CHAPTER III

SUM AND PRODUCTS

In this chapter, we shall give some definitions and theorems of sums and products of skewrings. For example, direct sum, subdirct sum, semi-direct sum, subdirect product, subdirectly irreducible and subdirectly reducible. Moreover, we shall generalize the Krull-Schmidt Theorem of group theory to skewrings.

Definition 3.1. *Let R be a skewring and $\{R_\alpha / \alpha \in I\}$ be a family of normal ideals of R. Then R is called a* **direct sum** *of $\{R_\alpha / \alpha \in I\}$ which is denoted by $R = \bigoplus_{\alpha \in I} R_\alpha$ if and only if*

(1) for every $x \in R$, there exists $x_{\alpha_i} \in R_{\alpha_i}$ where $i = 1,\ldots,n$ such that $x = x_{\alpha_1} + \ldots + x_{\alpha_n}$ and

(2) for all $\alpha, \beta \in I$, if $\alpha \neq \beta$ implies $R_\alpha \cap (\sum_{\beta \neq \alpha} R_\beta) = \{0\}$.

Remark 3.2. *Let a skewring R be a direct sum of $R_1,\ldots,R_n$ which are normal ideals of R. Then for all $x,y \in R$,*

(1) $x+y = x_1+y_1+\ldots+x_n+y_n$ and

(2) $xy = x_1y_1+\ldots+x_ny_n$.

where $x = x_1+\ldots+x_n$ and $y = y_1+\ldots+y_n$. for some $x_i,y_i \in R_i$ such that $i \in \{1,\ldots,n\}$.

Proof. It is well-known that (1) is true. We will prove (2) by math induction on n.

Let $n = 2$. Let $R = R_1 \oplus R_2$. Let $x,y \in R$. Then there exist $x_1,y_1 \in R_1$ and $x_2,y_2 \in R_2$ such that $x = x_1+x_2$ and $y = y_1+y_2$. Thus $xy = (x_1+x_2)(y_1+y_2) = x_1y_1+x_2y_1+x_1y_2+x_2y_2$. Since R_1,R_2 are normal ideals, $x_2y_1+x_1y_2 \in R_1 \cap R_2 = \{0\}$ which implies that $xy = x_1y_1+x_2y_2$.

Let $k \geq 2$. Assume that if $R = R_1 \oplus \ldots \oplus R_k$, then (2) is true. Suppose that $R = R_1 \oplus \ldots \oplus R_k \oplus R_{k+1}$. Let $x,y \in R$. Then there exist $x_i, y_i \in R_i$ where $i \in \{1,\ldots,k+1\}$ such that $x = x_1 + \ldots + x_{k+1}$ and $y = y_1 + \ldots + y_{k+1}$. Then

$$\begin{aligned} xy &= (x_1+\ldots+x_{k+1})(y_1+\ldots+y_{k+1}) \\ &= ((x_1+\ldots+x_k)+x_{k+1})+((y_1+\ldots+y_k)+y_{k+1}) \\ &= (x_1+\ldots+x_k)(y_1+\ldots+y_k)+x_{k+1}y_{k+1}, \text{ by basic step} \\ &= (x_1y_1+\ldots+x_ky_k)+x_{k+1}y_{k+1}, \text{ by induction hypothesis.} \end{aligned}$$

By math induction we have (2). #

Remark 3.3. *Let R be a skewring which is a direct sum of normal ideals $R_1,\ldots,R_n$. Then for all $i,j \in \{1,\ldots,n\}$ such that $i \neq j$, if $a \in R_i, b \in R_j$ implies $a+b = b+a$.*

Definition 3.4. *A skewring R is said to be* ***decomposable*** *if and only if $R = H \oplus K$ where H,K are nontrivial normal ideals of R.*

A skewring $R \neq \{0\}$ is said to be ***Indecomposable*** *if and only if $R = H \oplus K$ where H,K are normal ideals of R implies $H = R$ or $K = R$.*

Remark 3.5. *Let H,K be normal ideals of a skewring R such that $R = H \oplus K$. If N is a normal ideal of H, then N is a normal ideal of R.*

Proof. Suppose N is a normal ideal of H. It is well-known that N is a normal subgroup of $(R,+)$. Let $x \in N$, $r \in R$. Then there exist $h \in H$, $k \in K$ such that $r = h+k$. Then $rx = (h+k)x = hx+kx$. Since H,K are normal ideals of R, $rx-hx = kx \in H \cap K$. Since $R = H \oplus K$, $H \cap K = \{0\}$ and $rx = hx$. Since N is a normal ideal of H, $hx \in N$ and so $rx \in N$. Similarly, $xr \in N$. Hence N is a normal ideal of R. #

Definition 3.6. *Let R be a skewring.*

A decreasing sequence of left[right, two-sided] normal ideals of R, $R = R_0 \geq R_1 \geq \ldots$ *is called a* ***descending chain*** *of left[right, two-sided]normal ideal in R.*

R satisfies the ***descending chain condition (DCC)*** *for left[right, two-sided] normal ideals if and only if for any decreasing chain of left[right, two-sided] normal ideals of R,* $R = R_0 \geq R_1 \geq \ldots$, *there exists a positive integer N such that* $R_N = R_{N+1} = \ldots$.

An increasing sequence of left[right, two-sided] normal ideals of R, $R_0 \leq R_1 \leq \ldots$ *is called an* ***ascending chain*** *of left[right, two-sided] normal ideal in R.*

R satisfies the ***ascending chain condition (ACC)*** *for left[right, two-sided] normal ideals if and only if for any an* ***ascending chain*** *of left [right, two-sided] normal ideal in R,* $R_0 \leq R_1 \leq \ldots$, *there exists a positive integer N such that* $R_N = R_{N+1} = \ldots$.

Remark 3.7. *Every finite skewring satisfies the DCC for left[right, two-sided] normal ideals.*

Proposition 3.8. *Let R be a skewring. Then R satisfies the ACC for left[right, two-sided] normal ideals if and only if every nonempty family of left[right, two-sided] normal ideals has a maximal element.*

Propositin 3.9. *Let R be a skewring. Then R satisfies the ACC for left[right, two-sided] normal ideals if and only if every left[right, two-sided] normal ideals is finitely generated.*

Remark 3.10. *Let H,K be normal ideals of a skewring R such that* $R = H \oplus K$. *If R satisfies the ACC[DCC] for normal ideals, then so do H and K.*

Proof. We shall show that if R satisfies the ACC for normal ideals, then so do H and K. Suppose R satisfies the ACC on normal ideals. Let $H_0 \subseteq H_1 \subseteq \ldots$ be an increasing sequence of subskewrings of H such that for each i, H_i is a normal ideal in H. By Remark 3.5, H_i is a normal ideal in R for every i. Then this sequence is an ascending chain in R. Since R satisfies the ACC for normal ideals, there exists $n \in \mathbf{Z}^+$ such that $H_n = H_{n+1} = \ldots$. Hence H is satisfies the ACC for normal ideal. For K is similarly.

If R satisfies the DCC for normal ideals, we can prove similarly.#

Lemma 3.11. *For any skewring $R \neq \{0\}$ that satisfies the DCC for normal ideals has an indecomposable nonzero subskewring and $R = P \oplus K$ for some indecomposable normal ideal P of R and normal ideal K of R.*

Proof. If R is indecomposable, then we are done. Otherwise, there exist R_1, R'_1 which are nontrivial normal ideals of R such that $R = R_1 \oplus R'_1$.

If R_1 is indecomposable, then we are done. Otherwise, there exist R_2, R'_2 which are nontrivial normal ideals of R_1 such that $R_1 = R_2 \oplus R'_2$. By Remark 3.5, R_2, R'_2 are normal ideals of R. Then $R = R_2 \oplus R'_2 \oplus R'_1$ and $R \geq R_1 \geq R_2$. By Corollary 2.9 (4), $R'_2 \oplus R'_1$ is a normal ideal of R. Continue in this way. Then we have that $R \geq R_1 \geq R_2 \geq \ldots$ such that for each i, R_i is a normal ideal in R and $R = \ldots \oplus R'_n \oplus R'_{n-1} \oplus \ldots \oplus R'_1$. Since R satisfies the DCC for normal ideals, there exists $m \in \mathbf{Z}^+$ such that $R_m = R_{m+1} = \ldots$. Then $R = R_m \oplus R'_m \oplus R'_{m-1} \oplus \ldots \oplus R'_1$ such that R_m is indecomposable. By Remark 3.5, R_m, R'_i are normal ideals of R for every $i \in \{1, \ldots, m\}$. By Corollary 2.9 (4), $R'_m \oplus R'_{m-1} \oplus \ldots \oplus R'_1$ is a normal ideal of R. Hence the proof is finished. #

Theorem 3.12. *Any nontrivial skewring R that satisfies the DCC for normal ideals can be expressed as a direct sum of a finite number of indecomposable normal ideals of R.*

Proof. If R is indecomposable, then we are done. Otherwise, by Lemma 3.11, there exist an indecomposable nonzero normal ideal R_1 of R and a proper normal ideal R'_1 of R such that $R = R_1 \oplus R'_1$.

If R'_1 is indecomposable, then we are done. Otherwise, by Lemma 3.11, there exist an indecomposable nonzero normal ideal R_2 of R'_1 and proper normal ideal R'_2 of R'_1 such that $R'_1 = R_2 \oplus R'_2$. By Remark 3.5, R_2 and R'_2 are normal ideals in R and we have $R = R_1 \oplus R_2 \oplus R'_2$ such that $R \supset R'_1 \supset R'_2$. Continue in this way.

