1$ and $\gamma = \sup B_2$. Thus $0 < \gamma \leqslant \sqrt{2} \leqslant \delta$.

Subcase 4.1 : There exist $\alpha, \beta > 0$, $\alpha - \beta\sqrt{2} \in S$ and $\frac{\alpha}{\beta} = \delta$ and there exist $\alpha', \beta' > 0$, $-\alpha' + \beta'\sqrt{2} \in S$ and $\frac{\alpha'}{\beta'} = \gamma$. So $S = S_2 \cup S_3$.

Subcase 4.2 : There exist $\alpha, \beta > 0$, $\alpha - \beta\sqrt{2} \in S$ and $\frac{\alpha}{\beta} = \delta$ and for all $\alpha, \beta > 0$, $-\alpha + \beta\sqrt{2} \in S$ implies $\frac{\alpha}{\beta} < \gamma$. Thus $S = S_2 \cup S_3'$.

Subcase 4.3 : For all $\alpha, \beta > 0$, $\alpha - \beta\sqrt{2} \in S$ implies $\frac{\alpha}{\beta} > \delta$ and there exist $\alpha, \beta > 0$, $-\alpha + \beta\sqrt{2}$ and $\frac{\alpha}{\beta} = \gamma$. Hence $S = S_2' \cup S_3$.

Subcase 4.4 : For all $\alpha, \beta > 0$, $\alpha - \beta\sqrt{2} \in S$ implies $\frac{\alpha}{\beta} > \delta$ and for all $\alpha', \beta' > 0$, $-\alpha' + \beta'\sqrt{2} \in S$ implies $\frac{\alpha'}{\beta'} < \gamma$. Therefore $S = S_2' \cup S_3'$.

Therefore the theorem is completely proved. □

Corollary 3.9. Let S_2, S_2', S_3 and S_3' be defined as in Theorem 3.8. Assume that $\{1, \sqrt{2}\}$ is a $\mathbb{Q}$-linearly independent subset of S_i and S_i', for all $i \in \{2,3\}$. If $\delta, \gamma \in \mathbb{Q}^+$ and $0 < \gamma \leqslant \sqrt{2} \leqslant \delta$, then $S_2 \cong S_3$ and $S_2' \cong S_3'$.

Proof. Let $\delta, \gamma \in \mathbb{Q}^+$ and $0 < \gamma \leqslant \sqrt{2} \leqslant \delta$ be fixed. Define $f : S_2 \to S_3$ by $f(\alpha - \beta\sqrt{2}) = -\beta + \frac{\alpha}{\delta\gamma}\sqrt{2}$ where $\alpha \geqslant 0$ and ($\beta \leqslant 0$ or $\frac{\alpha}{\beta} \geqslant \delta$). Consider $\alpha - \beta\sqrt{2} \in S_2$ where $\alpha \geqslant 0$ and ($\beta \leqslant 0$ or $\frac{\alpha}{\beta} \geqslant \delta$). Since $\alpha \geqslant 0, \frac{\alpha}{\delta\gamma} \geqslant 0$. If $\beta \leqslant 0$, then $\beta - \frac{\alpha}{\delta\gamma}\sqrt{2} \in S_3$. If $\frac{\alpha}{\beta} \geqslant \delta$, then $\frac{\beta}{\frac{\alpha}{\delta\gamma}} = \frac{\beta}{\alpha}\delta\gamma \leqslant \frac{1}{\delta}\delta\gamma = \gamma$. Thus $-\beta + \frac{\alpha}{\delta\gamma}\sqrt{2} \in S_3$. This shows that f maps S_2 into S_3.

First, we prove that f is well-defined. Let $\alpha_1 - \beta_1\sqrt{2}, \alpha_2 - \beta_2\sqrt{2} \in S_2$ where $\alpha_1, \alpha_2 \geqslant 0$ and ($\beta_1 \leqslant 0$ or $\frac{\alpha_1}{\beta_1} \geqslant \delta$) and ($\beta_2 \leqslant 0$ or $\frac{\alpha_2}{\beta_2} \geqslant \delta$). Assume that $\alpha_1 - \beta_1\sqrt{2} = \alpha_2 - \beta_2\sqrt{2}$. Then $(\alpha_1 - \alpha_2) + (\beta_2 - \beta_1)\sqrt{2} = 0$. Thus $\alpha_1 - \alpha_2 = 0$ and $\beta_2 - \beta_1 = 0$ so that $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$. Hence $-\beta_1 + \frac{\alpha_1}{\delta\gamma}\sqrt{2} = -\beta_2 + \frac{\alpha_2}{\delta\gamma}\sqrt{2}$.

Next, we show that f is a homomorphism. Let $\alpha_1 - \beta_1\sqrt{2}, \alpha_2 - \beta_2\sqrt{2} \in S_2$ where $\alpha_1, \alpha_2 \geqslant 0$ and ($\beta_1 \leqslant 0$ or $\frac{\alpha_1}{\beta_1} \geqslant \delta$) and ($\beta_2 \leqslant 0$ or $\frac{\alpha_2}{\beta_2} \geqslant \delta$). Then

$$\begin{aligned} f((\alpha_1 - \beta_1\sqrt{2}) + (\alpha_2 - \beta_2\sqrt{2})) &= f((\alpha_1 + \alpha_2) - (\beta_1 + \beta_2)\sqrt{2}) \\ &= -(\beta_1 + \beta_2) + \frac{(\alpha_1 + \alpha_2)}{\delta\gamma}\sqrt{2} \\ &= -\beta_1 + \frac{\alpha_1}{\delta\gamma}\sqrt{2} + -\beta_2 + \frac{\alpha_2}{\delta\gamma}\sqrt{2} \\ &= f((\alpha_1 - \beta_1\sqrt{2}) + f(\alpha_2 - \beta_2\sqrt{2})). \end{aligned}$$

In order to show that f is one to one, let $\alpha_1 - \beta_1\sqrt{2}, \alpha_2 - \beta_2\sqrt{2} \in S_2$ where $\alpha_1, \alpha_2 \geqslant 0$ and ($\beta_1 \leqslant 0$ or $\frac{\alpha_1}{\beta_1} \geqslant \delta$) and ($\beta_2 \leqslant 0$ or $\frac{\alpha_2}{\beta_2} \geqslant \delta$). Suppose that $-\beta_1 + \frac{\alpha_1}{\delta\gamma}\sqrt{2} = -\beta_2 + \frac{\alpha_2}{\delta\gamma}\sqrt{2}$. Then $(-\beta_1 + \beta_2) + \frac{(\alpha_1 - \alpha_2)}{\delta\gamma}\sqrt{2} = 0$. Thus $-\beta_1 + \beta_2 = 0$ and $\frac{\alpha_1 - \alpha_2}{\delta\gamma} = 0$ so that $\beta_1 = \beta_2$ and $\alpha_1 = \alpha_2$. Hence $\alpha_1 - \beta_1\sqrt{2} = \alpha_2 - \beta_2\sqrt{2}$.

