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SOME PROPERTIES OF CONVERGENCE SETS

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ภรัณยู จันทร : สมบัติบางประการของเซตแห่งการลู่เข้า (SOME PROPERTIES OF CONVEGENCE SETS) อ. ที่ปรึกษา: ผศ.ดร. พิเชฐ ชาวหา อ. ที่ปรึกษาร่วม: ผศ.ดร. อิ่มจิตต์ เติมวุฒิพงษ์, 30 หน้า

เราศึกษาสมบัติบางประการทางทอพอโลยีและทางเรขาคณิตของเซตแห่งการลู่เข้าบนปริภูมิระยะทาง เราพิสูจน์ว่าการส่งแบบเวอร์ฌวลลีนอนเอกซ์แพนซีพบางชนิดมีเซตแห่งการลู่เข้าเป็นแบบสตาร์คอนเวกซ์ และมีเซตจุดตรึงเป็นแบบหดตัวได้

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We investigate some topological and geometric properties of convergence sets on metric spaces. We prove that, for a certain kind of virtually nonexpansive maps, their convergence sets are star-convex and their fixed point sets are contractible.

Department	...Mathematics...	Student's Signature............	
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CONTENTS

CHAPTER I

INTRODUCTION

In this chapter, we introduce the basic concepts and terminology used in our work.

Definition 1.1. *For a nonempty set X, a collection $\mathcal{T}$ of subsets of X is called a* **topology** *on X if it satisfies the following conditions.*

(i) $X \in \mathcal{T}$ *and* $\phi \in \mathcal{T}$.

(ii) *Any union of members of $\mathcal{T}$ is also a member of $\mathcal{T}$.*

(iii) *Any finite intersection of members of $\mathcal{T}$ is also a member of $\mathcal{T}$.*

The elements of $\mathcal{T}$ are called **open sets** *in X and $(X, \mathcal{T})$ is called a* **topological space***. A subset A of a topological space X is said to be* **closed** *if the set $X - A$ is open.*

Given a subset A of a topological space X, the **interior** of A is defined as the union of all open sets contained in A, and the **closure** of A is defined as the intersection of all closed sets containing A.

The interior of A is denoted by Int A, and the closure of A is denoted by $\overline{A}$. Obviously Int A is an open set and $\overline{A}$ is a closed set; furthermore,

$$\text{Int } A \subseteq A \subseteq \overline{A}.$$

If A is an open set, $A = \text{Int } A$; while if A is closed, $A = \overline{A}$.

Definition 1.2. *Let $(X, \mathcal{T})$ be a topological space and Y be a subset of X. The collection $\mathcal{T}_Y = \{Y \cap U : U \in \mathcal{T}\}$ is a topology on Y, called the* **subspace topology***. $(Y, \mathcal{T}_Y)$ is called a* **subspace** *of X.*

Remark 1.3. *If Y is an open subspace of X and $G \in \mathcal{T}_Y$, then $G \in \mathcal{T}$. If $O \in \mathcal{T}$ and $O \subseteq Y$, then $O \in \mathcal{T}_Y$.*

Definition 1.4. *A* **metric** *on a set X is a function*

$$d : X \times X \to \mathbb{R}$$

having the following properties:

(i) $d(x, y) \geq 0$ *for each* $x, y \in X$*; the equality holds if and only if* $x = y$*,*

(ii) $d(x, y) = d(y, x)$ *for each* $x, y \in X$*,*

(iii) $d(x, y) + d(y, z) \geq d(x, z)$*, for all* $x, y, z \in X$*.*

Given a metric d on X, the number $d(x, y)$ is often called the **distance** between x and y with respect to the metric d. Given $\varepsilon > 0$, the set

$$B_d(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$$

is called the **ε-ball centered at** x. Sometimes we omit the letter d from the notation and denote this ball simply by $B(x, \varepsilon)$, if no confusion will arise. A subset G of a metric space (X, d) is said to be **open** in X if for every point x in G, there is $\varepsilon > 0$ such that $B(x, \varepsilon) \subseteq G$. It is easy to show that $B(x, \varepsilon)$ is an open set. And by a **neighborhood of a point** x, we mean an open set containing x. A subset F of (X, d) is closed if $X - F$ is open. Let $\mathcal{T}_d$ be the collection of all open sets in (X, d). Then $\mathcal{T}_d$ has the following properties

(1) $X \in \mathcal{T}_d$ and $\phi \in \mathcal{T}_d$,

(2) Any union of members of $\mathcal{T}_d$ is also a member of $\mathcal{T}_d$,

(3) Any finite intersection of members of $\mathcal{T}_d$ is also a member of $\mathcal{T}_d$.

Thus $\mathcal{T}_d$ is a topology on X. The topology $\mathcal{T}_d$ is called the **topology induced** from the metric d on X.

Definition 1.5. *A topological space* $(X, \mathcal{T})$ *is called a* **metric space** *if* $\mathcal{T}$ *is a topology that induced by a metric on* X*, and in this case we denote* $(X, \mathcal{T})$ *by* (X, d)*, or simply* X *if no confusion arises.*

Example 1.6. *The* **standard metric** *or the* **usual metric** *on* $\mathbb{R}^n$ *is the metric* d *defined by*

$$d(x,y) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2},$$

for $x = (x_1, x_2, \ldots, x_n)$ *and* $y = (y_1, y_2, \ldots, y_n)$ *in* $\mathbb{R}^n$. *It is easy to see that* d *is a metric on* $\mathbb{R}^n$.

Definition 1.7. *Let* (X, d_X) *and* (Y, d_Y) *be metric spaces. We say that a function* $f : X \to Y$ *is* **continuous at a point** x *in* X *if for each* $\varepsilon > 0$ *there is* $\delta > 0$ *such that for every* $y \in X$ *if* $d_X(x,y) < \delta$, *then* $d_Y(f(x), f(y)) < \varepsilon$. *If* f *is continuous at every point* x *in a subset* A *of* X, *then* f *is said to be* **continuous on** A. *If* f *is continuous on* X, *then we simply say that* f *is* **continuous**.

Definition 1.8. *Let* X *and* Y *be metric spaces. A bijective function* $f : X \to Y$ *is called* **homeomorphism** *if* f *and* f^{-1} *are continuous.*

Definition 1.9. *By a* **linear topological space** *we mean a vector space* X *over* $\mathbb{R}$ *equipped with a Hausdorff topology such that the two functions* $+ : X \times X \to X$ *and* $\cdot : \mathbb{R} \times X \to X$ *are continuous.*

Definition 1.10. *Let* V *and* W *be vector spaces over* $\mathbb{R}$. *A function* $T : V \to W$ *is said to be a* **linear transformation** (**linear function**) *if*

$$T(ru + sv) = rT(u) + sT(v)$$

for each $r, s \in \mathbb{R}$ *and* $u, v \in V$.

Definition 1.11. *Let* X *be a vector space over* $\mathbb{R}$ *and* $x, y \in X$, *the set*

$$L(x,y) := \{ty + (1-t)x : 0 \le t \le 1\}$$

is called the **line segment** *from* x *to* y. *A subset* $C \subseteq X$ *is* **convex** *if* $L(x,y) \subseteq C$ *for every pair* $x, y \in C$. *A subset* C *of* X *is* **star-convex** *if* $L(0,y) \subseteq C$ *for each* $y \in C$.

Definition 1.12. *Let X be a vector space over $\mathbb{R}$. A function $\|\cdot\| : X \to \mathbb{R}$ is said to be a* **norm** *on X if*

(*i*) $\|x\| \geq 0$ *for all $x \in X$; the equality holds if and only if $x = 0$,*

(*ii*) $\|cx\| = |c|\|x\|$ *for all $x \in X$ and $c \in \mathbb{R}$,*

(*iii*) $\|x + y\| \leq \|x\| + \|y\|$ *for all $x, y \in X$.*

A vector space equipped with a norm is called a **normed linear space**

Theorem 1.13. *Let X be a normed linear space. Then the function $d : X \times X \to [0, \infty)$ defined by*

$$d(x, y) = \|x - y\|, \text{ for } x, y \in X,$$

is a metric on X.

Definition 1.14. *A subset A of a space X is said to be* **dense** *in X if $\overline{A} = X$.*

Example 1.15. *The set $\mathbb{Q}$ of all rational numbers is dense in the space $\mathbb{R}$.*

Definition 1.16. *A subset of a space X is called a* G_δ**-set** *in X if it is an intersection of a countable collection of open subsets of X.*

Remark 1.17. (1) *Every open subset of X is a G_δ-set.*

(2) *For a subset A of X, let $U(A, \varepsilon) = \bigcup_{x \in A} B(x, \varepsilon)$. Since $B(x, \varepsilon)$ is open for every $x \in A$, $U(A, \epsilon)$ is an open set. If A is closed then $A = \bigcap_{n \in \mathbb{Z}^+} U(A, \frac{1}{n})$. Therefore, every closed set is a G_δ-set.*

Definition 1.18. *Given a set X, we define a* **sequence** *in X to be a function $\mathbf{x} : \mathbb{N} \to X$. We usually denote $\mathbf{x}$ itself by the symbol $(x_1, x_2, \ldots)$ or (x_n).*

Definition 1.19. *Let X be a metric space. A sequence (x_n) in X is said to* **converge** *to a point y in X if for each $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that*

$$d(x_n, y) < \varepsilon \quad \text{whenever } n \geq N.$$

A sequence (x_n) *in* X *is said to be a* **Cauchy sequence** *if for each* $\varepsilon > 0$*, there is* $N \in \mathbb{N}$ *such that*

$$d(x_n, x_m) < \varepsilon \quad \textit{whenever } n, m \geq N.$$

A metric space (X, d) *is said to be* **complete** *if every Cauchy sequence in* X *converges (to a point) in* X*.*

Example 1.20. *The space* $\mathbb{R}$ *with the standard metric is a complete metric space, but its subspace* $\mathbb{Q}$ *is not.*

