เงื่อนไขที่เพียงพอบางประการสำหรับแนววิถีที่เป็นวัฏจักรที่คล้ายคลึงกับปัญหา 3x+1 ใน 2 มิติ

นางสาวอุมารินทร์ ปิ่นตบแต่ง

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรมหาบัณฑิต

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SOME SUFFICIENT CONDITIONS FOR CYCLIC

TRAJECTORIES IN A TWO-DIMENSIONAL ANALOG OF

THE $3X + 1$ PROBLEM

Miss Umarin Pintoptang

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By : Miss Umarin Pintoptang

Field of Study : Mathematics

Thesis Advisor : Assistant Professor Imchit Termwuttipong, Ph.D.

Thesis Co-advisor : Assistant Professor Mark E. Hall, Ph.D.

Accepted by the Faculty of Science, Chulalongkorn University, in Partial Fulfillment of the Requirements for the Master's Degree.

. . . . Dean of Faculty of Science
(Associate Professor Wanchai Phothiphichitr, Ph.D.)

THESIS COMMITTEE

. . . . Chairman
(Associate Professor Wanida Hemakul, Ph.D.)

. . . . Thesis Advisor
(Assistant Professor Imchit Termwuttipong, Ph.D.)

. . . . Thesis Co-advisor
(Assistant Professor Mark E. Hall, Ph.D.)

. . . . Member
(Assistant Professor Amorn Wasanawichit, Ph.D.)

. . . . Member
(Sajee Pianskool, Ph.D.)

อุมารินทร์ ปิ่นตบแต่ง : เงื่อนไขที่เพียงพอบางประการสำหรับแนววิถีที่เป็นวัฏจักรที่คล้ายคลึงกับปัญหา $3x+1$ ใน 2 มิติ (SOME SUFFICIENT CONDITIONS FOR CYCLIC TRAJECTORIES IN A TWO-DIMENSIONAL ANALOG OF THE $3x+1$ PROBLEM) อ.ที่ปรึกษา : ผู้ช่วยศาสตราจารย์ ดร. อิ่มจิตต์ เติมวุฒิพงษ์ , อ.ที่ปรึกษาร่วม : ผู้ช่วยศาสตราจารย์ ดร. มาร์ค เอ็ดวิน ฮอลล์, 47 หน้า ISBN 974-03-0188-6.

ปัญหา $3x+1$ เป็นปัญหาเกี่ยวกับพฤติกรรมของการดำเนินการซ้ำของฟังก์ชันซึ่งนิยามโดย

$$T(x)=\begin{cases}(3x+1)/2 & \text{เมื่อ } x \text{ เป็นจำนวนคี่} \\ x/2 & \text{เมื่อ } x \text{ เป็นจำนวนคู่}\end{cases}$$

ข้อความคาดการณ์ $3x+1$ กล่าวว่า ถ้าเริ่มต้นจากจำนวนเต็มบวก α ใด ๆ ดำเนินการส่งด้วยฟังก์ชันข้างต้นซ้ำ ๆ กันในที่สุดจะได้ค่าเป็น 1

ในวิทยานิพนธ์ฉบับนี้เราจะขยายการศึกษาปัญหาดังกล่าวดังนี้ ให้ Z_* เป็นเซตของจำนวนเต็มที่ไม่เป็นลบทั้งหมด ให้ k เป็นจำนวนเฉพาะคงที่และ $D=\begin{bmatrix}k & 0\\ 0 & k\end{bmatrix}$ ให้ A เป็นเมตริกซ์ของจำนวนเต็มบวกขนาด 2×2 ใด ๆ สำหรับแต่ละค่า β ที่คงที่ใน Z_*^2 ให้ $T: Z_*^2 \to Z_*^2$ กำหนดโดย สำหรับแต่ละ $\alpha \in Z_*^2$

$$T(\alpha)=\begin{cases}D^{-1}\alpha & \text{เมื่อ } D^{-1}\alpha \in Z_*^2 \\ A\alpha+\beta & \text{เมื่อ } D^{-1}\alpha \notin Z_*^2\end{cases}$$

ผลการวิจัยที่รายงานในวิทยานิพนธ์ฉบับนี้เกี่ยวกับการยืนยันว่าแนววิถี $\langle \alpha, T(\alpha), T^2(\alpha), \ldots \rangle$ จะเป็นวัฏจักรหรือไม่ เราพิสูจน์ว่าสำหรับเมตริกซ์ A บางรูปแบบแนววิถีไม่เป็นวัฏจักรไม่ว่าจะเลือก $\beta \in Z_*^2$ เป็นค่าใดก็ตามและสำหรับเมตริกซ์ A บางรูปแบบค่าของ β ที่กำหนดให้จะรับประกันได้ว่าแนววิถีจะเป็นวัฏจักร

ภาควิชา **คณิตศาสตร์**	ลายมือชื่อนิสิต..
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UMARIN PINTOPTANG : SOME SUFFICIENT CONDITIONS FOR CYCLIC TRAJECTORIES IN A TWO-DIMENSIONAL ANALOG OF THE $3x+1$ PROBLEM. THESIS ADVISOR : ASSISTANT PROFESSOR IMCHIT TERMWUTTIPONG, Ph.D., THESIS CO-ADVISOR : ASSISTANT PROFESSOR MARK E. HALL, Ph.D., 47 pp. ISBN 974-03-0188-6.

The $3x+1$ problem concerns the behavior of the iterates of the function defined by

$$T(x)=\begin{cases}(3x+1)/2 & \text{if } x \text{ is odd,}\\ x/2 & \text{if } x \text{ is even.}\end{cases}$$

The $3x+1$ Conjecture asserts that, starting from any positive integer α, repeated iteration of this function eventually produces the value 1.

In this thesis we study the following extended version of the above problem. Let Z_* be the set of all nonnegative integers. Let k be any fixed prime number and $D=\begin{bmatrix}k & 0\\ 0 & k\end{bmatrix}$. Let A be any 2×2 matrix of positive integers. For a fixed $\beta\in Z_*^2$, let $T: Z_*^2 \to Z_*^2$ be defined by, for each $\alpha\in Z_*^2$,

$$T(\alpha)=\begin{cases}D^{-1}\alpha & \text{if } D^{-1}\alpha\in Z_*^2,\\ A\alpha+\beta & \text{if } D^{-1}\alpha\notin Z_*^2.\end{cases}$$

The research reported in this thesis concerns determining whether or not the trajectory $\langle \alpha,\ T(\alpha),\ T^2(\alpha),\ldots\rangle$ is cyclic. For some forms of the matrix A it is proved that the trajectory cannot be cyclic for any choice of $\beta\in Z_*^2$. In some other cases values of β are given which ensure a cyclic trajectory.

Department **Mathematics**	Student's signature……………………………
Field of Study **Mathematics**	Advisor's signature…………………………...
Academic year 2001	Co-advisor's signature ………………………

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CONTENT

CHAPTER I

INTRODUCTION

The $3x + 1$ problem concerns the behavior of the iterates of the function which takes odd integers x to $(3x + 1)/2$ and even integers x to $x/2$:

$$T(x) = \begin{cases} (3x+1)/2 & \text{if } x \equiv 1 \pmod{2}, \\ x/2 & \text{if } x \equiv 0 \pmod{2}. \end{cases}$$

The $3x+1$ Conjecture asserts that, starting from any positive integer α, repeated iteration of this function eventually produces the value 1. We call the sequence of iterates $\langle \alpha, T(\alpha), T^2(\alpha), \ldots \rangle$ the **trajectory** of α. There are three possible behaviors for such trajectories when $\alpha > 0$.

(i) **Convergent trajectory**. The iterate $T^n(\alpha) = 1$ for some natural number n.

(ii) **Non-trivial cyclic trajectory**. The sequence $\big(T^n(\alpha)\big)$ eventually becomes periodic and $T^n(\alpha) \neq 1$ for any $n \geq 1$.

(iii) **Divergent trajectory**. $\lim_{n \to \infty} T^n(\alpha) = \infty$.

The $3x + 1$ Conjecture asserts that all trajectories of positive α are convergent. Note that in both cases (i) and (ii) the trajectory of α is cyclic . The difference is that in case (i) the trajectory of α contains the special value 1. ([1], Lagarias, J. C.)

At present, no one has been able to prove the $3x + 1$ Conjecture or find a counterexample. In order to gain new insights into this problem and make it more tractable as well, we will extend our study as follows, and consider all cyclic trajectories, instead of just convergent ones.

Let $\mathbb{Z}_*$ denote the set of all nonnegative integers. Let k be any fixed prime number and $D = \left[\begin{smallmatrix} k & 0 \\ 0 & k \end{smallmatrix}\right]$. Let A be any 2×2 matrix of positive integers. For a fixed $\beta \in \mathbb{Z}_*^2$, let $T : \mathbb{Z}_*^2 \longrightarrow \mathbb{Z}_*^2$ be defined by,
for each $\alpha \in \mathbb{Z}_*^2$,

$$T(\alpha) = \begin{cases} D^{-1}\alpha & \text{if } D^{-1}\alpha \in \mathbb{Z}_*^2, \\ A\alpha + \beta & \text{otherwise.} \end{cases}$$

The objective of this thesis is to find some sufficient conditions on A and/or α which ensure that for an appropriate β the trajectory $\langle \alpha, T(\alpha), T^2(\alpha), \ldots \rangle$ is cyclic.

The remainder of this thesis is organized as follows. In Chapter 2 we summarize some essential facts and give some notations which will be used in the succeeding chapters.

In Chapter 3 some conditions on A, α and β are investigated. In particular, a few theorems concerning situations guaranteeing that the trajectory is cyclic are proved in this chapter.

Finally, in Chapter 4 we give examples and conclude our work. The first and the second sections of the chapter provide some concrete examples, while the third one summarizes our results and discusses topics for further research.

CHAPTER II

BACKGROUND AND NOTATIONS

Notation. For any set X, let X^2 denote the set of all column vectors $\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right]$ where $x_1, x_2 \in X$ and let $M_2(X)$ denote the set of all 2×2 matrices whose entries are elements in X.

Definition 2.1. Let X be a nonempty set, and let (x_n) be a sequence in X. The sequence (x_n) is said to be **cyclic** if there exist $m, l \in \mathbb{N}$ such that $x_m = x_{m+nl}$ for all $n \in \mathbb{N}$.

Definition 2.2. Let X be a nonempty set and $f : X \longrightarrow X$. For each $\alpha_0 \in X$, the sequence $\langle \alpha_0, f(\alpha_0), f^2(\alpha_0), \ldots \rangle$ is called a **trajectory (of α_0)**.
For any $n \in \mathbb{N}$, we denote the value $f^n(\alpha_0)$ by α_n. In particular, the trajectory $\langle \alpha_0, f(\alpha_0), f^2(\alpha_0), \ldots \rangle$ will usually be written as $\langle \alpha_0, \alpha_1, \alpha_2, \ldots \rangle$.

Proposition 2.3. *Let X be a nonempty set, $f : X \longrightarrow X$ and $\langle \alpha_0, \alpha_1, \alpha_2, \ldots \rangle$ a trajectory of α_0. Then the following are equivalent:*

(i) $\langle \alpha_0, \alpha_1, \alpha_2, \ldots \rangle$ *is cyclic,*

(ii) *there exist $i, j \in \mathbb{N}$ such that $i < j$ and $\alpha_i = \alpha_j$.*

Proof. (i)$\Rightarrow$(ii) Assume that the trajectory $\langle \alpha_0, \alpha_1, \alpha_2, \ldots \rangle$ is cyclic. Then there exist $m, l \in \mathbb{N}$ such that $\alpha_m = \alpha_{m+nl}$ for all $n \in \mathbb{N}$. Since $l \in \mathbb{N}, m < m + l$. Hence there exist $i = m, j = m + l \in \mathbb{N}$ such that $i < j$ and $\alpha_i = \alpha_j$.
(i)$\Leftarrow$(ii) Assume that there exist $i, j \in \mathbb{N}$ such that $i < j$ and $\alpha_i = \alpha_j$. We will show that there exist $m, l \in \mathbb{N}$ such that $\alpha_m = \alpha_{m+nl}$ for all $n \in \mathbb{N}$. Since

$i < j, j - i \in \mathbb{N}$. We will prove by mathematical induction that $\alpha_i = \alpha_{i+n(j-i)}$ for all $n \in \mathbb{N}$. Since $\alpha_i = \alpha_j, \alpha_i = \alpha_{i+(j-i)}$. Let $k \in \mathbb{N}$. Assume that $\alpha_i = \alpha_{i+k(j-i)}$. We will show that $\alpha_i = \alpha_{i+(k+1)(j-i)}$. Since $\alpha_j = \alpha_{i+(j-i)} = f^{j-i}(\alpha_i) = f^{j-i}(\alpha_{i+k(j-i)})$ $= \alpha_{i+k(j-i)+(j-i)} = \alpha_{i+(k+1)(j-i)}, \alpha_i = \alpha_j = \alpha_{i+(k+1)(j-i)}$. By mathematical induction, $\alpha_i = \alpha_{i+n(j-i)}$ for all $n \in \mathbb{N}$. Hence there exist $m = i, l = j - i \in \mathbb{N}$ such that $\alpha_m = \alpha_{m+nl}$ for all $n \in \mathbb{N}$. $\square$

Notation. Let k be a prime number, $x \in \mathbb{Z}_*, \alpha = \left[\begin{smallmatrix} e_1 \\ e_2 \end{smallmatrix}\right] \in \mathbb{Z}_*^2$, and $A = [a_{ij}] \in M_2(\mathbb{N})$. We define the following notations:

$\bar{x}$ is the equivalent class of x in $\mathbb{Z}_k$,

$\bar{\alpha} = \left[\begin{smallmatrix} \bar{e}_1 \\ \bar{e}_2 \end{smallmatrix}\right]$ in $\mathbb{Z}_k^2$,

$\bar{A} = [\bar{a}_{ij}]$ in $M_2(\mathbb{Z}_k)$.

Notation. Let R be a ring. For any $A \in M_2(\mathrm{R})$, let $\mathrm{Im}(A) = \{Ax \mid x \in \mathrm{R}^2\}$.

Theorem 2.4 (Cayley-Hamilton Theorem [2], page 194 **).** *If A is a square matrix over a commutative ring with identity and $\chi(x)$ is its characteristic polynomial, then $\chi(A) = 0$.*

Definition 2.5. ([3], page 198) A **Fermat number** is an integer of the form $F_n = 2^{2^n} + 1$, where $n \geq 0$. If F_n is prime, F_n is called a **Fermat prime**.

Theorem 2.6 (Fermat's Theorem [3], page 74 **).** *Let p be any prime number, and a be an integer such that $p \nmid a$. Then $a^{p-1} \equiv 1 \ (mod\ p)$. Equivalently, if a is any integer such that $p \nmid a$, then $a^p \equiv a \ (mod\ p)$.*

Theorem 2.7. ([2], page 79) *Let V and W be vector spaces over field F and let $T : V \longrightarrow W$ be a linear transformation from V into W.*

T is $1-1$ if and only if for any $v \in V$, if $T(v) = 0$, then $v = 0$.

Lemma 2.8. *Let σ be an element of the symmetric group S_n and $b \in \{1, 2, \ldots, n\}$. Then $\{\sigma^l(b) \mid l \in \mathbb{N}\} = \{\sigma^{-l}(b) \mid l \in \mathbb{N}\}$.*

Proof. We will prove this by considering cases based on $|S_n|$.

Case 1. $|S_n| = 1$. Then $S_n = \{e\}$ and $\sigma^l = e = \sigma^{-l}$ for all $l \in \mathbb{N}$ where e is the identity map.

Case 2. $|S_n| > 1$.

($\subseteq$) Let $x \in \{\sigma^l(b) \mid l \in \mathbb{N}\}$. Then $x = \sigma^t(b)$ for some $t \in \mathbb{N}$. Since $\sigma^{|S_n|} = e$, it follows that $\sigma^{-1} = \sigma^{|S_n|-1}$, and thus

$$\begin{aligned} x &= \sigma^t(b) \\ &= (\sigma^{-1})^{-t}(b) \\ &= \left(\sigma^{|S_n|-1}\right)^{-t}(b) \\ &= \sigma^{-t(|S_n|-1)}(b). \end{aligned}$$

Because $t(|S_n|-1) \in \mathbb{N}$, this shows $x \in \{\sigma^{-l}(b) \mid l \in \mathbb{N}\}$.

($\supseteq$) Let $x \in \{\sigma^{-l}(b) \mid l \in \mathbb{N}\}$. Then $x = \sigma^{-t}(b)$ for some $t \in \mathbb{N}$. As above,

$$\begin{aligned} x &= \sigma^{-t}(b) \\ &= (\sigma^{-1})^{t}(b) \\ &= \left(\sigma^{|S_n|-1}\right)^{t}(b) \\ &= \sigma^{t(|S_n|-1)}(b). \end{aligned}$$

Hence $x \in \{\sigma^l(b) \mid l \in \mathbb{N}\}$. □

Definition 2.9. Let σ be an element of S_n. We say that σ **can be represented by a single cycle** if σ can be represented by a cycle $(i_1 \;\; i_2 \;\; \cdots \;\; i_n)$, where $i_1, i_2, \ldots, i_n$ are distinct elements of $\{1, 2, \ldots, n\}$.