Finally, we need to show that f is onto. Let $-\alpha + \beta\sqrt{2} \in S_3$ where $\beta \geqslant 0$ and ($\alpha \leqslant 0$ or $\frac{\alpha}{\beta} \leqslant \gamma$). Since $\beta \geqslant 0, \delta\gamma\beta \geqslant 0$. If $\frac{\alpha}{\beta} \leqslant \gamma$, then $\frac{\delta\gamma\beta}{\alpha} \geqslant \frac{1}{\gamma}\delta\gamma = \delta$. Then

$\delta\gamma\beta - \alpha\sqrt{2} \in S_2$. Thus $f(\delta\gamma\beta - \alpha\sqrt{2}) = -\alpha + \frac{\delta\gamma\beta}{\delta\gamma}\sqrt{2} = -\alpha + \beta\sqrt{2}$.

Hence $S_2 \cong S_3$.

Define $g : S_2' \to S_3'$ by $g(\alpha - \beta\sqrt{2}) = -\beta + \frac{\alpha}{\delta\gamma}\sqrt{2}$ where $\alpha \geqslant 0$ and ($\beta \leqslant 0$ or $\frac{\alpha}{\beta} > \delta$). Consider $\alpha - \beta\sqrt{2} \in S_2'$ where $\alpha \geqslant 0$ and ($\beta \leqslant 0$ or $\frac{\alpha}{\beta} > \delta$). Since $\alpha \geqslant 0, \frac{\alpha}{\delta\gamma} \geqslant 0$. If $\beta \leqslant 0$, then $\beta - \frac{\alpha}{\delta\gamma}\sqrt{2} \in S_3'$. If $\frac{\alpha}{\beta} > \delta$, then $\frac{\beta}{\frac{\alpha}{\delta\gamma}} = \frac{\beta}{\alpha}\delta\gamma < \frac{1}{\delta}\delta\gamma = \gamma$. Thus $-\beta + \frac{\alpha}{\delta\gamma}\sqrt{2} \in S_3'$. This shows that g maps S_2' into S_3'.

First, we prove that g is well-defined. Let $\alpha_1 - \beta_1\sqrt{2}, \alpha_2 - \beta_2\sqrt{2} \in S_2'$ where $\alpha_1, \alpha_2 \geqslant 0$ and ($\beta_1 \leqslant 0$ or $\frac{\alpha_1}{\beta_1} > \delta$) and ($\beta_2 \leqslant 0$ or $\frac{\alpha_2}{\beta_2} > \delta$). Assume that $\alpha_1 - \beta_1\sqrt{2} = \alpha_2 - \beta_2\sqrt{2}$. Then $(\alpha_1 - \alpha_2) + (\beta_2 - \beta_1)\sqrt{2} = 0$. Thus $\alpha_1 - \alpha_2 = 0$ and $\beta_2 - \beta_1 = 0$ so that $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$. Hence $-\beta_1 + \frac{\alpha_1}{\delta\gamma}\sqrt{2} = -\beta_2 + \frac{\alpha_2}{\delta\gamma}\sqrt{2}$.

Next, we show that g is a homomorphism. Let $\alpha_1 - \beta_1\sqrt{2}, \alpha_2 - \beta_2\sqrt{2} \in S_2'$ where $\alpha_1, \alpha_2 \geqslant 0$ and ($\beta_1 \leqslant 0$ or $\frac{\alpha_1}{\beta_1} > \delta$) and ($\beta_2 \leqslant 0$ or $\frac{\alpha_2}{\beta_2} > \delta$). Then

$$\begin{aligned} g((\alpha_1 - \beta_1\sqrt{2}) + (\alpha_2 - \beta_2\sqrt{2})) &= g((\alpha_1 + \alpha_2) - (\beta_1 + \beta_2)\sqrt{2}) \\ &= -(\beta_1 + \beta_2) + \frac{(\alpha_1 + \alpha_2)}{\delta\gamma}\sqrt{2} \\ &= -\beta_1 + \frac{\alpha_1}{\delta\gamma}\sqrt{2} + -\beta_2 + \frac{\alpha_2}{\delta\gamma}\sqrt{2} \\ &= g((\alpha_1 - \beta_1\sqrt{2}) + g(\alpha_2 - \beta_2\sqrt{2})). \end{aligned}$$

To prove that g is one to one, let $\alpha_1 - \beta_1\sqrt{2}, \alpha_2 - \beta_2\sqrt{2} \in S_2'$ where $\alpha_1, \alpha_2 \geqslant 0$ and ($\beta_1 \leqslant 0$ or $\frac{\alpha_1}{\beta_1} > \delta$) and ($\beta_2 \leqslant 0$ or $\frac{\alpha_2}{\beta_2} > \delta$). Suppose that $-\beta_1 + \frac{\alpha_1}{\delta\gamma}\sqrt{2} = -\beta_2 + \frac{\alpha_2}{\delta\gamma}\sqrt{2}$. Then $(-\beta_1 + \beta_2) + \frac{(\alpha_1 - \alpha_2)}{\delta\gamma}\sqrt{2} = 0$. Thus $-\beta_1 + \beta_2 = 0$ and $\frac{\alpha_1 - \alpha_2}{\delta\gamma} = 0$ so that $\beta_1 = \beta_2$ and $\alpha_1 = \alpha_2$. Hence $\alpha_1 - \beta_1\sqrt{2} = \alpha_2 - \beta_2\sqrt{2}$.

Finally, we need to show that g is onto. Let $-\alpha + \beta\sqrt{2} \in S_3'$ where $\beta \geqslant 0$ and ($\alpha \leqslant 0$ or $\frac{\alpha}{\beta} < \gamma$). Since $\beta \geqslant 0, \delta\gamma\beta \geqslant 0$. If $\frac{\alpha}{\beta} < \gamma$, then $\frac{\delta\gamma\beta}{\alpha} > \frac{1}{\gamma}\delta\gamma = \delta$. Then $\delta\gamma\beta - \alpha\sqrt{2} \in S_2'$. Thus $g(\delta\gamma\beta - \alpha\sqrt{2}) = -\alpha + \frac{\delta\gamma\beta}{\delta\gamma}\sqrt{2} = -\alpha + \beta\sqrt{2}$.

Hence $S_2' \cong S_3'$. □

Theorem 3.10. *T is a divisible subsemigroup of $\mathbb{Q}^+$ under addition if and only if $T = \mathbb{Q}^+$.*

Proof. Let T be a divisible subsemigroup of $(\mathbb{Q}^+, +)$ and let $x \in T$ be fixed. Since T is divisible, for each $n \in \mathbb{N}$ there exists $y \in T$ such that $x = ny$. Thus $y = \frac{1}{n}x \in T$ for all $n \in \mathbb{N}$. Since T is a semigroup under addition, $m(\frac{1}{n}x) \in T$ for all $m, n \in \mathbb{N}$. Thus $\frac{m}{n}x \in T$ for all $m, n \in \mathbb{N}$. This implies $\mathbb{Q}^+x \subseteq T$. Since $\mathbb{Q}^+x = \mathbb{Q}^+$, $\mathbb{Q}^+ \subseteq T$. Hence $T = \mathbb{Q}^+$. □

In Lemma 3.11 and Lemma 3.12, any semigroup may not be commutative.

Lemma 3.11. *Assume that S is a divisible semigroup. Define a relation $\sim$ on S as follows: for any $x, y \in S$,*

$$x \sim y \text{ if and only if } mx = ny \text{ for some } m, n \in \mathbb{N}.$$

Then $\sim$ is an equivalence relation.

Proof. Clearly, the relation $\sim$ is reflexive and symmetric.

