CHAPTER IV

THEOREMS FROM RING THEORY

In this chapter, we generalize theorems from ring theory to skewring theory. This is not easy because we must find theorems which do not assume the existence of a multiplicative identity or which do not use concepts from module theory.

In Chapter I, we gave the definitions of nilpotent elements, nilpotent normal ideals and normal nilideals. For this chapter, we shall generalize some theorems of ring theory to skewrings.

We first introduce the nilradicals of a skewring, among which there are the noether radical and the prime radical. Then we will consider the Jacobson radical of a skewring.

Theorem 4.1 *(1) Every nilpotent normal ideal is a normal nilideal.*

(2) If I,J are normal nilideals of a skewring R, then I+J is a normal nilideal in R.

If I,J are nilpotent left[right, two-sided] normal ideals of a skewring R, then the same is true for I+J.

Proof (1) Obvious.

(2) Let I,J be normal nilideals of a skewring R. then every element of I and J is nilpotent. By Corollary 2.9 (4), I+J is a normal ideal of R. By Fourth Isomorphism Theorem, $(I+J)/J \cong I/I\cap J$. Let f: $(I+J)/J \to I/I\cap J$ be an isomorphism. Let $x \in I+J$. Then there exists $i \in I, j \in J$ such that $x = i+j$. Then $f(x+J) = f(i+j+J) = f(i+J) = i+I\cap J$. Since $i \in I$, i is nilpotent. There is $n \in \mathbf{Z}^+$ such that $i^n = 0$. Thus $f((x+J)^n) = (f(x+J))^n = (i+I\cap J)^n = i^n+I\cap J = I\cap J$. Since f is a monomorphism, $(x^n+J) = (x+J)^n = J$. Thus $x^n \in J$ which is a normal nilideal. Then x^n is nilpotent, so there is an $m \in \mathbf{Z}^+$ such that $x^{nm} = (x^n)^m = 0$. Thus x is a nilpotent. Hence I+J is a normal nilideal.

Let I,J be nilpotent normal ideals of a skewring R. By Corollary 1.13 (4), I+J is a normal ideal in R. Then there exist $m,n\in\mathbf{Z}^+$ such that $I^m = J^n = \{0\}$. We shall show that $(I+J)^{m+n} = \{0\}$. Since I,J are ideals, for each $k\in\mathbf{Z}^+$, $x_i\in I$, $y_i\in J$ (where $i\in\{1,\ldots,k\}$), $(x_1+y_1)\ldots(x_k+y_k) = \sum z_1\ldots z_k$ for some $z_i\in I\cup J$. By Proposition 1.23, $(I+J)^{m+n} = \langle\{(x_1+y_1)..(x_{m+n}+y_{m+n})/x_i\in I, y_i\in J \text{ where } i\in\{1,..,m+n\}\}\rangle_n$. Then $(I+J)^{m+n} \subseteq \langle\{\sum z_1\ldots z_{m+n} / z_i\in I\cup J \text{ where } i\in\{1,..,m+n\}\}\rangle_n$. Suppose that $m\geq n$. Let $z_1,\ldots,z_{m+n}\in I\cup J$. If there exist $z_{i_1},\ldots,z_{i_m}\in I$ then $z_1\ldots z_{m+n}$ is of the form $(a_1 z_{i_1})\ldots(a_m z_{i_m})r$ for some $a_j, r\in R\cup\{1\}$ for all $j\in\{1,\ldots,m\}$. Since I is an ideal, $(a_j z_{i_j})\in I$ for all $j\in\{1,\ldots,m\}$. Since $I^m = \{0\}$, $(a_1 z_{i_1})\ldots(a_m z_{i_m}) = 0$ and hence $z_1\ldots z_{m+n} = 0$, (otherwise, there exist $z_{i_1},\ldots,z_{i_m}\in J$ and similary, $z_1\ldots z_{m+n} = 0$.) Therefore $(I+J)^{m+n} = \{0\}$. Hence I+J is a nilpotent normal ideal of R.

If I,J are nilpotent left normal ideals, by the proof of Corollary 2.9 (4), I+J is a normal subgroup of (R,+). Clearly, I+J is a left normal ideal of R. Similarly, I+J is a nilpotent left normal ideal. For right normal ideals, we can prove the theorem in the same way. #

Corollary 4.2. *Let $\{I_\alpha/\alpha\in A\}$ be a family of nilpotent left[right] normal ideals of a skewring R. Then* $\sum_{\alpha\in A} I_\alpha$ *is a left[right] normal nilideal*

Theorem 4.3. *Let $\{I_\alpha/\alpha\in A\}$ be the family of all nilpotent right normal ideals in a skewring R, $\{J_\beta/\beta\in B\}$ the family of all nilpotent left normal ideals in R, and $\{K_\gamma/\gamma\in C\}$ the family of all nilpotent normal ideals in R.*

Let $W_r = \sum_{\alpha\in A} I_\alpha$, $W_l = \sum_{\beta\in B} J_\beta$ *and* $W = \sum_{\gamma\in C} K_\gamma$. *Then* $W = W_r = W_l$.

Proof. Let I be a nilpotent left normal ideal. Then there exists an $n\in\mathbf{Z}^+$ such that $I^n = \{0\}$.

Claim1. I+IR is a normal ideal in R.

Clearly, I+IR is a left normal ideal in R. Let $x\in I$, $y\in IR$, $r\in R$. Then there exist $m\in\mathbf{Z}^+$, $y_i,z_i\in R$, $x_i\in I$, $r_i\in R\cup\mathbf{Z}$ where $i\in\{1,\ldots,m\}$ such that $y=\sum_{i=1}^{m}(z_i+r_ix_iy_i-z_i)$. Clearly, $r_ix_i\in I$, $y_ir\in R$ for every $i\in\{1,\ldots,m\}$ which implies that $(x+y)r = xr+yr = xr+\sum_{i=1}^{m} r_ix_iy_ir\in IR\subseteq I+IR$. Hence we have Claim1.

Claim2. $(IR)^n = \{0\}$.