Proposition 2.10. *Let $\sigma \in S_n$. If σ cannot be represented by a single cycle, then for any $a \in \{1, 2, \ldots, n\}$, there exists $b \in \{1, 2, \ldots, n\}$ such that $\sigma^l(b) \neq a$ for all $l \in \mathbb{N}$.*

Proof. Assume that σ cannot be represented by a single cycle. Let $a \in \{1, 2, \ldots, n\}$. Then $\{\sigma^l(a) \mid l \in \mathbb{N}\} \subsetneq \{1, 2, \ldots, n\}$. Thus there exists $b \in \{1, 2, \ldots, n\}$ such that $\sigma^l(a) \neq b$ for all $l \in \mathbb{N}$. Therefore for all $l \in \mathbb{N}$, $a \neq (\sigma^l)^{-1}(b)$ since $(\sigma^l)^{-1}$ is injective. Since $\{\sigma^l(b) \mid l \in \mathbb{N}\} = \{\sigma^{-l}(b) \mid l \in \mathbb{N}\}$, it follows that $a \neq \sigma^l(b)$ for all $l \in \mathbb{N}$. $\square$

CHAPTER III

SUFFICIENT CONDITIONS FOR CYCLIC TRAJECTORIES

Let $\mathbb{Z}_*$ denote the set of all nonnegative integers. Let k be any fixed prime number and $D = \left[\begin{smallmatrix} k & 0 \\ 0 & k \end{smallmatrix}\right]$. Let A be any 2×2 matrix of positive integers , i.e., $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$, where $a, b, c, d \in \mathbb{N}$. For a fixed $\beta \in \mathbb{Z}_*^2$, let $T : \mathbb{Z}_*^2 \longrightarrow \mathbb{Z}_*^2$ be defined by, for each $\alpha \in \mathbb{Z}_*^2$,

$$T(\alpha) = \begin{cases} D^{-1}\alpha & \text{if } D^{-1}\alpha \in \mathbb{Z}_*^2, \\ A\alpha + \beta & \text{otherwise.} \end{cases}$$

As stated above the objective of this thesis is to find some sufficient conditions on A and/or α which ensure that for an appropriate β the trajectory $\langle \alpha, T(\alpha), T^2(\alpha), \ldots \rangle$ is cyclic. In this chapter we will derive some general conditions of this type, then investigate a few more specific situations.

It is obvious that if $\alpha = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right] = \vec{0}$, then the trajectory is certainly cyclic , so we confine our investigation to the case $\alpha \neq \vec{0}$.

We first note a necessary condition for the trajectory $\langle \alpha_0, \alpha_1, \alpha_2, \ldots \rangle$ to be cyclic as follows:

Proposition 3.1. *If $T^n(\alpha_0) \neq D^{-1}\alpha_{n-1}$ for all $n \in \mathbb{N}$, then the trajectory $\langle \alpha_0, \alpha_1, \alpha_2, \ldots \rangle$ is not cyclic. Hence the trajectory $\langle \alpha_0, \alpha_1, \alpha_2, \ldots \rangle$ can be cyclic only if $T^n(\alpha_0) = D^{-1}\alpha_{n-1}$ for some $n \in \mathbb{N}$.*

Proof. Assume that $T^n(\alpha_0) \neq D^{-1}\alpha_{n-1}$ for all $n \in \mathbb{N}$, but the trajectory $\langle \alpha_0, \alpha_1, \alpha_2, \ldots \rangle$

is cyclic. Then $\alpha_0 \neq \vec{0}$ since otherwise $T^n(\alpha_0) = D^{-1}\alpha_0$ for all $n \in \mathbb{N}$, and $T^n(\alpha_0) = A^n\alpha_0 + A^{n-1}\beta + \cdots + \beta$ for all $n \in \mathbb{N}$. Since $\langle \alpha_0, \alpha_1, \alpha_2, \ldots \rangle$ is cyclic, there exist $l, m \in \mathbb{N}$ such that $l < m$ and $\alpha_l = \alpha_m$, so

$$A^l\alpha_0 + A^{l-1}\beta + \cdots + \beta = \alpha_l = \alpha_m = A^m\alpha_o + A^{m-1}\beta + \cdots + \beta,$$

and hence

$$\begin{aligned}\vec{0} &= A^m\alpha_o + A^{m-1}\beta + \cdots + A^{l+1}\beta + A^l\beta - A^l\alpha_0 \\ &= A^l(A^{m-l} - I_2)\alpha_0 + A^{m-1}\beta + \cdots + A^{l+1}\beta + A^l\beta. \qquad (3.1)\end{aligned}$$

Because $A \in M_2(\mathbb{N})$, $A^i \in M_2(\mathbb{N})$ for all $i \in \mathbb{N}$, so $A^{m-l} - I_2 \in M_2(\mathbb{N})$ or $A^{m-l} - I_2 = \left[\begin{smallmatrix} 0 & b \\ c & 0 \end{smallmatrix}\right]$ for some $b, c \in \mathbb{N}$. In either case equation (3.1) can be true only when $\alpha_0 = \vec{0}$ and $\beta = \vec{0}$. Hence we have a contradiction. □

We now consider the situation when $T^n(\alpha_0) = D^{-1}\alpha_{n-1}$ for some $n \in \mathbb{N}$. By simple verification we have the following assertions.

(a) The following are equivalent for any $n \in \mathbb{N}$:

- (i) $T^n(\alpha_0) = D^{-1}\alpha_{n-1}$,
- (ii) $D^{-1}\alpha_{n-1} \in \mathbb{Z}^2_*$,
- (iii) $\alpha_{n-1} \in (k\mathbb{Z})^2$,
- (iv) $\bar{\alpha}_{n-1} = \vec{0}$ in $\mathbb{Z}^2_k$.

(b) If $\alpha_0 \in (k\mathbb{Z})^2$, then there exists an $l \in \mathbb{N}$ such that $\alpha_l = T^l(\alpha_0) \notin (k\mathbb{Z})^2$, and $\langle \alpha_0, \alpha_1, \alpha_2, \ldots \rangle$ is cyclic if and only if $\langle \alpha_l, \alpha_{l+1}, \alpha_{l+2}, \ldots \rangle$ is cyclic.

According to the assertion (b), from now on we may assume that $\alpha_0 \notin (k\mathbb{Z})^2$.

Definition 3.2. For each $A \in M_2(\mathbb{N})$ and for each $\beta \in \mathbb{Z}_*^2$, define $\varphi : \mathbb{Z}_k^2 \longrightarrow \mathbb{Z}_k^2$ by $\varphi(v) = \bar{A}v + \bar{\beta}$ for all $v \in \mathbb{Z}_k^2$.

We say that the ordered pair (A, β) **satisfies the condition** $(*)$ if for any $v \in \mathbb{Z}_k^2$, there exists an $l \in \mathbb{N}$ such that $\varphi^l(v) = \vec{0}$.

Proposition 3.3. *If (A, β) satisfies the condition $(*)$, then*

(i) *for any $\alpha_0 \in \mathbb{Z}^2 - (k\mathbb{Z})^2$, there exists an $n \in \mathbb{N}$ such that $T^n(\alpha_0) = D^{-1}\alpha_{n-1}$, and*

(ii) $\bar{\beta} \in \mathrm{Im}(\bar{A})$.

Proof. (i) Assume that (A, β) satisfies the condition $(*)$, i.e., for any $v \in \mathbb{Z}_k^2$, there exists an $l \in \mathbb{N}$ such that $\varphi^l(v) = \vec{0}$. Let $\alpha_0 \in \mathbb{Z}^2 - (k\mathbb{Z})^2$. If there exists an $m \in \mathbb{N}$ such that $m \leq l$ and $T^m(\alpha_0) = D^{-1}\alpha_{m-1}$, then the proof is done. Suppose that $T^m(\alpha_0) \neq D^{-1}\alpha_{m-1}$ for all $m \leq l$. So $\alpha_m = T^m(\alpha_0) = A\alpha_{m-1} + \beta$ for $1 \leq m \leq l$, hence $\bar{\alpha}_l = \varphi^l(\bar{\alpha}_0) = \vec{0}$. Since $\bar{\alpha}_l = \vec{0}$ if and only if $\alpha_l \in (k\mathbb{Z})^2$, it follows that $\alpha_{l+1} = T^{l+1}(\alpha_0) = D^{-1}\alpha_l$. Hence there exists an $n = l + 1 \in \mathbb{N}$ such that $T^n(\alpha_0) = D^{-1}\alpha_{n-1}$.

(ii) Note that for any $v \in \mathbb{Z}_k^2$ and any $l \in \mathbb{N}$,

$$\vec{0} = \varphi^l(v) = \varphi\left(\varphi^{l-1}(v)\right) = \bar{A}\varphi^{l-1}(v) + \bar{\beta}.$$

This implies that $\bar{\beta} \in \mathrm{Im}(\bar{A})$, so (A, β) satisfies the condition $(*)$ implies $\bar{\beta} \in \mathrm{Im}(\bar{A})$. $\square$

The following results show the important role that $\det(\bar{A})$ plays in determining whether the ordered pair (A, β) satisfies the condition $(*)$.

Lemma 3.4. *If $\det(\bar{A}) \neq \bar{0}$ and φ is the identity map, then $T^n(\alpha_0) \neq D^{-1}\alpha_{n-1}$ for all $n \in \mathbb{N}$, and hence the trajectory $\langle \alpha_0, \alpha_1, \alpha_2, \ldots \rangle$ is not cyclic.*

Proof. Assume that $\det(\bar{A}) \neq \bar{0}$ and φ is the identity map. Then for any $v \in \mathbb{Z}_k^2$ and $l \in \mathbb{N}$, $\varphi^l(v) = v$, and $\varphi^l(v) = \vec{0}$ if and only if $v = \vec{0}$ and $\bar{\beta} = \vec{0}$. Hence (A, β) does not satisfy the condition $(*)$. Next we will prove by using mathematical induction that when $\bar{\beta} = \vec{0}$, for any $\alpha_0 \in \mathbb{Z}^2 - (k\mathbb{Z})^2$, $T^n(\alpha_0) \neq D^{-1}\alpha_{n-1}$ for all $n \in \mathbb{N}$, and hence by Proposition 3.1, the trajectory $\langle \alpha_0, \alpha_1, \alpha_2, \ldots \rangle$ is not cyclic. Let $\bar{\beta} = \vec{0}$ and $\alpha_0 \in \mathbb{Z}^2 - (k\mathbb{Z})^2$. Then $T(\alpha_0) \neq D^{-1}\alpha_0$ since $\bar{\alpha}_0 \neq \vec{0}$. Let $k \in \mathbb{N}$. Assume that $T^k(\alpha_0) \neq D^{-1}\alpha_{k-1}$. This means that $\bar{\alpha}_{k-1} \neq \vec{0}$. We will show that $T^{k+1}(\alpha_0) \neq D^{-1}\alpha_k$. By induction hypothesis we have, $\alpha_k = T^k(\alpha_0) = A\alpha_{k-1} + \beta$, so $\bar{\alpha}_k = \bar{A}\bar{\alpha}_{k-1} + \bar{\beta} = \varphi(\bar{\alpha}_{k-1}) = \bar{\alpha}_{k-1}$. Thus $\bar{\alpha}_k \neq \vec{0}$, and hence $\alpha_{k+1} \neq D^{-1}\alpha_k$. By mathematical induction, $T^n(\alpha_0) \neq D^{-1}\alpha_{n-1}$ for all $n \in \mathbb{N}$. □

Lemma 3.5. *If φ is a bijection that is not the identity map, then φ can be represented by a single cycle if and only if (A, β) satisfies the condition $(*)$.*

Proof. Assume that φ is a bijection that is not the identity map. We will prove that φ can be represented by a single cycle if and only if (A, β) satisfies the condition $(*)$.

($\Rightarrow$) Suppose φ can be represented by a single cycle. Then for any $v \in \mathbb{Z}_k^2$, $\left\{\varphi^l(v) \mid l \in \mathbb{N}\right\} = \mathbb{Z}_k^2$ and since $\vec{0} \in \mathbb{Z}_k^2$, there exists an $l \in \mathbb{N}$ such that $\varphi^l(v) = \vec{0}$. Hence (A, β) satisfies the condition $(*)$.

($\Leftarrow$) Suppose φ cannot be represented by a single cycle. By Proposition 2.10, there exists an element $u \in \mathbb{Z}_k^2$ such that $\varphi^l(u) \neq \vec{0}$ for all $l \in \mathbb{N}$. Hence (A, β) does not satisfy the condition $(*)$. □

Lemma 3.6. *If* $\det(\bar{A}) = \bar{0}$, *then* $(\bar{A})^2 = (\bar{a} + \bar{d})\bar{A}$, *where* $\bar{A} = \left[\begin{smallmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{smallmatrix}\right]$, *with* $\bar{a}, \bar{b}, \bar{c}, \bar{d} \in \mathbb{Z}_k$.

Proof. Assume that $\det(\bar{A}) = \bar{0}$. By definition, the characteristic polynomial of

$\bar{A}$ is $\chi_{\bar{A}}(x) = \det(xI_2 - \bar{A})$, where x is an indeterminate. Then

$$\begin{aligned} \chi_{\bar{A}}(x) &= \det(xI_2 - \bar{A}) \\ &= \det\left(\left[\begin{smallmatrix} x-\bar{a} & -\bar{b} \\ -\bar{c} & x-\bar{d} \end{smallmatrix}\right]\right) \\ &= (x-\bar{a})(x-\bar{d}) - \bar{b}\bar{c} \\ &= x^2 - (\bar{a}+\bar{d})x + (\bar{a}\bar{d} - \bar{b}\bar{c}) \\ &= x^2 - (\bar{a}+\bar{d})x \quad \text{since } \bar{a}\bar{d} - \bar{b}\bar{c} = \det(\bar{A}) = \bar{0}. \end{aligned}$$

By the Cayley-Hamilton Theorem, $\chi_{\bar{A}}(\bar{A}) = \bar{0}$, which implies $(\bar{A})^2 - (\bar{a}+\bar{d})\bar{A} = \bar{0}$, so $(\bar{A})^2 = (\bar{a}+\bar{d})\bar{A}$. □

Lemma 3.7. *If* $\det(\bar{A}) = \bar{0}$ *and* $(\bar{A})^2 \neq \bar{0}$*, then* $\bar{A}|_{\mathrm{Im}(\bar{A})}$ *and* $\varphi|_{\mathrm{Im}(\bar{A})}$ *are bijective. If in addition* $\bar{\beta} \in \mathrm{Im}(\bar{A})$*, then* $\varphi|_{\mathrm{Im}(\bar{A})} \in \mathrm{Sym}\left(\mathrm{Im}(\bar{A})\right)$.

Proof. Assume that $\det(\bar{A}) = \bar{0}$ and $(\bar{A})^2 \neq \bar{0}$. We will show that $\bar{A}|_{\mathrm{Im}(\bar{A})}$ is bijective. To simplify the notation we will write f for $\bar{A}|_{\mathrm{Im}(\bar{A})}$. Since $\mathrm{Im}(\bar{A})$ is finite, it suffices to show that f is injective. Since f is linear, it is enough to show that for any $w \in \mathrm{Im}(\bar{A})$, $f(w) = \vec{0}$ implies $w = \vec{0}$. Let $w \in \mathrm{Im}(\bar{A})$ be such that $f(w) = \vec{0}$. Since $w \in \mathrm{Im}(\bar{A})$, $w = \bar{A}v$ for some $v \in \mathbb{Z}_k^2$, so

$$\vec{0} = f(w) = \bar{A}w = (\bar{A})^2 v = (\bar{a}+\bar{d})\bar{A}v = (\bar{a}+\bar{d})w.$$

Because $(\bar{A})^2 \neq \bar{0}$ and $(\bar{A})^2 = (\bar{a}+\bar{d})\bar{A}$, $(\bar{a}+\bar{d}) \neq \bar{0}$, so we can conclude that w must be $\vec{0}$. Hence f is bijective.

Next, we will show that $\varphi|_{\mathrm{Im}(\bar{A})}$ is bijective. Since $\mathrm{Im}(\bar{A})$ is finite, again it suffices to show that $\varphi|_{\mathrm{Im}(\bar{A})}$ is injective. Let $w_1, w_2 \in \mathrm{Im}(\bar{A})$ be such that $\varphi|_{\mathrm{Im}(\bar{A})}(w_1) = \varphi|_{\mathrm{Im}(\bar{A})}(w_2)$. Then $\bar{A}w_1 + \bar{\beta} = \bar{A}w_2 + \bar{\beta}$, so $\bar{A}w_1 = \bar{A}w_2$. Since $\bar{A}|_{\mathrm{Im}(\bar{A})}$ is injective, $w_1 = w_2$. This implies that $\varphi|_{\mathrm{Im}(\bar{A})}$ is injective. Hence $\varphi|_{\mathrm{Im}(\bar{A})}$ is bijective. If in addition $\bar{\beta} \in \mathrm{Im}(\bar{A})$, then $\varphi|_{\mathrm{Im}(\bar{A})} : \mathrm{Im}(\bar{A}) \to \mathrm{Im}(\bar{A})$, and thus $\varphi|_{\mathrm{Im}(\bar{A})} \in \mathrm{Sym}\left(\mathrm{Im}(\bar{A})\right)$. □

Lemma 3.8. *If* $\det(\bar{A}) = \bar{0}$, $(\bar{A})^2 \neq \bar{0}$, $(\bar{A})^2 \neq \bar{A}$ *and* $\bar{\beta} \in \mathrm{Im}(\bar{A})$, *then* (A, β) *does not satisfy the condition* $(*)$.