If there exists $n \in \mathbb{Z}^+ \setminus \{1\}$ such that R'_n is indecomposable in R'_{n-1}, then $R = R_1 \oplus \ldots \oplus R_n \oplus R'_n$ such that R_i, R'_n are normal ideals in R, R_{i+1} is indecomposable in R'_i for every $i \in \{1,\ldots,n-1\}$. By Remark 3.5, R_i, R'_n are indecomposable in R. Otherwise, we have $R \supset R'_1 \supset R'_2 \supset \ldots$ which is a contradiction since R satisfies the DCC for normal ideals. #

Definition 3.13. *Let R be a skewring and let f be an endomorphism on R. Then f is a* ***normal ideal endomorphism*** *if and only if for all $x,y \in R$, $f(x+y-x) = x+f(y)-x$, $xf(y) = f(xy)$ and $f(y)x = f(yx)$.*

Example 3.14. *The zero fuction and the identity function on a skewring R are normal ideal endomorphisms.*

Lemma 3.15. *Let f and g be normal ideal endomorphisms of a skewring R. Then $f \circ g$ is a normal ideal endomorphism.*

Lemma 3.16. *Let a skewring $R = R_1 \oplus \ldots \oplus R_n$ where R_i is a normal ideal of R for every $i \in \{1,..,n\}$. For each $i \in \{1,\ldots,n\}$, let $\pi_i: R \to R_i$ be a projection map and define $\varphi_i: R \to R$ by $\varphi_i(x) = \pi_i(x)$ for every $x \in R$. Then the sum $\varphi_{i_1} + \ldots + \varphi_{i_k}$ of any distinct $\varphi_{i_1}, \ldots, \varphi_{i_k}$ where $i_1, \ldots, i_k \in \{1,\ldots,n\}$, is a normal ideal endomorphism on R.*

Proof. First, we shall show that φ_i is a normal ideal endomorphism of R for every $i\in\{1,\ldots,n\}$. It is well-known that φ_i is a normal endomorphism in (R,+) and clearly, φ_i is an endomorphism on R. Let $x,y\in R$. Then there exist $x_i,y_i\in R$ where $i\in\{1,\ldots,n\}$ be such that $x=x_1+\ldots+x_n$ and $y=y_1+\ldots+y_n$. Let $i\in\{1,\ldots,n\}$. Then $x\varphi_i(y)=(x_1+\ldots+x_n)\pi_i(y)=(x_1+\ldots+x_n)(0+\ldots+0+y_i+0+\ldots+0)=x_iy_i=\pi_i(xy)=\varphi_i(xy)$. Similarly, $(\varphi_i(x))y=\varphi_i(xy)$. Hence φ_i is a normal ideal endomorphism in R.

Next, we shall show that the sum $\varphi_{i_1}+\ldots+\varphi_{i_k}$ of any distinct $\varphi_{i_1},\ldots,\varphi_{i_k}$ where $i_1,\ldots,i_k\in\{1,\ldots,n\}$, is a normal ideal endomorphism on R. It is well-known that $\varphi_{i_1}+\ldots+\varphi_{i_k}$ is a normal endomorphism in (R,+). Consider,

$(0)(xy)=\varphi_{i_1}(xy)+\ldots+\varphi_{i_k}(xy)=\varphi_{i_1}(x)\varphi_{i_1}(y)+\ldots+\varphi_{i_k}(x)\varphi_{i_k}(y)=$

$x_{i_1}y_{i_1}+\ldots+x_{i_k}y_{i_k}=(x_{i_1}+\ldots+x_{i_k})(y_{i_1}+\ldots+y_{i_k})$

$=(\pi_{i_1}(x)+\ldots+\pi_{i_1}(x))(\pi_{i_1}(y)+\ldots+\pi_{i_k}(y))$

$=(\varphi_{i_1}(x)+\ldots+\varphi_{i_1}(x))(\varphi_{i_1}(y)+\ldots+\varphi_{i_k}(y))=(\varphi_{i_1}+\ldots+\varphi_{i_k})(x)(\varphi_{i_1}+\ldots+\varphi_{i_k})(y)$

and $x(\varphi_{i_1}+\ldots+\varphi_{i_k})(y)=x(\varphi_{i_1}(y)+\ldots+\varphi_{i_k}(y))=x\varphi_{i_1}(y)+\ldots+x\varphi_{i_k}(y)=$ $\varphi_{i_1}(xy)+\ldots+\varphi_{i_k}(xy)=(\varphi_{i_1}+\ldots+\varphi_{i_k})(xy)$. Similarly, $((\varphi_{i_1}+\ldots+\varphi_{i_k})(x))(y)=$ $(\varphi_{i_1}+\ldots+\varphi_{i_k})(xy)$. Hence $\varphi_{i_1}+\ldots+\varphi_{i_k}$ is a normal ideal endomorphism. #

Lemma 3.17. *Let R be a skewring that satisfies the ACC[DCC] for normal ideals and f is an [normal ideal] endomorphism of R. Then f is an automorphism if and only if f is an epimorphism[monomorphism].*

Proof. Step1. Assume that R satisfies the ACC for normal ideals and f is an endomorphism. We shall show that f is an automorphism if and only if f is an epimorphism.

Suppose f is an epimorphism. It is well-known that that for every $n\in\mathbf{Z}^+$, $Ker(f^n)\subseteq Ker(f^{n+1})$ where $f^n=f\circ f\circ\ldots\circ f$ (n terms). By Remark 1.34, $\{0\}\leq Ker(f)\leq Ker(f^2)\leq\ldots$ is an ascending chain in R. By assumption, there exists $n\in\mathbf{Z}^+$

such that $Ker(f^n) = Ker(f^{n+1})$. Since f is an epimorphism, f^n is an epimorphism.

To show that f is a monomorphism. Let $x \in Ker(f)$. Since f^n is an epimorphism, there exists $y \in R$ such that $f^n(y) = x$, that is $0 = f(x) = f^{n+1}(y)$. Thus $y \in Ker(f^{n+1}) = Ker(f^n)$ which implies that $x = f^n(y) = 0$. Thus $Ker(f) = \{0\}$. By Remark 1.33 (1), f is a monomorphism and hence f is an automorphism.

Step2. Assume that R satisfies the DCC for normal ideals and f is a normal ideal endomorphism. We shall show that f is an automorpism if and only if f is an monomorphism.

Suppose that f is a monomorphism. Let $n \in \mathbf{Z}^+$. By Lemma 3.15, f^n is a normal ideal endomorphism of R. By definition of normal ideal endomorphism, $Im(f^n)$ is a normal ideal of R. Thus we have $R \geq Im(f) \geq Im(f^2) \geq \ldots$ is a descending chain in R. By assumption, there exists $n \in \mathbf{Z}^+$ such that $Im(f^n) = Im(f^{n+1}) = \ldots$.

To show that f is an epimorphism. Let $x \in R$. Then $f^n(x) \in Im(f^n) = Im(f^{n+1})$ and there exists $y \in R$ such that $f^{n+1}(y) = f^n(x)$. Since f is a monomorphism, so is f^n and $f^n(x) = f^{n+1}(y) = f^n(f(y))$ implies $x = f(y)$. Therefore f is an epimorphism and hence f is an automorphism. #

The following Lemma is generalized from Fitting's Lemma.

Lemma 3.18. *If R is a skewring that satisfies both the ACC and DCC for normal ideals and f is a normal ideal endomorphism, then there exists an* $n \in$ Z^+ *such that* $R = Ker(f^n) \oplus Im(f^n)$.

Proof. By the proof in Lemma 3.17, we have $R \geq Im(f) \geq Im(f^2) \geq \ldots$ and $\{0\} \leq Ker(f) \leq Ker(f^2) \leq \ldots$ are descending and ascending chains respectively. By assumption, there exists $n \in \mathbf{Z}^+$ such that $Im(f^k) = Im(f^n)$ and $Ker(f^k) = Ker(f^n)$ for every $k \geq n$.

Let $a \in Ker(f^n) \cap Im(f^n)$. Then there exists $b \in R$ such that $f^n(b) = a$ and

$f^{2n}(b) = f^n(f^n(b)) = f^n(a) = 0$. Consequently, $b \in Ker(f^{2n}) = Ker(f^n)$, so that $a = f^n(b) = 0$. Hence $Ker(f^n) \cap Im(f^n) = \{0\}$.

Let $c \in R$. Then $f^n(c) \in Im(f^n) = Im(f^{2n})$. There exists a $d \in R$ such that $f^{2n}(d) = f^n(c)$. Therefore $f^n(c+f^n(-d)) = f^n(c)+f^{2n}(-d) = f^n(c)-f^n(c) = 0$ and hence $c+f^n(-d) \in Ker(f^n)$. Since $c = (c+f^n(-d))+f^n(d)$, we conclude that $R = Ker(f^n)+Im(f^n)$. Hence $R = Ker(f^n) \oplus Im(f^n)$. #

Definition 3.19. *An endomorphism f of a skewring R is said to be* ***nilpotent*** *if there exists a positive integer n such that $f^n(x) = 0$ for every $x \in R$.*

Lemma 3.20. *If $R \neq \{0\}$ is an indecomposable skewring that satisfies both the ACC and DCC for normal ideals and f is a normal ideal endomorphism of R, then either f is a nilpotent endomorphism or f is an automorphism.*

Proof. By Lemma 3.18, there exists $n \in \mathbf{Z}^+$ such that $R = Ker(f^n) \oplus Im(f^n)$. Since R is indecomposable, $Ker(f^n) = \{0\}$ or $Im(f^n) = \{0\}$. If $Im(f^n) = \{0\}$, then $f^n(x) = 0$ for every $x \in R$, so that f is nilpotent. If $Ker(f^n) = \{0\}$, then f is a monomorphism, since $Ker(f) \subseteq Ker(f^n)$. By Lemma 3.17, f is an automorphism. #

Lemma 3.21. *Let f and g be normal ideal endomorphisms of a skewring R. If $f+g$ is an endomorphism, then it is a normal ideal endomorphism.*

Proof. Suppose that f+g is an endomorphism. It is well-known that f+g is a normal endomorphism of (R,+). Let $x,y \in R$. Then $x(f+g)(y) = x(f(y)+g(y)) = xf(y)+xg(y) = f(xy)+g(xy) = (f+g)(xy)$. Similarly, $((f+g)(x))(y) = (f+g)(xy)$. Hence f+g is a normal ideal endomorphism.#

Lemma 3.22. *Let $R \neq \{0\}$ be an indecomposable skewring that satisfies both the ACC and the DCC for normal ideals.*

If f_1, f_2 are nilpotent normal ideal endomorphisms of R such that f_1+f_2

is an edomorphism, then f_1+f_2 is nilpotent.