By Proposition 1.23, $(IR)^n = \langle\{x_1(r_1x_2)\ldots(r_{n-1}x_n)r_n / x_i\in I, r_i\in R$ for every $i\in\{1,\ldots,n\}\}\rangle_n\subseteq\{y_1y_2\ldots y_nr / y_i\in I, r\in R$ for every $i\in\{1,\ldots,n\}\}\rangle_n$ (since I is an left ideal.) $= \{0\}$(since $I^n = \{0\}$.) Hence we have Claim2, so IR is nilpotent.

Since I and IR are nilpotent, by Theorem 4.1 (2), I+IR is nilpotent. Therefore $I+IR\subseteq W$ and since $I\subseteq I+IR$, $I\subseteq W$. Thus $W_l\subseteq W$. But each normal ideal is a left normal ideal, so $W\subseteq W_l$ and hence they are equal. Similarly, $W_r = W$. #

Theorem 4.4. *If R satisfies the ACC for left[right] normal ideals, then W is nilpotent.*

Proof. Let L be the family of all nilpotent left normal ideals of a skewring R which are contained in W. Since $\{0\}\in L$, L is not empty. Let $\{I_k / k\in\mathbf{Z}^+\}$ be a nonempty chain in L. Since R satisfies the ACC for left [right] normal ideals, there exists an $N\in\mathbf{Z}^+$ such that $I_k = I_N$ for all $k\geq N$. Then I_N is an upper bound of this chain. By Zorn's Lemma, L has a maximal element, say I. If $I = W$ we are finished, so suppose $I\neq W$. Let $a\in W\setminus I$. Then there are nilpotent left normal ideals $I_1,\ldots,I_n$ such that $a\in I_1+\ldots+I_n$. Let $I' = I+I_1+\ldots+I_n$. By Theorem 4.3, $I'\subseteq W$ and by Theorem 4.1 (2), I' is nilpotent, contradiction the maximal property of I. Hence $I = W$ which implies that W is nilpotent. #

Theorem 4.5. *Let R be a skewring. Then the following statements hold.:*

(1) The union of all normal nilideals of R is a normal nilideal and is denoted by UR(R).

(2) The union of all nilpotent normal ideals of R is a normal nilideal

and is denoted by NR(R).

(3) $NR(R) \subseteq UR(R) \subseteq N(R)$ *where N(R) is the set of all nilpotent elements in R.*

Proof. (1) Let $x,y \in UR(R)$. Then there exist normal nilideals I,J such that $x \in I$ and $y \in J$. Then $xy \in I$ and $x-y \in I+J$ which is a normal nilideal, by Theorem 4.1 (2). Thus UR(R) is an additive subgroup of R. Clearly, it is a normal nilideal of R.

(2) It follows from (1) and Theorem 4.1 (1).

(3) It follows from (2). #

In the above notation, UR(R) is called the ***upper nilradical*** *of R and NR(R) is called the* ***noether radical*** *of R.*

Definition 4.6. *A normal ideal I of a skewring R is called a* ***nilradical*** *of R if and only if I is a normal nilideal and the only nilpotent normal ideal of* R/I *is 0.*

Remark 4.7. *For any skewring R, UR(R) is the largest nilradical and* $UR(R/UR(R)) = 0$.

Proof. By theorem 4.5 (1), UR(R) is a normal nilideal. Claim that $R/UR(R)$ has no normal nilideal different from 0. Suppose not. Then there exists $I/UR(R)$ which is a normal nilideal of $R/UR(R)$ such that $I \not\subset UR(R)$. Thus I is a normal nilideal of R, so that $I \subseteq UR(R)$ which is a contradiction. Therefore we have the claim. By Theorem 4.1 (1) and the claim, $R/UR(R)$ has no nilpotent normal ideal different from 0. Therefore UR(R) is a nilradical and so UR(R) is the largest nilradical. Since 0 is the unique normal nilideal of $R/UR(R)$, $UR(R/UR(R)) = 0$. #

Definition 4.8. *Let S be a nonempty subset of a skewring R.*

S is called a ***semimultiplicative set*** *in a skewring R if and only if* $a,b\in S$ *implies that there exists* $x\in R\cup \mathbf{Z}$ *such that* $axb\in S$.

S is called a ***multiplicative set*** *if and only if* $a,b\in S$ *implies* $ab\in S$. *And S is generated by a nonempty subset X of R if and only if* $S = \{x_{i_1}\cdots x_{i_m} / m\in \mathbf{Z}^+, x_{i_1},\dots,x_{i_m}\in X\}$.

Remark 4.9. *A normal ideal P of a skewring R is a prime normal ideal if and only if the complement set S of P is a semimultiplicative set.*

Proof. Set $S = P^c$. Suppose that S is not a semimultiplicative set. Then there exist $a,b\in S$ such that $arb\notin S$ for every $r\in R\cup \mathbf{Z}$. Hence $arb\in P$ for every $r\in R\cup \mathbf{Z}$. To show that $\langle a\rangle_n\langle b\rangle_n\subseteq P$. Let $x\in\langle a\rangle_n$, $y\in\langle b\rangle_n$. Then there exist $m,n\in \mathbf{Z}^+$, $x_i,y_j\in R$, $r_i,p_i,q_j,s_j\in R\cup \mathbf{Z}$, where $i\in\{1,\dots,m\}$, $j\in\{1,\dots,n\}$, such that $x = \sum_{i=1}^{m}(x_i+r_iap_i-x_i)$ and $y = \sum_{j=1}^{n}(y_j+q_jbs_j-y_j)$. Then $xy = \sum_{i=1}^{m}\sum_{j=1}^{n} r_iap_iq_jbs_j\in P$ which implies that $\langle a\rangle_n\langle b\rangle_n\subseteq P$. Therefore $\langle a\rangle_n\subseteq P$ or $\langle b\rangle_n\subseteq P$, so $a\in P$ or $b\in P$ which is a contradiction. Hence S is a semimultiplicative set.