Proof. Assume that $\det(\bar{A}) = \bar{0}$, $(\bar{A})^2 \neq \bar{0}$, $(\bar{A})^2 \neq \bar{A}$ and $\bar{\beta} \in \mathrm{Im}(\bar{A})$. By Lemma 3.6, $\bar{a} + \bar{d} \neq \bar{1}$ and $\bar{a} + \bar{d} \neq \bar{0}$. Since k is prime and $(\bar{a} + \bar{d}) \in \mathbb{Z}_k - \{\bar{0}\}$, $k \nmid (a + d)$, hence by Fermat's Theorem,

$$(a+d)^k \equiv (a+d) \ (\bmod k),$$

therefore

$$(a+d)^k - 1 \equiv \Big((a+d) - 1\Big) \ (\bmod k).$$

Since $(a+d)^k - 1 = \Big((a+d) - 1\Big)\Big((a+d)^{k-1} + (a+d)^{k-2} + \cdots + 1\Big)$ and $\Big(k, (a+d) - 1\Big) = 1$,

$$(a+d)^{k-1} + (a+d)^{k-2} + \cdots + 1 \equiv 1 \ (\bmod k),$$

so $\overline{(a+d)^{k-1} + (a+d)^{k-2} + \cdots + 1} = \bar{1}$ in $\mathbb{Z}_k$, and hence

$$\Big((\bar{a} + \bar{d})^{k-1} + (\bar{a} + \bar{d})^{k-2} + \cdots + \bar{1}\Big)\bar{\beta} = \bar{\beta}. \tag{3.2}$$

It is easy to check that $(\varphi|_{\mathrm{Im}(\bar{A})})^l(\vec{0}) = \Big((\bar{a} + \bar{d})^{l-1} + (\bar{a} + \bar{d})^{l-2} + \cdots + \bar{1}\Big)\bar{\beta}$ for all $l \in \mathbb{N}$, and thus equation (3.2) implies that $\Big|\Big\{(\varphi|_{\mathrm{Im}(\bar{A})})^l(\vec{0})\Big|\ l \in \mathbb{N}\Big\}\Big| \leq k - 1 < k$, so $\varphi|_{\mathrm{Im}(\bar{A})}$ cannot be represented by a single cycle.

We will show that (A, β) does not satisfy the condition $(*)$, i.e., there exists $v \in \mathbb{Z}_k^2$ such that $\varphi^l(v) \neq \vec{0}$ for all $l \in \mathbb{N}$. Since $\varphi|_{\mathrm{Im}(\bar{A})}$ cannot be represented by a single cycle, by Proposition 2.10 there exists $w \in \mathrm{Im}(\bar{A})$ such that $(\varphi|_{\mathrm{Im}(\bar{A})})^l(w) \neq \vec{0}$ for all $l \in \mathbb{N}$. By Lemma 3.7 $\varphi|_{\mathrm{Im}(\bar{A})} \in \mathrm{Sym}\big(\mathrm{Im}(\bar{A})\big)$, so there exists $u \in \mathrm{Im}(\bar{A})$ such that $\varphi|_{\mathrm{Im}(\bar{A})}(u) = w$. If $w = \vec{0}$, then $(\varphi|_{\mathrm{Im}(\bar{A})})^{|\mathrm{Im}(\bar{A})|!}(w) = e(w) = w = \vec{0}$ where $e \in \mathrm{Sym}\big(\mathrm{Im}(\bar{A})\big)$ is the identity, a contradiction. Therefore $w \neq \vec{0}$. We will show that $\varphi^l(u) \neq \vec{0}$ for all $l \in \mathbb{N}$. Let $l \in \mathbb{N}$. If $l = 1$, then $\varphi^l(u) = \varphi(u) = w \neq \vec{0}$.

Now we assume $l > 1$. We have $\varphi^l(u)= (\varphi|_{\mathrm{Im}(\bar{A})})^{l-1}(\varphi(u))= (\varphi|_{\mathrm{Im}(\bar{A})})^{l-1}(w) \neq \vec{0}$. Hence (A, β) does not satisfy the condition $(*)$ in this case. □

Lemma 3.9. *If* $\det(\bar{A}) = \bar{0}$*, then we have the following.*

(i) *If* $\dim\big(\mathrm{Im}(\bar{A})\big) = 0$*, then* (A, β) *satisfies the condition* $(*)$ *if and only if* $\bar{\beta} = \vec{0}$.

(ii) *If* $\bar{\beta} \in \mathrm{Im}(\bar{A})$ *and* $(\bar{A})^2 = \bar{0}$*, then* (A, β) *satisfies the condition* $(*)$ *if and only if* $\bar{\beta} = \vec{0}$.

(iii) *If* $\dim\big(\mathrm{Im}(\bar{A})\big) = 1$*,* $\bar{\beta} \in \mathrm{Im}(\bar{A})$ *and* $(\bar{A})^2 = \bar{A}$*, then* (A, β) *satisfies the condition* $(*)$ *if and only if* $\bar{\beta} \neq \vec{0}$.

Proof. Assume that $\det(\bar{A}) = \bar{0}$.

(i) Suppose $\dim(\mathrm{Im}(\bar{A})) = 0$. Then $\bar{A} = \bar{0}$, so $\varphi(v) = \bar{\beta}$ for all $v \in \mathbb{Z}_k^2$, and hence (A, β) satisfies the condition $(*)$ if and only if $\bar{\beta} = \vec{0}$.

(ii) Suppose $\bar{\beta} \in \mathrm{Im}(\bar{A})$ and $(\bar{A})^2 = \bar{0}$ and observe that for any $v \in \mathbb{Z}_k^2$ we have $\varphi(v) = \bar{A}v + \bar{\beta} \in \mathrm{Im}(\bar{A})$. Furthermore, for any $w \in \mathrm{Im}(\bar{A})$ we can write w as $\bar{A}v$ for some $v \in \mathbb{Z}_k^2$, so $\varphi(w) = \bar{A}w + \bar{\beta} = (\bar{A})^2 v + \bar{\beta} = \bar{\beta}$. In particular, $\varphi^l(v) = \bar{\beta}$ for all $v \in \mathbb{Z}_k^2$ and all $l \in \mathbb{N}$ with $l \geq 2$. Hence (A, β) satisfies the condition $(*)$ if and only if $\bar{\beta} = \vec{0}$.

(iii) Suppose $\dim\big(\mathrm{Im}(\bar{A})\big) = 1$, $\bar{\beta} \in \mathrm{Im}(\bar{A})$ and $(\bar{A})^2 = \bar{A}$.

($\Rightarrow$) Suppose $\bar{\beta} = \vec{0}$. Then $\varphi|_{\mathrm{Im}(\bar{A})}$ is the identity map, since for any $w \in \mathrm{Im}(\bar{A})$ we have $w = \bar{A}v$ for some $v \in \mathbb{Z}_k^2$, so $\varphi|_{\mathrm{Im}(\bar{A})}(w) = \bar{A}w + \bar{\beta} = (\bar{A})^2 v = \bar{A}v = w$. Hence for any $w \in \mathrm{Im}(\bar{A})$ and any $l \in \mathbb{N}$, $(\varphi|_{\mathrm{Im}(\bar{A})})^l(w) = w$. Since $\dim\big(\mathrm{Im}(\bar{A})\big) = 1$, there exists an element $w \in \mathrm{Im}(\bar{A})$ such that $w \neq \vec{0}$, and hence $(\varphi|_{\mathrm{Im}(\bar{A})})^l(w) \neq \vec{0}$ for all $l \in \mathbb{N}$. Since $w \in \mathrm{Im}(\bar{A}), w = \bar{A}v = \varphi(v)$ for some $v \in \mathbb{Z}_k^2$. We have $\varphi^l(v) = (\varphi|_{\mathrm{Im}(\bar{A})})^{l-1}\left(\varphi(v)\right) = (\varphi|_{\mathrm{Im}(\bar{A})})^{l-1}(w) \neq \vec{0}$ for all $l \in \mathbb{N}$. Hence (A, β) does not satisfy the condition $(*)$.

($\Leftarrow$) Suppose $\bar{\beta} \neq \vec{0}$. Since $\dim\big(\mathrm{Im}(\bar{A})\big) = 1$ and $\bar{\beta} \in \mathrm{Im}(\bar{A})$, $\mathrm{Im}(\bar{A}) = \big\{l\bar{\beta} \mid l \in \mathbb{Z}_k\big\}$. An argument similar to the one used for the direction ($\Rightarrow$) shows that $\varphi(w) = w + \bar{\beta}$ for all $w \in \mathrm{Im}(\bar{A})$. In particular, $\varphi(l\bar{\beta}) = l\bar{\beta} + \bar{\beta} = (l+1)\bar{\beta}$ for all $l \in \mathbb{Z}_*$. It follows that $\varphi|_{\mathrm{Im}(\bar{A})} = (\bar{\beta}\ \ 2\bar{\beta}\ \cdots\ k\bar{\beta})$ as an element of $\mathrm{Sym}\big(\mathrm{Im}(\bar{A})\big)$. Since $\bar{\beta}, 2\bar{\beta}, \ldots, k\bar{\beta}$ are k distinct elements in $\mathrm{Im}(\bar{A})$ and $|\mathrm{Im}(\bar{A})| = k$, $\varphi|_{\mathrm{Im}(\bar{A})}$ can be represented by a single cycle. Thus for any $w \in \mathrm{Im}(\bar{A})$, $\big\{(\varphi|_{\mathrm{Im}(\bar{A})})^l(w) \mid l \in \mathbb{N}\big\} = \mathrm{Im}(\bar{A})$, and hence there exists an $l \in \mathbb{N}$ such that $(\varphi|_{\mathrm{Im}(\bar{A})})^l(w) = \vec{0}$. We can now show that (A, β) satisfies the condition $(*)$ as follows: Let $v \in \mathbb{Z}_k^2$. Since $\varphi(v) = \bar{A}v + \bar{\beta} \in \mathrm{Im}(\bar{A})$, there exists an $l \in \mathbb{N}$ such that $(\varphi|_{\mathrm{Im}(\bar{A})})^l\,(\varphi(v)) = \vec{0}$. But $\varphi^{l+1}(v) = (\varphi|_{\mathrm{Im}(\bar{A})})^l\,(\varphi(v))$, so we are done. □

Now we summarize all of the above lemmas as follows:

Theorem 3.10. *Let k be a given prime number, $A \in M_2(\mathbb{N})$ be arbitrary and β be any element in ${\mathbb{Z}_*}^2$. Then for φ defined as in Definition 3.2, we have the following.*

(i) *If $\det(\bar{A}) \neq \bar{0}$ and φ is the identity map, then $\langle \alpha_0, \alpha_1, \alpha_2, \ldots \rangle$ is not cyclic.*

(ii) *If φ is a bijection that is not the identity map, then φ can be represented by a single cycle if and only if (A, β) satisfies the condition $(*)$.*

(iii) *If $\det(\bar{A}) = \bar{0}$ and $\dim\big(\mathrm{Im}(\bar{A})\big) = 0$, then (A, β) satisfies the condition $(*)$ if and only if $\bar{\beta} = \vec{0}$.*

(iv) *If $\det(\bar{A}) = \bar{0}$ and $\dim\big(\mathrm{Im}(\bar{A})\big) = 1$, then (A, β) satisfies the condition $(*)$ only if $\bar{\beta} \in \mathrm{Im}(\bar{A})$.*

(v) *If $\det(\bar{A}) = \bar{0}, \dim\big(\mathrm{Im}(\bar{A})\big) = 1$ and $\bar{\beta} \in \mathrm{Im}(\bar{A})$, then*

(v.1) *$(\bar{A})^2 = \bar{0}$, implies (A, β) satisfies the condition $(*)$ if and only if $\bar{\beta} = \vec{0}$,* (v.2) $(\bar{A})^2 \neq \bar{0}$ *and* $(\bar{A})^2 \neq \bar{A}$, *implies* (A, β) *does not satisfies the condition*$(*)$,

(v.3) $(\bar{A})^2 \neq \bar{0}$ *and* $(\bar{A})^2 = \bar{A}$, *implies* (A, β) *satisfies the condition* $(*)$ *if and only if* $\bar{\beta} \neq \vec{0}$.

According to the results in Propositions 3.1 and 3.3(i) we will consider only the cases where (A, β) satisfies the condition $(*)$ because these are the cases most likely to yield success. By investigating those cases, we have found that if A has a positive integer eigenvalue λ, and β and α_m are positive multiples of the corresponding eigenvector $\vec{e}$ for some $m \in \mathbb{Z}_*$, then it has a greater chance that (A, β) might satisfy the condition $(*)$. Precisely, we will consider the following conditions on $A, \lambda, \vec{e}, \beta$ and α_m :

$$A\vec{e} = \lambda\vec{e}, \beta = k^j d\vec{e} \text{ and } \alpha_m = a\vec{e} \text{ for some } a, d, j \in \mathbb{Z}_*, k \nmid a, k \nmid d \quad \text{where in addition we write } \lambda \text{ as } k^i\lambda_1 \text{with } i \in \mathbb{Z}_* \text{ and } k \nmid \lambda_1. \tag{3.3}$$

We will proceed to investigate all possibilities for i, j, a and d.

Notation. For any $r \in X \subseteq \mathbb{Z}_*$, we denote $\{n \in X \mid n \leq r\}$ by $X(r)$. In particular, $\mathbb{N}(r) = \{n \in \mathbb{N} \mid n \leq r\}$ and $\mathbb{Z}_*(r) = \{n \in \mathbb{Z}_* \mid n \leq r\}$.

Theorem 3.11. *Let* $A, \lambda, \vec{e}, \beta$ *and* α_m *be as in* (3.3) *and* $j < i$. *Then for all* $t \in \mathbb{Z}_*$ *and* $l \in \mathbb{Z}_*(j)$ *we have*

$$\alpha_{m+tj+(t+1)+l} = k^{j-l}\left(a(\lambda_1 k^{i-j})^{t+1} + \frac{(\lambda_1 k^{i-j})^{t+1} - 1}{\lambda_1 k^{i-j} - 1}\, d\right)\vec{e}. \tag{3.4}$$

In particular, when $a = d$ *we have*

$$\alpha_{m+tj+(t+1)+l} = k^{j-l} a \left(\frac{(\lambda_1 k^{i-j})^{t+2} - 1}{\lambda_1 k^{i-j} - 1}\right)\vec{e}.$$

Therefore the trajectory $\langle \alpha_0, \alpha_1, \alpha_2, \ldots \rangle$ *is not cyclic.*

Proof. We first note that for each $s \in \mathbb{N}$, for any $n \in \mathbb{N}$,

$$k \nmid \left(a(\lambda_1 k^s)^n + \frac{(\lambda_1 k^s)^n - 1}{\lambda_1 k^s - 1} d \right), \tag{3.5}$$

since $k \nmid \left(\frac{(\lambda_1 k^s)^n - 1}{\lambda_1 k^s - 1} d \right)$. To prove equation (3.4) we will use induction on t as follows: For any $t \in \mathbb{Z}_*$, let $p(t)$ be the sentence: for all $l \in \mathbb{Z}_*(j)$,

$$\alpha_{m+tj+(t+1)+l} = k^{j-l} \left(a(\lambda_1 k^{i-j})^{t+1} + \frac{(\lambda_1 k^{i-j})^{t+1} - 1}{\lambda_1 k^{i-j} - 1} d \right) \vec{e}.$$

Basis step: We will show that $p(0)$ is true, i.e., for all $l \in \mathbb{Z}_*(j)$,

$$\alpha_{m+1+l} = k^{j-l} \Big(a(\lambda_1 k^{i-j}) + d \Big) \vec{e} \tag{3.6}$$

We will prove that equation (3.6) is true by induction on l. For $l = 0$: Since $\alpha_{m+1+l} = \alpha_{m+1}$ and $D^{-1}\alpha_m \notin \mathbb{Z}^2$, we have

$$\begin{aligned} \alpha_{m+1} &= A\alpha_m + \beta \\ &= aA\vec{e} + k^j d\vec{e} \\ &= a\lambda\vec{e} + k^j d\vec{e} \\ &= ak^i\lambda_1\vec{e} + k^j d\vec{e} \\ &= k^j \Big(a(\lambda_1 k^{i-j}) + d \Big) \vec{e}. \end{aligned}$$

Thus equation (3.6) is true when $l = 0$. Assume that equation (3.6) is true for $l \in \mathbb{Z}_*(j-1)$. We will show that equation (3.6) is true for $l+1$, i.e.,

$$\alpha_{m+1+(l+1)} = k^{j-(l+1)} \Big(a(\lambda_1 k^{i-j}) + d \Big) \vec{e}.$$

By the induction hypothesis for l,

$$\alpha_{m+1+l} = k^{j-l} \Big(a(\lambda_1 k^{i-j}) + d \Big) \vec{e}.$$

Since $l \in \mathbb{Z}_*(j-1)$, $j - l \in \mathbb{N}$, so $D^{-1}\alpha_{m+1+l} \in \mathbb{Z}^2$, and hence

$$\alpha_{m+1+(l+1)} = D^{-1}\alpha_{m+1+l} = k^{j-(l+1)} \Big(a(\lambda_1 k^{i-j}) + d \Big) \vec{e}.$$

Thus equation (3.6) is true for $l+1$. By induction on l, equation (3.6) is true for all $l \in \mathbb{Z}_*(j)$. Thus $p(0)$ is true.