Proof. Let f_1,f_2 be nilpotent normal ideal endomorphisms of R such that f_1+f_2 is an endomorphism. By Lemma 3.21, f_1+f_2 is a normal ideal endomorphism. Suppose f_1+f_2 is not nilpotent. By Lemma 3.20, f_1+f_2 is an automorphism. Then $(f_1+f_2)^{-1}$ is an automorphism. We shall show that $(f_1+f_2)^{-1}$ is a normal ideal automorphism. By group theory, $(f_1+f_2)^{-1}$ is a normal automorphism of (R,+). Let $x,y\in R$. Then

$$\begin{aligned}(f_1+f_2)(x(f_1+f_2)^{-1}(y)) &= f_1(x(f_1+f_2)^{-1}(y))+f_2(x(f_1+f_2)^{-1}(y))\\ &= xf_1(f_1+f_2)^{-1}(y)+xf_2(f_1+f_2)^{-1}(y)\\ &= x(f_1+f_2)(f_1+f_2)^{-1}(y) = xy.\end{aligned}$$

Then $(f_1+f_2)^{-1}(xy) = x(f_1+f_2)^{-1}(y)$. Similarly, $(f_1+f_2)^{-1}(yx) = y(f_1+f_2)^{-1}(x)$. Therefore $(f_1+f_2)^{-1}$ is a normal ideal automorphism.

Let $g = (f_1+f_2)^{-1}$ and define $g_1 = f_1\circ g$, $g_2 = f_2\circ g$. Then $g_1+g_2 = f_1\circ g + f_2\circ g = (f_1+f_2)\circ g = Id_R$ and for every $x\in R$, $-x = Id_R(-x) = (g_1+g_2)(-x) = g_1(-x)+g_2(-x)$. Hence $x = -(g_1(-x)+g_2(-x)) = -g_2(-x)-g_1(-x) = g_2(x)+g_1(x) = (g_2+g_1)(x)$ which implies that $g_2+g_1 = Id_R$. Therefore $g_1+g_2 = g_2+g_1$ and $g_1\circ(g_1+g_2) = g_1\circ Id_R = Id_R\circ g_1 = (g_1+g_2)\circ g_1$ which imply that $g_1\circ g_2 = g_2\circ g_1$. Thus for each $m\geq 1$, $(g_1+g_2)^m = g_1^m + \binom{m}{1} g_1^{m-1}\circ g_2+\ldots+\binom{m}{m-1} g_1\circ g_2^{m-1}+g_2^m$. Since f_1 is a nilpotent normal ideal endomorphism, by Lemma 3.20, f_1 is not an automorphism. By Lemma 3.17, f_1 is not an epimorphism and not a monomorphism.

Then $g_1 = f_1\circ g$ is not an automorphism. ....(i)

Since f_1 and g are normal ideal endomorphisms, by Lemma 3.15, $g_1 = f_1\circ g$ is a normal ideal endomorphism. ....(ii)

By (i),(ii) and Lemma 3.20, g_1 is nilpotent. Similarly, g_2 is nilpotent. Then there exist $m,n\in \mathbf{Z}^+$ such that $g_1^m = 0$ and $g_2^n = 0$. Then $(g_1+g_2)^{m+n} = g_1^{m+n} + \binom{m+n}{1} g_1^{m+n-1}\circ g_2+\ldots+\binom{m+n}{m+n-1} g_1\circ g_2^{m+n-1}+g_2^{m+n} = 0$. Thus for every $x\in R$, $(g_1+g_2)^{m+n}(x) = 0$ which contradicts $g_1+g_2 = Id_R$ and $R\neq\{0\}$. Hence f_1+f_2 is nilpotent.#

The following theorem is generalized from Krull-Schmidt Theorem.

Theorem 3.23. *Let R be a skewring that satisfies both the ACC and DCC for normal ideals.*

If $R = R_1 \oplus ... \oplus R_s$ and $R = H_1 \oplus ... \oplus H_t$ for some $s,t \in \mathbb{Z}^+$ and R_i, H_j are indecomposable normal ideals in R for all $i \in \{1,...,s\}, j \in \{1,...,t\}$. Then after reindexing $R_i \cong H_i$ for every $i \in \{1,...,r\}$ and $R = R_1 \oplus ... \oplus R_r \oplus H_{r+1} \oplus ... \oplus H_t$.

Proof. For cach $1 \le r \le \min\{s,t\}$, let P(r) be the statement : there is a reindexing of $H_1,\dots,H_t$ such that $R_i \cong H_i$ for every $i \in \{1,\dots,r\}$ and $R = R_1 \oplus \dots \oplus R_r \oplus H_{r+1} \oplus \dots \oplus H_t$ and (or $R = R_1 \oplus \dots \oplus R_r$ if $r = t$)

We will prove this by induction on r where $0 \le r \le \min\{s,t\}$.

If $r = 0$, then P(0) is the statement : $R = H_1 \oplus \dots \oplus H_t$ which is clear.

Let $r>0$. Assume that P(r-1) is true. Thus after reindexing $R_i \cong H_i$ for every $i \in \{1,\dots,r\text{-}1\}$ and $R = R_1 \oplus \dots \oplus R_{r-1} \oplus H_r \oplus \dots \oplus H_t$. We shall show that R(r) is true.

Let $\pi_1,\dots,\pi_s$[resp. $\pi'_1,\dots,\pi'_t$] be the projection determined by $R = R_1 \oplus \dots \oplus R_s$ [resp. $R = R_1 \oplus \dots \oplus R_{r-1} \oplus H_r \oplus \dots \oplus H_t$]. For each $i \in \{1,\dots,s\}$, let $\varphi_i : R \to R$ be defined by $\varphi_i(x) = \pi_i(x)$ for every $x \in R$ and for each $j \in \{1,\dots,t\}$, let $\psi_j : R \to R$ be defined by $\psi_j(x) = \pi_j'(x)$ for every $x \in R$. Then we have $\varphi_i|_{R_i} = Id_{R_i}$, $\varphi_i \circ \varphi_i = \varphi_i$, $\varphi_i \circ \varphi_j = 0$(where $i \ne j$), $\psi_1 + \dots + \psi_j = Id_R$, $\psi_j \circ \psi_j = \psi_j$, $\psi_i \circ \psi_j = 0$ (where $i \ne j$), $Im(\varphi_i) = R_i$, $Im(\psi_i) = R_i$ (where $i<r$) and $Im(\psi_i) = H_i$ (where $i \ge r$)

It follows that $\varphi_r \circ \psi_i = 0$ for every $i<r$. (Since for every $x \in R$, $\psi_i(x) \in R_i$, $\varphi_r \circ \psi_i(x) = \varphi_r \circ Id_{R_i} \circ \psi_i(x) = \varphi_r \circ \varphi_i \circ \psi_i(x) = 0$.) The preceding identities show that $\varphi_r = \varphi_r \circ Id_R = \varphi_r \circ (\psi_1 + \dots + \psi_t) = \varphi_r \circ \psi_1 + \dots + \varphi_r \circ \psi_t$. By Lemma 3.16, φ_r is a normal ideal endomorphism of R. By Lemma 3.15 and Lemma 3.16, every sum of distinct $(\varphi_r \circ \psi_j)|_{R_r}$ is a normal ideal endomorphism of R_r. ....**(i)**

By Remark 3.10, R_r satisfies both the ascending and descending chain conditions for normal ideals.

Claim1. There exists an j such that $r \le j \le t$ and $(\varphi_r \circ \psi_j)|_{R_r}$ is an automorphism of $R_r \neq \{0\}$. Suppose not.

Then for every $i \in \{r,\ldots,t\}$, $(\varphi_r \circ \psi_i)|_{R_r}$ is not an automorphism. ....**(ii)**

By **(i)**, for every $i \in \{r,\ldots,t\}$, $(\varphi_r \circ \psi_i)|_{R_r}$ is a normal ideal endomorphism of R_r.

By **(ii)** and Lemma 3.20, for every $i \in \{r,\ldots,t\}$, $(\varphi_r \circ \psi_i)|_{R_r}$ is nilpotent in R_r.

Since $\varphi_r = \varphi_r \circ \psi_r + \ldots + \varphi_r \circ \psi_t$, by **(i)** and Lemma 3.22, $\varphi_r|_{R_r}$ is nilpotent in R_r.

Thus $\varphi_r|_{R_r}$ is an automorphism and nilpotent on R_r which contradicts Lemma 3.20. Hence we have Claim1.

Therefore there exists $j \in \mathbf{Z}^+$ such that $r \le j \le t$ and $(\varphi_r \circ \psi_j)|_{R_r}$ is an automorphism. ....**(iii)**

So that, for each $n \in \mathbf{Z}^+$, $(\varphi_r \circ \psi_j)^{n+1}$ is also an automorphism of R_r. ....**(iv)**

By assumption and Remark 3.10, H_j satisfies the ACC and DCC for normal ideals for every $j \in \{1,\ldots,t\}$. By Lemma 3.15 and Lemma 3.16, $(\psi_j \circ \varphi_r)|_{H_j} : H_j \to H_j$ is a normal ideal endomorphism of H_j.

Claim2. $(\psi_j \circ \varphi_r)|_{H_j}$ is an automorphism of H_j. ....**(v)**

Suppose not. By Lemma 3.20, $(\psi_j \circ \varphi_r)|_{H_j}$ is nilpotent in H_j. Then there exists $m \in \mathbf{Z}^+$ such that $((\psi_j \circ \varphi_r)|_{H_j})^m = 0_{H_j}$. Then $(\varphi_r \circ \psi_j)^{m+1} = \varphi_r \circ ((\psi_j \circ \varphi_r))^m \circ \psi_j = \varphi_r \circ 0_{H_j} \circ \psi_j = 0_R$, so that $(\varphi_r \circ \psi_j)^{m+1}$ is a nilpotent automorphism of R_r (by **(iv)**) which contradicts Lemma 3.20. Hence we have Claim2.