Conversely, suppose that S is a semimultiplicative set. Let I and J be a normal ideal of R such that $IJ\subseteq P$. If $I\not\subseteq P$ and $J\not\subseteq P$, let $a\in I\setminus P$ and $b\in J\setminus P$. Then $a,b\in S$ and there exists $x\in R\cup Z$ such that $axb\in S$. On the other hand $axb = (ax)b\in IJ\subseteq P$ which is a contradiction. Hence P is a prime normal ideal of R. #

Definition 4.10. *The* ***prime radical*** *of a skewring R is that intersection of all prime normal ideals of R, it is denoted by PR(R).*

Remark 4.11. *For any skewring $R/PR(R)$ has no nilpotent normal ideal different from 0.*

Proof. Let $I/PR(R)$ be a nilpotent normal ideal of $R/PR(R)$. Then there exists $n\in\mathbf{Z}^+$ such that $I^n/PR(R) = (I/PR(R))^n = 0$. Then $I^n \subseteq PR(R)$, so $I^n \subseteq P$ for every prime normal ideal P of R. Thus $I\subseteq P$ for every prime normal ideal P of R, so that $I\subseteq PR(R)$. Hence $I/PR(R) = 0$. #

Theorem 4.12. *If R is any skewring, then the following sets are equal :*

(1) PR(R).

(2) The intersection of all normal ideals J of R such that R/J has no nilpotent normal ideal except 0.

(3) The set of all elements $x\in R$ such that if a semimultiplicative set S contains x, then it contains 0.

Proof. We denote the sets indicated in (1),(2) and (3), by E_1,E_2 and E_3 respectively. We shall show that $E_1 \subseteq E_3 \subseteq E_2 \subseteq E_1$.

Step1. We shal show that $E_2 \subseteq E_1$.

Let $\Im = \{J \,/\, J$ is a normal ideal of R and has R/J no nilpotent normal ideal except $0\}$. Then $E_2 = \cap\Im$. By Remark 4.11., $PR(R)\in\Im$, so that $\cap\Im\subseteq PR(R)$.

Step2. We shall show that $E_3 \subseteq E_2$.

Claim1. R/E_2 has no nonzero nilpotent normal ideal.

Suppose there exists a nilpotent normal ideal I/E_2 of R/E_2. Then there is an $n\in\mathbf{Z}^+$ such that $I^n/E_2 = (I/E_2)^n = 0$, so $I^n \subseteq E_2$. Thus for any normal ideal

J of R such that R/J has no nonzero nilpotent normal ideal, $I^n \subseteq J$ and I/J is a nilpotent normal ideal in R/J. Hence $I \subseteq J$ for every such J, and therefore $I \subseteq E_2$. Hence we have Claim1.

Let $x \in R$ be such that if S is a semimultiplicative set and $x \in S$, then $0 \in S$.

Claim2. $x \in E_2$.

Suppose not. Then $\langle x \rangle_n \not\subset E_2$. By Claim1, $\langle x \rangle_n \langle x \rangle_n \not\subset E_2$. We shall show that there exists $y \in R \cup Z$ such that $x_1 = xyx \notin E_2$. Suppose $xyx \in E_2$ for every $y \in R \cup Z$. A proof similar to the proof of Remark 4.9, $\langle x \rangle_n \langle x \rangle_n \subseteq E_2$ gives a contradiction. Then there exists a $y \in R \cup Z$ such that $x_1 = xyx \notin E_2$. By repeating the argument we obtain a set $S = \{x_0, x_1, \ldots\}$ of elements not in E_2 (with $x_0 = x$) and so each $x_i \neq 0$. We shall show that S is a semimultiplicative set. Let $x_i, x_j \in S$. If $i = j$ it is obvious. If $i > j$, then there exist $y_i, y_{i+1}, \ldots \in R \cup Z$ such that $x_{i+1} = x_i y_i x_i \in S$, $x_{i+2} = x_{i+1} y_{i+1} x_{i+1} = x_i (y_i x_i\, y_{i+1} x_i y_i) x_i \in S$. Continue in this way, then $x_j = x_i r x_i$ for some $r \in R$ and there is a $y_j \in R$ such that $x_{j+1} = x_j y_j x_j \in S$ for some $y_j \in R \cup Z$. Then $x_i (r x_i y_j) x_j = (x_i r x_i) y_j x_j = x_j y_j x_j = x_{j+1} \in S$. Therefore S is a semimultiplicative set and $x \in S$, but $0 \notin S$ which is a contradiction. Thus $x \in E_2$ and hence $E_3 \subseteq E_2$.

Step3. We shall show that $E_1 \subseteq E_3$.

Let $x \in PR(R)$ and S be a semimultiplicative set such that $x \in S$. To show that $0 \in S$, suppose not. Let $L = \{I / I$ is a normal ideal of R such that $I \cap S = \varnothing.\}$. Since $\{0\} \in L$, L is not empty. Let C be a nonempty chain in L. Clearly, $\cup L$ is an upper bound of C in L. By Zorn's Lemma, there exists a normal ideal P maximal among those such that $P \cap S = \varnothing$. Suppose P is not prime. Then there exist normal ideals I,J of R such that $IJ \subseteq P$, but $I \not\subset P$ and $J \not\subset P$. Then $P \neq I+P$ and $P \neq J+P$. By Corollary 1.13 (4), I+P and J+P are normal ideals of R. So by the maximality of P, $(I+P) \cap S \neq \varnothing$ and $(J+P) \cap S \neq \varnothing$. Then there exist $a, b \in S$ such that $a \in I+P$ and $b \in J+P$. Then there are $i \in I$, $j \in J$, $p, q \in P$ such that $a = i+p$ and $b = j+q$. Since S is a semimultiplicative set, there exists $y \in R$ such that $ayb \in S$. Then $ayb = (i+p)y(j+q) = (iy+py)(j+q) = iyj+pyj+iyq+pyq \in IJ+P \subseteq P$ which contradicts $S \cap P = \varnothing$. Hence P is prime. But $x \in S$, so $x \notin P$ which is impossible since $x \in PR(R)$. Thus $0 \in S$, so that $E_1 \subseteq E_3$. #

Corollary 4.13. *For any skewring R, $NR(R) \subseteq PR(R)$, the prime radical is the smallest nilradical of R, hence $PR(R) \subseteq UR(R)$, and $PR(R/PR(R)) = 0$.*

Proof. First, we shall show that $NR(R) \subseteq PR(R)$. Let I be any nilpotent normal ideal of R and assume that J is any normal ideal of R such that R/J has no nonzero nilpotent normal ideal. By the Fourth Isomorphism Theorem, $(I+J)/J \cong I/(I \cap J)$. Since I is a nilpotent normal ideal, $(I+J)/J$ is a nilpotent normal ideal of R/J. That is $(I+J)/J = 0$, so $I \subseteq J$. By Theorem 4.12, $I \subseteq E_2 = PR(R)$. Hence $NR(R) \subseteq PR(R)$.