Induction step: To simplify the notation, for any $t \in \mathbb{Z}_*$ let

$$C(t) = a(\lambda_1 k^{i-j})^{t+1} + \frac{(\lambda_1 k^{i-j})^{t+1} - 1}{\lambda_1 k^{i-j} - 1}\, d.$$

Assume that $p(t)$ is true. We will show that $p(t+1)$ is true, i.e., for all $l \in \mathbb{Z}_*(j)$,

$$\begin{aligned} \alpha_{m+(t+1)j+(t+2)+l} &= k^{j-l}\left(a(\lambda_1 k^{i-j})^{t+2} + \frac{(\lambda_1 k^{i-j})^{t+2} - 1}{\lambda_1 k^{i-j} - 1}\, d\right)\vec{e} \\ &= k^{j-l} C(t+1)\vec{e}. \end{aligned} \tag{3.7}$$

We will show that equation (3.7) is true by induction on l.

Basis step for l: $l = 0$. Since $p(t)$ is true, we have $\alpha_{m+tj+(t+1)+n} = k^{j-n}C(t)\vec{e}$ for all $n \in \mathbb{Z}_*(j)$. In particular, when $n = j$

$$\begin{aligned} \alpha_{m+tj+(t+1)+n} &= \alpha_{m+(t+1)j+(t+1)} \\ &= k^{j-j}C(t)\vec{e} \\ &= C(t)\vec{e}. \end{aligned}$$

From (3.5), $k \nmid C(t)$, so $D^{-1}\alpha_{m+(t+1)j+(t+1)} \notin \mathbb{Z}^2$, and hence

$$\begin{aligned} \alpha_{m+(t+1)j+(t+2)} &= A\alpha_{m+(t+1)j+(t+1)} + \beta \\ &= AC(t)\vec{e} + \beta \\ &= C(t)A\vec{e} + k^j d\vec{e} \\ &= C(t)\lambda\vec{e} + k^j d\vec{e} \\ &= C(t)k^i\lambda_1\vec{e} + k^j d\vec{e} \\ &= k^j\Big(\lambda_1 k^{i-j}C(t) + d\Big)\vec{e}. \end{aligned}$$

But

$$\lambda_1 k^{i-j}C(t) + d = a(\lambda_1 k^{i-j})^{t+2} + \lambda_1 k^{i-j}\frac{(\lambda_1 k^{i-j})^{t+1} - 1}{\lambda_1 k^{i-j} - 1}d + d$$

$$= a(\lambda_1 k^{i-j})^{t+2} + \left(\frac{(\lambda_1 k^{i-j})^{t+2} - \lambda_1 k^{i-j}}{\lambda_1 k^{i-j} - 1} + 1\right) d$$

$$= a(\lambda_1 k^{i-j})^{t+2} + \frac{(\lambda_1 k^{i-j})^{t+2} - 1}{\lambda_1 k^{i-j} - 1} d$$

$$= C(t+1),$$

so

$$\alpha_{m+(t+1)j+(t+2)} = k^j C(t+1)\vec{e}.$$

Thus equation (3.7) is true when $l = 0$.

Induction step for l: Assume that equation (3.7) is true for $l \in \mathbb{Z}_*(j-1)$. We will show that equation (3.7) is true for $l+1$, i.e.,

$$\alpha_{m+(t+1)j+(t+2)+(l+1)} = k^{j-(l+1)} C(t+1)\vec{e}.$$

By the induction hypothesis for l,

$$\alpha_{m+(t+1)j+(t+2)+l} = k^{j-l} C(t+1)\vec{e}.$$

Since $l \in \mathbb{Z}_*(j-1)$, $j - l \in \mathbb{N}$, so $D^{-1}\alpha_{m+(t+1)i+(t+2)+l} \in \mathbb{Z}^2$, and hence

$$\alpha_{m+(t+1)j+(t+2)+(l+1)} = k^{j-(l+1)} C(t+1)\vec{e}.$$

Thus equation (3.7) is true for $l+1$. By induction on l, equation (3.7) is true for all $l \in \mathbb{Z}_*(j)$. Hence $p(t+1)$ is true. By induction on t, $p(t)$ is true for all $t \in \mathbb{Z}_*$, i.e., for all $t \in \mathbb{Z}_*$ and all $l \in \mathbb{Z}_*(j)$,

$$\alpha_{m+tj+(t+1)+l} = k^{j-l} \left(a(\lambda_1 k^{i-j})^{t+1} + \frac{(\lambda_1 k^{i-j})^{t+1} - 1}{\lambda_1 k^{i-j} - 1} d \right) \vec{e}.$$

In particular, when $a = d$ we have

$$\alpha_{m+tj+(t+1)+l} = k^{j-l} a \left(\frac{(\lambda_1 k^{i-j})^{t+2} - 1}{\lambda_1 k^{i-j} - 1} \right) \vec{e}.$$

Next we will show that the α_{m+n} are distinct for all $n \in \mathbb{Z}_*$. It suffices to show that

(I) for each $t \in \mathbb{Z}_*$, the $\alpha_{m+tj+(t+1)+l}$ are distinct for all $l \in \mathbb{Z}_*(j)$ and

(II) for any $t_1, t_2 \in \mathbb{Z}_*$, with $t_1 \neq t_2$,

$$\left\{\alpha_{m+t_1j+(t_1+1)+l} \;\middle|\; l \in \mathbb{Z}_*(j)\right\} \cap \left\{\alpha_{m+t_2j+(t_2+1)+l} \;\middle|\; l \in \mathbb{Z}_*(j)\right\} = \varnothing.$$

We will prove (I) as follows: Let $t \in \mathbb{Z}_*$. Suppose that $\alpha_{m+tj+(t+1)+l_1} = \alpha_{m+tj+(t+1)+l_2}$ for some $l_1, l_2 \in \mathbb{Z}_*(j)$. Then $k^{j-l_1}C(t)\vec{e} = k^{j-l_2}C(t)\vec{e}$. This implies that $k^{j-l_1} = k^{j-l_2}$, and hence $l_1 = l_2$. Therefore (I) is true.

Next we will prove (II) as follows: Suppose that $\alpha_{m+t_1j+(t_1+1)+l_1} = \alpha_{m+t_2j+(t_2+1)+l_2}$ for some $l_1, l_2 \in \mathbb{Z}_*(j)$ and $t_1, t_2 \in \mathbb{Z}_*$ with $t_1 \neq t_2$. Then $k^{j-l_1}C(t_1)\vec{e} = k^{j-l_2}C(t_2)\vec{e}$. This implies that $k^{l_2}C(t_1) = k^{l_1}C(t_2)$. Suppose that $l_1 \neq l_2$. Without loss of generality, we may assume $l_1 < l_2$, so that $k^{l_2-l_1}C(t_1) = C(t_2)$. Thus $k \mid C(t_2)$, which contradicts (3.5). Hence $l_1 = l_2$. Therefore $C(t_1) = C(t_2)$, i.e.,

$$a\mu^{t_1+1} + \frac{\mu^{t_1+1} - 1}{\mu - 1}\, d = a\mu^{t_2+1} + \frac{\mu^{t_2+1} - 1}{\mu - 1}\, d,$$

where $\mu = \lambda_1 k^{i-j}$, so

$$a(\mu^{t_1+1} - \mu^{t_2+1}) = \frac{\mu^{t_2+1} - 1 - \mu^{t_1+1} + 1}{\mu - 1}\, d = \frac{\mu^{t_2+1} - \mu^{t_1+1}}{\mu - 1}\, d.$$

This implies that $a(\mu - 1) = -d$, a contradiction since $a(\mu - 1) > 0$ but $-d < 0$. Hence (II) is true.

Therefore the trajectory $\langle \alpha_m, \alpha_{m+1}, \alpha_{m+2}, \ldots \rangle$ is not cyclic, and hence the trajectory $\langle \alpha_0, \alpha_1, \alpha_2, \ldots \rangle$ is not cyclic. □

Theorem 3.12. *Let $A, \lambda, \vec{e}, \beta$ and α_m be as in (3.3) and $i < j$. Then for all $t \in \mathbb{Z}_*$ and $l \in \mathbb{Z}_*(i)$,*

$$\alpha_{m+ti+(t+1)+l} = \begin{cases} k^{i-l}\left(a\lambda_1^{t+1} + \dfrac{(\lambda_1^{t+1} - 1)}{\lambda_1 - 1}\, dk^{j-i}\right)\vec{e} & \text{if } \lambda_1 \neq 1, \\ k^{i-l}\Big(a + (t+1)dk^{j-i}\Big)\vec{e} & \text{otherwise.} \end{cases} \tag{3.8}$$

In particular, when $a = d$ *we have*

$$\alpha_{m+ti+(t+1)+l} = \begin{cases} k^{i-l}a\left(\lambda_1^{t+1} + \dfrac{(\lambda_1^{t+1}-1)}{\lambda_1 - 1}\, k^{j-i}\right)\vec{e} & \text{if } \lambda_1 \neq 1, \\ k^{i-l}a\Big(1 + (t+1)k^{j-i}\Big)\vec{e} & \text{otherwise.} \end{cases}$$

Therefore the trajectory $\langle \alpha_0, \alpha_1, \alpha_2, \ldots \rangle$ *is not cyclic.*

Proof. We first note that for each $s, n \in \mathbb{N}$,

$$k \nmid \Big(a\lambda_1^n + \big(\sum_{0 \leqslant \nu \leqslant n-1} \lambda_1^\nu\big) dk^s\Big), \tag{3.9}$$

since $k \nmid a\lambda_1^n$ and $k \mid \Big(\sum_{0\leqslant \nu \leqslant n-1} \lambda_1^\nu\Big) dk^s$. To prove equation (3.8) we will use induction on t as follows: For any $t \in \mathbb{Z}_*$, let $p(t)$ be the sentence: for all $l \in \mathbb{Z}_*(i)$,

$$\alpha_{m+ti+(t+1)+l} = k^{i-l}\Big(a\lambda_1^{t+1} + \big(\sum_{0\leqslant \nu \leqslant t} \lambda_1^\nu\big) dk^{j-i}\Big)\vec{e}.$$

To simplify the notation, for any $t \in \mathbb{Z}_*$, let $C(t) = a\lambda_1^{t+1} + \Big(\sum_{0\leqslant \nu\leqslant t} \lambda_1^\nu\Big) dk^{j-i}$.

Basis step: We will show that $p(0)$ is true, i.e., for all $l \in \mathbb{Z}_*(i)$,

$$\alpha_{m+1+l} = k^{i-l} C(0)\vec{e} \tag{3.10}$$

We will show that equation (3.10) is true by induction on l. For $l = 0$: Since $\alpha_{m+1+0} = \alpha_{m+1}$ and $D^{-1}\alpha_m \notin \mathbb{Z}^2$, we have

$$\begin{aligned} \alpha_{m+1} &= A\alpha_m + \beta \\ &= aA\vec{e} + k^j d\vec{e} \\ &= a\lambda\vec{e} + k^j d\vec{e} \\ &= ak^i\lambda_1\vec{e} + k^j d\vec{e} \\ &= k^i(a\lambda_1 + dk^{j-i})\vec{e} \\ &= k^i C(0)\vec{e}. \end{aligned}$$

Thus equation (3.10) is true when $l = 0$. Assume that equation (3.10) is true for $l \in \mathbb{Z}_*(i-1)$. We will show that equation (3.10) is true for $l+1$, i.e.,

$$\alpha_{m+1+(l+1)} = k^{i-(l+1)}C(0)\vec{e}.$$

By induction hypothesis for l,

$$\alpha_{m+1+l} = k^{i-l}C(0)\vec{e}.$$

Since $l \in \mathbb{Z}_*(i-1), i-l \in \mathbb{N}$, so $D^{-1}\alpha_{m+1+l} \in \mathbb{Z}^2$, and hence

$$\alpha_{m+1+(l+1)} = D^{-1}\alpha_{m+1+l} = k^{i-(l+1)}C(0)\vec{e}.$$

Thus equation (3.10) is true for $l+1$. By induction on l, equation (3.10) is true for all $l \in \mathbb{Z}_*(i)$. Thus $p(0)$ is true.

Induction step: Assume that $p(t)$ is true. We will show that $p(t+1)$ is true, i.e., for all $l \in \mathbb{Z}_*(i)$,

$$\alpha_{m+(t+1)i+(t+2)+l} = k^{i-l}C(t+1)\vec{e}. \tag{3.11}$$

We will show that equation (3.11) is true by induction on l.

Basis step for l : $l = 0$. Since $p(t)$ is true, we have

$$\alpha_{m+ti+(t+1)+n} = k^{i-n}C(t)\vec{e}$$

for all $n \in \mathbb{Z}_*(i)$. In particular, when $n = i$

$$\begin{aligned}\alpha_{m+ti+(t+1)+n} &= \alpha_{m+(t+1)i+(t+1)} \\ &= k^{i-i}C(t)\vec{e} \\ &= C(t)\vec{e}.\end{aligned}$$

From (3.9), $k \nmid C(t)$, so $D^{-1}\alpha_{m+(t+1)i+(t+1)} \notin \mathbb{Z}^2$, and hence

$$\alpha_{m+(t+1)i+(t+2)} = A\alpha_{m+(t+1)i+(t+1)} + \beta$$

$$
\begin{aligned}
&= AC(t)\vec{e} + k^j d\vec{e} \\
&= C(t)\lambda\vec{e} + k^j d\vec{e} \\
&= C(t)\lambda k^i \lambda_1 \vec{e} + k^j d\vec{e} \\
&= k^i \Big(a\lambda_1^{t+1} + \big(\sum_{0 \leqslant \nu \leqslant t} \lambda_1^{\nu} \big) dk^{j-i} \Big) \lambda_1 \vec{e} + k^j d\vec{e} \\
&= k^i \Big[\Big(a\lambda_1^{t+1} + \big(\sum_{0 \leqslant \nu \leqslant t} \lambda_1^{\nu} \big) dk^{j-i} \Big) \lambda_1 + dk^{j-i} \Big] \vec{e} \\
&= k^i \Big(a\lambda_1^{t+2} + \big(\sum_{0 \leqslant \nu \leqslant t+1} \lambda_1^{\nu} \big) dk^{j-i} \Big) \vec{e} \\
&= k^i C(t+1)\vec{e}.
\end{aligned}
$$

Then equation (3.11) is true when $l = 0$.

Induction step for l: Assume that equation (3.11) is true for $l \in \mathbb{Z}_*(i-1)$. We will show that equation (3.11) is true for $l+1$, i.e.,

$$\alpha_{m+(t+1)i+(t+2)+(l+1)} = k^{i-(l+1)} C(t+1)\vec{e}.$$

By induction hypothesis for l,

$$\alpha_{m+(t+1)i+(t+2)+l} = k^{i-l} C(t+1)\vec{e}.$$

Since $l \in \mathbb{Z}_*(i-1), i-l \in \mathbb{N}$, so $D^{-1}\alpha_{m+(t+1)i+(t+2)+l} \in \mathbb{Z}^2$, and hence

$$
\begin{aligned}
\alpha_{m+(t+1)i+(t+2)+(l+1)} &= D^{-1}\alpha_{m+(t+1)i+(t+2)+l} \\
&= k^{i-(l+1)} C(t+1)\vec{e}.
\end{aligned}
$$

Thus equation (3.11) is true for $l+1$. By induction on l, equation (3.11) is true for all $l \in \mathbb{Z}_*(i)$. Hence $p(t+1)$ is true. By induction on t, $p(t)$ is true for all $t \in \mathbb{Z}_*$. For any $t \in \mathbb{Z}_*$,

$$
\sum_{0 \leqslant \nu \leqslant t} \lambda_1^{\nu} = \begin{cases} \dfrac{\lambda_1^{t+1} - 1}{\lambda_1 - 1} & \text{if } \lambda_1 \neq 1, \\ t+1 & \text{if } \lambda_1 = 1. \end{cases}
$$

Hence for any $t \in \mathbb{Z}_*$ and all $l \in \mathbb{Z}_*(i)$,

$$\alpha_{m+ti+(t+1)+l} = \begin{cases} k^{i-l}\left(a\lambda_1^{t+1} + \dfrac{\lambda_1^{t+1}-1}{\lambda_2 - 1}\, dk^{j-i}\right)\vec{e} & \text{if } \ \lambda_1 \neq 1, \\ k^{i-l}\Big(a + (t+1)dk^{j-i}\Big)\vec{e} & \text{otherwise.} \end{cases}$$

In particular, when $a = d$ we have

$$\alpha_{m+ti+(t+1)+l} = \begin{cases} k^{i-l}a\left(\lambda_1^{t+1} + \dfrac{\lambda_1^{t+1}-1}{\lambda_1 - 1}\, k^{j-i}\right)\vec{e} & \text{if } \ \lambda_1 \neq 1, \\ k^{i-l}a\Big(1 + (t+1)k^{j-i}\Big)\vec{e} & \text{otherwise.} \end{cases}$$

Next we will show that the α_{m+n} are distinct for all $n \in \mathbb{Z}_*$. It suffices to show that

(I) for each $t \in \mathbb{Z}_*$, the $\alpha_{m+ti+(t+1)+l}$ are distinct for all $l \in \mathbb{Z}_*(i)$ and

(II) for any $t_1, t_2 \in \mathbb{Z}_*$ with $t_1 \neq t_2$,

$$\{\alpha_{m+t_1i+(t_1+1)+l} \mid l \in \mathbb{Z}_*(i)\} \cap \{\alpha_{m+t_2i+(t_2+1)+l} \mid l \in \mathbb{Z}_*(i)\} = \varnothing.$$

We will prove (I) as follows: Let $t \in \mathbb{Z}_*$. Suppose that $\alpha_{m+ti+(t+1)+l_1} = \alpha_{m+ti+(t+1)+l_2}$ for some $l_1, l_2 \in \mathbb{Z}_*(i)$. Then $k^{i-l_1}C(t)\vec{e} = k^{i-l_2}C(t)\vec{e}$. So $k^{i-l_1} = k^{i-l_2}$, and hence $l_1 = l_2$. Therefore (I) is true.