Definition 1.21. *A topology* $\mathcal{T}$ *on* X *is called* **Hausdorff** *if for each pair* x, y *of distinct points in* X*, there exist open sets* U_x *and* U_y *such that* $x \in U_x$*,* $y \in U_y$*, and* $U_x \cap U_y = \phi$*. A topological space* $(X, \mathcal{T})$ *is called a* **Hausdorff space** *if* $\mathcal{T}$ *is a Hausdorff topology.*

Definition 1.22 ([1], P. 295). *A space* X *is said to be a* **Baire space** *if the following condition holds: Given any countable collection* $\{A_n\}$ *of closed sets in* X *each of which has empty interior, their union* $\bigcup A_n$ *also has empty interior.*

Theorem 1.23 ([1], P. 296). *A space* X *is a Baire space if and only if given any countable collection* $\{U_n\}$ *of open sets in* X*, each of which is dense in* X*, their intersection* $\bigcap U_n$ *is also dense in* X*.*

Theorem 1.24 ([1], P. 296, Baire category theorem). *If* X *is a compact Hausdorff space or a complete metric space, then* X *is a Baire space.*

Corollary 1.25. *A countable dense subset of a complete metric space is not a* G_δ*-set.*

Proof. Let X be a complete metric space and A a countable dense subset of X. Suppose that $A = \bigcap_{i=1}^{\infty} G_i$, and G_i's are open in X. Then each G_i is also dense in X, since A is dense in X and $A \subseteq G_i$. Let $\mathcal{B} = \{G_i\}_{i \in \mathbb{N}} \bigcup \{X - \{a\} : a \in A\}$. Then $\bigcap_{G_\alpha \in \mathcal{B}} G_\alpha = \phi$. By Baire category theorem, X is a Baire space. By Theorem 1.23, $\bigcap_{G_\alpha \in \mathcal{B}} G_\alpha$ is dense in X, which is a contradiction. Therefore, A is not a G_δ-set. $\square$

Remark 1.26. *The set* $\mathbb{Q}$ *of all rational numbers is a countable dense subset of* $\mathbb{R}$ *and* $\mathbb{R}$ *is a complete metric space. By Corollary 1.25,* $\mathbb{Q}$ *is not a* G_δ*-set.*

Lemma 1.27. *Let* X *and* Y *be metric spaces and* $f : X \to Y$ *be a function. The set* $A = \{x \in X : f \text{ is continuous at } x\}$ *is a* G_δ*-set.*

Proof. Suppose A is nonempty. Let $n \in \mathbb{N}$ and $a \in A$ be arbitrary. Since f is continuous at a, there is a neighborhood $G_{n,a}$ of a such that $f(a) \in f(G_{n,a}) \subseteq B(f(a), \frac{1}{n})$.

Let $A_n = \bigcup_{a \in A} G_{n,a}$ and $B = \bigcap_{n \in \mathbb{N}} A_n$.

It is clear that $A \subseteq B$. Next, we will show that $B \subseteq A$. Let $b \in B$ and $\varepsilon > 0$ be arbitrary. There is $m \in \mathbb{N}$ such that $\frac{2}{m} < \varepsilon$. Since $b \in A_m$, there exists $a \in A$ such that $b \in G_{m,a}$. Let $g \in G_{m,a}$. So $f(b)$ and $f(g)$ are in $f(G_{m,a}) \subseteq B(f(a), \frac{1}{m})$. Thus

$$d_Y(f(b), f(g)) \leq d_Y(f(b), f(a)) + d_Y(f(a), f(g)) < \frac{1}{m} + \frac{1}{m} = \frac{2}{m} < \varepsilon.$$

Therefore, f is continuous at b, which implies $b \in A$. □

We will denote the set of all continuous functions from X to Y by $C(X,Y)$. We usually refer to an element in $C(X,Y)$ as a map from X to Y, and use 1_X to denote the identity map in $C(X,X)$.

Definition 1.28. *Let* (X, d_X) *and* (Y, d_Y) *be metric spaces. A subset* $\mathcal{F}$ *of* $C(X,Y)$ *is said to be* **equicontinuous** *at* $x \in X$ *if for each* $\varepsilon > 0$*, there is* $\delta > 0$ *such that for every* $y \in X$*,* $d_Y(f(x), f(y)) < \varepsilon$ *whenever* $d_X(x,y) < \delta$ *and* $f \in \mathcal{F}$*. The set of all points* x *in* X *at which* $\mathcal{F}$ *is equicontinuous, is denoted by* $E(\mathcal{F})$*.*

Remark 1.29. *Let* $\mathcal{F} \subseteq C(X,Y)$*. If* $\mathcal{F}$ *is finite, then* $E(\mathcal{F}) = X$*.*

Proof. Suppose $\mathcal{F} = \{f_1, f_2, \ldots, f_n\}$ for some $n \in \mathbb{N}$, and $f_i \in C(X,Y)$ for every $i \in \{1, \ldots, n\}$. Assume that $x \in X$ and $\varepsilon > 0$ be arbitrary. For every $i \in \{1, \ldots, n\}$, since $f_i \in C(X,Y)$, there is $\delta_i > 0$ such that $d_Y(f_i(x), f_i(x')) < \varepsilon$ whenever $d_X(x, x') < \delta_i$. Choose $\delta = \min\{\delta_1, \delta_2, \ldots, \delta_n\}$, so $\delta > 0$. For every $i \in \{1, \ldots, n\}$, we have

$d_Y(f_i(x), f_i(x')) < \varepsilon$ whenever $d_X(x, x') < \delta$. Therefore, $x \in E(\mathcal{F})$, which implies $E(\mathcal{F}) = X$. □

Let $f \in C(X, X)$. We define $f^n(x)$ to be $f^n(x) = \overbrace{f \circ f \circ f \cdots \circ f}^{n\text{-copies}}(x)$. If $F = \{f^n : n \in \mathbb{N}\}$, then we denote $E(F)$ by $E(f)$.

Example 1.30. *For each $m \in \mathbb{N} - \{1\}$ and $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = mx$, for $x \in \mathbb{R}$. Then $E(f) = \phi$.*

Proof. Suppose $f(x) = mx$ for some $m \in \mathbb{N} - \{1\}$. Let $x \in \mathbb{R}$, $\varepsilon = 1$ and $n \in \mathbb{N}$. Then

$$\begin{aligned} f^n((x - \frac{1}{n}, x + \frac{1}{n})) &= (m^n(x - \frac{1}{n}), m^n(x + \frac{1}{n})) \\ &= (m^n x - \frac{m^n}{n}, m^n x + \frac{m^n}{n}) \\ &\not\subseteq (m^n x - 1, m^n x + 1), \text{ by } m^n > n \text{ for all } n \in \mathbb{N} \text{ and } m \neq 1 \\ &= (f^n(x) - \varepsilon, f^n(x) + \varepsilon). \end{aligned}$$

Therefore, $x \notin E(f)$ for all $x \in \mathbb{R}$. □

Proposition 1.31. *Let $\mathcal{H}, \mathcal{F} \subseteq C(X, Y)$. Then the following conditions hold.*

(1) *If $\mathcal{H} \subseteq \mathcal{F}$, then $E(\mathcal{F}) \subseteq E(\mathcal{H})$.*

(2) *$E(\mathcal{H} \cup \mathcal{F}) \subseteq E(\mathcal{H}) \cap E(\mathcal{F})$.*

(2) *$E(\mathcal{H}) \cup E(\mathcal{F}) \subseteq E(\mathcal{H} \cap \mathcal{F})$.*

Proof. For (1). Suppose that $\mathcal{H} \subseteq \mathcal{F}$. Let $x \in E(\mathcal{F})$ and $\varepsilon > 0$ be arbitrary. Then there is a neighborhood U of x such that $f(U) \subseteq B(f(x), \varepsilon)$ for all $f \in \mathcal{F}$. Since $\mathcal{H} \subseteq \mathcal{F}$, $f(U) \subseteq B(f(x), \varepsilon)$ for all $f \in \mathcal{H}$, so $x \in E(\mathcal{H})$.

For (2). Since $\mathcal{H}, \mathcal{F} \subseteq \mathcal{H} \cup \mathcal{F}$ and by (1), we have $E(\mathcal{H} \cup \mathcal{F}) \subseteq E(\mathcal{H})$ and $E(\mathcal{H} \cup \mathcal{F}) \subseteq E(\mathcal{F})$. Therefore, $E(\mathcal{H} \cup \mathcal{F}) \subseteq E(\mathcal{H}) \cap E(\mathcal{F})$.