Next, we shall show that PR(R) is the smallest nilradical of R. By Remark 4.11, $R/PR(R)$ has no nilpotent normal ideal different from 0. To show that PR(R) is a normal nilideal, let I be any nilradical of R. Then R/I has no nilpotent normal ideal different from 0. By Theorem 4.12, $PR(R) \subseteq I$. Since I is a normal nilideal, $PR(R) \subseteq UR(R)$. By Theorem 4.5 (1), UR(R) is a normal nilideal, so is PR(R) and therefore PR(R) is a nilradical. By Theorem 4.12 (2), PR(R) is the smallest nilradical of R.

Finally, we shall show that $PR(R/PR(R)) = 0$. By Theorem 2.17, $P/PR(R)$ is prime in $R/PR(R)$ if and only if P is prime in R. Then $PR(R/PR(R)) = \cap\{P/PR(R) \,/\, P$ is a prime normal ideal of $R\} = K/PR(R)$ (where $K = \cap\{P/\ P$ is a prime normal ideal of $R\}) = PR(R)/PR(R) = 0$. #

*By Corollary 4.13, PR(R) is called the **lower nilradical** of R.*

Definition 4.14. *The center of a skewring R which is denoted by C(R), is the set $\{x \in R \mid xy = yx$ for every $y \in R\}$.*

Theorem 4.15. *If every nilpotent element of a skewring R is in the center of R, then NR(R) = PR(R) = UR(R) = N(R).*

Proof. We shall show that $N(R) \subseteq NR(R)$. Let $x \neq 0$ be nilpotent element. By assumption, $x \in C(R)$.

Claim that for every $t \in \mathbf{Z}^+$, $(\langle x\rangle_n)^t \subseteq \langle\{\sum_{i=1}^{n} r_i x^t \mid n \in \mathbf{Z}^+, r_i \in R \cup \mathbf{Z}$ where $i = 1,\ldots,n\}\rangle_n$ for every $t \in \mathbf{Z}^+ \setminus \{1\}$.

We will prove by induction on t. By Propositon 1.23, $(\langle x\rangle_n)^t = \{x_1 \ldots x_t \mid x_i \in \langle x\rangle_n$ for every $i \in \{1,\ldots,t\}\}$. If $t = 2$, let $y, z \in \langle x\rangle_n$. Then there exist $m, n \in \mathbf{Z}^+$, $y_i, z_j \in R$, $p_i, q_i, s_j, t_j \in R \cup \mathbf{Z}$, where $i \in \{1,\ldots,m\}$, $j \in \{1,\ldots,n\}$, such that $y = \sum_{i=1}^{m} (y_i + p_i x q_i - y_i)$ and $z = \sum_{j=1}^{n} (z_j + s_j x t_j - z_j)$. By Remark 1.5 (2), $yz = \sum_{i=1}^{m} \sum_{j=1}^{n} (p_i x q_i s_j x t_j)$ $= \sum_{i=1}^{m} \sum_{j=1}^{n} (p_i q_i s_j t_j) x^2$ since $x \in C(R)$. Then $yz = \sum_{i=1}^{k} r_i x^t$ for some $k \in \mathbf{Z}^+$, $r_i \in R \cup \mathbf{Z}$ where $i = 1,\ldots,k$. Therefore the basic step is true.

Let $t \geq 2$. Suppose that the claim is true for t. Then $(\langle x\rangle_n)^t \subseteq \langle\{\sum_{i=1}^{n} r_i x^t \mid n \in \mathbf{Z}^+, r_i \in R \cup \mathbf{Z}$ where $i = 1,\ldots,n\}\rangle_n$. Similarly, by the basic step, $(\langle x\rangle_n)^{t+1} \subseteq \langle\{\sum_{i=1}^{n} r_i x^{t+1} \mid n \in \mathbf{Z}^+, r_i \in R \cup \mathbf{Z}$ where $i = 1,\ldots,n\}\rangle_n$. By induction, we have the claim.

Clearly, $(\langle x\rangle_n)^t = \langle\{\sum_{i=1}^{n} r_i x^t \mid n \in \mathbf{Z}^+, r_i \in R \cup \mathbf{Z}$ where $i = 1,\ldots,n\}\rangle_n$ for all $t \in \mathbf{Z}^+$.

Since x is nilpotent, there exists a $t \in \mathbf{Z}^+$ such that $x^t = 0$. By the claim, $(\langle x\rangle_n)^t = \{0\}$, so $\langle x\rangle_n$ is nilpotent and $\langle x\rangle_n \subseteq NR(R)$. Then $x \in NR(R)$ and $N(R) \subseteq NR(R)$.

By Theorem 4.5 (3), $NR(R) \subseteq UR(R) \subseteq N(R)$. Hence the proof is finished. #

Definition 4.16. *For any multiplicative set S of a skewring R, S is called* ***nilpotent*** *if and only if there exists* $n \in Z^+$ *such that* $a_1 \ldots a_n = 0$ *for any* $a_1, \ldots, a_n \in S$.

Definition 4.17. *The* ***left[right] Annihilator of the subset S*** *of a skewring R is defined as* $Ann_l(S) = \{r \in R / rx = 0$ *for every* $x \in S\}$ *[* $Ann_r(S) = \{r \in R / xr = 0$ *for every* $x \in S\}$*]. The* ***Annihilator of S*** *is denoted by Ann(S) and Ann(S) =* $Ann_l(S) \cap Ann_r(S)$.