Next we will prove (II) as follows: Suppose that $\alpha_{m+t_1i+(t_1+1)+l_1} = \alpha_{m+t_2i+(t_2+1)+l_2}$ for some $l_1, l_2 \in \mathbb{Z}_*(i)$ and $t_1, t_2 \in \mathbb{Z}_*$ with $t_1 \neq t_2$. Without loss of generality we may assume $t_1 < t_2$. Then $k^{i-l_1}C(t_1)\vec{e} = k^{i-l_2}C(t_2)\vec{e}$, so $k^{l_2}C(t_1) = k^{l_1}C(t_2)$. Suppose that $l_1 \neq l_2$. Without loss of generality, we may assume $l_1 < l_2$, so that $k^{l_2-l_1}C(t_1) = C(t_2)$. Thus $k \mid C(t_2)$, which contradicts (3.9). Hence $l_1 = l_2$. Therefore $C(t_1) = C(t_2)$, i.e.,

$$a\lambda_1^{t_1+1} + \Big(\sum_{0 \leqslant \nu \leqslant t_1} \lambda_1^{\nu}\Big)dk^{j-i} = a\lambda_1^{t_2+1} + \Big(\sum_{0 \leqslant \nu \leqslant t_2} \lambda_1^{\nu}\Big)dk^{j-i},$$

so

$$a\lambda_1^{t+1}\big(1-{\lambda_1}^{t_2-t_1}\big)=\lambda_1^{t_1+1}\Big(\sum_{0\leqslant\nu\leqslant t_2-t_1-1}\lambda_1^{\nu}\Big)dk^{j-i}.$$

Thus

$$a\big(1-{\lambda_1}^{t_2-t_1}\big)=\Big(\sum_{0\leqslant\nu\leqslant t_2-t_1-1}\lambda_1^{\nu}\Big)dk^{j-i}.$$

Since ${\lambda_1}^{t_2-t_1}\geq 1$, $a\big(1-{\lambda_1}^{t_2-t_1}\big)\leq 0$ but $\Big(\sum_{0\leqslant\nu\leqslant t_2-t_1-1}\lambda_1^{\nu}\Big)dk^{j-i}\geq 1$, so we have a contradiction. Hence (II) is true.

Therefore the trajectory $\langle\alpha_m,\alpha_{m+1},\alpha_{m+2},\ldots\rangle$ is not cyclic, and hence the trajectory $\langle\alpha_0,\alpha_1,\alpha_2,\ldots\rangle$ is not cyclic. □

Theorem 3.13. *Let $A,\lambda,\vec{e},\beta$ and α_m be as in* (3.3). *Assume that $a=d$ and $i=j$. Then $\alpha_{m+1}=k^ia(\lambda_1+1)\vec{e}$. Write λ_1+1 as $k^r\lambda_2$ where $k\nmid\lambda_2$ and $r\in\mathbb{Z}_*$.*

(i) *For each $t\in\mathbb{N}$, if $k\nmid\Big(\lambda_2\lambda_1^l+\sum_{0\leq\nu\leq l-1}\lambda_1^{\nu}\Big)$ for all $l\in\mathbb{N}(t-2)$, then*

$$\alpha_{m+ti+(r+t)}=\begin{cases}a\left(\lambda_2\lambda_1^{t-1}+\dfrac{\lambda_1^{t-1}-1}{\lambda_1-1}\right)\vec{e} & \text{if } \lambda_1\neq 1,\\ a(\lambda_2+t-1)\vec{e} & \text{otherwise.}\end{cases}$$

(ii) *If there exists $t\in\mathbb{N}$ with the property that $k\nmid\Big(\lambda_2\lambda_1^l+\sum_{0\leq\nu\leq l-1}\lambda_1^{\nu}\Big)$ for all $l\in\mathbb{N}(t-2)$ and $\lambda_2\lambda_1^{t-1}+\sum_{0\leq\nu\leq t-2}\lambda_1^{\nu}=k^s$ for some $s\in\mathbb{N}$, then the trajectory $\langle\alpha_0,\alpha_1,\alpha_2,\ldots\rangle$ is cyclic.*

Proof. (i) For each $t\in\mathbb{N}$, let $p(t)$ be the sentence: if $k\nmid\Big(\lambda_2\lambda_1^l+\sum_{0\leq\nu\leq l-1}\lambda_1^{\nu}\Big)$ for all $l\in\mathbb{N}(t-2)$, then

$$\alpha_{m+ti+(r+t)}=a\Big(\lambda_2\lambda_1^{t-1}+\sum_{0\leq\nu\leq t-2}\lambda_1^{\nu}\Big)\vec{e}.$$

Basis step: We will show that $p(1)$ is true. It suffices to show that $\alpha_{m+i+(r+1)}=a\lambda_2\vec{e}$.

Since

$$\alpha_{m+1}=k^ia(\lambda_1+1)\vec{e}=k^{i+r}a\lambda_2\vec{e},$$

$D^{-1}\alpha_{m+1+n} \in \mathbb{Z}^2$ for all $n \in \mathbb{Z}_*(i+r-1)$, so

$$\alpha_{m+1+(n+1)} = D^{-1}\alpha_{m+1+n} = k^{(i+r)-(n+1)}a\lambda_2\vec{e}$$

for all $n \in \mathbb{Z}_*(i+r-1)$. In particular, when $n = i+r-1$

$$\alpha_{m+i+(r+1)} = \alpha_{m+1+(n+1)} = k^{(i+r)-(i+r)}a\lambda_2\vec{e} = a\lambda_2\vec{e}.$$

Induction step: Assume that $p(t)$ is true. We will show that $p(t+1)$ is true, i.e., if

$$k \nmid \Big(\lambda_2\lambda_1^l + \sum_{0\le\nu\le l-1}\lambda_1^\nu\Big) \text{ for all } \quad l \in \mathbb{N}(t-1),$$

then

$$\alpha_{m+(t+1)i+(r+t+1)} = a\Big(\lambda_2\lambda_1^t + \sum_{0\le\nu\le t-1}\lambda_1^\nu\Big)\vec{e}.$$

Suppose that

$$k \nmid \Big(\lambda_2\lambda_1^l + \sum_{0\le\nu\le l-1}\lambda_1^\nu\Big) \text{ for all } \quad l \in \mathbb{N}(t-1).$$

Then

$$k \nmid \Big(\lambda_2\lambda_1^l + \sum_{0\le\nu\le l-1}\lambda_1^\nu\Big) \text{ for all } \quad l \in \mathbb{N}(t-2),$$

and since $p(t)$ is true we have

$$\alpha_{m+ti+(r+t)} = a\Big(\lambda_2\lambda_1^{t-1} + \sum_{0\le\nu\le t-2}\lambda_1^\nu\Big)\vec{e}.$$

By assumption, $k \nmid \Big(\lambda_2\lambda_1^{t-1} + \sum_{0\le\nu\le t-2}\lambda_1^\nu\Big)$, and since $k \nmid a$ we have

$$k \nmid a\Big(\lambda_2\lambda_1^{t-1} + \sum_{0\le\nu\le t-2}\lambda_1^\nu\Big),$$

so $D^{-1}\alpha_{m+ti+(r+t)} \notin \mathbb{Z}^2$, and hence

$$\begin{aligned}\alpha_{m+ti+(r+t+1)} &= A\alpha_{m+ti+(r+t)} + \beta \\ &= Aa\Big(\lambda_2\lambda_1^{t-1} + \sum_{0\le\nu\le t-2}\lambda_1^\nu\Big)\vec{e} + k^i a\vec{e}\end{aligned}$$

$$= a\Big(\lambda_2\lambda_1^{t-1} + \sum_{0\le\nu\le t-2}\lambda_1^{\nu}\Big)\lambda\vec{e} + k^i a\vec{e}$$

$$= a\Big(\lambda_2\lambda_1^{t-1} + \sum_{0\le\nu\le t-2}\lambda_1^{\nu}\Big)k^i\lambda_1\vec{e} + k^i a\vec{e}$$

$$= k^i a\Big(\big(\lambda_2\lambda_1^{t-1} + \sum_{0\le\nu\le t-2}\lambda_1^{\nu}\big)\lambda_1 + 1\Big)\vec{e}$$

$$= k^i a\Big(\lambda_2\lambda_1^{t} + \sum_{0\le\nu\le t-1}\lambda_1^{\nu}\Big)\vec{e}.$$

Since $D^{-1}\alpha_{m+ti+(r+t+1)+n} \in \mathbb{Z}^2$ for all $n \in \mathbb{Z}_*(i-1)$,

$$\begin{aligned}\alpha_{m+ti+(r+t+1)+(n+1)} &= D^{-1}\alpha_{m+ti+(r+t+1)+n}\\ &= k^{i-(n+1)}a\Big(\lambda_2\lambda_1^{t} + \sum_{0\le\nu\le t-1}\lambda_1^{\nu}\Big)\vec{e}\end{aligned}$$

for all $n \in \mathbb{Z}_*(i-1)$. In particular, when $n = i-1$ we have

$$\begin{aligned}\alpha_{m+(t+1)i+(r+t+1)} &= D^{-1}\alpha_{m+(t+1)i+(r+t)}\\ &= k^{i-i}a\Big(\lambda_2\lambda_1^{t} + \sum_{0\le\nu\le t-1}\lambda_1^{\nu}\Big)\vec{e}\\ &= a\Big(\lambda_2\lambda_1^{t} + \sum_{0\le\nu\le t-1}\lambda_1^{\nu}\Big)\vec{e}.\end{aligned}$$

Thus $p(t+1)$ is true. By mathematical induction, $p(t)$ is true for all $t \in \mathbb{N}$. For any $n \in \mathbb{N}$, we have

$$\sum_{0\le\nu\le n}\lambda_1^{\nu} = \begin{cases}\dfrac{\lambda_1^{n+1}-1}{\lambda_1 - 1} & \text{if } \lambda_1 \ne 1,\\[2ex] n+1 & \text{if } \lambda_1 = 1.\end{cases}$$

Hence for each $t \in \mathbb{N}$, if $k \nmid \Big(\lambda_2\lambda_1^{l} + \sum_{0\le\nu\le l-1}\lambda_1^{\nu}\Big)$ for all $l \in \mathbb{N}(t-2)$, then

$$\alpha_{m+ti+(r+t)} = \begin{cases} a\left(\lambda_2\lambda_1^{t-1} + \dfrac{\lambda_1^{t-1}-1}{\lambda_1 - 1}\right)\vec{e} & \text{if } \lambda_1 \ne 1,\\[2ex] a(\lambda_2 + t - 1)\vec{e} & \text{otherwise.}\end{cases}$$

(ii) Assume that there exists $t \in \mathbb{N}$ with the property that

$$k \nmid \Big(\lambda_2\lambda_1^{l} + \sum_{0\le\nu\le l-1}\lambda_1^{\nu}\Big)$$

for all $l \in \mathbb{N}(t-2)$ and $\lambda_2\lambda_1^{t-1} + \sum\limits_{0\le\nu\le t-2} \lambda_1^{\nu} = k^s$ for some $s \in \mathbb{N}$. From (i) we have

$$\alpha_{m+ti+(r+t)} = a\Big(\lambda_2\lambda_1^{t-1} + \sum_{0\le\nu\le t-2} \lambda_1^{\nu}\Big)\vec{e} = ak^s\vec{e}.$$

Since $D^{-1}\alpha_{m+ti+(r+t)+n} \in \mathbb{Z}^2$ for all $n \in \mathbb{Z}_*(s-1)$,

$$\begin{aligned}\alpha_{m+ti+(r+t)+(n+1)} &= D^{-1}\alpha_{m+ti+(r+t)+n}\\ &= k^{s-(n+1)}a\vec{e}\end{aligned}$$

for all $n \in \mathbb{Z}_*(s-1)$. In particular, when $n = s-1$

$$\begin{aligned}\alpha_{m+ti+(r+t)+(n+1)} &= \alpha_{m+ti+(r+t)+s}\\ &= ak^{s-s}\vec{e}\\ &= a\vec{e}\\ &= \alpha_m,\end{aligned}$$

and hence the trajectory $\langle \alpha_0, \alpha_1, \alpha_2, \ldots\rangle$ is cyclic. □

Theorem 3.14. *Let $A, \lambda, \vec{e}, \beta$ and α_m be as in (3.3). Assume that $a \neq d$ and $i = j$. Then $\alpha_{m+1} = k^i(a\lambda_1 + d)\vec{e}$. Write $a\lambda_1 + d$ as $k^r\lambda_2$, where $k \nmid \lambda_2$ and $r \in \mathbb{Z}_*$.*

(i) *For each $t \in \mathbb{N}$, if $k \nmid \Big(\lambda_2\lambda_1^l + (\sum\limits_{0\le\nu\le l-1} \lambda_1^{\nu})d\Big)$ for all $l \in \mathbb{N}(t-2)$, then*

$$\alpha_{m+ti+(r+t)} = \begin{cases} \Big(\lambda_2\lambda_1^{t-1} + \dfrac{\lambda_1^{t-1}-1}{\lambda_1 - 1}\, d\Big)\vec{e} & \text{if } \lambda_1 \neq 1,\\ \Big(\lambda_2 + (t-1)d\Big)\vec{e} & \text{otherwise.}\end{cases}$$

(ii) *If there exists $t \in \mathbb{N}$ with the property that $k \nmid \Big(\lambda_2\lambda_1^l + (\sum\limits_{0\le\nu\le l-1} \lambda_1^{\nu})d\Big)$ for all $l \in \mathbb{N}(t-2)$ and $\lambda_2\lambda_1^{t-1} + \Big(\sum\limits_{0\le\nu\le t-2} \lambda_1^{\nu}\Big)d = ak^s$ for some $s \in \mathbb{Z}_*$, then the trajectory $\langle \alpha_0, \alpha_1, \alpha_2, \ldots\rangle$ is cyclic.*

Proof. (i) For each $t \in \mathbb{N}$, let $p(t)$ be the sentence: if $k \nmid \Big(\lambda_2\lambda_1^l + (\sum\limits_{0\le\nu\le l-1} \lambda_1^\nu)d\Big)$ for all $l \in \mathbb{N}(t-2)$, then

$$\alpha_{m+ti+(r+t)} = \Big(\lambda_2\lambda_1^{t-1} + (\sum_{0\le\nu\le t-2} \lambda_1^\nu)d\Big)\vec{e}.$$

Basis step: We will show that $p(1)$ is true. It suffices to show that $\alpha_{m+i+(r+1)} = \lambda_2\vec{e}$. Since

$$\alpha_{m+1} = k^i(a\lambda_1 + d)\vec{e} = k^{i+r}\lambda_2\vec{e},$$

$D^{-1}\alpha_{m+1+n} \in \mathbb{Z}^2$ for all $n \in \mathbb{Z}_*(i+r-1)$, so

$$\alpha_{m+1+(n+1)} = D^{-1}\alpha_{m+1+n} = k^{(i+r)-(n+1)}\lambda_2\vec{e}$$

for all $n \in \mathbb{Z}_*(i+r-1)$. In particular, when $n = i+r-1$

$$\alpha_{m+1+(n+1)} = \alpha_{m+i+(r+1)} = k^{(i+r)-(i+r)}\lambda_2\vec{e} = \lambda_2\vec{e}.$$

Induction step: Assume that $p(t)$ is true. We will show that $p(t+1)$ is true, i.e., if

$$k \nmid \Big(\lambda_2\lambda_1^l + (\sum_{0\le\nu\le l-1} \lambda_1^\nu)d\Big) \quad \text{for all} \quad l \in \mathbb{N}(t-1),$$

then

$$\alpha_{m+(t+1)i+(r+t+1)} = \Big(\lambda_2\lambda_1^t + (\sum_{0\le\nu\le t-1} \lambda_1^\nu)d\Big)\vec{e}.$$

Suppose that

$$k \nmid \Big(\lambda_2\lambda_1^l + (\sum_{0\le\nu\le l-1} \lambda_1^\nu)d\Big) \quad \text{for all} \quad l \in \mathbb{N}(t-1).$$

Then

$$k \nmid \Big(\lambda_2\lambda_1^l + (\sum_{0\le\nu\le l-1} \lambda_1^\nu)d\Big) \quad \text{for all} \quad l \in \mathbb{N}(t-2),$$

and since $p(t)$ is true we have

$$\alpha_{m+ti+(r+t)} = \Big(\lambda_2\lambda_1^{t-1} + (\sum_{0\le\nu\le t-2} \lambda_1^\nu)d\Big)\vec{e}.$$