For (3) $\mathcal{H} \cap \mathcal{F} \subseteq \mathcal{H}$ and $\mathcal{H} \cap \mathcal{F} \subseteq \mathcal{F}$ and by (1), we have $E(\mathcal{H}), E(\mathcal{F}) \subseteq E(\mathcal{H} \cap \mathcal{F})$. Thus $E(\mathcal{H}) \cup E(\mathcal{F}) \subseteq E(\mathcal{H} \cap \mathcal{F})$. □

Remark 1.32. *By Example 1.30 and Proposition 1.31* (1)*, we have* $E(C(\mathbb{R},\mathbb{R})) \subseteq E(f)$ *where* $f : \mathbb{R} \to \mathbb{R}$ *is defined by* $f(x) = 2x$ *and* $E(C(\mathbb{R},\mathbb{R})) = \phi$*.*

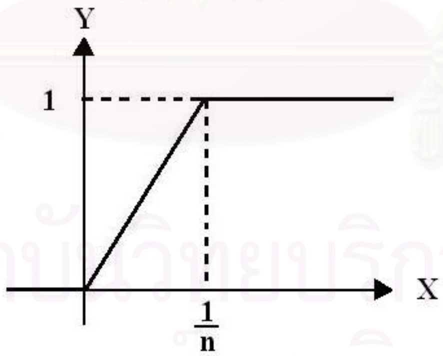
Theorem 1.33. *Let* $\{U_\alpha : \alpha \in \Lambda\}$ *be a collection of open subsets of* X*. If* $\mathcal{F}$ *is equicontinuous on* U_α *for all* $\alpha \in \Lambda$*, then* $\bigcup_{\alpha\in\Lambda} U_\alpha \subseteq E(\mathcal{F})$*.*

Proof. Let $u \in \bigcup_{\alpha\in\Lambda} U_\alpha$ and $\varepsilon > 0$ be arbitrary. So $u \in U_\beta$ for some $\beta \in \Lambda$. There is a neighborhood V_β of u such that $f(V_\beta) \subseteq B(f(u),\varepsilon)$ for every $f \in \mathcal{F}$. Since V_β is open in U_β, V_β is open in X. Hence $u \in E(\mathcal{F})$. □

Example 1.34. *Let* $\mathcal{F} = \{f_n : n \in \mathbb{N}\}$*, where for each* $n \in \mathbb{N}$*,* $f_n \in C(\mathbb{R},\mathbb{R})$ *is defined by*

$$f_n(x) = \begin{cases} 0 & , if \;\; x < 0 \\ nx & , if \;\; 0 \le x \le \frac{1}{n} \\ 1 & , if \;\; \frac{1}{n} < x. \end{cases} \tag{1.1}$$

Then $E(\mathcal{F}) = \mathbb{R} - \{0\}$*.*



Proof. First, we will show that $0 \notin E(\mathcal{F})$. Choose $\varepsilon = \frac{1}{2}$ and let U be an open set containing 0, and $n \in \mathbb{N}$ be such that $(0 - \frac{1}{n}, 0 + \frac{1}{n}) \subseteq U$. Then

$$f_n((0 - \frac{1}{n}, 0 + \frac{1}{n})) = [0,1) \nsubseteq (-\frac{1}{2}, \frac{1}{2}) = (f(0) - \frac{1}{2}, f(0) + \frac{1}{2}) = (f(0) - \varepsilon, f(0) + \varepsilon)$$

for every $n \in \mathbb{N}$. Thus $0 \notin E(f)$, which implies $E(F) \subseteq \mathbb{R} - \{0\}$. Next, we will show that $\mathcal{F}$ is equicontinuous on $\mathbb{R} - \{0\}$. For every n, if we restrict the domain of f_n to

$(-\infty,0)\cup(\frac{1}{n},\infty)$ then the class $\mathcal{F}$ defined as in (1.1) is finite. Therefore, $(-\infty,0)$ and $(\frac{1}{n},\infty)$ are subset of $E(\mathcal{F})$ for every $n\in\mathbb{N}$. By Theorem 1.33,

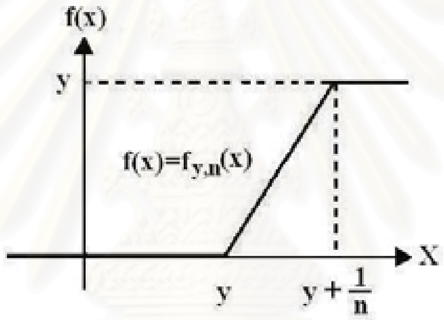
$$\mathbb{R}-\{0\}=(-\infty,0)\cup\bigcup_{n\in\mathbb{N}}(\frac{1}{n},\infty)\subseteq E(\mathcal{F}),$$

so $E(\mathcal{F})=\mathbb{R}-\{0\}$. $\square$

Example 1.35. *Let $y\in\mathbb{R}$ and $n\in\mathbb{N}$, define $f_{y,n}\in C(\mathbb{R},\mathbb{R})$ by*

$$f_{y,n}(x)=y(f_n(x-y)),\ \textit{for } x\in\mathbb{R},$$

where f_n's are defined as in Example 1.34. Then $E(\mathcal{F})=(-\infty,0]$ where $\mathcal{F}=\{f_{y,n}:y\in\mathbb{R}^+,\ n\in\mathbb{N}\}$.



Proof. As in the previous example, $(-\infty,0)\subseteq E(\mathcal{F})$. Next, we show that $0\in E(\mathcal{F})$. Let $\varepsilon>0$ be arbitrary. We choose $\delta=\varepsilon$ and let $f_{y,n}\in\mathcal{F}$. If $y\geq\varepsilon$, then

$$|f_{y,n}(x)-0|=y(f_n(x-y))=y\cdot 0=0$$

for every $x\in(-\varepsilon,\varepsilon)$. If $0<y<\varepsilon$, then

$$|f_{y,n}(x)-0|=y(f_n(x-y))=yf_n(x-y)\leq y<\varepsilon$$

for every $x\in\mathbb{R}$. Thus $0\in E(\mathcal{F})$. Finally, we show that $\mathcal{F}$ is not equicontinuous on $\mathbb{R}^+$. Let $y\in\mathbb{R}^+$. We will show that $y\notin E(\mathcal{F})$. Choose $\varepsilon=\frac{y}{2}$ and let U be a neighborhood of y. There is $n\in\mathbb{N}$ such that $(y-\frac{1}{n},y+\frac{1}{n})\subseteq U$. Choose $f_{y,n}\in\mathcal{F}$. Then

$$f_{y,n}((y-\frac{1}{n},y+\frac{1}{n}))=[0,y)\not\subseteq(\frac{y}{2},\frac{y}{2})=(f_{y,n}(y)-\frac{y}{2},f_{y,n}(y)+\frac{y}{2})=(f(y)-\varepsilon,f(y)+\varepsilon).$$

Then $\mathcal{F}$ is not equicontinuous at y. Therefore, $E(\mathcal{F})=(-\infty,0]$. $\square$

Theorem 1.36. *For each $\mathcal{F} \subseteq C(X,Y)$, the set $E(\mathcal{F})$ is a G_δ-set.*

Proof. For each $n \in \mathbb{N}$ and $x \in E(\mathcal{F})$, there is a neighborhood $G_{n,x}$ of x such that $f(x) \in f(G_{n,x}) \subseteq B(f(x), \frac{1}{n})$ for every $f \in \mathcal{F}$, since $\mathcal{F}$ is equicontinuous at x.
Let $A_n = \bigcup_{x \in E(\mathcal{F})} G_{n,x}$ and $B = \bigcap_{n \in \mathbb{N}} A_n$.
It is clear that $E(\mathcal{F}) \subseteq B$. Next, we will show that $B \subseteq E(\mathcal{F})$. Let $b \in B$ and $\varepsilon > 0$ be arbitrary, there is $m \in \mathbb{N}$ such that $\frac{2}{m} < \varepsilon$. Since $b \in A_n$ for every $n \in \mathbb{N}$, $b \in A_m$. There is a point $a \in E(\mathcal{F})$ such that $b \in G_{m,a}$. Let $c \in G_{m,a}$ and $f \in \mathcal{F}$. Hence $f(b)$ and $f(c)$ are in $f(G_{m,a}) \subseteq B(f(x), \frac{1}{m})$, so

$$d(f(b), f(c)) \leq d(f(b), f(a)) + d(f(a), f(c)) < \frac{1}{m} + \frac{1}{m} = \frac{2}{m} < \varepsilon.$$

Therefore, there is a neighborhood $G_{m,a}$ of b such that $f(G_{m,a}) \subseteq B(f(b), \varepsilon)$ for all $f \in \mathcal{F}$, so $\mathcal{F}$ is equicontinuous at b. Thus $b \in E(\mathcal{F})$. □

Corollary 1.37. *The set $E(f)$ is a G_δ-set, for every $f \in C(X,X)$.*

Definition 1.38. *Let X and Y be two spaces, I the unit interval $[0,1]$ and $f, g \in C(X,Y)$. We say that f and g are* **homotopic** *if there exists a map $H : X \times I \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for each $x \in X$.*

Definition 1.39. *Let X be a space and $A \subseteq X$. A* **retraction** *of X onto A is a map $r : X \to A$ such that $r|_A$ is the identity map of A. If such a map r exists, we say that A is a retract of X.*

Definition 1.40. *A map $f \in C(X,Y)$ is called* **nullhomotopic** *if f is homotopic to a constant map. A space X is called* **contractible** *if 1_X is nullhomotopic.*

Theorem 1.41. *Let A be a retract of X. If X is contractible, then so is A.*

Remark 1.42. *Any star-convex subset of a linear topological space is contractible.*

Proof. Let X be a star-convex subset of a linear topological space. Define a homotopy $H : X \times [0,1] \to X$ by $H(x,t) = tx$ for each $x \in X$ and $t \in [0,1]$. Thus 1_X is nullhomotopic, so X is contractible. □

CHAPTER II

SOME PROPERTIES OF CONVERGENCE SETS

For an nonempty Hausdorff space X and $f \in C(X, X)$, the **convergence set of** f is defined to be the set

$$C(f) := \{x \in X : \text{ the sequence } (f^n(x)) \text{ converges in } X\},$$

and the **fixed point set of** f is the set $F(f)$ of all fixed points of f. That is $F(f) := \{x \in X : f(x) = x\}$. Note that $F(f)$ is closed for every $f \in C(X, X)$.

Remark 2.1. *Let X be a metric space and $f \in C(X, X)$. We clearly have:*

(1) $F(f) \subseteq C(f)$,

(2) $\lim\limits_{n\to\infty} f^n(x)$ *is unique and belongs to* $F(f)$ *for each* $x \in C(f)$,

(3) $C(f) = \phi$ *if and only if* $F(f) = \phi$.

From now on, we will assume that $F(f) \neq \phi$.