By **(iii)** and **(v)**, $\psi_j|_{R_r} : R_r \to H_j$ is an isomorphism and so is $\varphi_r|_{H_j} : H_j \to R_r$. Reindexing the H_k, so that we may assume $j = r$ and $R_r \cong H_r$. We have proved the first half of statement P(r).

Since $R = R_1 \oplus \ldots \oplus R_{r-1} \oplus H_r \oplus \ldots \oplus H_t$ by the induction hypothesis, the subskewring $R_1 + \ldots + R_{r-1} + H_{r+1} + \ldots + H_t$ is the direct sum of $R_1 \oplus \ldots \oplus R_{r-1} \oplus H_{r+1} \oplus \ldots \oplus H_t$. Observe that for every $i < r$, $\psi_r[R_i] = \psi_r \circ \psi_i[R] = \{0\}$ and for every $i > r$,

$\psi_r[H_i] = \psi_r \circ \psi_i[R] = \{0\}$. So $\psi_r[R_1+\ldots+R_{r-1}+H_{r+1}+\ldots+H_t] = \{0\}$. Let $x \in R_r \cap (R_1+\ldots+R_{r-1}+H_{r+1}+\ldots+H_t)$. Since $\psi_r[R_1+\ldots+R_{r-1}+H_{r+1}+\ldots+H_t] = \{0\}$, $\psi_r(x) = \{0\}$. Since $\psi_r|_{R_r}$ is an isomorphism, $x = 0$. Therefore $R_r \cap (R_1+\ldots+R_{r-1}+H_{r+1}+\ldots+H_t) = \{0\}$. It follows that the skewring $R^* = R_1+\ldots+R_r+H_{r+1}+\ldots+H_t$ is the direct sum. Hence $R^* = R_1 \oplus \ldots \oplus R_r \oplus H_{r+1} \oplus \ldots \oplus H_t$.

Define $\theta: R \to R$ as follows :

By the induction hypothesis, we have that $R = R_1 \oplus \ldots \oplus R_{r-1} \oplus H_r \oplus \ldots \oplus H_t$. Then every element $x \in R$ may be written in the form $x = x_1+\ldots+x_{r-1}+h_r+\ldots+h_t$ with $x_i \in R_i$ and $h_j \in H_j$. Let $\theta(x) = x_1+\ldots+x_{r-1}+\varphi_r(h_r)+h_{r+1}+\ldots+h_t$. Since $\varphi_r|_{H_r}: H_r \to R_r$ is an isomorphism, $Im(\theta) = R^*$ and θ is a monomorphism.

Claim3. θ is a normal ideal endomorphism.

It is well-known that θ is a normal endomorphism of $(R,+)$. Let $x,y \in R$. Then there exist $x_i, y_i \in R_i$, $h_j, k_j \in H_j$ where $i \in \{1,\ldots,r-1\}$ and $j \in \{r,\ldots,t\}$ such that $x = x_1+\ldots+x_{r-1}+h_r+\ldots+h_t$ and $y = y_1+\ldots+y_{r-1}+k_r+\ldots+k_t$. Then

$$\begin{aligned} x\theta(y) &= (x_1+\ldots+x_{r-1}+h_r+\ldots+h_t)(y_1+\ldots+y_{r-1}+\varphi_r(k_r)+k_{r+1}+\ldots+k_t) \\ &= x_1y_1+\ldots+x_{r-1}y_{r-1}+h_r\varphi_r(k_r)+h_{r+1}k_{r+1}+\ldots+h_tk_t \\ &= x_1y_1+\ldots+x_{r-1}y_{r-1}+\varphi_r(h_rk_r)+h_{r+1}k_{r+1}+\ldots+h_tk_t = \theta(xy). \end{aligned}$$

Similarly, $\theta(x)y = \theta(xy)$. Hence θ is a normal ideal endomorphism. So we have Claim3.

Since θ is a monomorphism, by Lemma 3.17, θ is an automorphism. So that $R = Im(\theta) = R^* = R_1 \oplus \ldots \oplus R_r \oplus H_{r+1} \oplus \ldots \oplus H_t$. This proves the second part of P(r) and complete the induction argument. Therefore, after reindexing we have that $R_i \cong H_i$ for every $1 \le i \le \min\{s,t\}$. If $\min\{s,t\} = s$, then $R_1 \oplus \ldots \oplus R_s = R = R_1 \oplus \ldots \oplus R_s \oplus H_{s+1} \oplus \ldots \oplus H_t$ and if $\min\{s,t\} = t$, then $R_1 \oplus \ldots \oplus R_s = R = R_1 \oplus \ldots \oplus R_t$. Since $R_i \neq \{0\}$ and $H_j \neq \{0\}$ for all i,j, we must have $s = t$ in either case. #

Definition 3.24. *Let R be a skewring, S be a subskewring of R and I be a normal ideal of R. Then R is called a* ***semi-direct sum*** *of S and I if and only if $R = S+I$ and $S \cap I = \{0\}$. We denote this by $R = S \otimes I$.*

Definition 3.25. *Let R be a skewring. For any additive endomorphism f of R is called* ***left[right] translation*** *if and only if f(xy) = f(x)y[f(xy) = xf(y)] for all x,y∈R and we denote the set of all left[right] translations by LT(R)[RT(R)].*

Definition 3.26. *Let R,S be skewrings, f:R→S. Then f is called an* ***additive anti-homomorphism*** *if and only if f(x+y) = f(y)+f(x) for all x,y∈R and f is called a* ***multiplicative anti-homomorphism*** *if and only if f(xy) = f(y)f(x) for all x,y∈R.*

Theorem 3.27. *Let* $0 \longrightarrow R \xrightarrow{f} S \xrightarrow{g} T \longrightarrow 0$ *be an exact sequence of skewrings. If there exists a homomorphism h:T→S such that* $g \circ h = Id_T$, *then* $S = f[R] \otimes h[T]$.

Proof. By definition of exact sequcnce, f[R] = Im(f) = Ker(g) which is a normal ideal in S. Suppose that there exists a homomorphism h:T→S such that $g \circ h = Id_T$. Then h is injective. Moreover, $T \cong h[T]$ which is a subskewring of S, by Proposition 1.36 (1). We shall show that $S = f[R] \otimes h[T]$.

Claim1. $f[R] \cap h[T] = \{0\}$.

Let $x \in f[R] \cap h[T]$. Since $x \in f[R] = Ker(g)$, g(x) = 0. Since $x \in h[T]$, there exists $y \in T$ such that h(y) = x. Therefore $0 = g(x) = g(h(y)) = Id_T(y) = y$. Since h is a homomorphism, 0 = h(y) = x. Hence $f[R] \cap h[T] = \{0\}$ and we have Claim1.

Claim2. S = f[R]+h[T].

Clearly, f[R]+h[T] is contained in S. Conversely, let $x \in S$. Then $g(x) \in T$, so that $h(g(x)) \in h[T]$. We have that x = x-h(g(x))+h(g(x)). We shall show that $x-h(g(x)) \in f[R]$ (= Ker(g)), consider $g(x-h(g(x))) = g(x)-g(h(g(x))) = g(x)-Id_T(g(x)) = g(x)-g(x) = 0$. Thus $x-h(g(x)) \in Ker(g) = f[R]$ which implies that $x = x-h(g(x))+h(g(x)) \in f[R]+h[T]$. So $S \subseteq f[R]+h[T]$. Hence S = f[R]+h[T] and we have Claim2. By Claim1 and Claim2, $S = f[R] \otimes h[T]$. #

Theorem 3.28. *Let S and I be skewrings. Then there exist* $\alpha:S\to GAut(I)$ $(=\{f:I\to I\,/\,f$ *is an additive automorphism.*$\})$ *which is an additive anti-homomorphism,* $l:S\to LT(I)$ *which is a homomorphism, and* $r:S\to RT(I)$ *which is a multiplicative anti-homomorphism and additive homomorphism which have the following properties : for all* $s_1,s_2,s_3\in S$, $i_1,i_2,i_3\in I$,

(1) $r(s_1)\circ l(s_2) = l(s_2)\circ r(s_1)$ *and* $[r(s_1)](i_1)+[l(s_2)](i_2) = [l(s_2)](i_2)+[r(s_1)](i_1)$,

(2) $[r(s_1)](i_1)+ i_2i_3 = i_2i_3+[r(s_1)](i_1)$ *and* $[l(s_1)](i_1)+ i_2i_3 = i_2i_3+[l(s_1)](i_1)$,

(3) $[\alpha(s_1s_2)]i_1i_2 = i_1i_2$, $[\alpha(s_1s_2)]\circ[l(s_3)](i_1) = [l(s_3)](i_1)$ *and* $[\alpha(s_1s_2)]\circ[r(s_3)](i_1) = [r(s_3)](i_1)$,

(4) $i_1[\alpha(s_1)](i_2) = i_1i_2$ *and* $[\alpha(s_1)](i_1)i_2 = i_2[\alpha(s_1)](i_1) = i_2i_1$,

(5) $[l(s_1)]\circ[\alpha(s_2)](i_1) = [l(s_1)](i_1)$ *and* $[r(s_1)]\circ[\alpha(s_2)](i_1) = [r(s_1)](i_1)$ *and*

(6) $i_1[l(s_1)](i_2) = [r(s_1)](i_1)i_2$

if and only if there exists a skewring R such that S is isomorphic to some subskewring S′ of R, I is isomorphic to some normal ideal I′ of R and $R = S'\otimes I'$. *(i.e. R is a semi-direct sum of S′ and I′.)*

Proof. Let $R = S\times I$ and define the binary operations $+,\cdot$ on R as follows : For all $(s_1,i_1),(s_2,i_2)\in R$, $(s_1,i_1)+(s_2,i_2) = (s_1+s_2\,,\,[\alpha(s_2)](i_1)+i_2)$ and

$$(s_1,i_1)(s_2,i_2) = (s_1s_2\,,\,[r(s_2)](i_1)+[l(s_1)](i_2)+ i_1i_2).$$

Step1. We shall show that R is a skewring.