Let $a \sim b$ and $b \sim c$ where $a, b, c \in S$. Then $m_1 a = n_1 b$ and $m_2 b = n_2 c$ for some $m_1, n_1, m_2, n_2 \in \mathbb{N}$. Thus $(m_2 m_1)a = m_2(m_1 a) = m_2(n_1 b) = (m_2 n_1)b = (n_1 m_2)b = n_1(m_2 b) = n_1(n_2 c) = (n_1 n_2)c$. So $a \sim c$. Hence $\sim$ is transitive. Therefore $\sim$ is an equivalence relation. □

From Lemma 3.11, if $mx = ny$ where $x, y \in S$ and $m, n \in \mathbb{N}$, then we write $y = \frac{m}{n}x$. For each $x \in S$, let $\overline{x}$ be the equivalence class of $\sim$ containing x where $\sim$ is defined in Lemma 3.11. Then for $x \in S$, $\overline{x} = \{\frac{m}{n}x | m, n \in \mathbb{N}\}$ which is clearly a subsemigroup of S and $\overline{x} \cup \{0\}$ is a semigroup.

Lemma 3.12. *Suppose that S is a divisible power cancellative semigroup. Let $x \in S$ and define $\varphi_{\overline{x}} : \mathbb{Q}_0^+ \to \overline{x} \cup \{0\}$ by $\varphi_{\overline{x}}(\frac{m}{n}) = \frac{m}{n}x$ where $m, n \in \mathbb{N}$ and $\varphi_{\overline{x}}(0) = 0$. Then $\varphi_{\overline{x}}$ is an epimorphism.*

Proof. Since S is divisible and power cancellative, for each $x \in S$ and for each positive integer n, there exists a unique element y in S such that $x = ny$.

First, we need to show that $\frac{m}{n}x = m(\frac{1}{n}x)$ for all $m, n \in \mathbb{N}$ for all $x \in S$. Let $m, n \in \mathbb{N}$ and $x \in S$. Since S is divisible, there exists $y \in S$ such that $x = ny$. So $y = \frac{1}{n}x$. Thus $my = m(\frac{1}{n}x)$. Since $x = ny, mx = m(ny) = (mn)y = (nm)y = n(my)$. Hence $my = \frac{m}{n}x$. So $\frac{m}{n}x = m(\frac{1}{n}x)$.

Next, we prove that $\varphi_{\overline{x}}$ is well-defined. Assume that $\frac{m_1}{n_1} = \frac{m_2}{n_2}$ where $m_1, m_2, n_1, n_2 \in \mathbb{N}$. So $\frac{m_1 n_2}{n_1} = m_2 \in \mathbb{N}$. Thus $m_2 x = \frac{m_1 n_2}{n_1}x = m_1 n_2(\frac{1}{n_1}x) = n_2 m_1(\frac{1}{n_1}x) = n_2(m_1(\frac{1}{n_1}x)) = n_2(\frac{m_1}{n_1}x)$. Hence $\frac{m_1}{n_1}x = \frac{m_2}{n_2}x$.

We want to show that $(\frac{ps+qr}{qs})x = \frac{p}{q}x + \frac{r}{s}x$ for all $p, q, r, s \in \mathbb{N}$ and for all $x \in S$. Let $p, q, r, s \in \mathbb{N}$ and $x \in S$. Since S is divisible, there exist $y_1, y_2, y_3 \in S$ such that $x = qy_1, x = sy_2$ and $x = (qs)y_3$. So $y_1 = \frac{1}{q}x, y_2 = \frac{1}{s}x$ and $y_3 = \frac{1}{qs}x$. Hence

$$\begin{aligned}\left(\frac{ps+qr}{qs}\right)x &= (ps+qr)\left(\frac{1}{qs}x\right) = (ps+qr)y_3 = psy_3 + qry_3 \text{ and}\\ \frac{p}{q}x + \frac{r}{s}x &= p\left(\frac{1}{q}x\right) + r\left(\frac{1}{s}x\right) = py_1 + ry_2.\end{aligned}$$

Thus

$$\begin{aligned}psy_3 &= ps\left(\frac{1}{qs}x\right) = \frac{ps}{qs}x = \frac{p}{q}x = p\left(\frac{1}{q}x\right) = py_1 \text{ and}\\ qry_3 &= qr\left(\frac{1}{qs}x\right) = \frac{qr}{qs}x = \frac{r}{s}x = r\left(\frac{1}{s}x\right) = ry_2.\end{aligned}$$

So

$$\left(\frac{ps+qr}{qs}\right)x = psy_3 + qry_3 = py_1 + ry_2 = \frac{p}{q}x + \frac{r}{s}x.$$

In order to show that $\varphi_{\overline{x}}$ is a homomorphism, let $\alpha, \beta \in \mathbb{Q}_0^+$.

Case 1 : $\alpha = 0$.

$$\varphi_{\overline{x}}(\alpha+\beta) = \varphi_{\overline{x}}(0+\beta) = \varphi_{\overline{x}}(\beta) = 0 + \varphi_{\overline{x}}(\beta) = \varphi_{\overline{x}}(0) + \varphi_{\overline{x}}(\beta) = \varphi_{\overline{x}}(\alpha) + \varphi_{\overline{x}}(\beta).$$

Case 2 : $\alpha \neq 0$ and $\beta \neq 0$. So $\alpha = \frac{p_1}{q_1}$ and $\beta = \frac{p_2}{q_2}$ for some $p_1, q_1, p_2, q_2 \in \mathbb{N}$. Thus

$$\begin{aligned}
\varphi_{\overline{x}}(\alpha+\beta) &= \varphi_{\overline{x}}\left(\frac{p_1}{q_1} + \frac{p_2}{q_2}\right) \\
&= \varphi_{\overline{x}}\left(\frac{p_1q_2 + p_2q_1}{q_1q_2}\right) \\
&= \left(\frac{p_1q_2 + p_2q_1}{q_1q_2}\right)x \\
&= \frac{p_1}{q_1}x + \frac{p_2}{q_2}x \\
&= \varphi_{\overline{x}}\left(\frac{p_1}{q_1}\right) + \varphi_{\overline{x}}\left(\frac{p_2}{q_2}\right) \\
&= \varphi_{\overline{x}}(\alpha) + \varphi_{\overline{x}}(\beta).
\end{aligned}$$

Finally, we want to show that $\varphi_{\overline{x}}$ is onto. Let $a \in \overline{x}$. So $a \sim x$. Thus there exist $n, m \in \mathbb{N}$ such that $na = mx$. Hence $a = \frac{m}{n}x$. Choose $\frac{m}{n} \in \mathbb{Q}^+$. So $\varphi_{\overline{x}}(\frac{m}{n}) = \frac{m}{n}x = a$.

Therefore $\varphi_{\overline{x}}$ is an epimorphism. □

Recall that the annexed sum $\widetilde{\sum_{\alpha\in\Gamma}} S_\alpha$ is obtained as the direct sum $\sum_{\alpha\in\Gamma} S_\alpha^0$ excluding the identity, where for each $\alpha \in \Gamma$, S_α^0 is the semigroup S_α with two-sided identity 0 adjoined.

Next, we prove a theroem on commutative power cancellative divisible semigroups.