Remark 4.18. *The left[right] Annihilator of the subset S of a skewring R is a left[right] normal ideal of R.*

Proof. If $S = \varnothing$, then $Ann_l(S) = R$. Suppose $S \neq \varnothing$. Since $0 \in Ann_l(S)$, $Ann_l(S) \neq \varnothing$. Let $x, y \in Ann_l(S)$, $r \in R$, $s \in S$. Then $(x-y)s = xs-ys = 0$, $(rx)s = r(xs) = 0$ and $(r+x-r)s = 0$, so $x-y$, rx, $r+x-r \in Ann_l(S)$. Hence $Ann_l(S)$ is a left normal ideal of R. Similarly, $Ann_r(S)$ and Ann(s) are right and two-sided normal ideals respectively.#

The following theorem is generalized from Levitzki Theorem

Theorem 4.19. *If a skewring R satisfies the ACC for left normal ideals, then every normal nilideal is a nilpotent normal ideal hence UR(R) = NR(R) is the largest nilpotent normal ideal and* $NR(R/NR(R)) = 0$.

Proof. Let J be a normal nilideal of R. By Proposition 3.9, J is a finitely generated left normal ideal, say by the elements $r_1, \ldots, r_n$. Let S denote the semimultiplicative set generated by $\{r_1, \ldots, r_n\}$, that is the set of products of the element $r_1,\ldots,r_n$. Since $S \subsetneq J$ which is a normal nilideal, every element of S is nilpotent.

Step1. We shall show that S is nilpotent.

Suppose S is not nilpotent. Since r_1 is a nilpotent element, the multiplicative set generated by $\{r_1\}$ is nilpotent. Then there exists an integer m, $1 \le m \le n$ such that the multiplicative set S_{m-1}, generated by $\{r_1,\ldots,r_{m-1}\}$ is nilpotent, but the multiplicative set S_m, generated by $\{r_1,\ldots,r_m\}$ is not nilpotent. Since S_{m-1} is nilpotent and finitely generated, S_{m-1} is finite.

Set $T := \{r_m{}^k c \neq 0 \,/\, k \ge 1, c \in S_{m-1}\}$. Since r_m is nilpotent and S_{m-1} is finite, T is a finite subset of S_m. Let S′ be the semimultiplicative set generated by T.

Claim1. S′ is not nilpotent.

Since r_m is nilpotent, there is a smallest integer k such that $r_m{}^k = 0$....(i) Since S_{m-1} is nilpotent, there is a smallest integer l such that $c_1\ldots c_l = 0$ for any $c_1,\ldots,c_l \in S_{m-1}$. ....(ii)

By (i) and (ii), any nonzero product of $k+l-1$ of the element $r_1,\ldots,r_m$ contains r_m at least once, and some $r_i(1 \le i \le m-1)$ at least once. ....(iii)

Let $s \in Z^+ \backslash \{1\}$. Since S_m is not nilpotent, there exists a nonzero product of $s(k+l-1)$ elements in $\{r_1,\ldots,r_m\}$. We may write it as a product $b_1\ldots b_s$ where $b_i \neq 0$ is a product of $k+l-1$ elements in $\{r_1,\ldots,r_m\}$. By (iii), b_i has r_m and some $r_j(1 \le j \le m-1)$ as factors. A regrouping by the associative law leads to the fact that $b_1\ldots b_s$ may be written as $d({r_m}^{k_1} c_1)({r_m}^{k_2} c_2)\ldots({r_m}^{k_t} d')$ where $t \ge s$, each $c_i \in S_{m-1}$ and $d,d' \in \in S_{m-1} \cup \{1\}$. Then $({r_m}^{k_1} c_1)({r_m}^{k_2} c_2)\ldots({r_m}^{k_{t-1}} c_{t-1}) \neq 0$ which show that S′ contains nonzero product of s-1 elements in T. Since s is arbitraly, S′ is not nilpotent and hence we have Claim1.

Thus the finitely generated multiplicative set S′ is not nilpotent. Since $S' \subseteq S_m \subseteq S$, the elements in S′ are nilpotent. By repeating the same procedure, we may determine a decreasing chain of finitely generated multiplicative sets $S \supseteq S' \supseteq S'' \supseteq \ldots$, each not nilpotent.

Let $I = \mathrm{Ann}(S), I' = \mathrm{Ann}(S'),\ldots$ be the Annihilators of the subsets S,S′,

$S'',\ldots$ of the skewring R, hence $I\subseteq I'\subseteq I''\ldots$.

Claim2. This sequence is strictly increasing.

By the construction, it is enough to show that $I\subset I'$. Since every element of S' has the left factor r_m and $r_m{}^k = 0$, $r_m{}^{k-1}S' = \{0\}$, that is $r_m{}^{k-1}\in I'$. If $r_m{}^{k-1}\notin I$, then we are done. If $r_m{}^{k-1}\in I$, then $r_m{}^{k-1}r_i = 0$ for every $i = 1,\ldots,m$. We shall show that $k>2$. Suppose that $k = 2$. Then $r_m r_i = 0$ for every $i = 1,\ldots,m$. Let $c_1,\ldots,c_{l+1}\in S_m$. Consider $c_1\ldots c_{l+1}$.

Case1. For every i, $c_i\in S_{m-1}$. Then $c_1\ldots c_{l+1} = 0$, by **(ii)**.

Case2. There exist $c_i\in S_m\setminus S_{m-1}$ and $c_j\in S_{m-1}$. Then r_m is some factor of c_i.

Subcase2.1. $i\leq l$. Then there exists $j\in\{1,\ldots,m\}$ such that $r_m r_j$ is some factor of $c_1\ldots c_l$. Since $r_m r_j = 0$, $c_1\ldots c_{l+1} = 0$.

Subcase2.2. $i = l+1$. Then $c_1,\ldots,c_l\in S_{m-1}$. Then $c_1\ldots c_l = 0$, by **(ii)**, so that $c_1\ldots c_{l+1} = 0$.

Case3. For every i, $c_i\in S_m\setminus S_{m-1}$. Then $c_i = r_m$ for every I and so $c_1\ldots c_{l+1} = r_m{}^{l+1} = 0$, since $k = 2$.