By assumption, $k \nmid \Big(\lambda_2\lambda_1^{t-1} + (\sum_{0\le\nu\le t-2} \lambda_1^{\nu})d\Big)$, so $D^{-1}\alpha_{m+ti+(r+t)} \notin \mathbb{Z}^2$, and hence

$$\begin{aligned}
\alpha_{m+ti+(r+t+1)} &= A\alpha_{m+ti+(r+t)} + \beta \\
&= \Big(\lambda_2\lambda_1^{t-1} + (\sum_{0\le\nu\le t-2} \lambda_1^{\nu})d\Big)\lambda\vec{e} + k^i d\vec{e} \\
&= k^i\Big(\lambda_2\lambda_1^{t-1} + (\sum_{0\le\nu\le t-2} \lambda_1^{\nu})d\Big)\lambda_1\vec{e} + k^i\vec{e} \\
&= k^i\Big(\lambda_2\lambda_1^{t} + (\sum_{0\le\nu\le t-1} \lambda_1^{\nu})d\Big)\vec{e}.
\end{aligned}$$

Since $D^{-1}\alpha_{m+ti+(r+t+1)+n} \in \mathbb{Z}^2$ for all $n \in \mathbb{Z}_*(i-1)$,

$$\begin{aligned}
\alpha_{m+ti+(r+t+1)+(n+1)} &= D^{-1}\alpha_{m+ti+(r+t+1)+n} \\
&= k^{i-(n+1)}\Big(\lambda_2\lambda_1^{t} + (\sum_{0\le\nu\le t-1} \lambda_1^{\nu})d\Big)\vec{e}
\end{aligned}$$

for all $n \in \mathbb{Z}_*(i-1)$. In particular, when $n = i-1$ we have

$$\begin{aligned}
\alpha_{m+(t+1)i+(r+t+1)} &= k^{i-i}\Big(\lambda_2\lambda_1^{t} + (\sum_{0\le\nu\le t-1} \lambda_1^{\nu})d\Big)\vec{e} \\
&= \Big(\lambda_2\lambda_1^{t} + (\sum_{0\le\nu\le t-1} \lambda_1^{\nu})d\Big)\vec{e}.
\end{aligned}$$

Thus $p(t+1)$ is true. By mathematical induction, $p(t)$ is true for all $t \in \mathbb{N}$. For any $n \in \mathbb{N}$, we have

$$\sum_{0\le\nu\le n} \lambda_1^{\nu} = \begin{cases} \dfrac{\lambda_1^{n+1}-1}{\lambda_1-1} & \text{if } \lambda_1 \neq 1, \\ n+1 & \text{if } \lambda_1 = 1. \end{cases}$$

Hence for each $t \in \mathbb{N}$, if $k \nmid \Big(\lambda_2\lambda_1^{l} + (\sum_{0\le\nu\le l-1} \lambda_1^{\nu})d\Big)$ for all $l \in \mathbb{N}(t-2)$, then

$$\alpha_{m+ti+(r+t)} = \begin{cases} \Big(\lambda_2\lambda_1^{t-1} + \dfrac{\lambda_1^{t-1}-1}{\lambda_1-1}\,d\Big)\,\vec{e} & \text{if } \lambda_1 \neq 1, \\ \Big(\lambda_2 + (t-1)d\Big)\vec{e} & \text{otherwise.} \end{cases}$$

(ii) Assume that there exists $t \in \mathbb{N}$ with the property that

$$k \nmid \Big(\lambda_2\lambda_1^{l} + (\sum_{0\le\nu\le l-1} \lambda_1^{\nu})d\Big)$$

for all $l \in \mathbb{N}(t-2)$ and $\lambda_2\lambda_1^{t-1} + \Big(\sum_{0\le\nu\le t-2} \lambda_1^{\nu}\Big)d = ak^s$ for some $s \in \mathbb{N}$. By (i) we have

$$\alpha_{m+ti+(r+t)} = \Big(\lambda_2\lambda_1^{t-1} + \big(\sum_{0\le\nu\le t-2} \lambda_1^{\nu}\big)d\Big)\vec{e} = ak^s\vec{e}.$$

Since $D^{-1}\alpha_{m+ti+(r+t)+n} \in \mathbb{Z}^2$ for all $n \in \mathbb{Z}_*(s-1)$,

$$\begin{aligned}\alpha_{m+ti+(r+t)+(n+1)} &= D^{-1}\alpha_{m+ti+(r+t)+n} \\ &= k^{s-(n+1)}a\vec{e}\end{aligned}$$

for all $n \in \mathbb{Z}_*(s-1)$. In particular, when $n = s-1$

$$\begin{aligned}\alpha_{m+ti+(r+t)+(n+1)} &= \alpha_{m+ti+(r+t)+s} \\ &= k^{s-s}a\vec{e} \\ &= a\vec{e} \\ &= \alpha_m,\end{aligned}$$

and hence the trajectory $\langle \alpha_0, \alpha_1, \alpha_2, \ldots \rangle$ is cyclic. □

CHAPTER IV

EXAMPLES AND CONCLUSION

In this chapter we will provide a few concrete examples of matrices A and vectors α_0 and β that satisfy the conditions in each of the Theorems 3.13 and 3.14 and the hypotheses in Theorem 3.10.

4.1 Existence of Matrices as in Theorems 3.13–3.14 and 3.10(iii)

We will provide concrete examples of matrices A and vectors α_0 and β that satisfy the conditions in Theorem 3.13 and satisfy the hypotheses in Theorem 3.10(iii).

Proposition 4.1.1. *Let $A = k^j \left[\begin{smallmatrix} b & c \\ u & b+c-u \end{smallmatrix}\right]$ and $\beta = k^j a \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$ for some $a, b, c, u, j \in \mathbb{N}$, where $k \nmid abc$, $u < b+c$, $k \nmid (b+c)$ and $k \nmid (b+c+1)$; and let $\alpha_0 \in \mathbb{Z}^2 - (k\mathbb{Z})^2$ be such that $\alpha_m = a\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$ for some $m \in \mathbb{Z}_*$. Then A satisfies the hypotheses in Theorem 3.10(iii). Note that $k \neq 2$. If there exists $t \in \mathbb{N}$ such that*

$$k = (b+c+1)(b+c)^{t-1} + \frac{(b+c)^{t-1} - 1}{b+c-1},$$

then the trajectory $\langle \alpha_0, \alpha_1, \alpha_2, \ldots \rangle$ is cyclic. In particular, if k is the Fermat prime $k = 2^{2^n} + 1$ for some $n \in \mathbb{Z}_*$ and $a = b = c = 2^{2^n - 1}$, we have that the trajectory $\langle \alpha_0, \alpha_1, \alpha_2, \ldots \rangle$ is cyclic.

Proof. Clearly, $\bar{A} = \bar{0}$, so A trivially satisfies the hypotheses in Theorem 3.10(iii). Furthermore, since $b+c$ and $b+c+1$ are consecutive integers, one of them must be even, and thus the assumptions $k \nmid (b+c)$ and $k \nmid (b+c+1)$ implies $k \neq 2$.

Assume that there exists $t \in \mathbb{N}$ such that $k = (b+c+1)(b+c)^{t-1} + \dfrac{(b+c)^{t-1}-1}{b+c-1}$. We will show that

$$k \nmid \left((b+c+1)(b+c)^l + \frac{(b+c)^l - 1}{b+c-1}\right)$$

for all $l \in \mathbb{N}(t-2)$. Since

$$k = (b+c+1)(b+c)^{t-1} + \frac{(b+c)^{t-1}-1}{b+c-1} > (b+c+1)(b+c)^l + \frac{(b+c)^l - 1}{b+c-1}$$

and

$$(b+c+1)(b+c)^l + \frac{(b+c)^l - 1}{b+c-1} \in \mathbb{N}$$

for all $l \in \mathbb{N}(t-2)$,

$$(b+c+1)(b+c)^l + \frac{(b+c)^l - 1}{b+c-1} \in \{1, 2, \ldots, k-1\}$$

for all $l \in \mathbb{N}(t-2)$, so

$$k \nmid \left((b+c+1)(b+c)^l + \frac{(b+c)^l - 1}{b+c-1}\right)$$

for all $l \in \mathbb{N}(t-2)$. Given $\alpha_0 \in \mathbb{Z}^2 - (k\mathbb{Z})^2$ such that $\alpha_m = a\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$ for some $m \in \mathbb{Z}_*$, it is straightforward to check that $\vec{e} = \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$, $\lambda = k^j(b+c)$, $i = j$, $\lambda_1 = b+c$ and $\lambda_2 = b+c+1$, so by Theorem 3.13(ii) we have that the trajectory $\langle \alpha_0, \alpha_1, \alpha_2, \ldots \rangle$ is cyclic.

In particular, if k is the Fermat prime $k = 2^{2^n} + 1$ for some $n \in \mathbb{Z}_*$ and $a = b = c = 2^{2^n - 1}$, we have

$$k = 2^{2^n} + 1 = 2 \cdot 2^{2^n - 1} + 1 = 2a + 1 = (b+c+1)(b+c)^{1-1} + \frac{\left((b+c)^{1-1} - 1\right)}{b+c-1},$$

and hence by the above result the trajectory $\langle \alpha_0, \alpha_1, \alpha_2, \ldots \rangle$ is cyclic. $\square$

Lemma 4.1.2. *Let $k = 2$, let $\lambda_1, \lambda_2 \in \mathbb{N}$ be such that $k \nmid \lambda_1$ and $k \nmid \lambda_2$ and let $t \in \mathbb{N}$.*

(i) *If $k \nmid \Big(\lambda_2\lambda_1^l + \sum\limits_{0 \leqslant \nu \leqslant l-1} \lambda_1^\nu\Big)$ for all $l \in \mathbb{N}(t-2)$, then $t = 1$ or $t = 2$.*

(ii) *If $k \nmid \Big(\lambda_2\lambda_1^l + \sum\limits_{0 \leqslant \nu \leqslant l-1} \lambda_1^\nu\Big)$ for all $l \in \mathbb{N}(t-2)$ and $\lambda_2\lambda_1^{t-1} + \sum\limits_{0 \leqslant \nu \leqslant t-2} \lambda_1^\nu = k^s$ for some $s \in \mathbb{N}$, then $t = 2$.*

Proof. (i) Assume that $k \nmid \Big(\lambda_2\lambda_1^l + \sum\limits_{0 \leqslant \nu \leqslant l-1} \lambda_1^\nu\Big)$ for all $l \in \mathbb{N}(t-2)$. Suppose $t \geq 3$. Then $t - 2 \geq 1$. By assumption, $k \nmid (\lambda_2\lambda_1 + 1)$. Since $k = 2$, $k \nmid \lambda_1$ and $k \nmid \lambda_2$, the product $\lambda_2\lambda_1$ is odd, so $\lambda_2\lambda_1 + 1$ is even, and hence $k \mid (\lambda_2\lambda_1 + 1)$, a contradiction. Therefore $t = 1$ or 2.

(ii) Assume that $k \nmid \Big(\lambda_2\lambda_1^l + \sum\limits_{0 \leqslant \nu \leqslant l-1} \lambda_1^\nu\Big)$ for all $l \in \mathbb{N}(t-2)$ and $\lambda_2\lambda_1^{t-1} + \sum\limits_{0 \leqslant \nu \leqslant t-2} \lambda_1^\nu = k^s$ for some $s \in \mathbb{N}$. By part (i), t must be 1 or 2. If $t = 1$, then $\lambda_2\lambda_1^{t-1} + \sum\limits_{0 \leqslant \nu \leqslant t-2} \lambda_1^\nu = \lambda_2$, so $\lambda_2 = k^s$, contrary to $k \nmid \lambda_2$. Hence $t = 2$. □

Proposition 4.1.3. *Let $k = 2$, $A = k^{j-1}\left[\begin{smallmatrix} 2^n+1 & 2^n+1 \\ u & 2(2^n+1)-u \end{smallmatrix}\right]$ and $\beta = k^j a \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$ for some $a, j, n, u \in \mathbb{N}$, where $j > 1$, $k \nmid a$ and $u < 2(2^n + 1)$; and let $\alpha_0 \in \mathbb{Z}^2 - (k\mathbb{Z})^2$ be such that $\alpha_m = a\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$ for some $m \in \mathbb{Z}_*$. Let λ, λ_1, λ_2, i and r be as in Theorem 3.13. Then $\lambda = 2^j(2^n+1)$, $\lambda_1 = 2^n + 1$ and $i = j$, and furthermore $\lambda, \lambda_1, \lambda_2, i$ and r satisfy the hypotheses in Theorem 3.13(ii) precisely when $n \in \{1, 2\}$.*

Proof. Let $\vec{e} = \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$. Since $A\vec{e} = 2^j(2^n + 1)\vec{e}$, $\lambda = 2^j(2^n + 1)$. Since $2 \nmid (2^n + 1)$, $\lambda_1 = 2^n + 1$ and $i = j$.

We will prove that $\lambda, \lambda_1, \lambda_2, i$ and r satisfy the hypotheses in Theorem 3.13(ii) precisely when $n \in \{1, 2\}$.

Assume that λ, λ_1, λ_2, i and r satisfy the hypotheses in Theorem 3.13(ii), i.e., there exists $t \in \mathbb{N}$ with the property that $2 \nmid \Big(\lambda_2\lambda_1^l + \sum\limits_{0 \leqslant \nu \leqslant l-1} \lambda_1^\nu\Big)$ for all $l \in \mathbb{N}(t-2)$ and $\lambda_2\lambda_1^{t-1} + \sum\limits_{0 \leqslant \nu \leqslant t-2} \lambda_1^\nu = 2^s$ for some $s \in \mathbb{N}$. By Lemma 4.1.2(ii), $t = 2$. Thus

$2^s = \lambda_2\lambda_1 + 1$. Suppose $n \geq 3$. Then $\lambda_1 + 1 = 2^n + 2 = 2(2^{n-1}+1)$. Since $2 \nmid (2^{n-1}+1)$, $\lambda_2 = 2^{n-1}+1$ and $r = 1$. Thus

$$\begin{aligned}\lambda_2\lambda_1 + 1 &= (2^{n-1}+1)(2^n+1)+1 \\ &= 2^{2n-1} + 2^{n-1} + 2^n + 2 \\ &= 2(2^{2n-2} + 2^{n-2} + 2^{n-1} + 1).\end{aligned}$$

Since $2n-2,\ n-2,\ n-1 \in \mathbb{N}$,

$$2 \nmid (2^{2n-2} + 2^{n-2} + 2^{n-1} + 1) \quad \text{and} \quad 2^{2n-2} + 2^{n-2} + 2^{n-1} + 1 > 1,$$

contrary to $\lambda_2\lambda_1 + 1 = 2^s$. Hence the only possibilities for n are 1 and 2.

Next we will show that the hypotheses in Theorem 3.13(ii) are satisfied in both cases.

Case 1. $n = 1$. Then $\lambda = 2^j \cdot 3$, so $\lambda_1 = 3$. Since $\lambda_1 + 1 = 4 = 2^2$, $\lambda_2 = 1$ and $r = 2$. We choose $t = 2$. Clearly, $2 \nmid \left(\lambda_2\lambda_1^l + \sum\limits_{0 \leqslant \nu \leqslant l-1} \lambda_1^\nu\right)$ for all $l \in \mathbb{N}(t-2)$, and since $\lambda_2\lambda_1 + 1 = 1 \cdot 3 + 1 = 4 = 2^2$, the hypotheses in Theorem 3.13(ii) are satisfied.

Case 2. $n = 2$. Then $\lambda = 2^j \cdot 5$, so $\lambda_1 = 5$. Since $\lambda_1 + 1 = 6 = 2 \cdot 3$, $\lambda_2 = 3$ and $r = 1$. Again we choose $t = 2$. Clearly, $2 \nmid \left(\lambda_2\lambda_1^l + \sum\limits_{0 \leqslant \nu \leqslant l-1} \lambda_1^\nu\right)$ for all $l \in \mathbb{N}(t-2)$, and since $\lambda_2\lambda_1 + 1 = 3 \cdot 5 + 1 = 16 = 2^4$, the hypotheses in Theorem 3.13(ii) are satisfied. □

Proposition 4.1.4. *Let* $k = 2, A = k^{j_1}\left[\begin{smallmatrix} 2^{n_1}+1 & 2^{n_2}+1 \\ u & 2^{n_1}+2^{n_2}+2-u \end{smallmatrix}\right]$ *and* $\beta = k^j a \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$ *for some* $a, j_1, j, n_1, n_2, u \in \mathbb{N}$, *where* $k \nmid a$ *and* $u < 2^{n_1} + 2^{n_2} + 2$; *and let* $\alpha_0 \in \mathbb{Z}^2 - (k\mathbb{Z})^2$ *be such that* $\alpha_m = a\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$ *for some* $m \in \mathbb{Z}_*$. *Let* λ, λ_1, λ_2, i *and* r *be as in Theorem 3.13. Suppose* $n_1 \neq n_2$. *Then* $\lambda = 2^{j_1+1}(2^{n_1-1} + 2^{n_2-1} + 1)$, *and furthermore* λ, λ_1, λ_2, i *and* r *satisfy the hypotheses in Theorem 3.13(ii) for appropriate values of* j_1 *precisely when* n_1, n_2 *satisfy* $3 \leq n_1 + n_2 \leq 5$.

Proof. Suppose $n_1 < n_2$. Let $\vec{e} = \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$. Since $A\vec{e} = 2^{j_1+1}(2^{n_1-1} + 2^{n_2-1} + 1)\vec{e}$, we have $\lambda = 2^{j_1+1}(2^{n_1-1} + 2^{n_2-1} + 1)$.