Theorem 3.13. *Let S be a power cancellative semigroup. Then S is divisible if and only if there is a set Γ such that S is a homomorphic image of the annexed sum $\widetilde{\sum_{\alpha\in\Gamma}} R_\alpha$ where each R_α is isomorphic to the additive semigroup of all positive rational numbers.*

Proof. Assume that S is divisible. Define a relation $\sim$ as in Lemma 3.11. Then $\sim$ is an equivalence relation.

Let $\Gamma = S/\sim$ and for each $\alpha \in \Gamma$, let $R_\alpha^0 = \mathbb{Q}_0^+$. Define $\psi : \sum_{\alpha\in\Gamma} R_\alpha^0 \rightarrow S^0$ by $\psi(<r_\alpha>) = \sum_{\alpha\in\Gamma} \varphi_\alpha(r_\alpha)$ for all $<r_\alpha> \in \sum_{\alpha\in\Gamma} R_\alpha^0$ (note that for each $\alpha \in \Gamma, \alpha$ is an equivalence class of $\sim$ in S and φ_α is an epimorphism defined in Lemma 3.12).

Now, we want to show that ψ is an epimorphism. We can see that ψ maps the identity of $\sum_{\alpha\in\Gamma} R_\alpha^0$ to 0 in S^0. Since for all $\alpha \in \Gamma, \varphi_\alpha$ is a function, ψ is well-defined.

Next, we prove that ψ is a homomorphism. Let $<r_\alpha>, <s_\alpha> \in \sum_{\alpha\in\Gamma} R_\alpha^0$. Thus

$$\begin{aligned}
\psi(<r_\alpha> + <s_\alpha>) &= \psi(<r_\alpha + s_\alpha>) \\
&= \sum_{\alpha\in\Gamma} \varphi_\alpha(r_\alpha + s_\alpha) \\
&= \sum_{\alpha\in\Gamma} (\varphi_\alpha(r_\alpha) + \varphi_\alpha(s_\alpha)) \\
&= \sum_{\alpha\in\Gamma} \varphi_\alpha(r_\alpha) + \sum_{\alpha\in\Gamma} \varphi_\alpha(s_\alpha) \\
&= \psi(<r_\alpha>) + \psi(<s_\alpha>).
\end{aligned}$$

To prove that ψ is onto, let $a \in S^0$.

Case 1 : $a = 0$. Let b be the identity of $\sum_{\alpha\in\Gamma} R_\alpha^0$. Then $\psi(b) = 0 = a$.

Case 2 : $a \neq 0$. Since $a \in \overline{a}$, choose $<r_\beta> \in \sum_{\beta\in\Gamma} R_\beta^0$ where $r_\beta = \begin{cases} 1, & \text{if } \beta = \overline{a}, \\ 0, & \text{if } \beta \neq \overline{a}. \end{cases}$

Thus $\psi(<r_\beta>) = \sum_{\beta\in\Gamma} \varphi_\beta(r_\beta) = \varphi_{\overline{a}}(1) = 1 \cdot a = a$.

Hence ψ is an epimorphism from $\sum_{\alpha\in\Gamma} R_\alpha^0$ onto S^0. Next, we need to show that $\psi(<r_\beta>) \neq 0$ for all $<r_\beta>$ such that $<r_\beta>$ is not the identity of $\sum_{\alpha\in\Gamma} R_\alpha^0$.

Let $<r_\beta> \in \sum_{\alpha\in\Gamma} R_\alpha^0$ be such that $<r_\beta>$ is not the identity. So there exists $\overline{a} \in \Gamma$ such that $r_{\overline{a}} \neq 0$, we may assume that $r_{\overline{a}} = \frac{p}{q} \in R_\alpha$ for some $p, q \in \mathbb{N}$. Then

$\psi(< r_\beta >) = \sum_{\beta\in\Gamma} \varphi_\beta(r_\beta)$. Since $\frac{p}{q} \neq 0, \frac{p}{q}a \neq 0$. Thus $\sum_{\beta\in\Gamma \text{ and } \beta\neq\overline{a}} \varphi_\beta(r_\beta) + \frac{p}{q}a \neq 0$. Hence $\sum_{\beta\in\Gamma} \varphi_\beta(r_\beta) \neq 0$. So $\psi(< r_\beta >) \neq 0$.

This proves that ψ is an epimorphism from $\tilde{\sum_{\alpha\in\Gamma}} R_\alpha$ onto S, as required.

Conversely, we assume that there is a set Γ such that S is a homomorphic image of the annexed sum $\tilde{\sum_{\alpha\in\Gamma}} R_\alpha$ where each R_α is isomorphic to the additive semigroup of all positive rational numbers. Since S is a homomorphic image of $\tilde{\sum_{\alpha\in\Gamma}} R_\alpha$ which is divisible, S is divisible. □

We note that Theorem 3.13 without assuming power cancellative of S was introduced by T.Tamura in [11], without proof.

The following example shows that there are a divisible semigroup S and a set Γ such that S is a homomorphic image of the annexed sum $\tilde{\sum_{\alpha\in\Gamma}} R_\alpha$ where each R_α is isomorphic to the additive semigroup of all positive rational numbers.

Example 3.14. Let $T = \mathbb{Q}_0^+ \times \mathbb{Q}_0^+ \setminus \{(0,0)\}$. Define $f : T \to (\mathbb{Q}, +)$ by $f(x,y) = x - y$ for all $x, y \in \mathbb{Q}_0^+$. Clearly, f is well-defined.

We want to show that f is a homomorphism. Let $(x_1, y_1), (x_2, y_2) \in T$. Then

$$\begin{aligned} f((x_1,y_1)+(x_2,y_2)) &= f(x_1+x_2, y_1+y_2) \\ &= (x_1+x_2)-(y_1+y_2) \\ &= (x_1-y_1)+(x_2-y_2) \\ &= f(x_1-y_1)+f(x_2-y_2). \end{aligned}$$

So f is a homomorphism.

In order to show that f is onto, let $y \in (\mathbb{Q}, +)$. Choose

$$x = \begin{cases} (0, -y), & \text{if } y < 0, \\ (1, 1), & \text{if } y = 0, \\ (y, 0), & \text{if } y > 0. \end{cases}$$

Then $x \in T$. So

$$f(x) = \begin{cases} f(0, -y) = 0 - (-y) = y, & \text{if } y < 0, \\ f(1, 1) = 1 - 1 = 0 = y, & \text{if } y = 0, \\ f(y, 0) = y - 0 = y, & \text{if } y > 0. \end{cases}$$

Thus f is onto.

Hence there is a set $\{1, 2\}$ such that $(\mathbb{Q}, +)$ is a homomorphic image of T where $(\mathbb{Q}, +)$ is a divisible semigroup.

Example 3.15. Let $R = \mathbb{Q}_0^+ \times \mathbb{Q}_0^+ \times \mathbb{Q}_0^+ \setminus \{(0, 0, 0)\}$. Define $g : R \to (\mathbb{Q}, +)$ by $g(x, y, z) = x + y - z$ where $x, y, z \in \mathbb{Q}_0^+$. Clearly, g is well-defined.

We prove that g is a homomorphism. Let $(x_1, y_1, z_1), (x_2, y_2, z_2) \in R$. Then

$$\begin{aligned} g((x_1, y_1, z_1) + (x_2, y_2, z_2)) &= g(x_1 + x_2, y_1 + y_2, z_1 + z_2) \\ &= (x_1 + x_2) + (y_1 + y_2) - (z_1 + z_2) \\ &= (x_1 + y_1 - z_1) + (x_2 + y_2 - z_2) \\ &= g(x_1, y_1, z_1) + g(x_2, y_2, z_2). \end{aligned}$$

So g is a homomorphism.