Clearly, $(R,+)$ and $(R,\cdot)$ are closed. Let $(s_1,i_1),(s_2,i_2),(s_3,i_3)\in R$. Then

$$\begin{aligned}(s_1,i_1)+[(s_2,i_2)+(s_3,i_3)] &= (s_1,i_1)+(s_2+s_3\,,\,[\alpha(s_3)](i_2)+i_3)\\ &= (s_1+(s_2+s_3)\,,\,[\alpha(s_2+s_3)](i_1)+ [\alpha(s_3)](i_2)+i_3)\\ &= ((s_1+s_2)+s_3\,,\,[\alpha(s_3)]\circ[\alpha(s_2)](i_1)+ [\alpha(s_3)](i_2)+i_3)\\ &= ((s_1+s_2)+s_3\,,\,[\alpha(s_3)]([\alpha(s_2)](i_1)+i_2)+i_3)\\ &= (s_1+s_2\,,\,[\alpha(s_2)](i_1)+i_2)+(s_3,i_3)\\ &= [(s_1,i_1)+(s_2,i_2)]+(s_3,i_3)\end{aligned}$$

Therefore the associative law is true for $(R,+)$. Since $(s_1,i_1)+(0,0) = (s_1+0\,,\,[\alpha(0)](i_1)+0) = (s_1\,,\,Id_I(i_1)) = (s_1,i_1)$. Therefore $(0,0)$ is a right identity of

$(R,+)$. Since $(s_1,i_1)+(-s_1, -[\alpha(-s_1)](i_1)) = (s_1-s_1, [\alpha(-s_1)](i_1)+(-[\alpha(-s_1)](i_1))) = (0,0)$, $(-s_1, -[\alpha(-s_1)](i_1))$ is a right inverse of (s_1,i_1). Hence $(R,+)$ is a group. Consider

$(s_1,i_1)[(s_2,i_2)(s_3,i_3)] = (s_1,i_1)(s_2s_3, [r(s_3)](i_2) + [l(s_2)](i_3) + i_2i_3)$

$= (s_1(s_2s_3), [r(s_2s_3)](i_1) + [l(s_1)]([r(s_3)](i_2)+[l(s_2)](i_3))+i_2i_3) +$
$i_1([r(s_3)](i_2)+[l(s_2)](i_3) + i_2i_3))$

$= ((s_1s_2)s_3, [r(s_3)]\circ[r(s_2)](i_1) + [l(s_1)]\circ[r(s_3)](i_2) + [l(s_1)]\circ[l(s_2)](i_3) + [l(s_1)](i_2i_3) +$
$i_1[r(s_3)](i_2) + i_1[l(s_2)](i_3) + i_1(i_2i_3))$

$= ((s_1s_2)s_3, [r(s_3)]\circ[r(s_2)](i_1) + [r(s_3)]\circ[l(s_1)](i_2) + [l(s_1s_2)](i_3) + [l(s_1)](i_2)i_3 +$
$[r(s_3)](i_1i_2) + [r(s_2)](i_1)i_3+ (i_1i_2)i_3)$

$= ((s_1s_2)s_3, [r(s_3)]\circ[r(s_2)](i_1) + [r(s_3)]\circ[l(s_1)](i_2) + [r(s_3)](i_1i_2) + [l(s_1s_2)](i_3) +$
$[r(s_2)](i_1)i_3+ [l(s_1)](i_2)i_3 + (i_1i_2)i_3)$

$= ((s_1s_2)s_3, [r(s_3)]([r(s_2)](i_1)+[l(s_1)](i_2)+(i_1i_2)) + [l(s_1s_2)](i_3) + [r(s_2)](i_1)i_3+ [l(s_1)](i_2)i_3$
$+ (i_1i_2)i_3)$

$= (s_1s_2, [r(s_2)](i_1)+[l(s_1)](i_2)+ i_1i_2)(s_3,i_3)$.

Therefore $(R,\cdot)$ is a semigroup.

$(s_1,i_1)[(s_2,i_2)+(s_3,i_3)] = (s_1,i_1)(s_2+s_3, [\alpha(s_3)](i_2)+i_3)$

$= (s_1(s_2+s_3), [r(s_2+s_3)](i_1) + [l(s_1)]([\alpha(s_3)](i_2)+i_3) + i_1([\alpha(s_3)](i_2)+i_3))$

$= (s_1s_2+s_1s_3, [r(s_2)+r(s_3)](i_1) + [l(s_1)]\circ[\alpha(s_3)](i_2) + [l(s_1)](i_3) + i_1[\alpha(s_3)](i_2) + i_1i_3)$

$= (s_1s_2+s_1s_3, [r(s_2)](i_1) + [r(s_3)](i_1) + [l(s_1)](i_2) + [l(s_1)](i_3) + i_1i_2 + i_1i_3)$

$= (s_1s_2+s_1s_3, [r(s_2)](i_1) + [l(s_1)](i_2) + i_1i_2 + [r(s_3)](i_1) + [l(s_1)](i_3) + i_1i_3)$

$= (s_1s_2+s_1s_3, [\alpha(s_1s_3)]\circ[r(s_2)](i_1) + [\alpha(s_1s_3)]\circ[l(s_1)](i_2) + [\alpha(s_1s_3)](i_1i_2) + [r(s_3)](i_1) +$
$[l(s_1)](i_3) + i_1i_3)$

$= (s_1s_2+s_1s_3, [\alpha(s_1s_3)]([r(s_2)](i_1)+[l(s_1)](i_2)+i_1i_2) + [r(s_3)](i_1) + [l(s_1)](i_3) + i_1i_3)$

$= (s_1s_2, [r(s_2)](i_1)+[l(s_1)](i_2)+ i_1i_2) + (s_1s_3, [r(s_3)](i_1)+[l(s_1)](i_3)+ i_1i_3)$

$= (s_1,i_1)(s_2,i_2) + (s_1,i_1)(s_3,i_3)$ and

$[(s_1,i_1)+(s_2,i_2)](s_3,i_3) = (s_1+s_2, [\alpha(s_2)](i_1)+i_2)(s_3,i_3)$

$= ((s_1+s_2)s_3, [r(s_3)]([\alpha(s_2)](i_1)+i_2) + [l(s_1+s_2)](i_3) + ([\alpha(s_2)](i_1)+i_2)i_3)$

$= (s_1s_3+s_2s_3, [r(s_3)]\circ[\alpha(s_2)](i_1) + [r(s_3)](i_2) + [l(s_1)+l(s_2)](i_3) + [\alpha(s_2)](i_1)i_3 + i_2i_3)$

$= (s_1s_3+s_2s_3, [r(s_3)](i_1) + [r(s_3)](i_2) + [l(s_1)](i_3) + [l(s_2)](i_3) + i_1i_3 + i_2i_3)$

$= (s_1s_3+s_2s_3, [r(s_3)](i_1) + [l(s_1)](i_3) + i_1i_3 + [r(s_3)](i_2) + [l(s_2)](i_3) + i_2i_3)$

$= (s_1s_3+s_2s_3, [\alpha(s_2s_3)]\circ[r(s_3)](i_1) + [\alpha(s_2s_3)]\circ[l(s_1)](i_3) + [\alpha(s_2s_3)](i_1i_3) + [r(s_3)](i_2) + [l(s_2)](i_3) + i_2i_3)$

$= (s_1s_3+s_2s_3, [\alpha(s_2s_3)]([r(s_3)](i_1) + [l(s_1)](i_3) + i_1i_3) + [r(s_3)](i_2) + [l(s_2)](i_3) + i_2i_3)$

$= (s_1s_3, [r(s_3)](i_1)+[l(s_1)](i_3)+ i_1i_3) + (s_2s_3, [r(s_3)](i_2) + [l(s_2)](i_3) + i_2i_3)$

$= (s_1,i_1)(s_3,i_3) + (s_2,i_2)(s_3,i_3)$.

Therefore the distributive law is true for $(R,+,\cdot)$ and hence R is a skewring.

Step2. We shall show that S isomorphic to some subskewring of R and I isomorphic to some normal ideal of R.

Let $S' = \{(s,0) / s\in S\}$ and let $(s_1,0),(s_2,0)\in S'$. Then $(s_1,0)-(s_2,0) = (s_1-s_2, [\alpha(-s_2)](0)-0) = (s_1-s_2, 0)\in S'$ and $(s_1,0)(s_2,0) = (s_1s_2, [r(s_2)](0)+[l(s_1)](0)+ 0) = (s_1s_2,0)\in S'$. Therefore S' is a subskewring of R and hence $S \cong S\times\{0\}$.

Let $I' = \{(0,i) / i\in I\}$ and let $(0,i_1),(0,i_2)\in I'$, $(s,i)\in R$. Then $(0,i_1)-(0,i_2) = (0, [\alpha(-0)](i_1)-i_2)$, $(0,i_1)(0,i_2) = (0, [r(0)](i_1) + [l(0)](i_2)+ i_1i_2)$, $(s,i)(0,i_1) = (0, [r(0)](i) + [l(s)](i_1)+ ii_1)$, $(0,i_1)(s,i) = (0, [r(s)](i_1) + [l(0)](i)+ i_1i)\in I'$ and $(s,i)+(0,i_1)-(s,i) = [(s,i)+(0,i_1)]-(s,i) = (s, [\alpha(0)](i)+i_1)-(s,i) = (s, i+i_1)-(s,i) = (0, [\alpha(-s)](i+i_1)-i)\in I'$. Therefore I' is a normal ideal of R hence $I \cong \{0\}\times I$ and clearly, $R = S'\otimes I'$. Hence we have the first statement.