From 3 cases, we get that S_m is nilpotent which is a contradiction. Then $k>2$. Let C be a product of $r_1,\ldots,r_m$ containing at least some factor $r_i(1\leq i\leq m)$ Then $(r_m{}^{k-2})(r_m c) = r_m{}^{k-1}c = 0$. By definition of S', $r_m{}^{k-2}\in I'$. Since $r_m{}^{k-2}r_m = r_m{}^{k-1}\neq 0$, we get that $r_m{}^{k-2}\notin I$. Therefore $I\subset I'$. Similarly, $I\subset I'\subset I''\ldots$ which contradicts the fact that R satisfies the ACC for left normal ideals. Hence S is nilpotent.

Step2. We shall show that J is nilpotent.

Since S is nilpotent, there exists a $t\in\mathbf{Z}^+$ such that $c_1\ldots c_t = 0$ for any $c_i\in S$. We shall show that $J^t = \{0\}$.

Claim3. For any $m\in\mathbf{Z}^+$, $J^m\subseteq\langle\{\sum_{j=1}^{k} x_i r_{i_1}\cdots r_{i_m} / x_i\in R\cup\mathbf{Z}, i = 1,\ldots,k\}\rangle_n$.

By Proposition 1.23, $J^m = \langle\{x_1\ldots x_m / x_i\in J, i = 1,\ldots,m\}\rangle_n$. Let $x,y\in J$. Then there exist $m,n\in\mathbf{Z}^+$, $a_i,b_j\in R$, $x_i,y_j\in R\cup\mathbf{Z}$, where $i,j = 1,\ldots,n$ are such that

$x=\sum_{i=1}^{n}(a_i+x_ir_i-a_i)$ and $y=\sum_{j=1}^{n}(b_j+y_jr_j-b_j)$. By Remark 1.5 (2), $xy=\sum_{i=1}^{n}\sum_{j=1}^{n}(x_ir_iy_jr_j)$.

Since J is an ideal and $r_i\in J$ for all $i=1,\dots,n$, $r_iy_j\in J$ for all $i,j=1,\dots,n$. Then there exist $a_l^{(i,j)}\in R$, $y_l^{(i,j)}\in R\cup Z$ where $l=1,\dots,n$ such that $r_iy_j=\sum_{l=1}^{n}(a_l^{(i,j)}+y_l^{(i,j)}r_l-a_l^{(i,j)})$. By Remark 1.5 (2) $xy=\sum_{i,j,l=1}^{n}x_iy_l^{(i,j)}r_lr_j$. Thus the basic step is true.

Regarding the induction step, we can prove it similarly. Then we have Claim3.

Hence $J^t=\{0\}$ which implies J is nilpotent. By Remark 4.5 (2), (3) and Remark 4.7, UR(R) = NR(R) is the largest nilpoent normal ideal and $NR(R/NR(R))=0$. #

Definition 4.20. *Let R be a skewring. The intersection of all the maximal normal ideal of R is called the **Jacobson radical** of R and it is denoted by JR(R).*

Hence JR(R) = R exactly when R has no maximal normal ideal. By the above definition, JR(R) is a normal ideal of R.

Remark 4.21. *For any skewring R, $JR(R)=\cap Ker(f)$ where f is any epimorphism of R to some simple skewring.*

Proof. Consider 2 cases.

Case1. R has a maximal normal ideal.

Let $x\in JR(R)$. So $x\in M$ for every maximal normal ideal M of R. Let $g:R\to S$ be an epimorphism of R to a nonzero simple skewring S. To show that $x\in Ker(g)$. By First Isomorphism Theorem, $R/Ker(g)\cong S$. Then there exists a bijection between the set of all normal ideals of S and the set of all normal ideals of R containing Ker(g). Since S is simple, Ker(g) is a maximal normal ideal of R. Since $S\neq\{0\}$ and g is surjective, $Ker(g)\neq R$. Therefore $x\in Ker(g)$.

Since g and S are arbitraly, $x \in Ker(f)$ for every epimorphism of R to some simple skewring. Hence $JR(R) \subseteq \cap Ker(f)$.

Conversely, let $x \in Ker(f)$ where f is an epimorphism of R to some simple skewring S. Suppose I is a maximal normal ideal of R. To show that $x \in I$. Let $S = R/I$ and define $g: R \to R/I$ to be the canonical epimorphism. Then $Ker(g) = I$. By Theorem 2.14, there exist a bijection between the set of all normal ideals of R containing I and the set of all normal ideals of R/I. Since I is maximal, R/I is simple. Since $Ker(g) = I$, $x \in I$. Hence the converse is true.

Case2. R has no maximal normal ideal.

Then $JR(R) = R$. Suppose there exists an epimorphism f of R to some simple skewring. Similarly Case1, Ker(f) is a maximal normal ideal of R which is a contradiction. Thus the assumption is not true. Then there is no such f. That is, $\cap Ker(f) = R$. therefore $JR(R) = \cap Ker(f)$. #

Remark 4.22. *If f is a homomorphism of a skewring R to some skewring R', then $f[JR(R)] \subseteq JR(R')$.*

Proof. Let g be an epimorphism of R′ to some simple skewring S. Then $g \circ f$ is an epimorphism of R to S. By Remark 4.21, $JR(R) \subseteq Ker(g \circ f)$. That is $g \circ f[JR(R)] = \{0\}$. Hence $f[JR(R)] \subseteq Ker(g)$ where g is an epimorphism of R′ to some simple skewring S′. by Remark 4.21, $f[JR(R)] \subseteq JR(R')$. #

Remark 4.23. *If I is a normal ideal of a skewring R, then $JR(I) \subseteq JR(R)$ and $(JR(R)+I)/I \subseteq JR(R/I)$. Furthermore, if $I \subseteq JR(R)$ then $JR(R)/I = JR(R/I)$.*

Proof. Let $i: I \to R$ be the inclusion map. By Remark 4.22, $JR(I) = i[JR(I)] \subseteq JR(R)$. Let $\pi: R \to R/I$ be the canonical epimorphism. By Remark 4.22,

$\pi[JR(R)] \subseteq JR(R/I)$.

Claim that $(JR(R)+I)/I \subseteq \pi[JR(R)]$.