We need to show that g is onto, let $y \in (\mathbb{Q}, +)$. Choose

$$x = \begin{cases} (0, 0, -y), & \text{if } y < 0, \\ (0, 1, 1), & \text{if } y = 0, \\ (0, y, 0), & \text{if } y > 0. \end{cases}$$

Then $x \in R$. So

$$g(x) = \begin{cases} g(0,0,-y) = 0+0-(-y) = y, & \text{if } y < 0, \\ g(0,1,1) = 0+1-1 = 0 = y, & \text{if } y = 0, \\ g(0,y,0) = 0+y-0 = y, & \text{if } y > 0. \end{cases}$$

Thus g is onto.

Hence there is a set $\{1,2,3\}$ such that $(\mathbb{Q},+)$ is a homomorphic image of R where $(\mathbb{Q},+)$ is a divisible semigroup.

Example 3.16. If φ is a homomorphism from the semigroup $(\mathbb{Q}^+,+)$ into the semigroup $(\mathbb{Q},+)$ such that $0 \in Im\ \varphi$, then φ is the zero map.

Proof. Assume that $\varphi : (\mathbb{Q}^+,+) \to (\mathbb{Q},+)$ is a homomorphism. Suppose that there exists $x \in (\mathbb{Q}^+,+)$ such that $\varphi(x) \neq 0$. Thus

$$\begin{aligned} \varphi(nx) &= \varphi(x+\cdots+x) \quad (n \text{ times}) \\ &= \varphi(x)+\cdots+\varphi(x) \quad (n \text{ times}) \\ &= n\varphi(x) \neq 0 \text{ for all } n \in \mathbb{N}. \end{aligned}$$

For $n \in \mathbb{N}$,

$$\begin{aligned} \varphi(x) &= \varphi\left(n \cdot \frac{1}{n}x\right) \\ &= \varphi\left(\frac{1}{n}x+\cdots+\frac{1}{n}x\right) \quad (n \text{ times}) \\ &= \varphi\left(\frac{1}{n}x\right)+\cdots+\varphi\left(\frac{1}{n}x\right) \quad (n \text{ times}) \\ &= n\varphi\left(\frac{1}{n}x\right). \end{aligned}$$

So $\frac{\varphi(x)}{n} = \varphi(\frac{1}{n}x)$ for all $n \in \mathbb{N}$.

For $p, q \in \mathbb{N}$,

$$\begin{aligned}
\varphi\left(\frac{p}{q}x\right) &= \varphi\left(p \cdot \frac{1}{q}x\right) \\
&= \varphi\left(\frac{1}{q}x + \cdots + \frac{1}{q}x\right) \quad (p \text{ times}) \\
&= \varphi\left(\frac{1}{q}x\right) + \cdots + \varphi\left(\frac{1}{q}x\right) \quad (p \text{ times}) \\
&= \frac{\varphi(x)}{q} + \cdots + \frac{\varphi(x)}{q} \quad (p \text{ times}) \\
&= p\frac{\varphi(x)}{q} \\
&= \frac{p}{q}\varphi(x).
\end{aligned}$$

Since $\varphi(x) \neq 0$ and $p, q \in \mathbb{N}$, $\frac{p}{q}\varphi(x) \neq 0$. Hence $\varphi(\frac{p}{q}x) \neq 0$ for all $p, q \in \mathbb{N}$. Thus $0 \notin \varphi(\mathbb{Q}^+x)$. So $0 \notin \varphi(\mathbb{Q}^+)$. Hence $0 \notin Im\ \varphi$.

Therefore $(\mathbb{Q}, +)$ is not a homomorphic image of $(\mathbb{Q}^+, +)$. □

Theorem 3.17. *If S is a finite divisible semigroup, then S is a band.*

Proof. Assume that S is a finite divisible semigroup. Let $S = \{a_1, a_2, \ldots, a_n\}$ when $n \in \mathbb{N}$. Suppose there exists $i \in \{1, 2, \ldots, n\}$ such that $2a_i \neq a_i$. Thus $| < a_i > | > 1$. Choose $a \in S$ such that $| < a > | \geqslant | < a_j > |$ for all j. Then $| < a > | > 1$ and so $2a \neq a$. Since $a \in S$, $< a >$ is finite. So there exist $m \in \mathbb{N}$ and the least element $r \in \mathbb{N}$ such that $(m + r)a = ra$ and $< a >= \{a, 2a, \ldots, ra, (r + 1)a, \ldots, (m + r - 1)a\}$. Since S is divisible, $a = 2b$ for some $b \in S$ and $b \neq a$. So $a = 2b \in< b >$. Since $< b >$ is a semigroup, for all $k \in \mathbb{N}$, $ka \in< b >$. Thus $< a > \subseteq < b >$. By the property of $< a >$, we have $< a > = < b >$. So $b = ia$ for some $i \in \{2, 3, \ldots, m + r - 1\}$. Thus $a = 2b = 2(ia) = (2i)a$. Thus $r = 1$. Consequently, $< a >$ is a subgroup of S of order m. Since $a \in S$ and S is divisible, $a = mc$ for some $c \in S$. So $< a > \subseteq < c >$. By the property of a, $< a > = < c >$ which is a subgroup of S of order m. Let e be the identity of $< a >$. So $a = mc = e$. Thus $2a = 2e = e = a$, a contradiction. □

CHAPTER IV

SOME NONCOMMUTATIVE DIVISIBLE SEMIGROUPS

Recall that $M_2(\mathbb{R})$ under usual multiplication is a noncommutative semigroup. The purpose of this chapter is finding some subsemigroups of $M_2(\mathbb{R})$ which are divisible.

To show that $M_2(\mathbb{R})$ is not divisible, consider $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in M_2(\mathbb{R})$.

Suppose that there exists $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{R})$ such that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

So $\begin{bmatrix} a^2 + bc & b(a+d) \\ c(a+d) & bc + d^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Thus

$$a^2 + bc = 0 \quad \text{....................(1)}$$

$$b(a+d) = 1 \quad \text{....................(2)}$$

$$c(a+d) = 0 \quad \text{....................(3)}$$

$$bc + d^2 = 0 \quad \text{....................(4).}$$

By $(3), c = 0$ or $a + d = 0$. From (2), $a + d \neq 0$. Thus $c = 0$. From $(1), a = 0$. By $(3), d = 0$. Then $a + d = 0$, it is impossible. Thus there is no $a \in \mathbb{R}$ satisfying the equation (1). Hence there is no $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{R})$ such that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.
Therefore $M_2(\mathbb{R})$ is a noncommutative semigroup which is not divisible.