Conversely, suppose there exists a skewring R such that S is isomorphic to some subskewring S' of R, and I is isomorphic to some normal ideal I' of R and $R = S'\otimes I'$. Let $\varphi:S\to S'$ and $\psi:I\to I'$ be such that φ and ψ are isomorphisms. For any $s'\in S'$, define $\alpha_{s'}:I'\to I'$ by $\alpha_{s'}(i') = -s'+i'+s'$, $l_{s'}: I'\to I'$ by $l_{s'}(i') = s'i'$ and $r_{s'}:I'\to I'$ by $r_{s'}(i') = i's'$ for every $i'\in I'$. Then we have $\psi^{-1}\circ\alpha_{s'}\circ\psi\in GAut(I)$, $\psi^{-1}\circ l_{s'}\circ\psi\in LT(I)$, $\psi^{-1}\circ r_{s'}\circ\psi\in RT(I)$ for every $s'\in S'$. Define $\alpha:S\to GAut(I)$ by $\alpha(s) = \psi^{-1}\circ\alpha_{\varphi(s)}\circ\psi$, $l:S\to LT(I)$ by $l(s) = \psi^{-1}\circ l_{\varphi(s)}\circ\psi$ and $r:S\to RT(I)$ by $r(s) = \psi^{-1}\circ r_{\varphi(s)}\circ\psi$ for every $s\in S$. Hence we have the converse. #

Corollary 3.29. *Let S and I be rings. Suppose that there exist maps $l:S\to LT(I)$ which is a ring homomorphism and a multiplicative anti-homomorphism $r:S\to RT(I)$ which is also an additive homomorphism which satisfy (1) $r(s_1)\circ l(s_2) = l(s_2)\circ r(s_1)$ and (2) $i_1[l(s_1)](i_2) = [r(s_1)](i_1)i_2$ for all $s_1,s_2\in S$, $i_1,i_2\in I$. Then there exists a ring R such that S is isomorphic to some subring S' of R, I is isomorphic to an ideal I' of R and R is the semi-direct sum of S' and I'.*

Theorem 3.30. *Let S and I be skewrings. Suppose that there exist $\alpha,\alpha':S\to GAut(I)$ which are additive anti-homomorphisms, $l,l':S\to LT(I)$ which are homomorphisms, and $r,r':S\to RT(I)$ which are multiplicative anti-homomorphisms and additive homomorphisms which satisfy the same properties as those in Theorem 3.28. By Theoem 3.28, we get skewrings $R_{\alpha,l,r}$ and $R_{\alpha',l',r'}$. Let $\varphi:S\to S$ and $\psi:I\to I$ be isomorphisms. If the following conditions hold :*

For every $s\in S$, (1) $\alpha(s) = \psi^{-1}\circ\alpha'(\varphi(s))\circ\psi$,

(2) $l(s) = \psi^{-1}\circ l'(\varphi(s))\circ\psi$ and

(3) $r(s) = \psi^{-1}\circ r'(\varphi(s))\circ\psi$.

then $\varphi\times\psi:R_{\alpha,l,r}\to R_{\alpha',l',r'}$ is an isomorphism where $\varphi\times\psi(x,y) = (\varphi(x),\psi(y))$ for all $(x,y)\in R_{\alpha,l,r}$.

Proof. Assume the conditions. Let $(s_1,i_1),(s_2,i_2)\in R_{\alpha,l,r}$. Then

$\varphi\times\psi((s_1,i_1)+(s_2,i_2)) = \varphi\times\psi((s_1+s_2\,,\,[\alpha(s_2)](i_1)+i_2))$

$= (\varphi(s_1+s_2)\,,\,\psi([\alpha(s_2)](i_1)+i_2))$

$= (\varphi(s_1)+\varphi(s_2)\,,\,\psi([\alpha(s_2)](i_1)) + \psi(i_2))$

$= (\varphi(s_1)+\varphi(s_2)\,,\,\psi((\psi^{-1}\circ\alpha'(\varphi(s_2))\circ\psi)(i_1)) + \psi(i_2))$

$= (\varphi(s_1)+\varphi(s_2)\,,\,(\alpha'(\varphi(s_2))\circ\psi)(i_1) + \psi(i_2))$

$= (\varphi(s_1)+\varphi(s_2)\,,\,[\alpha'(\varphi(s_2))](\psi(i_1)) + \psi(i_2))$

$= (\varphi(s_1),\psi(i_1)) + (\varphi(s_2),\psi(i_2))$

$= \varphi\times\psi((s_1,i_1)) + \varphi\times\psi((s_2,i_2))$ and

$$\varphi\times\psi((s_1,i_1)(s_2,i_2)) = \varphi\times\psi((s_1s_2, [r(s_2)](i_1)+[l(s_1)](i_2)+ i_1i_2)$$
$$= (\varphi(s_1s_2), \psi([r(s_2)](i_1)+[l(s_1)](i_2)+ i_1i_2))$$
$$= (\varphi(s_1)\varphi(s_2), \psi\circ[r(s_2)](i_1) + \psi\circ[l(s_1)](i_2)+ \psi(i_1i_2))$$
$$= (\varphi(s_1)\varphi(s_2), \psi((\psi^{-1}\circ r'(\varphi(s_2))\circ\psi)(i_1)) + \psi((\psi^{-1}\circ l'(\varphi(s_1))\circ\psi))(i_2)+ \psi(i_1)\psi(i_2))$$
$$= (\varphi(s_1)\varphi(s_2), (r'(\varphi(s_2))\circ\psi)(i_1)) + (l'(\varphi(s_1))\circ\psi)(i_2)+ \psi(i_1)\psi(i_2))$$
$$= (\varphi(s_1)\varphi(s_2), [r'(\varphi(s_2)](\psi(i_1)) + [l'(\varphi(s_1)](\psi(i_2))+ \psi(i_1)\psi(i_2))$$
$$= (\varphi(s_1),\psi(i_1))(\varphi(s_2),\psi(i_2))$$
$$= \varphi\times\psi((s_1,i_1))\varphi\times\psi((s_2,i_2)).$$

Therefore $\varphi\times\psi$ is a homomorphism. Since φ and ψ are isomorphisms, $\varphi\times\psi$ is an isomorphism. #

Corollary 3.31. *Let S and I be rings. Suppose that there exist $l,l':S\to LT(I)$ which are homomorphisms, and $r,r':S\to RT(I)$ which are multiplicative anti-homomorphisms and additive homomorphisms which satisfy the same properties as those in Corollary 3.29. By Corollary 3.29, we get rings $R_{l,r}$ and $R_{l',r'}$. Let $\varphi:S\to S$ and $\psi:I\to I$ be isomorphisms. If the following conditions hold : For every $s\in S$, (1) $l(s) = \psi^{-1}\circ l'(\varphi(s))\circ\psi$ and (2) $r(s) = \psi^{-1}\circ r'(\varphi(s))\circ\psi$. Then $\varphi\times\psi:R_{l,r}\to R_{l',r'}$ is an isomorphism where $\varphi\times\psi(x,y) = (\varphi(x),\psi(y))$ for all $(x,y)\in R_{l,r}$.*

Definition 3.32. *Let R be a skewring and $\{R_\alpha / \alpha\in A\}$ be a family of skewrings. Then R is said to be a* ***subdirect sum*** *of $\{R_\alpha / \alpha\in A\}$ if and only if there exists a monomorphism $f:R\to \prod_{\alpha\in A} R_\alpha$ such that for each $\alpha\in A$, $\pi_\alpha\circ f:R\to R_\alpha$ is an epimorphism where π_α is the projection map.*

Definition 3.33. *Let R be a subskewring of a direct product of family of skewrings $\{R_\alpha / \alpha\in A\}$. R is said to be* ***subdirect product*** *of $\{R_\alpha / \alpha\in A\}$ if and only if for every $\alpha\in A$, $\pi_\alpha(R) = R_\alpha$ where π_α is the projection map.*

Definition 3.34. *Let $\{R_\alpha / \alpha\in A\}$ be a family of skewrings, and R a skewring. A* ***representation of R as a subdirect product of $\{R_\alpha / \alpha\in A\}$*** *is a homomorphism $g:R\to \prod_{\alpha\in A} R_\alpha$ such that for each $\alpha\in A$, $\pi_\alpha \circ g:R\to R_\alpha$ is an epimorphism where π_α is a projection map. Then Im(g) is a subdirect product of $\{R_\alpha / \alpha\in A\}$.*

Definition 3.35. *Let R be a skewring. Then R is said to be a* ***subdirectly irreducible*** *if and only if for every family of skewrings $\{R_\alpha / \alpha\in A\}$ and for every monomorphism representation $g:R\to \prod_{\alpha\in A} R_\alpha$ there exists $\beta\in A$ such $\pi_\beta \circ g:R\to R_\beta$ is an isomorphism where π_β is the projection map.*

If R is not a subdirectly irreducible, we shall call R a ***subdirectly reducible skewring.***

Theorem 3.36. *Let R be a skewring, $\{R_\alpha / \alpha\in A\}$ be a family of skewrings. Then R is a subdirect sum of $\{R_\alpha / \alpha\in A\}$ if and only if for each $\beta\in A$, there exists an epimorphism $g_\beta:R\to R_\beta$ such that $\bigcap_{\alpha\in A} Ker(g_\alpha) = \{0\}$.*

Proof. Suppose that R is a subdirect sum of $\{R_\alpha / \alpha\in A\}$. Then there exists a monomorphism $f:R\to \prod_{\alpha\in A} R_\alpha$ such that for each $\beta\in A$, $\pi_\beta \circ f:R\to R_\beta$ is an epimorphism. For each $\beta\in A$, let $g_\beta = \pi_\beta \circ f$. Let $r\in \bigcap_{\alpha\in A} Ker(g_\alpha)$. Suppose $r \neq 0$. Then $f(r)\neq 0$ which implies that there exists $\alpha_0\in A$ such that $0\neq \pi_{\alpha_0}\circ f(r) = g_{\alpha_0}(r)$. Therefore $r\notin Ker(g_{\alpha_0})$, so $r\notin \bigcap_{\alpha\in A} Ker(g_\alpha)$ which is a contradiction. Hence $\bigcap_{\alpha\in A} Ker(g_\alpha) = \{0\}$.