Let $x \in (JR(R)+I)/I$. Then there exist $a \in JR(R)$, $b \in I$ such that $x = (a+b)+I = a+I \in JR(R)/I = \pi[JR(R)]$. Then we have the claim and so $(JR(R)+I)/I \subseteq JR(R/I)$.

Suppose $I \subseteq JR(R)$. By Corollary 2.16, $JR(R)/I = \pi[JR(R)] = \pi[\cap\{$maximal normal ideal M of R containing $I\}] = \cap(M/I) = JR(R/I)$.#

Remark 4.24. *For any skewring R, $JR(R/JR(R)) = 0$ and JR(R) is the smallest normal ideal of R with this property.*

Proof. Since $JR(R) \subseteq JR(R)$, by Remark 4.23, $0 = JR(R)/JR(R) = JR(R/JR(R))$. Let I be the normal ideal of R such that $JR(R/I) = 0$. By Remark 4.23, $(JR(R)+I)/I \subseteq JR(R/I) = 0$. Then $JR(R)+I \subseteq I$, that is $JR(R) \subseteq I$. #

Lemma 4.25. *If R is a finitely generated skewring, if I is a normal ideal of R and $JR(R)+I = R$, then $R = I$.*

Proof. By Remark 4.23, $(JR(R)+I)/I \subseteq JR(R/I)$. Since $JR(R)+I = R$, $R/I \subseteq JR(R/I)$ which implies that $JR(R/I) = R/I$. Therefore R/I has no maximal proper normal ideal. Suppose $R \neq I$. Since R is a finitely generated, by Remark 1.26, I is contained in a maximal normal ideal of R which is a contradiction. Hence $R = I$. #

The elements in the skewring R will be characterized as ***nongenerators*** *in the following sense : The element* $x \in R$ *is a nongenerator of R when the following property holds : If* $B \subseteq R$ *and* $B \cup \{x\}$ *is a set of generators of R as a normal ideal of R, then B is also a set of generators of R as a normal ideal.*

Theorem 4.26. *For any finitely generated skewring R (finitely generated as normal ideal), JR(R) is the set of nongenerators of R.*

Proof. Let $x \in JR(R)$ and $B \subseteq R$ such that $B \cup \{x\}$ is a set of generators of R and denote by I, the normal ideal of R which is generated by B. Then $R = \langle x \rangle_n + I \subseteq JR(R) + I \subseteq R$. Therefore $JR(R)+I = R$. By Lemma 4.25, $R = I$. Thus B is a set of generators of R. Hence x is a nongenerator of R.

Conversely, let x be a nongenerator of R. If $x \notin JR(R)$, then there exists a maximal normal ideal I of R such that $x \notin I$. Thus $I + \langle x \rangle_n = R$, hence $I \cup \{x\}$ is a system of generators (as normal ideal) of R. Since x is a nongenerator, I generates R and hence $I = R$ which contradicts the fact that I is a maximal normal ideal of R. This shows that $x \in JR(R)$ and the proof is finished. #

A subset M of a skewring R is an ***m-system*** *(generalized multiplication system) if* $c, d \in M$ *implies that there exists an* $x \in R$ *such that* $cxd \in M$*. Then the (Mc Coy)* ***radical of normal ideal*** *I of a skewring R is the set of all elements* $r \in R$ *such that every m-system which contains r contains an element of I, and is denoted by M(I). The* ***radical*** *M(R) of a skewring R is the radical of the zero normal ideal.*

We recall that a prime normal ideal P in a skew ring R is said to be a ***minimal prime normal ideal belonging to a normal ideal I*** *of R if* $I \subseteq P$ *and there does not exist a prime normal ideal Q in R such that* $I \subseteq Q \subset P$*.*

We now connect this concept with that of an m-system.

Remark 4.27. *Every ideal of a skewring R is an m-system.*

Lemma 4.28. *Let I be a normal ideal in a skewring R and M an m-system which does not intersect I. Then M is contained in an m-system M′ which is maximal in the class of m-systems which do not intersect I.*

Proof. Let $L = \{S \subseteq R / S$ is an m-system such that $M \subseteq S$ and $I \cap S = \varnothing.\}$. Since $M \in L$, L is not empty. Let C be a nonempty chain in L. Clearly, $\cup C$ is an m-system and hence $\cup C$ is an upper bound of C in L. By Zorn's Lemma, L has a maximal element, say M′. #

Lemma 4.29. *Let M be an m-system in a skewring R and I a normal ideal of R which does not intersect M. Then I is contained in a normal ideal P* which is maximal in the class of normal ideals which do not intersect M. The normal ideal P* is necessarily a prime normal ideal.*

Proof. Let $L = \{S \subseteq R / S$ is an m-system such that $I \subseteq S$ and $M \cap S = \varnothing.\}$. Since $I \in L$, L is not empty. Let C be a nonempty chain in L. Clearly, $\cup C$ is an upper bound of C in L. By Zorn's Lemma, L has a maximal element, say P^*.

Next, we shall show that P^* is a prime normal ideal. Let A,B be normal ideals in R such that $AB \subseteq P^*$ and suppose $A \not\subset P^*$, $B \not\subset P^*$. Then the maximal property of P^* implies that P^*+A contains an element $m_1 \in M$ and P^*+B contains an element $m_2 \in M$ such that for some $a \in A$, $b \in B$, $p_1, p_2 \in P^*$, $m_1 = p_1+a$ and $m_2 = p_2+b$. Since M is an m-system, there exists an $x \in R$ such that $m_1 x m_2 \in M$. Moreover, $m_1 x m_2 \notin P^*$. Then $(p_1+a)x(p_2+b) = p_1x(p_2+b)+ax(p_2+b) = p_1x(p_2+b)+axp_2+axb \notin P^*$. Since P^* is a normal ideal, $axb \notin P^*$. However, $axb \in AB \subseteq P^*$ which is a contradiction. Hence P^* is prime. #

Lemma 4.30. *The complement of a prime normal ideal P in a skewring R is an m-system.*

Proof. Let M be the complement of P. Let $c,d \in M$. Then $c,d \notin P$. Suppose that for every $x \in R$, $cxd \notin M$ which implies that $cxd \in P$. ....(*)