Let

$$A = \left\{ \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \;\middle|\; a, b \in [0, \infty) \right\},$$

$$B = \left\{ \begin{bmatrix} a & 0 \\ b & 1 \end{bmatrix} \;\middle|\; a, b \in [0, \infty) \right\},$$

$$C = \left\{ \begin{bmatrix} 1 & b \\ 0 & a \end{bmatrix} \;\middle|\; a, b \in [0, \infty) \right\},$$

$$D = \left\{ \begin{bmatrix} 1 & 0 \\ b & a \end{bmatrix} \;\middle|\; a, b \in [0, \infty) \right\},$$

$$R = \left\{ \begin{bmatrix} a & b(1-a) \\ 0 & 1 \end{bmatrix} \;\middle|\; a, b \in [0, 1] \right\},$$

$$S = \left\{ \begin{bmatrix} a & 0 \\ b(1-a) & 1 \end{bmatrix} \;\middle|\; a, b \in [0, 1] \right\},$$

$$U = \left\{ \begin{bmatrix} 1 & b(1-a) \\ 0 & a \end{bmatrix} \;\middle|\; a, b \in [0, 1] \right\} \quad \text{and}$$

$$V = \left\{ \begin{bmatrix} 1 & 0 \\ b(1-a) & a \end{bmatrix} \;\middle|\; a, b \in [0, 1] \right\}.$$

Proof of A being a noncommutative divisible subsemigroup of $M_2(\mathbb{R})$ under usual multiplication. Since $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in A, A \neq \varnothing$. Let $\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} c & d \\ 0 & 1 \end{bmatrix} \in A$. Thus $\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c & d \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} ac & ad+b \\ 0 & 1 \end{bmatrix}$. Since $a, c \in [0, \infty), ac \in [0, \infty)$. Since $a, b, d \in [0, \infty), ad + b \in [0, \infty)$. So $\begin{bmatrix} ac & ad+b \\ 0 & 1 \end{bmatrix} \in A$. Thus $\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c & d \\ 0 & 1 \end{bmatrix} \in A$. Hence A is a semigroup. Since $A \subseteq M_2(\mathbb{R}), A$ is a subsemigroup of $M_2(\mathbb{R})$ under usual multiplication. Since $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in A$, $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. So A is not commutative.

To prove that A is divisible, let $\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \in A$ where $a, b \in [0, \infty)$ and $n \in \mathbb{N}$.

Consider $\begin{bmatrix} \sqrt[n]{a} & \frac{b}{\sqrt[n]{a^{n-1}} + \sqrt[n]{a^{n-2}} + \cdots + \sqrt[n]{a} + 1} \\ 0 & 1 \end{bmatrix} \in A$.

$$\begin{bmatrix} \sqrt[n]{a} & \frac{b}{\sqrt[n]{a^{n-1}} + \sqrt[n]{a^{n-2}} + \cdots + \sqrt[n]{a} + 1} \\ 0 & 1 \end{bmatrix}^n$$

$$= \begin{bmatrix} \sqrt[n]{a} & \frac{b}{\sqrt[n]{a^{n-1}} + \sqrt[n]{a^{n-2}} + \cdots + \sqrt[n]{a} + 1} \\ 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} \sqrt[n]{a} & \frac{b}{\sqrt[n]{a^{n-1}} + \sqrt[n]{a^{n-2}} + \cdots + \sqrt[n]{a} + 1} \\ 0 & 1 \end{bmatrix} \cdots \begin{bmatrix} \sqrt[n]{a} & \frac{b}{\sqrt[n]{a^{n-1}} + \sqrt[n]{a^{n-2}} + \cdots + \sqrt[n]{a} + 1} \\ 0 & 1 \end{bmatrix}}_{(n-1 \text{ terms})}$$

$$= \begin{bmatrix} \sqrt[n]{a^2} & \frac{b(\sqrt[n]{a} + 1)}{\sqrt[n]{a^{n-1}} + \sqrt[n]{a^{n-2}} + \cdots + \sqrt[n]{a} + 1} \\ 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} \sqrt[n]{a} & \frac{b}{\sqrt[n]{a^{n-1}} + \sqrt[n]{a^{n-2}} + \cdots + \sqrt[n]{a} + 1} \\ 0 & 1 \end{bmatrix} \cdots \begin{bmatrix} \sqrt[n]{a} & \frac{b}{\sqrt[n]{a^{n-1}} + \sqrt[n]{a^{n-2}} + \cdots + \sqrt[n]{a} + 1} \\ 0 & 1 \end{bmatrix}}_{(n-2 \text{ terms})}$$

$$= \begin{bmatrix} \sqrt[n]{a^3} & \frac{b(\sqrt[n]{a^2} + \sqrt[n]{a} + 1)}{\sqrt[n]{a^{n-1}} + \sqrt[n]{a^{n-2}} + \cdots + \sqrt[n]{a} + 1} \\ 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} \sqrt[n]{a} & \frac{b}{\sqrt[n]{a^{n-1}} + \sqrt[n]{a^{n-2}} + \cdots + \sqrt[n]{a} + 1} \\ 0 & 1 \end{bmatrix} \cdots \begin{bmatrix} \sqrt[n]{a} & \frac{b}{\sqrt[n]{a^{n-1}} + \sqrt[n]{a^{n-2}} + \cdots + \sqrt[n]{a} + 1} \\ 0 & 1 \end{bmatrix}}_{(n-3 \text{ terms})}$$

$$= \ldots$$

$$= \begin{bmatrix} \sqrt[n]{a^n} & \frac{b(\sqrt[n]{a^{n-1}} + \sqrt[n]{a^{n-2}} + \cdots + \sqrt[n]{a} + 1)}{\sqrt[n]{a^{n-1}} + \sqrt[n]{a^{n-2}} + \cdots + \sqrt[n]{a} + 1} \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}.$$

Thus A is divisible. Hence A is a noncommutative divisible subsemigroup of $M_2(\mathbb{R})$ under usual multiplication.

Proof of B being a noncommutative divisible subsemigroup of $M_2(\mathbb{R})$ under usual multiplication. Obviously, B is a subsemigroup of $M_2(\mathbb{R})$ under usual multiplication. Since $\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in B$, $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$. Thus B is not commutative.

To show that B is divisible, let $x \in B$ and $n \in \mathbb{N}$. So $x^t \in A$. Since A is divisible, there exists $y \in A$ such that $y^n = x^t$. Then $y^t \in B$. Thus $(y^t)^n = \underbrace{(y^t) \cdots (y^t)}_{(n \text{ terms})} = \underbrace{(y \cdots y)^t}_{(n \text{ terms})} = (y^n)^t = (x^t)^t = x$. Hence B is divisible.

Therefore B is a noncommutative divisible subsemigroup of $M_2(\mathbb{R})$ under usual multiplication.

Proof of C being a noncommutative divisible subsemigroup of $M_2(\mathbb{R})$ under usual multiplication. Clearly, C is a subsemigroup of $M_2(\mathbb{R})$ under usual multiplication. Since $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in C$, $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. So C is not commutative.