Conversely, assume that for each $\beta\in A$, there exists an epimorphism $g_\beta:R\to R_\beta$ such that $\bigcap_{\alpha\in A} Ker(g_\alpha) = \{0\}$. We define $f:R\to \prod_{\alpha\in A} R_\alpha$ by $f(r) = \{g_\alpha(r)\}_{\alpha\in A}$ for every $r\in R$. From the above, for each $\beta\in A$, $\pi_\beta \circ f = g_\beta$. Since g_β is surjective, $\pi_\beta \circ f$ is surjective. Let $r,s \in R$. Then $f(rs) = \{g_\alpha(rs)\}_{\alpha\in A} = \{g_\alpha(r)g_\alpha(s)\}_{\alpha\in A}$ $= \{g_\alpha(r)\}_{\alpha\in A}\{g_\alpha(s)\}_{\alpha\in A} = f(r)f(s)$ and similarly, $f(r+s) = f(r)+f(s)$. Therefore f is a homomorphism. Let $r \in Ker(f)$. Then $0 = f(r) = \{g_\alpha(r)\}_{\alpha\in A}$. Then $g_\alpha(r) = 0$ for every $\alpha \in A$. Therefore $r \in \bigcap_{\alpha\in A} Ker(g_\alpha) = \{0\}$, so that $Ker(f) = \{0\}$. Therefore f is a monomorphism and hence R is a subdirect sum of $\{R_\alpha / \alpha\in A\}$. #

Corollary 3.37. *Let R be a skewring and $\{I_\alpha / \alpha\in A\}$ be a family of normal ideals of R. If $\bigcap_{\alpha\in A} I_\alpha = \{0\}$, then R is a subdirect sum of the family of skew rings $\{R/I_\alpha / \alpha\in A\}$.*

Proof. For each $\alpha\in A$, let $\pi_\alpha : R \to R/I_\alpha$ be the canonical epimorphism. Since for each $\beta\in A$, π_β is an epimorphism and $\bigcap_{\alpha\in A} Ker(\pi_\alpha) = \bigcap_{\alpha\in A} I_\alpha = \{0\}$, by Theorem 3.36, R is a subdirect sum of $\{R/I_\alpha / \alpha\in A\}$. #

Theorem 3.38. *Let R be a subskewring of the Cartesian product $\prod_{\alpha\in A} R_\alpha$ of skewrings. Then there exists a natural epimorphism θ from R to a subdirect product of the family of skewrings $\{R'_\alpha / \alpha\in A\}$ where $R'_\alpha = R/(R\cap j_\alpha[R_\alpha])$ and for every $\alpha\in A$, $j_\alpha : R \to \prod_{\alpha\in A} R_\alpha$ which is defined by $j_\alpha(r) = (r_\beta)_{\beta\in A}$ where*

$$r_\beta = \begin{cases} 0 & \text{if } \beta \neq \alpha, \\ r & \text{if } \beta = \alpha, \end{cases}$$

in order that θ be an isomorphism, it is necessary and sufficient that $\bigcap_{\alpha\in A}(R\cap j_\alpha[R_\alpha]) = \{0\}$.

Proof. For every $r\in R$, we define $\theta(r) = (r_\alpha)_{\alpha\in A}$ where $r_\alpha = r+(R\cap j_\alpha[R_\alpha])$

is the coset of r in $R/(R\cap j_\alpha[R_\alpha])$ for every $\alpha\in A$. Then θ is a homomorphism and $\theta[R]$ is a subdirect product of $\{R'_\alpha / \alpha\in A\}$. The $\mathrm{Ker}(\theta) = \bigcap_{\alpha\in A}(R\cap j_\alpha[R_\alpha])$. If θ is an isomorphism, then $\bigcap_{\alpha\in A}(R\cap j_\alpha[R_\alpha]) = \{0\}$. #

Theorem 3.39. *Let R be a skewring and $\{R_\alpha / \alpha\in A\}$ be a family of skewrings. Let $g:R\rightarrow \prod_{\alpha\in A} R_\alpha$ be a representation of R as a subdirect product of $\{R_\alpha / \alpha\in A\}$. Then $Im(g) \cong R/\bigcap_{\alpha\in A} Ker(\pi_\alpha\circ g)$.*

Proof. Define $\varphi:R\rightarrow \mathrm{Im}(g)$ by $\varphi(x) = g(x)$ for every $x\in R$. Then φ is an epimorphism. We shall show that $\mathrm{Ker}(\varphi) = \bigcap_{\alpha\in A}\mathrm{Ker}(\pi_\alpha\circ g)$. Let $x\in \mathrm{Ker}(\varphi)$. Then $\varphi(x) = (0_\alpha)_{\alpha\in A}$, so $g(x) = (0_\alpha)_{\alpha\in A}$. For each $\alpha\in A$, $\pi_\alpha\circ g(x) = 0_\alpha$, then $x\in \mathrm{Ker}(\pi_\alpha\circ g)$. Hence $x\in \bigcap_{\alpha\in A}\mathrm{Ker}(\pi_\alpha\circ g)$. Thus $\mathrm{Ker}(\varphi)\subseteq \bigcap_{\alpha\in A}\mathrm{Ker}(\pi_\alpha\circ g)$.

Next, let $x\in \bigcap_{\alpha\in A}\mathrm{Ker}(\pi_\alpha\circ g)$. Then $\pi_\alpha\circ g(x) = 0_\alpha$ for every $\alpha\in A$ which implies that $g(x) = (0_\alpha)_{\alpha\in A}$. Since $\varphi(x) = g(x) = (0_\alpha)_{\alpha\in A}$, $x\in \mathrm{Ker}(\varphi)$. Hence $\bigcap_{\alpha\in A}\mathrm{Ker}(\pi_\alpha\circ g) \subseteq \mathrm{Ker}(\varphi)$ and $\mathrm{Ker}(\varphi) = \bigcap_{\alpha\in A}\mathrm{Ker}(\pi_\alpha\circ g)$. By the First Isomorphism Theorem, $\mathrm{Im}(g) \cong R/\bigcap_{\alpha\in A}\mathrm{Ker}(\pi_\alpha\circ g)$. #

Corollary 3.40. *Let R be a skewring and $\{R_\alpha / \alpha\in A\}$ be a family of skewrings. Let $g:R\rightarrow \prod_{\alpha\in A} R_\alpha$ be a monomorphic representation of R as a subdirect product of $\{R_\alpha / \alpha\in A\}$. Then $\bigcap_{\alpha\in A} Ker(\pi_\alpha\circ g) = \{0\}$, hence $Im(g) \cong R$.*

Proof. We shall show that $\bigcap_{\alpha\in A}\mathrm{Ker}(\pi_\alpha\circ g) = \{0\}$, let $x\in \bigcap_{\alpha\in A}\mathrm{Ker}(\pi_\alpha\circ g)$. Then $\pi_\alpha\circ g(x) = 0_\alpha$ for every $\alpha\in A$. This implies that $g(x) = (0_\alpha)_{\alpha\in A}$. Since g is a monomorphism, $x = 0$ and $\bigcap_{\alpha\in A}\mathrm{Ker}(\pi_\alpha \circ g) = \{0\}$. By Theorem 3.39, $\mathrm{Im}(g) \cong R$. #

Proposition 3.41. *Let R be a skewring and $L = \{I_\alpha / \alpha\in A\}$ be a family of nonzero normal ideals of R. Define $f: R \to \prod_{\alpha\in A} R/I_\alpha$ by $f(x) = (x+I_\alpha)_{\alpha\in A}$ for every $x\in R$. Then f is a representation of R as a subdirect product of $\{R/I_\alpha / \alpha\in A\}$. Furthermore, if $\bigcap_{\alpha\in A} I_\alpha = \{0\}$, then f is a monomorphic representation of R.*

Proof. Clearly, f is a homomorphism of R. We shall show that Im(f) is a subdirect product of $\{R/I_\alpha / \alpha\in A\}$. It is clear that for every $\alpha\in A$, $\pi_\alpha(\mathrm{Im}(f)) \subseteq R/I_\alpha$. Let $\alpha\in A$, $x\in R$. Then $x+I_\alpha \in R/I_\alpha$, so $f(x)\in \prod_{\alpha\in A} R/I_\alpha$ and $x+I_\alpha = \pi_\alpha(f(x)) \in \pi_\alpha(\mathrm{Im}(f))$. Hence $R/I_\alpha \subseteq \pi_\alpha(\mathrm{Im}(f))$. Therefore $\pi_\alpha \circ f[R] = \pi_\alpha(\mathrm{Im}(f)) = R/I_\alpha$. Hence f is a representation of R as a subdirect product of $\{R/I_\alpha / \alpha\in A\}$.

Next, assume that $\bigcap_{\alpha\in A} I_\alpha = \{0\}$. We shall show that f is a monomorphism. Let $x\in R$ be such that $f(x) = (I_\alpha)_{\alpha\in A}$. Then $(x+I_\alpha)_{\alpha\in A} = (I_\alpha)_{\alpha\in A}$, so $x\in I_\alpha$ for all $\alpha\in A$. By assumption, $x = 0$. Hence f is an injective and is a monomorphism.#

Proposition 3.42. *Let R be a skewring and L the set of all normal ideals of R except {0}. Then R is a subdirectly irreducible if and only if L has a minimum element.*

Proof. Assume that R is a subdirectly irreducible. Suppose L has no minimum element. Then $\cap L = \{0\}$. By Proposition 3.41, we have that $f: R \to \prod_{I\in L} R/I$ defined by $f(x) = (x+I)_{I\in L}$ for every $x\in R$ which it is a monomorphic representation of R as a subdirect product of $\left\{ R/_I \,/\, I \in L \right\}$. By assumption, there exists $I_0 \in L$ such that $\pi_{I_0} \circ f$ is an isomorphism. We shall show that $I_0 = \{0\}$. Let $x \in I_0$. Then $\pi_{I_0} \circ f(x) = \pi_{I_0}((x+I)_{I \in L}) = x + I_0$. Since $x \in I_0$, $x \in Ker(\pi_{I_0} \circ f)$. Since $\pi_{I_0} \circ f$ is an isomorphism, $x = 0$. So $I_0 = \{0\}$ which is contradiction since $\{0\} = I_0 \in L$. Therefore L has a minimum element.