Let $x \in \langle c \rangle_n$, $y \in \langle d \rangle_n$. Then there exist $m,n \in \mathbb{Z}^+$, $x_i, x'_j \in R$, $r_i, r'_j, s_i, s'_j \in R \cup \mathbb{Z}$, where $i = 1,\dots,m$ and $j = 1,\dots,n$ such that $x = \sum_{i=1}^{m} (x_i + r_i c s_i - x_i)$ and $y = \sum_{j=1}^{n} (x'_j + r'_j d s'_j - x'_j)$. By Remark 1.5 (2), $xy = \sum_{i=1}^{m} \sum_{j=1}^{n} r_i c s_i r'_j d s'_j$. Since P is a normal ideal and by (*), $xy \in P$. Then $\langle c \rangle_n \langle d \rangle_n \subseteq P$. Since P is prime, $\langle c \rangle_n \subseteq P$ or $\langle d \rangle_n \subseteq P$ which contradicts $c,d \notin P$. Then there exists an $x \in R$ such that $cxd \in M$. Hence M is an m-system. #

Lemma 4.31. *Let P be a subset of a skewring R. Then the complement of P is a maximal in the class of m-systems which do not intersect a normal ideal I if and only if P is a minimal prime normal ideal belonging to I.*

Proof. Let P be a subset of R such that $M = P^c$ is a maximal m-system which does not intersect I. If P^* is the prime normal ideal whose existence is shown in Lemma 4.29, by Lemma 4.30, the complement of P^* is an m-system which contains M and which does not intersect I. The maximality of M implies that the complement of P^* is contained in M and so it is equal to M. Then $P = P^*$. Thus P is a prime normal ideal containing I. Clearly, there does not exist a prime normal ideal Q such that $I \subseteq Q \subset P$ since this would imply that the complement of Q is an m-system which does not intersect I and which properly contains M. Hence P is a minimal prime normal ideal belonging to I.

Conversely, if P is a minimal prime normal ideal belonging to I, by Lemma 4.30, the complement M of R is an m-system which does not intersect I, and Lemma 4.28 shows the existence of a maximal m-system M′ which contains M and does not intersect I. Let P′ be the complement of M′. From above, P′ is a minimal prime normal ideal belonging to I. Since $M \subseteq M'$, it follows that $P' \subseteq P$ and thus $I \subseteq P' \subseteq P$. By the minimality of P, $P = P'$ and $M = M'$. Thus the complement of P is a maximal m-system which does not intersect I. #

Theorem 4.32. *The Mc Coy radical of a normal ideal I in a skewring R is the intersection of all prime normal ideals belonging to I.*

Proof. Claim that M(I) is contained in the same prime normal ideals as I. Let P be the prime normal ideal in R such that $I \subset P$. Let $r \in M(I)$. Suppose that $r \notin P$. Then $r \in P^c$ which is an m-system and it does not intersect I. By definition of M(I), there exists $x \in I$ such that $x \in P^c$ which is a contradiction. Thus we have the claim.

Then M(I) is contained in the intersection of all the minimal prime normal ideals belonging to I. Now let $a \in R \setminus M(I)$. There exists an m-system M which contains a but does not intersect I. By Lemma 4.28, M is contained in a maximal m-system M′ which does not intersect I. By Lemma 4.31, the complement of M′ is a minimal prime normal ideal belonging to I and clearly the complement of M′ does not contain a. Hence a cannot be in the intersection of all the minimal prime normal ideal belonging to I and the theorem is proved.#

Corollary 4.33. *The Mc Coy radical of a skewring R is the intersection of all minimal prime normal ideals of R.*

Theorem 4.34. *If M is the Mc Coy radical of a skewring R, then R/M has zero Mc Coy radical.*

Proof. Let $\alpha = a+M$ be an element of the radical of R/M. By Corollary

4.33, α is contained in all prime normal ideals of R/M. If $\alpha \neq 0$, $a \notin M$ and hence a is not contained in some prime normal ideal P belonging to M. By Theorem 4.32, $M \subsetneq P$ and clearly, P/M is prime. Furthermore, P/M does not contain α since a is not in P. This contradiction shows that we must have $\alpha = 0$. Hence R/M has zero Mc Coy radical. #

Theorem 4.35. *If I is a normal ideal of a skewring R, then the Mc Coy radical of the skewring I is $I \cap M(R)$.*

Proof. Let M(I) be the radical of a skewring I. Clearly, $M(I) \subseteq I$. Let $r \in M(I) = M(\{0\})$ and M be an m-system in R such that $r \in M$. Then $r \in I \cap M$ which is an m-system in I. Since $r \in M(\{0\}) = M(I)$, $0 \in I \cap M$ which implies that $r \in M(R)$. Therefore $M(I) \subseteq I \cap M(R)$. On the other hand, if $b \in I \cap M(R)$, then every m-systems in R containing b contains 0. In particular, every m-systems in I containing b which is an m-system in R contains 0. Thus $b \in M(I)$ and $I \cap M(R) \subseteq M(I)$. #

Definition 4.36. *A skewring R is a* ***p-skewring*** *if and only if {0} is a prime normal ideal in R.*

Theorem 4.37. *Let R be a skewring. Then P is a prime normal ideal of R if and only if R/P is a p-skewring.*

Proof. Clearly, if P is a prime normal ideal then R/P is a p-skewring. Conversely, suppose that R/P is a p-skewring. Let A,B be normal ideals in R such that $AB \subseteq P$. Then $(A/P)(B/P) = AB/P = 0$ which is prime in R/P. Then $A/P = 0$ or $B/P = 0$, that is $A \subseteq P$ or $B \subseteq P$. Hence P is prime. #

Theorem 4.38. *A skewring R is isomorphic to the subdirect sum of p-skewrings if M(R) = {0}.*

Proof. Suppose that $M(R) = \{0\}$. Let $\{P_\alpha / \alpha \in A\}$ be a family of prime normal ideals of R. By Corollary 4.33, $\bigcap_{\alpha \in A} P_\alpha = \{0\}$. By Corollary 3.37, R is a subdirect sum of $\{R/P_\alpha \,/\, \alpha \in A\}$ which is a set of p-skewrings. #