In order to show that C is divisible, let $\begin{bmatrix} 1 & b \\ 0 & a \end{bmatrix} \in C$ where $a, b \in [0, \infty)$ and $n \in \mathbb{N}$. Consider $\begin{bmatrix} 1 & \frac{b}{\sqrt[n]{a^{n-1}} + \sqrt[n]{a^{n-2}} + \cdots + \sqrt[n]{a} + 1} \\ 0 & \sqrt[n]{a} \end{bmatrix} \in C$.

$$\begin{bmatrix} 1 & \frac{b}{\sqrt[n]{a^{n-1}}+\sqrt[n]{a^{n-2}}+\cdots+\sqrt[n]{a}+1} \\ 0 & \sqrt[n]{a} \end{bmatrix}^n$$

$$= \begin{bmatrix} 1 & \frac{b}{\sqrt[n]{a^{n-1}}+\sqrt[n]{a^{n-2}}+\cdots+\sqrt[n]{a}+1} \\ 0 & \sqrt[n]{a} \end{bmatrix} \underbrace{\begin{bmatrix} 1 & \frac{b}{\sqrt[n]{a^{n-1}}+\sqrt[n]{a^{n-2}}+\cdots+\sqrt[n]{a}+1} \\ 0 & \sqrt[n]{a} \end{bmatrix} \cdots \begin{bmatrix} 1 & \frac{b}{\sqrt[n]{a^{n-1}}+\sqrt[n]{a^{n-2}}+\cdots+\sqrt[n]{a}+1} \\ 0 & \sqrt[n]{a} \end{bmatrix}}_{(n-1 \text{ terms})}$$

$$= \begin{bmatrix} 1 & \frac{b(\sqrt[n]{a}+1)}{\sqrt[n]{a^{n-1}}+\sqrt[n]{a^{n-2}}+\cdots+\sqrt[n]{a}+1} \\ 0 & \sqrt[n]{a^2} \end{bmatrix} \underbrace{\begin{bmatrix} 1 & \frac{b}{\sqrt[n]{a^{n-1}}+\sqrt[n]{a^{n-2}}+\cdots+\sqrt[n]{a}+1} \\ 0 & \sqrt[n]{a} \end{bmatrix} \cdots \begin{bmatrix} 1 & \frac{b}{\sqrt[n]{a^{n-1}}+\sqrt[n]{a^{n-2}}+\cdots+\sqrt[n]{a}+1} \\ 0 & \sqrt[n]{a} \end{bmatrix}}_{(n-2 \text{ terms})}$$

$$= \begin{bmatrix} 1 & \frac{b(\sqrt[n]{a^2}+\sqrt[n]{a}+1)}{\sqrt[n]{a^{n-1}}+\sqrt[n]{a^{n-2}}+\cdots+\sqrt[n]{a}+1} \\ 0 & \sqrt[n]{a^3} \end{bmatrix} \underbrace{\begin{bmatrix} 1 & \frac{b}{\sqrt[n]{a^{n-1}}+\sqrt[n]{a^{n-2}}+\cdots+\sqrt[n]{a}+1} \\ 0 & \sqrt[n]{a} \end{bmatrix} \cdots \begin{bmatrix} 1 & \frac{b}{\sqrt[n]{a^{n-1}}+\sqrt[n]{a^{n-2}}+\cdots+\sqrt[n]{a}+1} \\ 0 & \sqrt[n]{a} \end{bmatrix}}_{(n-3 \text{ terms})}$$

$$= \ldots$$

$$= \begin{bmatrix} 1 & \frac{b(\sqrt[n]{a^{n-1}}+\sqrt[n]{a^{n-2}}+\cdots+\sqrt[n]{a}+1)}{\sqrt[n]{a^{n-1}}+\sqrt[n]{a^{n-2}}+\cdots+\sqrt[n]{a}+1} \\ 0 & \sqrt[n]{a^n} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & b \\ 0 & a \end{bmatrix}.$$

Thus C is divisible. Hence C is a noncommutative divisible subsemigroup of $M_2(\mathbb{R})$ under usual multiplication.

Proof of D being a noncommutative divisible subsemigroup of $M_2(\mathbb{R})$ under usual multiplication. Obviously, D is a subsemigroup of $M_2(\mathbb{R})$ under usual multiplication. Since $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in D$, $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Thus D is not commutative. The divisibility of D

can be proved similarly to that of B. Therefore D is a noncommutative divisible subsemigroup of $M_2(\mathbb{R})$ under usual multiplication.

Proof of R being a noncommutative divisible subsemigroup of $M_2(\mathbb{R})$ under usual multiplication. Since $\begin{bmatrix} 0 & 1(1-0) \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \in R, R \neq \varnothing$.

Let $\begin{bmatrix} a & b(1-a) \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} c & d(1-c) \\ 0 & 1 \end{bmatrix} \in R$ where $a, b, c, d \in [0,1]$.

Thus $\begin{bmatrix} a & b(1-a) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c & d(1-c) \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} ac & ad(1-c)+b(1-a) \\ 0 & 1 \end{bmatrix}$.

Since $a, c \in [0,1], ac \in [0,1]$. To show that $ad(1-c)+b(1-a) = t(1-ac)$ for some $t \in [0,1]$, let $\alpha = ad(1-c)+b(1-a)$. So

$$\begin{aligned} 0 &\leqslant ad(1-c)+b(1-a) \\ &\leqslant a(1-c)+(1-a) \\ &= a - ac + 1 - a \\ &= 1 - ac. \end{aligned}$$

Thus $0 \leqslant \alpha \leqslant 1-ac$. Hence there exists $t \in [0,1]$ such that $\alpha = t(1-ac)$. So R is a semigroup. Since $R \subseteq M_2(\mathbb{R})$, is a subsemigroup of $M_2(\mathbb{R})$ under usual multiplication. Since $\begin{bmatrix} 0 & 1(1-0) \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0(1-0) \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in R$,
$\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. So R is not commutative.

We want to show that R is divisible. Let $\begin{bmatrix} a & b(1-a) \\ 0 & 1 \end{bmatrix} \in R$ where $a, b \in [0,1]$

and $n \in \mathbb{N}$. Consider $\begin{bmatrix} \sqrt[n]{a} & \frac{b(1-a)}{\sqrt[n]{a^{n-1}}+\sqrt[n]{a^{n-2}}+\cdots+\sqrt[n]{a}+1} \\ 0 & 1 \end{bmatrix} \in R$.

$$\begin{aligned}
&\begin{bmatrix} \sqrt[n]{a} & \frac{b(1-a)}{\sqrt[n]{a^{n-1}}+\sqrt[n]{a^{n-2}}+\cdots+\sqrt[n]{a}+1} \\ 0 & 1 \end{bmatrix}^n \\
&= \begin{bmatrix} \sqrt[n]{a} & \frac{b(1-a)}{\sqrt[n]{a^{n-1}}+\sqrt[n]{a^{n-2}}+\cdots+\sqrt[n]{a}+1} \\ 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} \sqrt[n]{a} & \frac{b(1-a)}{\sqrt[n]{a^{n-1}}+\sqrt[n]{a^{n-2}}+\cdots+\sqrt[n]{a}+1} \\ 0 & 1 \end{bmatrix} \cdots \begin{bmatrix} \sqrt[n]{a} & \frac{b(1-a)}{\sqrt[n]{a^{n-1}}+\sqrt[n]{a^{n-2}}+\cdots+\sqrt[n]{a}+1} \\ 0 & 1 \end{bmatrix}}_{(n-1 \text{ terms})} \\
&= \begin{bmatrix} \sqrt[n]{a^2} & \frac{b(1-a)(\sqrt[n]{a}+1)}{\sqrt[n]{a^{n-1}}+\sqrt[n]{a^{n-2}}+\cdots+\sqrt[n]{a}+1} \\ 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} \sqrt[n]{a} & \frac{b(1-a)}{\sqrt[n]{a^{n-1}}+\sqrt[n]{a^{n-2}}+\cdots+\sqrt[n]{a}+1} \\ 0 & 1 \end{bmatrix} \cdots \begin{bmatrix} \sqrt[n]{a} & \frac{b(1-a)}{\sqrt[n]{a^{n-1}}+\sqrt[n]{a^{n-2}}+\cdots+\sqrt[n]{a}+1} \\ 0 & 1 \end{bmatrix}}_{(n-2 \text{ terms})} \\
&= \begin{bmatrix} \sqrt[n]{a^3} & \frac{b(1-a)(\sqrt[n]{a^2}+\sqrt[n]{a}+1)}{\sqrt[n]{a^{n-1}}+\sqrt[n]{a^{n-2}}+\cdots+\sqrt[n]{a}+1} \\ 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} \sqrt[n]{a} & \frac{b(1-a)}{\sqrt[n]{a^{n-1}}+\sqrt[n]{a^{n-2}}+\cdots+\sqrt[n]{a}+1} \\ 0 & 1 \end{bmatrix} \cdots \begin{bmatrix} \sqrt[n]{a} & \frac{b(1-a)}{\sqrt[n]{a^{n-1}}+\sqrt[n]{a^{n-2}}+\cdots+\sqrt[n]{a}+1} \\ 0 & 1 \end{bmatrix}}_{(n-3 \text{ terms})} \\
&= \cdots \\
&= \begin{bmatrix} \sqrt[n]{a^n} & \frac{b(1-a)(\sqrt[n]{a^{n-1}}+\sqrt[n]{a^{n-2}}+\cdots+\sqrt[n]{a}+1)}{\sqrt[n]{a^{n-1}}+\sqrt[n]{a^{n-2}}+\cdots+\sqrt[n]{a}+1} \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} a & b(1-a) \\ 0 & 1 \end{bmatrix}.
\end{aligned}$$