Conversely, assume that L has a minimum element say I_m. Let $\{R_\alpha / \alpha \in A\}$ be a family of skewrings and $f: R \to \prod_{\alpha \in A} R_\alpha$ a monomorphic representation of R as a subdirect product of $\{R_\alpha / \alpha \in A\}$. By Corollary 3.40, $\bigcap_{\alpha \in A} Ker(\pi_\alpha \circ f) = \{0\}$. Suppose that for every $\alpha \in A$, $Ker(\pi_\alpha \circ f) \neq 0$. Then $\{Ker(\pi_\alpha \circ f) / \alpha \in A\} \subseteq L$. Therefore $I_m \subseteq \bigcap_{\alpha \in A} Ker(\pi_\alpha \circ f) = \{0\}$ which is a contradiction. Therefore there exists a $\beta \in A$ such that $Ker(\pi_\alpha \circ f) = 0$, so $\pi_\beta \circ f$ is an isomorphism. Hence R is a subdirectly irreducible. #

Next, we want to show that every skewring is a subdirect product of subdirectly irreducible skewrings. First we need three Lemmas.

Lemma 3.43. *Let R be a nontivial skewring and $x \in R \setminus \{0\}$. Then there exists a maximal normal ideal M of R such that $x \notin M$.*

Proof. Let $L = \{I / I$ is a normal ideal of R and $x \notin I\}$. Since $\{0\} \in L$, L is not empty. Let C be a nonempty chain in L. Clearly, $\cup C$ is a normal ideal of R and $\cup C$ is an upper bound of C. By Zorn's Lemma, L has a maximal element. #

Lemma 3.44. *Using the assumptions of Lemma 3.43, let $\mathcal{I} = \{I / I$ is a normal ideal of R such that $M \subset I\}$. Then $\mathcal{I}$ has a minimum element.*

Proof. Since $R \in \mathfrak{I}$, $\mathfrak{I}$ is not empty. If there exists $I \in \mathfrak{I}$ and $x \notin I$, then this contradicts the maximality of M. Therefore for every $I \in \mathfrak{I}$, $x \in I$. Then we have that $\cap \mathfrak{I}$ is a normal ideal of R which is the minimum element and $x \in \cap \mathfrak{I}$. Hence $\cap \mathfrak{I} \neq M$. #

Lemma 3.45. *Using the assumptions of Lemma 3.43, R/M is a subdirectly irreducible skewring.*

Proof. Let L be the set of normal ideals of R/M except $\{M\}$. By Corollary 2.15, L is isomorphic to the set of normal ideals of R strictly containing M. By Lemma 3.44, L has a minimum element. By Proposition 3.42, R/M is a subdirectly irreducible skewring. #

Theorem 3.46. *Let R be a skewring. Then R is a subdirect product of subdirectly irreducible skewrings.*

Proof. By Lemma 3.43, for all $x \in R \setminus \{0\}$, we have that I_x is a maximal normal ideal of R such that $x \notin I_x$. By Lemma 3.45, R/I_x is subdirectly irreducible. Let $L = \{I_x / x \in R \setminus \{0\}\}$. Let $x \in \cap L$. Suppose that $x \neq 0$. Then $x \notin I_x$ which is a contradiction since $x \in \cap L$. So $\cap L = \{0\}$. By Proposition 3.41, we have that $f: R \rightarrow \prod_{I \in L} R/I$ is a monomorphic representation of R as a subdirect product of $\left\{ R/I \,/\, I \in L \right\}$. Therefore f[R] is a subdirect product of $\left\{ R/I \,/\, I \in L \right\}$. Since $R \cong f[R]$, R is a subdirect product of subdirectly irreducible skewrings. #

Definition 3.47. *A skewring R is* ***semisimple*** *if and only if it is a direct sum of simple normal ideals of R.*

Remark 3.48. *The Cartesian product of finite number of semisimple skewrings is a semisimple skewring.*

Definition 3.49. *A normal ideal I of a skewring R is a* ***direct summand*** *of R if and only if there exists a normal ideal J of R such that* $R = I \oplus J$*.*

Definition 3.50. *A skewring R is* ***completely reducible*** *if and only if every normal ideal of R is a direct summand of R.*

Lemma 3.51. *If U is a set of normal ideals of a skewring R and H is a normal ideal in R, then there exists a subset V of U which is maximal with respect to the existence of* $H \oplus (\oplus\{K / K \in V\})$*.*

Proof. Denote the direct sum in the theorem by X(V). Let L be the set of subsets V of U for which X(V) exists. Since $X(\varnothing) = H$, $\varnothing \in L$ and L is not empty. Partially order P by inclusion. Let C be a nonempty chain in L. let W $= \cup C$. Then W is a subset of U and is an upper bound of C. We shall show that $W \in L$, that is we shall show that X(W) exists.

Claim that for all $K, K' \in W$ such that $K \neq K'$, $K \cap K' = \{0\}$ and for every $K \in W$ such that $K \neq H$, $K \cap H = \{0\}$.

If $K, K' \in W$ and $K \neq K'$, then there exists a $V \in C$ such that $K, K' \in V$. Since X(V) exists, $K \cap K' = \{0\}$. If $K \in W$ and $K \neq H$, then there exists $V \in C$ such that $K \in V$. Since X(V) exists, $H \cap K = \{0\}$. Therefore the claim is true

Hence X(W) exists. Thus $W \in L$ and W is an upper bound of C in L. By Zorn's Lemma, L has a maximal element. #

Lemma 3.52. *let R be a skewring and I,J be normal ideals of R such that R = I⊕J. If H is a normal ideal of R such that I⊆H⊆R, then H = I⊕(J∩H).*

Proof. Suppose thst H is a normal ideal of R such that I⊆H⊆R. Clearly, I+(J∩H)⊆H. Let h∈H. Since R = I⊕J, there exist x∈I and y∈J such that h = x+y and we have x∈H. Since y = h-x∈H, h = x+y∈I+(J∩H), so H⊆I+(J∩H). Therefore H = I+(J∩H). Since R = I⊕J, I∩(J∩H)⊆I∩J = {0} which implies that H = I⊕(J∩H).#

Theorem 3.53. *A skewring R is completely reducible if and only if it is semisimple.*

Proof. Let R be completely reducible. Let L = {S / S is a set of simple normal ideals of R such that X(S) = ⊕{H/H∈S} exists}. By Lemma 3.51, there exists a maximal set of simple normal ideals S such that X(S) = ⊕{H/H∈S} exists. By completely reducibility, R = X(S)⊕K for some normal ideal K of R. If K = {0}, we are done. Suppose that K ≠ {0}.

Claim that K is completely reducible.

Let M be a normal ideal in K. By Remark 3.5 and R = X(S)⊕K, M is a normal ideal in R. Since R is a completely reducible, there exists a normal ideal P of R such that R = M⊕P. Since M⊆K⊆R, by Lemma 3.52, K = M⊕(P∩K). Hence K is completely reducible and the claim is true.

By the maximal property of S and Remark 3.5, K has no nontrivial simple normal ideal. Let 0 ≠ x∈K and $M = \langle x \rangle_n$. be a normal ideal in K which is generated by x. Then M is not simple. Since K is completely reducible and Remark 3.5, there exists a normal ideal P of R such that K = M⊕P. By Remark 3.5, every normal ideal of M is a normal ideal of K. Since K has no nontrivial simple normal ideal, this is true for M. Similarly, by the proof of the claim, M is completely reducible.

Let x^M be a smallest normal ideal in M which is generated by x.

Clearly, $M = \langle x \rangle_n = x^M$. Since M is not simple, there exists a nontrivial normal ideal A_1 of M. Since M is completely reducible, there exists a nontrivial normal ideal B_1 of M such that $M = A_1 \oplus B_1$. Similarly, B_1 has no simple normal ideal and so B_1 is completely reducible. By induction, we have $M = A_1 \oplus B_1 = A_1 \oplus A_2 \oplus B_2 = A_1 \oplus \ldots \oplus A_n \oplus B_n \oplus \ldots$ where $A_i \neq \{0\}$ and $B_i \neq \{0\}$ for every $i \in \mathbf{Z}^+$. Then $\oplus A_i$ exists and it is a normal ideal in M. Since M is completely reducible, there exists a normal ideal D of M such that $M = (\oplus A_i) \oplus D$. Let $D = A_0$, $M = \oplus A_i$. Then there exist an $r \in \mathbf{Z}^+$ and $a_i \in A_i$ such that $x = a_0 + a_1 + \ldots + a_r$. Hence $x^M \subsetneq A_1 \oplus \ldots \oplus A_r \subset M$ which is a contradiction. Thus $K = \{0\}$.

Conversely, let S be the set of simple normal ideals of R such that $R = \oplus\{H / H \in S\}$ and let M be a normal ideal of R. By Lemma 3.51, there exists a maximal subset T of S such that $X(T) = M \oplus (\oplus\{H / H \in T\})$ exists. Suppose X(T) is a proper subskewring of R. If for every $H \in S$, $X(T) \cap H = H$, then $H \subseteq X(T)$ for every $H \in S$ which implies that $X(T) = R$ which is a contradiction. Then there exists an $H \in S$ such that $X(T) \cap H$ is a proper normal ideal of H. Since H is simple, $X(T) \cap H = \{0\}$. Then $X(T \cup \{H\})$ exists which contradicts the maximal property of T. Therefore $X(T) = R$. Hence R is completely reducible. #