Definition 2.2. *Let X be a metric space and $f \in C(X, X)$.*

(*i*) *The map f is called* **nonexpansive** *if for each $x, y \in X$,*

$$d(f(x), f(y)) \leq d(x, y).$$

(*ii*) *The map f is called* **quasi-nonexpansive** *if for each $x \in X$ and $y \in F(f)$,*

$$d(f(x), y) \leq d(x, y).$$

(*iii*) *The map f is called* **virtually nonexpansive** *if $C(f) \subseteq E(f)$.*

It is obvious that every nonexpansive map is quasi-nonexpansive. It is known that every quasi-nonexpansive maps is virtually nonexpansive and $F(f)$ is a retract of $C(f)$[2].

Proposition 2.3. *Let $f \in C(X,X)$. If $\{f^n : n \in \mathbb{N}\}$ is a finite set, then f is virtually nonexpansive.*

Proof. Assume that $\{f^n : n \in \mathbb{N}\}$ is a finite set. Then $E(f) = X$, by Remark 1.29, and so f is virtually nonexpansive. □

Here is an example to show that a map may be nonexpansive relative to a metric but not nonexpansive relative to another metric even though the two metrics are equivalent.

Example 2.4. *Consider $\mathbb{R}^2$ with the metric induced by the norm*

$$\|(x,y)\|_\infty = \max\{|x|,|y|\},$$

and define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$f(x,y) = (x,|x|).$$

Then f is nonexpansive relative to $\|\cdot\|_\infty$, not nonexpansive relative to the standard metric. However it is virtually nonexpansive relative to any metric on $\mathbb{R}^2$ even though the two metrics are equivalent.

Proof. To show that f is nonexpansive relative to $\|\cdot\|_\infty$, let $(x_1,y_1),(x_2,y_2) \in \mathbb{R}^2$. We have

$$\begin{aligned}
\|f(x_1,y_1) - f(x_2,y_2)\|_\infty &= \|(x_1,|x_1|) - (x_2,|x_2|)\|_\infty \\
&= \|(x_1 - x_2, |x_1| - |x_2|)\|_\infty \\
&= \max\{|x_1 - x_2|, ||x_1| - |x_2||\} \\
&= |x_1 - x_2| \\
&\leq \max\{|x_1 - x_2|, |y_1 - y_2|\} \\
&= \|(x_1,y_1) - (x_2,y_2)\|_\infty .
\end{aligned}$$

Next, we note that for $(0,1),(1,1)$ in $\mathbb{R}^2$, we have

$$\|f(0,1) - f(1,1)\| = \sqrt{2} > 1 = \|(0,1) - (1,1)\| .$$

That is f is not nonexpansive relative to the standard metric. Since $f^n = f$ for any $n \in \mathbb{N}$, $\{f^n : n \in \mathbb{N}\}$ is finite and by Proposition 2.3 implies that f is virtually nonexpansive. $\square$

Example 2.5. *Consider $C([0, 2\pi], [0, 2\pi])$ with the norm*

$$\|x\| = \int_0^{2\pi} |x(t)|dt$$

for $x \in C([0, 2\pi], [0, 2\pi])$. The map $f : C([0, 2\pi], [0, 2\pi]) \to C([0, 2\pi], [0, 2\pi])$ defined by

$$(f(x))(t) = sin(t)|x(t)| \qquad (x \in C([0, 2\pi], [0, 2\pi]))$$

is nonexpansive.

Proof. To show that f is nonexpansive, let $x, y \in C([0, 2\pi], [0, 2\pi])$. We have

$$\begin{aligned} \|f(x) - f(y)\| &= \int_0^{2\pi} |(f(x))(t) - (f(y))(t)|dt \\ &= \int_0^{2\pi} |sin(t)x(t) - sin(t)y(t)|dt \\ &= \int_0^{2\pi} |sin(t)||(x(t) - y(t))|dt \\ &\le \int_0^{2\pi} |x(t) - y(t)|dt \\ &= \|x - y\| . \end{aligned}$$

This implies that f is a nonexpansive map. $\square$

The followings are examples of virtually nonexpansive maps on $\mathbb{C}$.

Example 2.6. *Let $f : \mathbb{C} \to \mathbb{C}$ be defined by $f(z) = \overline{z}$ for each $z \in \mathbb{C}$.*

Let $z = x + yi \in \mathbb{C}$. Then $f(x + yi) = x - yi$ and

$$F(f) = \{x + yi : y = 0\}.$$

Also, for $n \in \mathbb{N}$, $f^n(x + yi) = x + (-1)^n y$ and

$$C(f) = \{x + yi : y = 0\} = F(f).$$

Since $\{f^n : n \in \mathbb{N}\} = \{f, 1_{\mathbb{C}}\}$ and by Proposition 2.3, f is virtually nonexpansive.

Example 2.7. *Let $f : \mathbb{C} \to \mathbb{C}$ be defined by $f(x+iy) = x+i\frac{1}{2}\left(y+|x|\right)$ for each $z = x+iy$ in $\mathbb{C}$.*

It is easy to see that $F(f) = \{x+iy : y = |x|\}$ and $C(f) = \mathbb{C}$. Note that $f^n(x+iy) = x + i\dfrac{1}{2^n}(y + \sum_{i=0}^{n-1} 2^i|x|)$ and $\dfrac{1}{2^n}\sum_{i=0}^{n-1} 2^i < 1$ for all $n \in \mathbb{N}$. We will show that $E(f) = \mathbb{C}$. Let $x+iy \in \mathbb{C}$ and $\varepsilon > 0$ be arbitrary. Choose $\delta = \frac{\varepsilon}{3}$. Hence for each $x_1 + iy_1 \in \mathbb{C}$ and $n \in \mathbb{N}$ such that $\|x+iy-(x_1+iy_1)\| < \delta$, we have

$$\begin{aligned}
\|f^n(x+iy) - f^n(x_1+iy_1)\| &= \left\|(x-x_1) + i\frac{y-y_1}{2^n} + i\frac{1}{2^n}(|x|-|x_1|)\sum_{i=0}^{n-1}2^i\right\| \\
&\le \|(x-x_1)\| + \left\|i\frac{y-y_1}{2^n}\right\| + \left\|i(|x|-|x_1|)\frac{1}{2^n}\sum_{i=0}^{n-1}2^i\right\| \\
&\le \delta + \delta + \|(|x|-|x_1|)\| \\
&\le \delta + \delta + \delta = \varepsilon.
\end{aligned}$$

Thus $x+iy \in E(f)$. Therefore, $E(f) = \mathbb{C}$.

Example 2.8. *Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $f(x,y,z) = \left(x, y, \frac{1}{2}(z+|y|)\right)$ for each $(x,y,z) \in \mathbb{R}^3$.*

It is easy to see that $F(f) = \{(x,y,z) : z = |y|\}$ and $C(f) = \mathbb{R}^3$. Similar to Example 2.7, we can show that f is virtually nonexpansive.

The followings are examples of maps that are not virtually nonexpansive.

Example 2.9. *Let $f : \mathbb{C} \to \mathbb{C}$ be defined by $f(z) = z|z|$ for each $z \in \mathbb{C}$.*

It is easy to see that for each $n \in \mathbb{N}$, $f^n(z) = z|z|^{2^n-1}$. Then

$$F(f) = \{z \in \mathbb{C} : |z| = 1\} \cup \{0\} \text{ and}$$

$$C(f) = \{z \in \mathbb{C} : |z| \le 1\}.$$

Next, we will show that f is not virtually nonexpansive. Suppose that f is virtually nonexpansive. Since $1 \in F(f)$, there exists $\delta > 0$ such that for every $y \in \mathbb{C}$ and $n \in \mathbb{N}$, $\|1 - f^n(y)\| < \frac{1}{2}$ whenever $\|1-y\| < \delta$. Let $k \in \mathbb{N}$ be such that $1 - \left(\frac{1}{2}\right)^{\frac{1}{2^k}} < \delta$. Then

$\left\|1 - f^k((\frac{1}{2})^{\frac{1}{2k}})\right\| = \|1 - \frac{1}{2}\| = \frac{1}{2}$ which leads to a contradiction. So f is not virtually nonexpansive.

Next, we let $\ell^\infty(\mathbb{R})$ be the set of all bounded sequences of real numbers. That is

$$\ell^\infty(\mathbb{R}) = \{(x_1, x_2, \ldots) : \sup_i |x_i| < \infty\}.$$

Then $\ell^\infty(\mathbb{R})$ is a vector space under the usual addition and scalar multiplication. That is for each $(x_n), (y_n) \in \ell^\infty(\mathbb{R})$ and $c \in \mathbb{R}$,

$$(x_n) + (y_n) = (x_n + y_n),$$
$$c(x_n) = (cx_n).$$

Define

$$\|x\|_\infty = \sup_i |x_i|,$$

for $x = (x_n) \in \ell^\infty(\mathbb{R})$. It is easy to verify that $\|\cdot\|_\infty$ is a norm on $\ell^\infty(\mathbb{R})$.

Example 2.10. *Let $f : \ell^\infty(\mathbb{R}) \to \ell^\infty(\mathbb{R})$ be defined by $f(x_1, x_2, x_3, \ldots) = (x_2^2, x_3^2, x_4^2, \ldots)$ for each $(x_1, x_2, x_3, \ldots) \in \ell^\infty(\mathbb{R})$.*

Proof. We will show that f in not virtually nonexpansive. Suppose that f is a virtually nonexpansive map. Since $(1, 1, 1, \ldots) \in F(f)$, there exists $\delta > 0$ such that if

$$\|(1, 1, 1, \ldots) - (y_1, y_2, y_3, \ldots)\|_\infty < \delta,$$

then

$$\|(1, 1, 1, \ldots) - f^n(y_1, y_2, y_3, \ldots)\|_\infty = \|f^n(1, 1, 1, \ldots) - f^n(y_1, y_2, y_3, \ldots)\|_\infty < \frac{1}{2}$$

for every $n \in \mathbb{N}$. Let $k \in \mathbb{N}$ be such that $1 - \left(\frac{1}{2}\right)^{\frac{1}{2k}} < \delta$. Hence

$$\left\|(1, 1, 1, \ldots) - (1, 1, \ldots, \overset{k^{th}-term}{\left(\frac{1}{2}\right)^{\frac{1}{2k}}}, 1, \ldots)\right\|_\infty < \delta$$

but

$$\left\|(1,1,1,\ldots) - f^k(1,1,\ldots,\overbrace{\left(\frac{1}{2}\right)^{\frac{1}{2k}}}^{k^{th}-term},1,\ldots)\right\|_\infty = \left\|(1,1,1,\ldots) - (\frac{1}{2},1,1,\ldots)\right\|_\infty = \frac{1}{2},$$

which is a contradiction. It is easy to see that $F(f) = \{(x, x^{\frac{1}{2}}, x^{\frac{1}{4}}, \ldots) : x \in \mathbb{R}^+ \bigcup \{0\}\}$.