We will prove that λ, λ_1, λ_2, i and r satisfy the hypotheses in Theorem 3.13(ii) precisely when n_1, n_2 satisfy $3 \le n_1 + n_2 \le 5$, i.e., when

$$(n_1, n_2) \in \big\{(1,2),(1,3),(1,4),(2,3),(3,2),(4,1),(3,1),(2,1)\big\}.$$

Assume that λ, λ_1, λ_2, i and r satisfy the hypotheses in Theorem 3.13(ii). As can be seen from the proof of Proposition 4.1.3, the only possible choice for t is 2, and thus it suffices to show that $\lambda_2\lambda_1 + 1$ is a power of 2 precisely when $3 \le n_1 + n_2 \le 5$.

Case 1. $n_1 = 1$. Then $n_2 \ge 2$ and $\lambda = 2^{j_1+1}(1 + 2^{n_2-2} + 1) = 2^{j_1+2}(2^{n_2-1} + 1)$.

Case 1.1. $n_2 = 2$. Then $\lambda = 2^{j_1+2}(1+1) = 2^{j_1+3}$, so $\lambda_1 = 1$ and $i = j_1 + 3$. Since $\lambda_1 + 1 = 1 + 1 = 2$, $\lambda_2 = 1$ and $r = 1$. Thus $\lambda_2\lambda_1 + 1 = 1 \cdot 1 + 1 = 2$.

Case 1.2. $n_2 = 3$. Then $\lambda = 2^{j_1+2}(2+1) = 2^{j_1+2} \cdot 3$, so $\lambda_1 = 3$ and $i = j_1 + 2$. Since $\lambda_1 + 1 = 3 + 1 = 4 = 2^2$, $\lambda_2 = 1$ and $r = 2$. Thus $\lambda_2\lambda_1 + 1 = 1 \cdot 3 + 1 = 4 = 2^2$.

Case 1.3. $n_2 = 4$. Then $\lambda = 2^{j_1+2}(4+1) = 2^{j_1+2} \cdot 5$, so $\lambda_1 = 5$ and $i = j_1 + 2$. Since $\lambda_1 + 1 = 6 = 2 \cdot 3$, $\lambda_2 = 3$ and $r = 1$. Thus $\lambda_2\lambda_1 + 1 = 3 \cdot 5 + 1 = 16 = 2^4$.

Case 1.4. $n_2 \ge 5$. Then $\lambda = 2^{j_1+2}(2^{n_2-2} + 1)$ and since $2 \nmid (2^{n_2-2} + 1)$, $\lambda_1 = 2^{n_2-2} + 1$ and $i = j_1 + 2$. Since $\lambda_1 + 1 = 2^{n_2-2} + 2 = 2(2^{n_2-3} + 1)$ and $2 \nmid (2^{n_2-3} + 1)$, $\lambda_2 = 2^{n_2-3} + 1$ and $r = 1$. Thus

$$\begin{aligned}\lambda_2\lambda_1 + 1 &= (2^{n_2-3} + 1)(2^{n_2-2} + 1) + 1\\ &= 2^{2n_2-5} + 2^{n_2-3} + 2^{n_2-2} + 2\\ &= 2(2^{2n_2-6} + 2^{n_2-4} + 2^{n_2-3} + 1).\end{aligned}$$

Since $2n_2 - 6, n_2 - 4, n_2 - 3 \in \mathbb{N}$,

$$2 \nmid (2^{2n_2-6} + 2^{n_2-4} + 2^{n_2-3} + 1) \quad\text{and}\quad 2^{2n_2-6} + 2^{n_2-4} + 2^{n_2-3} + 1 > 1,$$

and hence $\lambda_2\lambda_1 + 1$ is not a power of 2.

Case 2. $n_1 = 2$. Then $n_2 \geq 3$ and $\lambda = 2^{j_1+1}(2 + 2^{n_2-1} + 1) = 2^{j_1+1}(2^{n_2-1} + 3)$.

Case 2.1. $n_2 = 3$. Then $\lambda = 2^{j_1+1}(2^2+3) = 2^{j_1+1}\cdot 7$, so $\lambda_1 = 7$ and $i = j_1 + 1$. Since $\lambda_1+1 = 7+1 = 8 = 2^3$, $\lambda_2 = 1$ and $r = 3$. Thus $\lambda_2\lambda_1+1 = 1\cdot 7+1 = 8 = 2^3$.

Case 2.2. $n_2 \geq 4$. Then $\lambda_1 = 2^{n_2-1}+3$ and $i = j_1 + 1$ since $\lambda = 2^{j_1+1}(2^{n_2-1} + 3)$ and $2 \nmid (2^{n_2-1} + 3)$. Since $\lambda_1 + 1 = 2^{n_2-1} + 4 = 2^2(2^{n_2-3} + 1)$, $\lambda_2 = 2^{n_2-3} + 1$ and $r = 2$. Thus

$$\begin{aligned}\lambda_2\lambda_1 + 1 &= (2^{n_2-3} + 1)(2^{n_2-1} + 3) + 1\\ &= 2^{2n_2-4} + 3 \cdot 2^{n_2-3} + 2^{n_2-1} + 3 + 1\\ &= 2(2^{2n_2-5} + 3 \cdot 2^{n_2-4} + 2^{n_2-2} + 2).\end{aligned}$$

If $n_2 = 4$, then $\lambda_2\lambda_1 + 1 = 34 = 2 \cdot 17$ which is not a power of 2. If $n_2 = 5$, then $\lambda_2\lambda_1 + 1 = 96 = 2^5 \cdot 3$ which is not a power of 2. If $n_2 \geq 6$, then $\lambda_2\lambda_1 + 1$ $= 2^2(2^{2n_2-6} + 3 \cdot 2^{n_2-5} + 2^{n_2-3} + 1)$ and since $2n_2 - 6, n_2 - 5, n_2 - 3 \in \mathbb{N}$,

$$2 \nmid (2^{2n_2-6} + 3 \cdot 2^{n_2-5} + 2^{n_2-3} + 1) \quad \text{and} \quad (2^{2n_2-6} + 3 \cdot 2^{n_2-5} + 2^{n_2-3} + 1) > 1,$$

$\lambda_2\lambda_1 + 1$ is not a power of 2.

Case 3. $n_1 \geq 3$. Then $n_2 \geq 4$. Since $\lambda = 2^{j_1+1}(2^{n_1-1} + 2^{n_2-1} + 1)$ and $2 \nmid (2^{n_1-1} + 2^{n_2-1} + 1)$, $\lambda_1 = 2^{n_1-1} + 2^{n_2-1} + 1$ and $i = j_1 + 1$. Then

$$\lambda_1 + 1 = 2^{n_1-1} + 2^{n_2-1} + 2 = 2(2^{n_1-2} + 2^{n_2-2} + 1)$$

and

$$2 \nmid (2^{n_1-2} + 2^{n_2-2} + 1),$$

so $\lambda_2 = 2^{n_1-2} + 2^{n_2-2} + 1$ and $r = 1$. Thus

$$\lambda_2\lambda_1 + 1 = (2^{n_1-2} + 2^{n_2-2} + 1)(2^{n_1-1} + 2^{n_2-1} + 1) + 1$$

$$= 2^{2n_1-3} + 2^{n_1+n_2-3} + 2^{n_1-2} + 2^{n_1+n_2-3} + 2^{2n_2-3} + 2^{n_2-2}$$
$$+ 2^{n_1-1} + 2^{n_2-1} + 2$$
$$= 2^{2n_1-3} + 2^{n_1+n_2-2} + 2^{n_1-2} + 2^{2n_2-3} + 2^{n_2-2} + 2^{n_1-1} + 2^{n_2-1} + 2$$
$$= 2(2^{2n_1-4} + 2^{n_1+n_2-3} + 2^{n_1-3} + 2^{2n_2-4} + 2^{n_2-3} + 2^{n_1-2} + 2^{n_2-2} + 1).$$

Case 3.1. $n_1 = 3$. Then

$$\lambda_2\lambda_1 + 1 = 2(2^2 + 2^{n_2} + 1 + 2^{2n_2-4} + 2^{n_2-3} + 2 + 2^{n_2-2} + 1)$$
$$= 2(2^{n_2} + 2^{2n_2-4} + 2^{n_2-3} + 2^{n_2-2} + 8)$$
$$= 2^2(2^{n_2-1} + 2^{2n_2-5} + 2^{n_2-4} + 2^{n_2-3} + 4).$$

Since $n_2 - 1, 2n_2 - 5, n_2 - 3 \in \mathbb{N}$, $2^{n_2-1} + 2^{2n_2-5} + 2^{n_2-4} + 2^{n_2-3} + 4$ is even only if $n_2 - 4 \in \mathbb{N}$, and thus $\lambda_2\lambda_1 + 1$ can be a power of 2 only if $n_2 > 4$. If $n_2 = 5$, then $\lambda_2\lambda_1 + 1 = 232 = 2^3 \cdot 29$ which is not a power of 2. If $n_2 = 6$, then $\lambda_2\lambda_1 + 1 = 704 = 2^6 \cdot 11$ which is not a power of 2. If $n_2 \geq 7$, then $\lambda_2\lambda_1 + 1$ $= 2^4(2^{n_2-3} + 2^{2n_2-7} + 2^{n_2-6} + 2^{n_2-5} + 1)$ and since $n_2 - 3, 2n_2 - 7, n_2 - 6, n_2 - 5 \in \mathbb{N}$, $2 \nmid (2^{n_2-3} + 2^{2n_2-7} + 2^{n_2-6} + 2^{n_2-5} + 1)$ and $2^{n_2-3} + 2^{2n_2-7} + 2^{n_2-6} + 2^{n_2-5} + 1 > 1$, $\lambda_2\lambda_1 + 1$ is not a power of 2.

Case 3.2. $n_1 \geq 4$. Then $n_2 \geq 5$. Since

$$\lambda_2\lambda_1 + 1 = 2(2^{2n_1-4} + 2^{n_1+n_2-3} + 2^{n_1-3} + 2^{2n_2-4} + 2^{n_2-3} + 2^{n_1-2} + 2^{n_2-2} + 1)$$

and

$$2n_2 - 4, n_1 + n_2 - 3, n_1 - 3, 2n_2 - 4, n_2 - 3, n_1 - 2, n_2 - 2 \in \mathbb{N},$$
$$2 \nmid (2^{2n_1-4} + 2^{n_1+n_2-3} + 2^{n_1-3} + 2^{2n_2-4} + 2^{n_2-3} + 2^{n_1-2} + 2^{n_2-2} + 1)$$

and

$$2^{2n_1-4} + 2^{n_1+n_2-3} + 2^{n_1-3} + 2^{2n_2-4} + 2^{n_2-3} + 2^{n_1-2} + 2^{n_2-2} + 1 > 1,$$

$\lambda_2\lambda_1 + 1$ is not a power of 2.

By cases 1–3, we have $(n_1, n_2) \in \{(1,2),(1,3),(1,4),(2,3)\}$. As can be seen from above, interchanging the roles of n_1 and n_2 does not affect the proof. Therefore if $n_2 < n_1$, then $\lambda, \lambda_1, \lambda_2, i$ and r satisfy the hypotheses in Theorem 3.13(ii) precisely when $(n_1, n_2) \in \{(2,1),(3,1),(4,1),(3,2)\}$. □

Now we summarize Propositions 4.1.3 and 4.1.4 as follows:

Corollary 4.1.5. *Let $k = 2, A = k^{j_1}\left[\begin{smallmatrix} 2^{n_1}+1 & 2^{n_2}+1 \\ u & 2^{n_1}+2^{n_2}+2-u \end{smallmatrix}\right]$ and $\beta = k^j a \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$ for some $a, j_1, j, n_1, n_2, u \in \mathbb{N}$, where $k \nmid a$ and $u < 2^{n_1}+2^{n_2}+2$; and let $\alpha_0 \in \mathbb{Z}^2-(k\mathbb{Z})^2$ be such that $\alpha_m = a\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$ for some $m \in \mathbb{Z}_*$. Then A satisfies the hypotheses in Theorem 3.10(iii). Let λ, λ_1, λ_2, i and r be as in Theorem 3.13. Then λ, λ_1, λ_2, i and r satisfy the hypotheses in Theorem 3.13(ii) for appropriate values of j_1 ($j_1 = j-1$ when $n_1 = n_2$) precisely when n_1, n_2 satisfy $2 \le n_1 + n_2 \le 5$. In particular, for appropriate values of j_1 ($j_1 = j-1$ when $n_1 = n_2$) the trajectory $\langle \alpha_0, \alpha_1, \alpha_2, \ldots \rangle$ is cyclic for all pairs $(n_1, n_2) \in \mathbb{N} \times \mathbb{N}$ such that n_1, n_2 satisfy $2 \le n_1 + n_2 \le 5$.*

Proof. This follows directly from Propositions 4.1.3 and 4.1.4 and Theorem 3.13(ii), since $n_1 + n_2 = 2$ implies $n_1 = n_2 = 1$ and $n_1 = n_2$ together with $n_1 + n_2 \le 5$ implies $n_1 = 1$ or $n_1 = 2$. □

Next we will provide concrete examples for the existence of matrices A and vectors α_0 and β that satisfy the conditions in Theorem 3.14 and satisfy the hypotheses in Theorem 3.10(iii).

Proposition 4.1.6. *Let $A = k^j\left[\begin{smallmatrix} b & c \\ u & b+c-u \end{smallmatrix}\right]$ and $\beta = k^j d\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$ for some $b, c, d, u, j \in \mathbb{N}$, where $k \nmid bc$, $u < b+c$, $k \nmid (b+c)$, $d \ne 1$ and $k \nmid (b+c+d)$; and let $\alpha_0 \in \mathbb{Z}^2-(k\mathbb{Z})^2$ be* *such that* $\alpha_m = \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$ *for some* $m \in \mathbb{Z}_*$. *Then* A *satisfies the hypotheses in Theorem 3.10(iii). Note that* $k \neq 2$. *If there exists* $t \in \mathbb{N}$ *such that*

$$k = (b+c+d)(b+c)^{t-1} + \frac{\Big((b+c)^{t-1} - 1\Big)}{b+c-1} \ ,$$

then the trajectory $\langle \alpha_0, \alpha_1, \alpha_2, \ldots \rangle$ *is cyclic. In particular, if* k *is the Fermat prime* $k = 2^{2^n} + 1$ *for some* $n \in \mathbb{N}$, $b = 1$ *and* $c = d = 2^{2^n - 1}$, *we have that the trajectory* $\langle \alpha_0, \alpha_1, \alpha_2, \ldots \rangle$ *is cyclic.*

Proof. As in Proposition 4.1.1, it easy to check that A satisfies the hypotheses in Theorem 3.10(iii) and k cannot equal 2. Assume that there exists $t \in \mathbb{N}$ such that

$$k = (b+c+d)(b+c)^{t-1} + \frac{\Big((b+c)^{t-1} - 1\Big)}{b+c-1}.$$

We will show that $k \nmid \left((b+c+d)(b+c)^l + \dfrac{(b+c)^l - 1}{b+c-1}\right)$ for all $l \in \mathbb{N}(t-2)$. Since

$$k = (b+c+d)(b+c)^{t-1} + \frac{(b+c)^{t-1} - 1}{b+c-1} > (b+c+d)(b+c)^l + \frac{(b+c)^l - 1}{b+c-1}$$

and

$$(b+c+d)(b+c)^l + \frac{(b+c)^l - 1}{b+c-1} \in \mathbb{N}$$

for all $l \in \mathbb{N}(t-2)$, $(b+c+d)(b+c)^l + \dfrac{(b+c)^l - 1}{b+c-1} \in \{1, 2, \ldots, k-1\}$ for all $l \in \mathbb{N}(t-2)$, so

$$k \nmid \left((b+c+d)(b+c)^l + \frac{(b+c)^l - 1}{b+c-1}\right)$$

for all $l \in \mathbb{N}(t-2)$. By Theorem 3.14(ii), since $\vec{e} = \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$, $\lambda = k^j(b+c)$, $i = j$, $\lambda_1 = b+c$ and $\lambda_2 = b+c+d$, we have that the trajectory $\langle \alpha_0, \alpha_1, \alpha_2, \ldots \rangle$ is cyclic.