Thus R is divisible. Hence R is a noncommutative divisible subsemigroup of $M_2(\mathbb{R})$ under usual multiplication.

Proof of S being a noncommutative divisible subsemigroup of $M_2(\mathbb{R})$ under usual multiplication. Clearly, S is a subsemigroup of $M_2(\mathbb{R})$ under usual multiplication. Since $\begin{bmatrix} 0 & 0 \\ 1(1-0) & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0(1-0) & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in S$,

$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$. Thus S is not commutative. The divisibility of S can be proved similarly to that of B. Therefore S is a noncommutative divisible subsemigroup of $M_2(\mathbb{R})$ under usual multiplication.

Proof of U being a noncommutative divisible subsemigroup of $M_2(\mathbb{R})$ under usual multiplication. Obviously, U is a subsemigroup of $M_2(\mathbb{R})$ under usual multiplication. Since $\begin{bmatrix} 1 & 1(1-0) \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0(1-0) \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in U$, $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.

Thus U is not commutative.

To prove that U is divisible, let $\begin{bmatrix} 1 & b(1-a) \\ 0 & a \end{bmatrix} \in U$ where $a, b \in [0,1]$ and $n \in \mathbb{N}$. Consider $\begin{bmatrix} 1 & \frac{b(1-a)}{\sqrt[n]{a^{n-1}} + \sqrt[n]{a^{n-2}} + \cdots + \sqrt[n]{a} + 1} \\ 0 & \sqrt[n]{a} \end{bmatrix} \in U$.

$$\begin{bmatrix} 1 & \frac{b(1-a)}{\sqrt[n]{a^{n-1}} + \sqrt[n]{a^{n-2}} + \cdots + \sqrt[n]{a} + 1} \\ 0 & \sqrt[n]{a} \end{bmatrix}^n$$

$$= \begin{bmatrix} 1 & \frac{b(1-a)}{\sqrt[n]{a^{n-1}} + \sqrt[n]{a^{n-2}} + \cdots + \sqrt[n]{a} + 1} \\ 0 & \sqrt[n]{a} \end{bmatrix} \underbrace{\begin{bmatrix} 1 & \frac{b(1-a)}{\sqrt[n]{a^{n-1}} + \sqrt[n]{a^{n-2}} + \cdots + \sqrt[n]{a} + 1} \\ 0 & \sqrt[n]{a} \end{bmatrix} \cdots \begin{bmatrix} 1 & \frac{b(1-a)}{\sqrt[n]{a^{n-1}} + \sqrt[n]{a^{n-2}} + \cdots + \sqrt[n]{a} + 1} \\ 0 & \sqrt[n]{a} \end{bmatrix}}_{(n-1 \text{ terms})}$$

$$= \begin{bmatrix} 1 & \frac{b(1-a)(\sqrt[n]{a}+1)}{\sqrt[n]{a^{n-1}} + \sqrt[n]{a^{n-2}} + \cdots + \sqrt[n]{a} + 1} \\ 0 & \sqrt[n]{a^2} \end{bmatrix} \underbrace{\begin{bmatrix} 1 & \frac{b(1-a)}{\sqrt[n]{a^{n-1}} + \sqrt[n]{a^{n-2}} + \cdots + \sqrt[n]{a} + 1} \\ 0 & \sqrt[n]{a} \end{bmatrix} \cdots \begin{bmatrix} 1 & \frac{b(1-a)}{\sqrt[n]{a^{n-1}} + \sqrt[n]{a^{n-2}} + \cdots + \sqrt[n]{a} + 1} \\ 0 & \sqrt[n]{a} \end{bmatrix}}_{(n-2 \text{ terms})}$$

$$= \begin{bmatrix} 1 & \frac{b(1-a)(\sqrt[n]{a^2}+\sqrt[n]{a}+1)}{\sqrt[n]{a^{n-1}}+\sqrt[n]{a^{n-2}}+\cdots+\sqrt[n]{a}+1} \\ 0 & \sqrt[n]{a^3} \end{bmatrix} \underbrace{\begin{bmatrix} 1 & \frac{b(1-a)}{\sqrt[n]{a^{n-1}}+\sqrt[n]{a^{n-2}}+\cdots+\sqrt[n]{a}+1} \\ 0 & \sqrt[n]{a} \end{bmatrix} \cdots \begin{bmatrix} 1 & \frac{b(1-a)}{\sqrt[n]{a^{n-1}}+\sqrt[n]{a^{n-2}}+\cdots+\sqrt[n]{a}+1} \\ 0 & \sqrt[n]{a} \end{bmatrix}}_{(n-3 \text{ terms})}$$

$$= \ldots$$

$$= \begin{bmatrix} 1 & \frac{b(1-a)(\sqrt[n]{a^{n-1}}+\sqrt[n]{a^{n-2}}+\cdots+\sqrt[n]{a}+1)}{\sqrt[n]{a^{n-1}}+\sqrt[n]{a^{n-2}}+\cdots+\sqrt[n]{a}+1} \\ 0 & \sqrt[n]{a^n} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & b(1-a) \\ 0 & a \end{bmatrix}.$$

Thus U is divisible. Hence U is a noncommutative divisible subsemigroup of $M_2(\mathbb{R})$ under usual multiplication.

Proof of V being a noncommutative divisible subsemigroup of $M_2(\mathbb{R})$ under usual multiplication. Clearly, V is a subsemigroup of $M_2(\mathbb{R})$ under usual multiplication. Since $\begin{bmatrix} 1 & 0 \\ 1(1-0) & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0(1-0) & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in V$, $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Thus V is not commutative. The divisibility of V can be proved similarly to that of B. Therefore V is a noncommutative divisible subsemigroup of $M_2(\mathbb{R})$ under usual multiplication.

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