□

Example 2.11. *Let $f : \ell^\infty(\mathbb{R}^+) \to \ell^\infty(\mathbb{R}^+)$ be defined by $f(x_1, x_2, x_3, \ldots) = (x_2^{\frac{1}{2}}, x_3^{\frac{1}{2}}, x_4^{\frac{1}{2}}, \ldots)$ for each $(x_1, x_2, x_3, \ldots) \in \ell^\infty(\mathbb{R}^+)$.*

Proof. Suppose that f is virtually nonexpansive. Since $(0,0,0,\ldots) \in F(f)$, there exists $\delta > 0$ such that if

$$\|(0,0,0,\ldots) - (y_1,y_2,y_3,\ldots)\|_\infty < \delta,$$

then

$$\|(0,0,0,\ldots) - f^n(y_1,y_2,y_3,\ldots)\|_\infty = \|f^n(0,0,0,\ldots) - f^n(y_1,y_2,y_3,\ldots)\|_\infty < \frac{1}{2}$$

for every $n \in \mathbb{N}$. Let $k \in \mathbb{N}$ be such that $\frac{1}{2^{2k}} < \delta$. Hence

$$\left\|(0,0,0,\ldots) - (0,0,\ldots,\overbrace{\frac{1}{2^{2k}}}^{k^{th}-term},0,\ldots)\right\|_\infty < \delta$$

but

$$\left\|(0,0,0,\ldots) - f^k(0,0,\ldots,\overbrace{\frac{1}{2^{2k}}}^{k^{th}-term},0,\ldots)\right\|_\infty = \left\|(0,0,0,\ldots) - (\frac{1}{2},0,0,\ldots)\right\|_\infty = \frac{1}{2}.$$

which a contradiction. It is easy to see that $F(f) = \{(x, x^2, x^4, \ldots) : x \in \mathbb{R}^+ \bigcup \{0\}\}$. □

The next example shows that if $f \in C(X,X)$ is a virtually nonexpansive map and $p \in C(X,X)$ is a homeomorphism, then $p \circ f$ and $f \circ p$ need not to be a virtually nonexpansive map.

Example 2.12. *Let $p : \mathbb{R} \to \mathbb{R}$ defined by $p(x) = 2x$ for each $x \in \mathbb{R}$. It is easy to see that p is a homeomorphism. By Example 1.30, $E(p) = \phi$ and $F(f) = \{0\}$, so p is not* *virtually nonexpansive. Since* $1_{\mathbb{R}}$ *is virtually nonexpansive,* $p \circ 1_{\mathbb{R}} = 1_{\mathbb{R}} \circ p = p$ *is not virtually nonexpansive.*

Theorem 2.13. *Let* $f, p \in C(X, X)$*. Then* f *is virtually nonexpansive if and only if for every homeomorphism* p *on* X*,* $p \circ f \circ p^{-1}$ *is virtually nonexpansive.*

Proof. For only if part. Let $x \in F(p \circ f \circ p^{-1})$ and $\varepsilon > 0$ be arbitrary. Note that

$$f^n = \overbrace{(p \circ f \circ p^{-1}) \circ (p \circ f \circ p^{-1}) \circ \ldots \circ (p \circ f \circ p^{-1})}^{n-\text{time}} = p \circ f^n \circ p^{-1}$$

and $f(p^{-1}(x)) = p^{-1}(x)$, since $p \circ f \circ p^{-1}(x) = x$. Therefore, $p^{-1}(x) \in F(f)$.

Since p is continuous, for each $z \in X$, there is $\delta_1 > 0$ such that for every $y \in X$, if $\|z - y\| < \delta_1$, then $\|p(z) - p(y)\| < \varepsilon$. Since f is virtually nonexpansive and by [2], for each $z \in F(f)$, there is $\delta_2 > 0$ such that for every $y \in X$, $\|f^n(z) - f^n(y)\| < \delta_1$ where $\|z - y\| < \delta_2$ and for every $n \in \mathbb{N}$. Since p^{-1} is continuous, for each $z \in X$, there is $\delta_3 > 0$ such that for every $y \in X$, if $\|z - y\| < \delta_3$, then $\left\|p^{-1}(z) - p^{-1}(y)\right\| < \delta_2$. Since $p^{-1}(x) \in F(f)$, for every $y \in X$, such that $\|x - y\| < \delta_3$ implies

$$\left\|p \circ f^n \circ p^{-1}(x) - p \circ f^n \circ p^{-1}(y)\right\| < \varepsilon$$

for any $n \in \mathbb{N}$. Thus $F(f) \in E(f)$ and by [2], which implies $p \circ f \circ p^{-1}$ is virtually nonexpansive. For if part, the conclusion is obvious. □

Lemma 2.14 ([4]). *If* $f \in C(\mathbb{R}, \mathbb{R})$ *is quasi-nonexpansive, then* $F(f)$ *is a convex subset of* $\mathbb{R}$*.*

Theorem 2.15. *Let* X *be a convex subspace of* $\mathbb{R}$ *and* $f \in C(\mathbb{R}, \mathbb{R})$ *quasi-nonexpansive. If* $|F(f)| > 1$*, then* $C(f) = X$*.*

Proof. Let $c \in X$. Since f is quasi-nonexpansive, by Theorem 2.14, $F(f)$ is a closed convex subset of X.

Case 1. $F(f) = X$. Then $F(f) \subseteq C(f) \subseteq X$.

Case 2. $F(f) = (-\infty, x] \cap X$ for some $x \in X$. Since $(-\infty, x] \cap X = F(f) \subseteq C(f)$, it suffices to show that $(x, \infty) \cap X \subseteq C(f)$ and let $z = x - |x - c| \in F(f)$. Since f is quasi-nonexpansive, we have $c \geq f^1(c) \geq f^2(c) \geq \ldots \geq z$, it follows that $(f^n(c))$ is decreasing and bounded below by z. Hence it is a convergent sequence.

Case 3. $F(f) = [x, \infty) \cap X$ for some $x \in \mathbb{R}$. The proof is similar to case2.

Case 4. $F(f) = [x, y]$ for some $x, y \in \mathbb{R}$. Suppose $c \notin [x, y]$. Then there are 3 possibilities:

(4.1) There exists $z \in F(f)$ and $m \in \mathbb{N}$ such that for each $n \geq m$, $f^n(c) \geq z$. Thus $f^n(c) \geq z$ for each $n \geq m$. Since f is quasi-nonexpansive,

$$c \geq f^1(c) \geq f^2(c) \geq \ldots \geq z.$$

Therefore, $(f^n(c))$ is a convergent sequence.

(4.2) There exists $z \in F(f)$ and $m \in \mathbb{N}$ such that for each $n \geq m$, $f^n(c) \leq z$. The proof is similar to the case (4.1.).

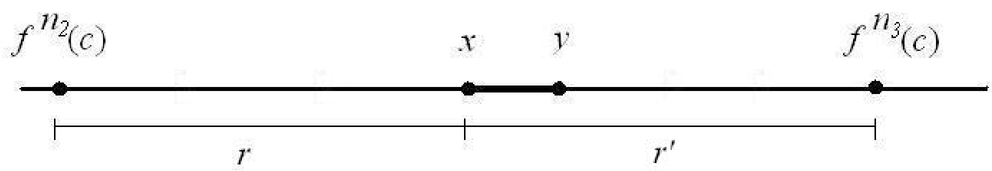
(4.3) For each $z \in F(f)$ and each $m \in \mathbb{N}$, there exist $n, k \geq m$ such that

$$f^n(c) < z \text{ and } f^k(c) > z.$$

We will show that this case is impossible. To do this, let define a subsequence $(f^{n_k}(c))$ as follows:

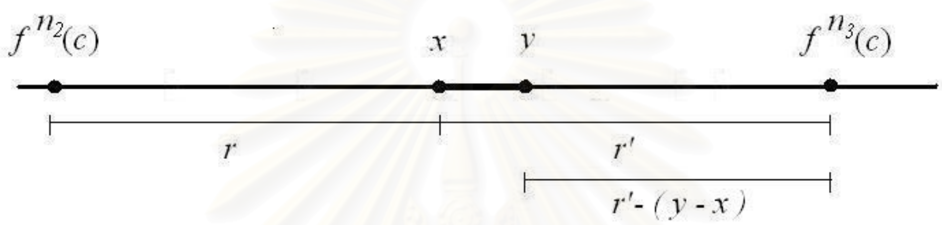
$$\begin{aligned}
&f^{n_1}(c) = f(c), \\
&f^{n_2}(c) < x \text{ for some } n_2 \geq n_1, \\
&f^{n_3}(c) > x \text{ for some } n_3 \geq n_2, \\
&\qquad\qquad \vdots \\
&\text{for } k \text{ is even, } f^{n_k}(c) < x, \\
&\text{for } k \text{ is odd, } f^{n_k}(c) > x.
\end{aligned}$$

Note: $0 < x - f^{n_k}(c)$ for every even number k. Let $r = |x - f^{n_2}(c)| > 0$ and $r' = |f^{n_3}(c) - x|$.

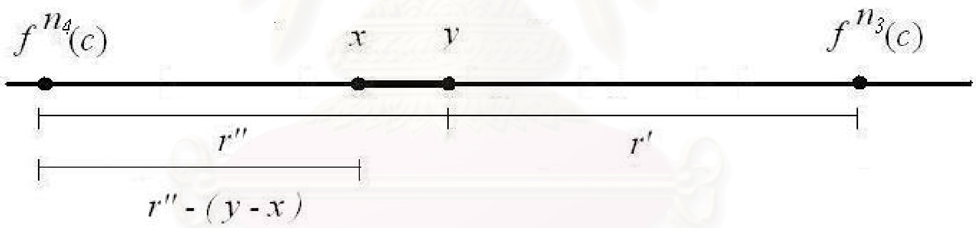


Since $r = |f^{n_2}(c) - x| \geq |f^{n_3}(c) - x| = r'$ and $f^{n_3}(c) \geq y$, we have

$$|f^{n_3}(c) - y| = r' - (y - x) \leq r - (y - x).$$



Next, let $r'' = |f^{n_4}(c) - y|$. Since $r - (y - x) \geq |f^{n_3}(c) - y| \geq |f^{n_4}(c) - y| = r''$ and $f^{n_4}(c) \leq x$, we have $x - f^{n_4}(c) = r'' - (y - x) \leq r - 2(y - x)$.



Follow this process, we have

$$f^{n_5}(c) - y \leq r - 3(y - x),$$

$$x - f^{n_6}(c) \leq r - 4(y - x), \ldots,$$

$$x - f^{n_i}(c) \leq r - (i - 2)(y - x), \text{ if } i \text{ is even,}$$

$$f^{n_i}(c) - y \leq r - (i - 2)(y - x), \text{ if } i \text{ is odd.}$$

There is an even number $m \in \mathbb{N}$ such that $x - f^{n_m}(c) \leq r - (m - 2)(y - x) \leq 0$ which leads to a contradiction. $\square$

Definition 2.16. *Let X be a metric space and $f \in C(X,X)$.*

(*i*) *The map f is called* **periodic** *if there is $n \in \mathbb{N}$ such that $f^n = 1_X$.*

(*ii*) *The map f is called* **recurrent** *if for each $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that for each $x \in X$, $d(f^N(x), x) < \varepsilon$.*

(*iii*) *The map f is called* **pointwise recurrent** *if for each $x \in X$ and $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that $d(f^N(x), x) < \varepsilon$.*

Remark 2.17. *Every periodic map is recurrent, and every recurrent map is pointwise recurrent.*

Lemma 2.18. *Let $f : X \to X$ be pointwise recurrent. Then for each $x \in X$ and $\varepsilon > 0$, the set $A_{x,\varepsilon} := \{n \in \mathbb{N} : d(f^n(x), x) < \varepsilon\}$ is infinite.*

Proof. Let $x \in X$ and $\varepsilon > 0$ be arbitrary. We suppose that $A_{x,\varepsilon}$ is a finite set. It is easy to see that f is not periodic. Since f is not periodic, $d(f^n(x), x) > 0$ for all $n \in \mathbb{N}$. Thus

$$0 < \min\{d(f^n(x), x) : n \in A_{x,\varepsilon}\} < \varepsilon.$$

Since f is pointwise recurrent, there is $m \in \mathbb{N}$ such that

$$d(f^m(x), x) < \min\{d(f^n(x), x) : n \in A_{x,\varepsilon}\} < \varepsilon.$$

It follows that $m \in A_{x,\varepsilon}$. Hence

$$d(f^m(x), x) < \min\{d(f^n(x), x) : n \in A_{x,\varepsilon}\} \leq d(f^m(x), x),$$

which leads to a contradiction. Therefore, $A_{x,\varepsilon}$ is infinite. $\square$

Theorem 2.19. *If $f \in C(X,X)$ is pointwise recurrent, then $C(f) = F(f)$.*

Proof. It suffices to show that $C(f) \subseteq F(f)$. Let $x \in C(f)$, and $\lim_{n\to\infty} f^n(x) = y$ for some $y \in F(f)$. Let $\varepsilon > 0$ be arbitrary. There is $N \in \mathbb{N}$ such that $d(f^n(x), y) < \frac{\varepsilon}{2}$ for each $n \geq N$. By Lemma 2.18, we know that $\{n \in \mathbb{N} : d(f^n(x), x) < \frac{\varepsilon}{2}\}$ is infinite, so there is $k \geq N$ such that $d(f^k(x), x) < \frac{\varepsilon}{2}$. Hence

$$d(x, y) < d(x, f^k(x)) + d(f^k(x), y) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since ε is arbitrary, $d(x,y)=0$. That is $x=y=\lim_{n\to\infty} f^n(x)$ and

$$f(x)=f(\lim_{n\to\infty} f^n(x))=\lim_{n\to\infty} f^{n+1}(x)=x.$$

Therefore, $x\in F(f)$. □

The next theorem describes $C(f)$ when $f\in C(X,X)$ is a virtually nonexpansive map on a complete metric space. The proof generalizes the result in [2].

Theorem 2.20. *Let X be a complete metric space. If $f\in C(X,X)$ is virtually nonexpansive, then $C(f)$ is a G_δ−set.*

Proof. Let $f\in C(X,X)$ is virtually nonexpansive. Since $C(f)\subseteq E(f)$, f is equicontinuous for every $\alpha\in C(f)$. That is for every $\alpha\in C(f)$ and $m\in\mathbb{N}$ there exists $\delta_{\alpha,m}>0$ such that if $d(y,\alpha)<\delta_{\alpha,m}$, then

$$d(f^n(y),f^n(\alpha))<\frac{1}{m} \text{ for every } n\in\mathbb{N}.$$

Let $A_m=\bigcup_{\alpha\in C(f)} B(\alpha,\delta_{\alpha,m})$, for each $m\in\mathbb{N}$ and $B=\bigcap_{m\in\mathbb{N}} A_m$.

We will claim that $B=C(f)$. It is clear that $C(f)\subseteq B$. To show that $B\subseteq C(f)$. Let $b\in B$ and $\varepsilon>0$ be arbitrary. There exists $k\in\mathbb{N}$ such that $\frac{1}{k}\leq\frac{\varepsilon}{4}$. Since $b\in A_m$ for every $m\in\mathbb{N}$, there is $\alpha\in C(f)$ and $\delta_{\alpha,k}>0$ such that $d(b,\alpha)<\delta_{\alpha,k}$, so

$$d(f^n(b),f^n(\alpha))<\frac{1}{k}\leq\frac{\varepsilon}{4} \quad \text{for all } n\in\mathbb{N}.$$

Since $\alpha\in C(f)$, there exist $x\in X$ and $N\in\mathbb{N}$ such that $d(f^n(\alpha),x)<\frac{\varepsilon}{4}$ for every $n\geq N$. Hence

$$d(f^n(b),x)\leq d(f^n(b),f^n(\alpha))+d(f^n(\alpha),x)<\frac{\varepsilon}{4}+\frac{\varepsilon}{4}=\frac{\varepsilon}{2}$$

for every $n\geq N$. And

$$d(f^i(b),f^j(b))\leq d(f^i(b),x)+d(x,f^j(b))<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$$

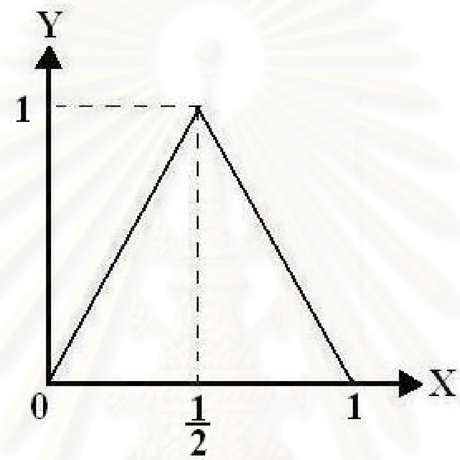
for every $i,j\geq N$. Therefore, $(f^n(b))$ is a Cauchy sequence. Since X is complete, $(f^n(b))$ converges to a point in X. That is $b\in C(f)$. □

The next example shows that there is a map such that $C(f)$ is not a G_δ-set.

Example 2.21. *Define* $T : [0,1] \to [0,1]$ *by*

$$T(x) = \begin{cases} 2x & \text{if } 0 \le x \le \frac{1}{2}, \\ 2-2x & \text{if } \frac{1}{2} < x \le 1. \end{cases}$$

This map is called the **tent map**.



Note $F(T) = \{0, \frac{2}{3}\}$. We consider the composition of T as follows:

$$T \circ T(x) = T^2(x) = \begin{cases} 4x & \text{if } 0 \le x \le \frac{1}{4}, \\ 2-4x & \text{if } \frac{1}{4} < x \le \frac{2}{4}, \\ 4x-2 & \text{if } \frac{2}{4} < x \le \frac{3}{4}, \\ 4-4x & \text{if } \frac{3}{4} < x \le 1. \end{cases}$$

$$T^n(x) = \begin{cases} 2^n x & \text{if } 0 \le x \le \frac{1}{2^n}, \\ 2 - 2^n x & \text{if } \frac{1}{2^n} < x \le \frac{2}{2^n}, \\ 2^n x - 2 & \text{if } \frac{2}{2^n} \le x \le \frac{3}{2^n}, \\ 4 - 2^n x & \text{if } \frac{3}{2^n} < x \le \frac{4}{2^n}, \\ 2^n x - 4 & \text{if } \frac{4}{2^n} \le x \le \frac{5}{2^n}, \\ 6 - 2^n x & \text{if } \frac{5}{2^n} < x \le \frac{6}{2^n}, \\ 2^n x - 6 & \text{if } \frac{6}{2^n} \le x \le \frac{7}{2^n}, \\ & \vdots \\ k + 1 - 2^n x & \text{if } \frac{k}{2^n} < x \le \frac{k+1}{2^n} \text{ where } k \text{ is odd}, \\ 2^n x - k + 1 & \text{if } \frac{k+1}{2^n} \le x \le \frac{k+2}{2^n}, \\ & \vdots \\ 2^n - 2^n x & \text{if } \frac{2^n - 1}{2^n} < x \le 1. \end{cases}$$

Remark 2.22. *The tent map T has the following properties*

(1) *for $x \in (0, \frac{1}{2}]$, there is $k \in \mathbb{N}$ such that $\frac{1}{2} \le T^k(x) \le 1$,*

(2) *for $x \in \left[\frac{1}{2}, 1\right]$, there is $k \in \mathbb{N}$ such that $0 \le T^k(x) \le \frac{1}{2}$.*

Proof. (1) Let $x \in (0, \frac{1}{2}]$. There is $k \in \mathbb{N}$ such that

$$\frac{1}{2^{k+1}} \le x \le \frac{1}{2^k}$$

and

$$\frac{1}{2^k} \le 2x = T(x) \le \frac{1}{2^{k-1}}.$$

Thus

$$\frac{1}{2} \le 2^k x = T^k(x) \le 1.$$

(2) Define $g : [0, 1] \to [0, 1]$ by $g(y) = \dfrac{2 - y}{2}$ for each $y \in [0, 1]$. We consider the set

$$A = \{g^0(1) = 1, g(1), g^2(1), g^3(1), \ldots\}.$$

We claim that $g^k(1) < g^{k+2}(1)$ if k is odd and $g^{k+2}(1) < g^k(1)$ if k is even.

Since $g^n(y) = \frac{1}{2^n}\sum_{i=1}^{n}(-1)^{n-i}2^i + (-1)^n(y)$, we obtain

$$g^n(1) = \frac{1}{2^n}\sum_{i=0}^{n}(-1)^{n-i}2^i \quad \text{and} \quad g^{n+2}(1) = \frac{1}{2^{n+2}}\sum_{i=0}^{n+2}(-1)^{n+2-i}2^i.$$

Consider $g^n(1) - g^{n+2}(1)$. We note that

$$\begin{aligned}
g^n(1) - g^{n+2}(1) &= \frac{1}{2^n}\sum_{i=0}^{n}(-1)^{n-i}2^i - \frac{1}{2^{n+2}}\sum_{i=0}^{n+2}(-1)^{n+2-i}2^i \\
&= \frac{1}{2^{n+2}}\left(2^2\sum_{i=0}^{n}(-1)^{n-i}2^i\right) - \frac{1}{2^{n+2}}\left((-1)^{n-1}2 + (-1)^n + \sum_{i=2}^{n+2}(-1)^{n+2-i}2^i\right) \\
&= \frac{1}{2^{n+2}}\left((\sum_{i=0}^{n}(-1)^{n-i}2^{i+2}) - (\sum_{i=2}^{n+2}(-1)^{n+2-i}2^i) - ((-1)^{n+2-1}2 + (-1)^{n+2})\right) \\
&= \frac{1}{2^{n+2}}\left((\sum_{i=0}^{n}(-1)^{n-i}2^{i+2}) - (\sum_{i=0}^{n}(-1)^{n-i}2^{i+2}) - ((-1)^{n-1}2 + (-1)^n)\right) \\
&= \frac{1}{2^{n+2}}\left(-(-1)^{n-1}2 + (-1)^n\right) = (-1)^{n+1}\frac{-2+1}{2^{n+2}} \\
&= \frac{(-1)^n}{2^{n+2}}.
\end{aligned}$$

So if n is odd, then $g^n(1) - g^{n+2}(1) < 0$, otherwise $g^n(1) - g^{n+2}(1) > 0$.

Let $x \in \left[\frac{1}{2}, 1\right] - \{\frac{2}{3}\}$. We have $g^k(1) \le x \le g^{k+2}(1)$ for some odd number k or $g^{k+2}(1) \le x \le g^k(1)$ for some even number k.

Since $g^n(x) \in [\frac{1}{2}, 1]$ for every $n \in \mathbb{N}$ and $x \in [0,1]$,

$$T \circ g^n = f\left(\frac{2 - g^{n-1}}{2}\right) = 2 - 2\left(\frac{2 - g^{n-1}}{2}\right) = g^{n-1} \text{ for every } n \in \mathbb{N}.$$

By composition, we have

$$T^k \circ g^k(1) = g^{k-k}(1) = g^0(1) = 1 \ge T^k(x) \ge T^k \circ g^{k+2}(1) = g^{k+2-k}(1) = g^2(1) = \frac{3}{4}$$

and then $0 \le T^{k+1}(x) \le \frac{1}{2}$. □

Next we will determine the convergence set, $C(T)$, of the tent map T. Define the set

$$T^{-\infty}(x) = \bigcup_{n=1}^{\infty} T^{-n}(x)$$

where $T^{-1}(x)$ is the inverse image of $\{x\}$ and the set $T^{-n}(x)$ is the inverse image of the set $T^{-n+1}(x)$.

By the definition of T, we have $T^{-1}(x) = \{\frac{x}{2}, \frac{2-x}{2}\}$.
Then $T^{-1}(0) = \{0, 1\}$, $T^{-2}(0) = \{0, \frac{1}{2}, 1\}$, $T^{-3}(0) = \{0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1\}$, $\ldots$, $T^{-n}(0) = \left\{\frac{m}{2^{n-1}} : m = 0, 1, 2 \ldots, 2^{n-1}\right\}$, and

$$T^{-\infty}(0) = \bigcup_{n=1}^{\infty} \left\{\frac{m}{2^{n-1}} : m = 0, 1, 2 \ldots, 2^{n-1}\right\},$$

which is dense in $[0, 1]$. We claim that $C(T) = T^{-\infty}(0) \bigcup T^{-\infty}(\frac{2}{3})$. It is easy to see that $T^{-\infty}(0) \bigcup T^{-\infty}(\frac{2}{3}) \subseteq C(T)$. Now suppose that there is $x \in C(T)$ such that $x \notin T^{-\infty}(0) \bigcup T^{-\infty}(\frac{2}{3})$.

Case 1. $\lim\limits_{n\to\infty} T^n(x) = 0$ but $T^n(x) \neq 0$ for every $n \in \mathbb{N}$. Choose $\varepsilon = \frac{1}{2}$, so there is $N \in \mathbb{N}$ such that $|T^n(x) - 0| < \frac{1}{2}$ for every $n \geq N$. Hence

$$0 < T^N(x) < \frac{1}{2}.$$

By the property (1) of the tent map in Remark 2.22, there is $k \in \mathbb{N}$ such that

$$\frac{1}{2} < T^{N+k}(x) < 1.$$

Case 2. $\lim\limits_{n\to\infty} T^n(x) = \dfrac{2}{3}$ but $T^n(x) \neq \dfrac{2}{3}$ for every $n \in \mathbb{N}$. Choose $\varepsilon = \dfrac{1}{6}$, so there is $N \in \mathbb{N}$ such that $|T^N(x) - \dfrac{2}{3}| < \dfrac{1}{6}$ for every $n \geq N$. Hence

$$\frac{1}{2} < T^N(x) < \frac{5}{6} < 1.$$

By the property (2) of the tent map in Remark 2.22, there is $k \in \mathbb{N}$ such that

$$0 < T^{N+k}(x) < \frac{1}{2}.$$

Hence it is a contradiction. Therefore,

$$C(f) = T^{-\infty}(0) \bigcup T^{-\infty}(\frac{2}{3}).$$

Since $C(T) = f^{-\infty}(0) \cup T^{-\infty}(\frac{2}{3})$ is a countable dense subset of $[0, 1]$ and by Lemma 1.25, $C(T)$ is not a G_δ-set.

Now we show that the map T is not virtually nonexpansive. By Theorem 1.8 in [2], it suffices to show that $F(T) \nsubseteq E(T)$. Note that $0 \in F(T)$. We will show that $0 \notin E(T)$. Suppose that $0 \in E(T)$. That is every $\varepsilon > 0$, there exists $\delta > 0$ such that if $|x| < \delta$, then $|T^n(x)| < \varepsilon$ for every $n \in \mathbb{N}$. For $\varepsilon = \frac{1}{2}$, there exists $\delta > 0$ such that if $|x| < \delta$, then

$$|T^n(x)| < \frac{1}{2} \text{ for every } n \in \mathbb{N}$$

which contradicts to the property (1) of the tent map in Remark 2.22. For when $n \in \mathbb{N}$ is fixed, there is $k \in \mathbb{N}$ such that

$$T^{n+k}(x) > \frac{1}{2}, \text{ whenever } |x| < \delta.$$

Therefore, the tent map is not virtually nonexpansive and the convergence set of the tent map is not a G_δ-set.

CHAPTER III

STAR-CONVEXITY OF CONVERGENCE SETS

In this chapter, we investigate a geometric property of the convergence set of virtually nonexpansive maps. More precisely, we show that the convergence set of special virtually nonexpansive maps is star-convex and its fixed point set is contractible.

Theorem 3.1. *Let X be a linear topological space. If $f : X \to X$ is a linear map, then $C(f)$ is a convex subset of X.*

Proof. Let $f \in C(X, X)$ be a linear map and $x, y \in C(f)$, say $\lim_{n\to\infty} f^n(x) = a$ and $\lim_{n\to\infty} f^n(y) = b$ for some $a, b \in F(f)$. Since X is convex, $L(x, y) \subseteq X$. Then

$$f^n(tx + (1-t)y) = f^n(tx) + f^n((1-t)y) = tf^n(x) + (1-t)f^n(y)$$

for every point $tx + (1-t)y \in L(x, y)$ and $n \in \mathbb{N}$. Hence

$$\lim_{n\to\infty} f^n(tx + (1-t)y) = t \lim_{n\to\infty} f^n(x) + (1-t) \lim_{n\to\infty} f^n(y) = ta + (1-t)b,$$

so $tx + (1-t)y \in C(f)$. Then $C(f)$ is a convex subset of X. □

Proposition 3.2. *Let X be a linear topological space and $f \in C(X, X)$ such that $f(x + y) = f(x) + f(y)$ for every $x, y \in X$. Then $f(tx) = tf(x)$ for every $t \in \mathbb{R}$ and $x \in X$, and hence f is a linear map.*

Proof. Let $x \in X$. Since $f(0) = 0$, we have $0 = f(0) = f(x + (-x)) = f(x) + f(-x)$, i.e., $f(-x) = -f(x)$. For every $n \in \mathbb{Z}$, $f(nx) = \overbrace{f(x) + \ldots + f(x)}^{n-\text{time}} = nf(x)$ and then $f(x) = f(\frac{n}{n}x) = nf(\frac{1}{n}x)$. That is $\frac{1}{n}f(x) = f(\frac{1}{n}x)$ for every $n \in \mathbb{N}$. Let $q = \frac{m}{n} \in \mathbb{Q}$, so $f(qx) = f(\frac{m}{n}x) = mf(\frac{1}{n}x) = \frac{m}{n}f(x) = qf(x)$. Now let $t \in \mathbb{R}$. There exists a sequence

(q_n) in $\mathbb{Q}$ such that $\lim_{n\to\infty} q_n = t$. Hence

$$f(tx) = f(\lim_{n\to\infty} q_n x) = \lim_{n\to\infty} f(q_n x) = \lim_{n\to\infty} q_n f(x) = tf(x).$$

Therefore, $f(tx) = tf(x)$ for every $t \in \mathbb{R}$ and $x \in X$. □

Theorem 3.3. *Let X be a star-convex subset of a linear topological space and $f \in C(X, X)$. If there is a map $\Phi \in C([0,1],[0,1])$ such that for every $t \in [0,1]$, $f(tx) = \Phi(t)f(x)$ for each $x \in X$ and $\lim_{n\to\infty} \Phi^n(t)$ exists, then $C(f)$ is a star-convex subset of X.*

Proof. Let $x \in C(f)$ and $t \in [0,1]$. From $f(tx) = \Phi(t)f(x)$, we have $f^n(tx) = \Phi^n(t)f^n(x)$ for every $n \in \mathbb{N}$. Therefore,

$$\lim_{n\to\infty} f^n(tx) = \lim_{n\to\infty} \Phi^n(t)f^n(x) = \lim_{n\to\infty} \Phi^n(t) \lim_{n\to\infty} f^n(x).$$

Since $\lim_{n\to\infty} \Phi^n(t)$ exists, $tx \in C(f)$. Thus $C(f)$ is a star-convex subset of X. □

Example 3.4. *Let X be a star-convex subset of a linear topological space Y and $f \in C(X, X)$ with $f(tx) = t^q x$ for some $q \in \mathbb{R}^+$ for every $x \in X, t \in \mathbb{R}$. Then $C(f)$ is a star-convex subset of X.*

Theorem 3.5. *Let X be a linear topological space and $f \in C(X, X)$. Suppose f is not constant and $\Phi \in C([0,1],[0,1])$ is such that $f(tx) = \Phi(t)f(x)$ for each $x \in X$ and $t \in [0,1]$. Then the following properties hold.*

(1) $\Phi(1) = 1$.

(2) $\Phi(st) = \Phi(s)\Phi(t)$ *for every* $s, t \in [0,1]$.

(3) $\Phi(0) = 0$.

(4) $f(0) = 0$.

(5) $|F(\Phi)| \geq 2$.

Proof. Let $x \in X$ be such that $f(x) \neq 0$. Then $f(x) = f(1x) = \Phi(1)f(x)$. Thus $(1 - \Phi(1))f(x) = 0$, so $\Phi(1) = 1$. That is (1) holds. Let $s, t \in \mathbb{R}$. Since

$$\Phi(st)f(x) = f((st)x) = f(s(tx)) = \Phi(s)f(tx) = \Phi(s)\Phi(t)f(x),$$

we have $(\Phi(st) - \Phi(s)\Phi(t))f(x) = 0$. This implies $\Phi(st) = \Phi(s)\Phi(t)$.

Let $y, z \in X$ be such that $f(y) - f(z) \neq 0$. Then

$$\Phi(0)f(y) = f(0y) = f(0) = f(0z) = \Phi(0)f(z).$$

Thus $\Phi(0)(f(y) - f(z)) = 0$, so $\Phi(0) = 0$. This implies (3). And (4) follows from $f(0) = \Phi(0)f(0) = 0$. (5) is obtained from (1) and (3). □

Theorem 3.6. *Let X be a linear topological space and $f \in C(X,X)$, if f is a quasi-nonexpansive map with $|F(f)| > 1$ and a map $\Phi \in C([0,1],[0,1])$ is such that $f(tx) = \Phi(t)f(x)$ for each $x \in X$ and $t \in [0,1]$, then Φ is the identity map on $[0,1]$.*

Proof. Let $t \in \mathbb{R}$, $s \in F(\Phi)$ and $y \in F(f) - \{0\}$. It follows that $sy \in F(f)$ and

$$\begin{aligned} |t-s|\,\|y\| &= \|ty - sy\| \\ &\geq \|f(ty) - f(sy)\| \\ &= \|\Phi(t)f(y) - \Phi(s)f(y)\| \\ &= |\Phi(t) - \Phi(s)|\,\|f(y)\| \\ &= |\Phi(t) - s|\,\|y\|\,. \end{aligned}$$

Thus Φ is quasi-nonexpansive. Since 0 and 1 are in $F(\Phi)$, by Lemma 2.14 $F(\Phi)$ is convex. Therefore, $F(\Phi) = [0,1]$ implies that $\Phi(t) = t$ for every $t \in [0,1]$. □

Theorem 3.7. *Let X be a star-convex subset of a linear topological space Y and $f \in C(X,X)$ virtually nonexpansive. If a map $\Phi \in C([0,1],[0,1])$ is such that for every $t \in [0,1]$, $f(tx) = \Phi(t)f(x)$ for each $x \in X$ and $\lim_{n\to\infty} \Phi^n(t)$ exists, then $F(f)$ is contractible.*

Proof. By Theorem 3.3, $C(f)$ is a star-convex subset of X. By Remark 1.42, $C(f)$ is contractible. But from [2] we know that $F(f)$ is a retract of $C(f)$, so $F(f)$ is contractible. □

In the following example, Theorem 3.7 is used to determine that the fixed point set of f is contractible.

Example 3.8. *Let* $f:\mathbb{R}^3\to\mathbb{R}^3$ *be defined by*

$$f(\overline{x})=\left(x,\frac{5}{6}y+\frac{1}{3\sqrt{2}}z-\left|\frac{\sqrt{3}}{6\sqrt{2}}x+\frac{1}{6}y+\frac{1}{6\sqrt{2}}z\right|,\frac{1}{2\sqrt{3}}x-\frac{\sqrt{2}}{6}y+\frac{1}{3}z+\left|\frac{\sqrt{3}}{6}x+\frac{\sqrt{2}}{6}y+\frac{1}{6}z\right|\right)$$

where $\overline{x}=(x,y,z)\in\mathbb{R}^3$.

This map satisfies the property that $f(t(x,y,z))=tf(x,y,z)$ for every $(x,y,z)\in\mathbb{R}^3$ and $t\in[0,1]$. Note that $f=PTP^{-1}$ where $P:\mathbb{R}^3\to\mathbb{R}^3$ is the linear transformation represented by the matrix

$$\begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}}\\ 0 & -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}}\end{bmatrix}$$

and $T:\mathbb{R}^3\to\mathbb{R}^3$ given by $T(x,y,z)=(x,y,\frac{1}{2}(z+|y|))$ for each $(x,y,z)\in\mathbb{R}^3$. By Example 2.8, T is virtually nonexpansive. The map f is virtually nonexpansive, since P is homeomorphism and by Theorm 2.13. Therefore, $F(f)$ is contractible, by Theorem 3.7. We will determine $F(f)$ and $C(f)$. We claim that $F(f)=P(F(T))$. We first show that $F(f)\subseteq P(F(T))$. Let $x\in F(f)$. Then $PTP^{-1}(x)=x$, so $T(P^{-1}(x))=P^{-1}(x)$. Thus $P^{-1}(x)\in F(T)$. This means $x\in P(F(T))$ or $F(f)\subseteq P(F(T))$. To show that $F(f)\supseteq P(F(T))$, let $x\in P(F(T))$. Then $x=P(y)$ for some $y\in F(T)$. Hence $f(x)=PTP^{-1}(x)=PTP^{-1}(Py)=PT(y)=P(y)=x$. This implies $F(f)\supseteq P(F(T))$. Therefore,

$$\begin{aligned}F(f)&=P(F(T))\\&=P(\{(x,y,|y|):(x,y,z)\in\mathbb{R}^3\})\\&=\left\{\left(\frac{1}{\sqrt{2}}(x+y),\frac{1}{\sqrt{3}}(-x+y-|y|)),\frac{1}{\sqrt{6}}(-x-y+2|y|)\right):x,y\in\mathbb{R}\right\}.\end{aligned}$$

Since $f^n=\overbrace{(PTP^{-1})(PTP^{-1})\ldots(PTP^{-1})}^{n-\text{time}}=PT^nP^{-1}$ and $C(T)=\mathbb{R}^3$,

$$\begin{aligned}\lim_{n\to\infty}f^n(x)&=\lim_{n\to\infty}(PTP^{-1})^n(x)=\lim_{n\to\infty}PT^nP^{-1}(x)\\&=\lim_{n\to\infty}PT^nP^{-1}(x)=P(\lim_{n\to\infty}T^n(P^{-1}(x)))\end{aligned}$$

exists for every $x\in\mathbb{R}^3$. Therefore, $C(f)=\mathbb{R}^3$.

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