In particular, if k is the Fermat prime $k = 2^{2^n} + 1$ for some $n \in \mathbb{N}$, $b = 1$, and $c = d = 2^{2^n - 1}$, we have

$$k = 2^{2^n} + 1 = 1 + 2 \cdot 2^{2^n - 1} = 1 + 2c = (b+c+d)(b+c)^{1-1} + \frac{\Big((b+c)^{1-1} - 1\Big)}{b+c-1},$$

and hence by the above result the trajectory $\langle \alpha_0, \alpha_1, \alpha_2, \ldots \rangle$ is cyclic. $\square$

Proposition 4.1.7. *Let* $k = 2$, $A = k^{j-1}\left[\begin{smallmatrix} 2^n+1 & 2^n+1 \\ u & 2(2^n+1)-u \end{smallmatrix}\right]$ *and* $\beta = k^j d\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$ *for some* $d, j, n, u \in \mathbb{N}$, *where* $j > 1$, $k \nmid d$ *and* $u < 2(2^n+1)$; *and let* $\alpha_0 \in \mathbb{Z}^2 - (k\mathbb{Z})^2$ *be such that* $\alpha_m = a\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$ *for some* $a, m \in Z_*$, *where* $a \neq d$, $k \nmid a$ *and* $a(2^n+1) + d$ $= a \cdot 2^s$ *for some* $s \in \mathbb{N}$. *Then* A *satisfies the hypotheses in Theorem 3.10(iii). Let* λ, λ_1, λ_2, i *and* r *be as in Theorem 3.14. Then* $\lambda = 2^j(2^n+1)$, $\lambda_1 = 2^n+1$ *and* $i = j$, *and furthermore the trajectory* $\langle \alpha_0, \alpha_1, \alpha_2, \ldots \rangle$ *is cyclic.*

Proof. It is easy to check that A satisfies the hypotheses in Theorem3.10(iii). Let $\vec{e} = \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$. Since $A\vec{e} = 2^j(2^n+1)\vec{e}$, $\lambda = 2^j(2^n+1)$. Since $2 \nmid (2^n+1)$, $\lambda_1 = 2^n+1$ and $i = j$. Since $a\lambda_1 + d = a(2^n+1) + d = a \cdot 2^s$, $\lambda_2 = a$ and $r = s$. For $t = 2$, we have

$$2 \nmid \Big(\lambda_2 {\lambda_1}^l + \big(\sum_{0 \leqslant \nu \leqslant l-1} \lambda_1^\nu\big) d\Big)$$

for all $l \in \mathbb{N}(t-2)$ and

$$\lambda_2 {\lambda_1}^{t-1} + \Big(\sum_{0 \leqslant \nu \leqslant t-2} \lambda_1^\nu\Big) d = \lambda_2\lambda_1 + d = a \cdot (2^n+1) + d = a \cdot 2^s.$$

Thus by Theorem 3.14(ii) the trajectory $\langle \alpha_0, \alpha_1, \alpha_2, \ldots \rangle$ is cyclic. $\square$

4.2 Existence of Matrices as in Theorems 3.13–3.14 and 3.10(v)

Before we provide concrete examples for the existence of matrices A and vectors α_0 and β that satisfy the conditions in Theorems 3.13 and 3.14 and satisfy the hypotheses in Theorem 3.10(v), we will prove the following lemma.

Lemma 4.2.1. *Let* $A \in M_2(\mathbb{N})$ *be such that* $\det(\bar{A}) = \bar{0}, \dim(\mathrm{Im}(\bar{A})) = 1$ *and* $(\bar{A})^2 = \bar{0}$.

(i) $\mathrm{Im}(\bar{A}) = \mathrm{span}\left[\begin{smallmatrix}\bar{1}\\ \bar{0}\end{smallmatrix}\right]$ *if and only if* $\bar{A} = \left[\begin{smallmatrix}\bar{0} & \bar{n}\\ \bar{0} & \bar{0}\end{smallmatrix}\right]$ *for some* $n \in \mathbb{N}$ *with* $k \nmid n$. *In this case we have* $A = \left[\begin{smallmatrix}k^{i_1}m_1 & m_2\\ k^{i_2}m_3 & k^{i_3}m_4\end{smallmatrix}\right]$, *where* $i_1, i_2, i_3, m_1, m_2, m_3, m_4 \in \mathbb{N}$ *and* $k \nmid m_1 \cdot m_2 \cdot m_3 \cdot m_4$.

(ii) $\mathrm{Im}(\bar{A}) = \mathrm{span}\left[\begin{smallmatrix}\bar{0}\\ \bar{1}\end{smallmatrix}\right]$ *if and only if* $\bar{A} = \left[\begin{smallmatrix}\bar{0} & \bar{0}\\ \bar{n} & \bar{0}\end{smallmatrix}\right]$ *for some* $n \in \mathbb{N}$ *with* $k \nmid n$. *In this case we have* $A = \left[\begin{smallmatrix}k^{i_1}m_1 & k^{i_2}m_2\\ m_3 & k^{i_3}m_4\end{smallmatrix}\right]$, *where* $i_1, i_2, i_3, m_1, m_2, m_3, m_4 \in \mathbb{N}$ *and* $k \nmid m_1 \cdot m_2 \cdot m_3 \cdot m_4$.

(iii) $\mathrm{Im}(\bar{A}) = \mathrm{span}\left[\begin{smallmatrix}\bar{1}\\ \bar{1}\end{smallmatrix}\right]$ *if and only if* $\bar{A} = \left[\begin{smallmatrix}\bar{n} & \overline{k-n}\\ \bar{n} & \overline{k-n}\end{smallmatrix}\right]$ *for some* $n \in \mathbb{N}$ *with* $k \nmid n$. *In this case we have* $A = \left[\begin{smallmatrix}m_1 & m_2\\ m_3 & m_4\end{smallmatrix}\right]$, *where* $m_1, m_2, m_3, m_4 \in \mathbb{N}$, $k \nmid m_1 \cdot m_2 \cdot m_3 \cdot m_4$, $\bar{n} = \bar{m}_1 = \bar{m}_3$ *and* $\overline{k-n} = \bar{m}_2 = \bar{m}_4$.

Proof. (i) We will show that $\mathrm{Im}(\bar{A}) = \mathrm{span}\left[\begin{smallmatrix}\bar{1}\\ \bar{0}\end{smallmatrix}\right]$ if and only if $\bar{A} = \left[\begin{smallmatrix}\bar{0} & \bar{n}\\ \bar{0} & \bar{0}\end{smallmatrix}\right]$ for some $n \in \mathbb{N}$ with $k \nmid n$. The form of A follows easily from this result.

($\Rightarrow$) Assume that $\mathrm{Im}(\bar{A}) = \mathrm{span}\left[\begin{smallmatrix}\bar{1}\\ \bar{0}\end{smallmatrix}\right]$. Write A as $\left[\begin{smallmatrix}n_1 & n_2\\ n_3 & n_4\end{smallmatrix}\right]$, where $n_1, n_2, n_3, n_4 \in \mathbb{N}$. Since $\left[\begin{smallmatrix}\bar{1}\\ \bar{0}\end{smallmatrix}\right] \in {\mathbb{Z}_k}^2$,

$$\left[\begin{smallmatrix}\bar{n}_1\\ \bar{n}_3\end{smallmatrix}\right] = \left[\begin{smallmatrix}\bar{n}_1 & \bar{n}_2\\ \bar{n}_3 & \bar{n}_4\end{smallmatrix}\right]\left[\begin{smallmatrix}\bar{1}\\ \bar{0}\end{smallmatrix}\right] \in \mathrm{Im}(\bar{A}) = \mathrm{span}\left[\begin{smallmatrix}\bar{1}\\ \bar{0}\end{smallmatrix}\right].$$

This implies that $\bar{n}_3 = \bar{0}$. Similarly, since $\left[\begin{smallmatrix}\bar{0}\\ \bar{1}\end{smallmatrix}\right] \in (\mathbb{Z}_k)^2$, $\bar{n}_4 = \bar{0}$. From $(\bar{A})^2 = \bar{0}$, by Lemma 3.6 we have $\bar{n}_1 + \bar{n}_4 = \bar{0}$, and hence $\bar{n}_1 = \bar{0}$. Thus $\bar{A} = \left[\begin{smallmatrix}\bar{0} & \bar{n}_2\\ \bar{0} & \bar{0}\end{smallmatrix}\right]$. Since $\dim(\mathrm{Im}(\bar{A})) = 1$, $\bar{A} \neq \bar{0}$, so $\bar{n}_2 \neq \bar{0}$. Hence $\bar{A} = \left[\begin{smallmatrix}\bar{0} & \bar{n}_2\\ \bar{0} & \bar{0}\end{smallmatrix}\right]$, where $n_2 \in \mathbb{N}$ and $k \nmid n_2$.

($\Leftarrow$) Assume that $\bar{A} = \left[\begin{smallmatrix}\bar{0} & \bar{n}\\ \bar{0} & \bar{0}\end{smallmatrix}\right]$, where $n \in \mathbb{N}$ and $k \nmid n$. For any $\bar{p}, \bar{q} \in \mathbb{Z}_k$,

$$\left[\begin{smallmatrix}\bar{0} & \bar{n}\\ \bar{0} & \bar{0}\end{smallmatrix}\right]\left[\begin{smallmatrix}\bar{p}\\ \bar{q}\end{smallmatrix}\right] = \left[\begin{smallmatrix}\bar{n}\bar{q}\\ \bar{0}\end{smallmatrix}\right] = \bar{n}\bar{q}\left[\begin{smallmatrix}\bar{1}\\ \bar{0}\end{smallmatrix}\right] \in \mathrm{span}\left[\begin{smallmatrix}\bar{1}\\ \bar{0}\end{smallmatrix}\right].$$

Thus $\mathrm{Im}(\bar{A}) \subseteq \mathrm{span}\left[\begin{smallmatrix}\bar{1}\\ \bar{0}\end{smallmatrix}\right]$. To prove that $\mathrm{Im}(\bar{A}) = \mathrm{span}\left[\begin{smallmatrix}\bar{1}\\ \bar{0}\end{smallmatrix}\right]$, it suffices to show that $\mathrm{Im}(\bar{A}) \neq \left\{\vec{0}\right\}$, because $\mathrm{span}\left[\begin{smallmatrix}\bar{1}\\ \bar{0}\end{smallmatrix}\right]$ is one-dimensional vector space and $\mathrm{Im}(\bar{A})$ is a subspace. Since $A\left[\begin{smallmatrix}\bar{0}\\ \bar{1}\end{smallmatrix}\right] = \left[\begin{smallmatrix}\bar{n}\\ \bar{0}\end{smallmatrix}\right] \neq \vec{0}$, $\mathrm{Im}(\bar{A}) \neq \left\{\vec{0}\right\}$. Thus $\mathrm{Im}(\bar{A}) = \mathrm{span}\left[\begin{smallmatrix}\bar{1}\\ \bar{0}\end{smallmatrix}\right]$.

(ii) This is similar to the proof of (i).

(iii) We will show that $\mathrm{Im}(\bar{A}) = \mathrm{span}\left[\begin{smallmatrix}\bar{1}\\ \bar{1}\end{smallmatrix}\right]$ if and only if $\bar{A} = \begin{bmatrix}\bar{n} & \overline{k-n}\\ \bar{n} & \overline{k-n}\end{bmatrix}$ for some $n \in \mathbb{N}$ with $k \nmid n$. Again, the form of A follows easily from this result.

($\Rightarrow$) Assume that $\mathrm{Im}(\bar{A}) = \mathrm{span}\left[\begin{smallmatrix}\bar{1}\\ \bar{1}\end{smallmatrix}\right]$. Write A as $\left[\begin{smallmatrix}n_1 & n_2\\ n_3 & n_4\end{smallmatrix}\right]$, where $n_1, n_2, n_3, n_4 \in \mathbb{N}$. Since $\left[\begin{smallmatrix}\bar{1}\\ \bar{0}\end{smallmatrix}\right] \in {\mathbb{Z}_k}^2$,

$$\left[\begin{smallmatrix}\bar{n}_1\\ \bar{n}_3\end{smallmatrix}\right] = \left[\begin{smallmatrix}\bar{n}_1 & \bar{n}_2\\ \bar{n}_3 & \bar{n}_4\end{smallmatrix}\right]\left[\begin{smallmatrix}\bar{1}\\ \bar{0}\end{smallmatrix}\right] \in \mathrm{Im}(\bar{A}) = \mathrm{span}\left[\begin{smallmatrix}\bar{1}\\ \bar{1}\end{smallmatrix}\right].$$

This implies $\bar{n}_1 = \bar{n}_3$. Similarly, since $\left[\begin{smallmatrix}\bar{0}\\ \bar{1}\end{smallmatrix}\right] \in (\mathbb{Z}_k)^2$, $\bar{n}_2 = \bar{n}_4$. Thus $\bar{A} = \left[\begin{smallmatrix}\bar{n}_1 & \bar{n}_2\\ \bar{n}_1 & \bar{n}_2\end{smallmatrix}\right]$. Suppose $\bar{n}_1 = \bar{0}$. If $\bar{n}_2 = \bar{0}$, then $\bar{A} = \bar{0}$, so $\dim(\mathrm{Im}(\bar{A})) = 0$, a contradiction. Thus $\bar{n}_2 \neq \bar{0}$ which implies $\bar{n}_2^2 \neq \bar{0}$. Hence

$$(\bar{A})^2 = \begin{bmatrix}\bar{0} & \bar{n}_2\\ \bar{0} & \bar{n}_2\end{bmatrix}\begin{bmatrix}\bar{0} & \bar{n}_2\\ \bar{0} & \bar{n}_2\end{bmatrix} = \begin{bmatrix}\bar{0} & \bar{n}_2^2\\ \bar{0} & \bar{n}_2^2\end{bmatrix} \neq \bar{0},$$

a contradiction. Therefore $\bar{n}_1 \neq \bar{0}$. Similarly, we can show that $\bar{n}_2 \neq \bar{0}$. Thus $\bar{A} = \left[\begin{smallmatrix}\bar{n}_1 & \bar{n}_2\\ \bar{n}_1 & \bar{n}_2\end{smallmatrix}\right]$, where $k \nmid n_1, k \nmid n_2$. To find the relationship between $\bar{n}_1$ and $\bar{n}_2$, observe that

$$\bar{0} = (\bar{A})^2 = \left[\begin{smallmatrix}\bar{n}_1 & \bar{n}_2\\ \bar{n}_1 & \bar{n}_2\end{smallmatrix}\right]\left[\begin{smallmatrix}\bar{n}_1 & \bar{n}_2\\ \bar{n}_1 & \bar{n}_2\end{smallmatrix}\right] = \begin{bmatrix}(\bar{n}_1)^2+\bar{n}_1\bar{n}_2 & \bar{n}_1\bar{n}_2+(\bar{n}_2)^2\\ (\bar{n}_1)^2+\bar{n}_1\bar{n}_2 & \bar{n}_1\bar{n}_2+(\bar{n}_2)^2\end{bmatrix}.$$

Thus $\bar{0} = (\bar{n}_1)^2 + \bar{n}_1\bar{n}_2 = \bar{n}_1(\bar{n}_1 + \bar{n}_2)$. Since $\bar{n}_1 \neq \bar{0}$, $\bar{0} = \bar{n}_1 + \bar{n}_2$, which implies $\bar{n}_2 = -\bar{n}_1 = \overline{k - n_1}$. Hence $\bar{A} = \begin{bmatrix}\bar{n}_1 & \overline{k-n_1}\\ \bar{n}_1 & \overline{k-n_1}\end{bmatrix}$, where $k \nmid n_1$.

($\Leftarrow$) Assume that $\bar{A} = \begin{bmatrix}\bar{n} & \overline{k-n}\\ \bar{n} & \overline{k-n}\end{bmatrix}$ for some $n \in \mathbb{N}$ with $k \nmid n$. For any $\bar{p}, \bar{q} \in \mathbb{Z}_k$,

$$\begin{bmatrix}\bar{n} & \overline{k-n}\\ \bar{n} & \overline{k-n}\end{bmatrix}\left[\begin{smallmatrix}\bar{p}\\ \bar{q}\end{smallmatrix}\right] = \begin{bmatrix}\bar{n}\bar{p}+ \overline{k-n}\bar{q}\\ \bar{n}\bar{p}+ \overline{k-n}\bar{q}\end{bmatrix} = (\bar{n}\bar{p} + \overline{k-n}\bar{q})\left[\begin{smallmatrix}\bar{1}\\ \bar{1}\end{smallmatrix}\right] \in \mathrm{span}\left[\begin{smallmatrix}\bar{1}\\ \bar{1}\end{smallmatrix}\right].$$

Thus $\mathrm{Im}(\bar{A}) \subseteq \mathrm{span}\left[\begin{smallmatrix}\bar{1}\\ \bar{1}\end{smallmatrix}\right]$. As above, to prove equality it suffices to show that $\mathrm{Im}(\bar{A}) \neq \left\{\vec{0}\right\}$. Since $A\left[\begin{smallmatrix}\bar{1}\\ \bar{0}\end{smallmatrix}\right] = \left[\begin{smallmatrix}\bar{n}\\ \bar{n}\end{smallmatrix}\right] \neq \vec{0}$, we have that $\mathrm{Im}(\bar{A}) = \mathrm{span}\left[\begin{smallmatrix}\bar{1}\\ \bar{1}\end{smallmatrix}\right]$. $\square$

Proposition 4.2.2. *Let* $A = \left[\begin{smallmatrix}k & 1\\ k^2 & k\end{smallmatrix}\right]$ *and* $\beta = ka\left[\begin{smallmatrix}1\\ k\end{smallmatrix}\right]$ *for some* $a \in \mathbb{N}$ *with* $k \nmid a$*; and let* $\alpha_0 \in \mathbb{Z}^2 - (k\mathbb{Z})^2$ *be such that* $\alpha_m = a\left[\begin{smallmatrix}1\\ k\end{smallmatrix}\right]$ *for some* $m \in \mathbb{Z}_*$*. Then* A *satisfies the hypotheses in Theorem 3.10(v.1). Let* λ, λ_1, λ_2, i *and* r *be as in Theorem 3.13. Then* $\lambda = 2k$*, and the trajectory* $\langle \alpha_0, \alpha_1, \alpha_2, \ldots\rangle$ *is cyclic if* $k = 2$ *or* $k = 3$*.*

Proof. It is easy to check that A satisfies the hypotheses in Theorem 3.10(v.1). Let $\vec{e} = \left[\begin{smallmatrix}1\\k\end{smallmatrix}\right]$. Since $A\vec{e} = 2k\left[\begin{smallmatrix}1\\k\end{smallmatrix}\right]$, $\lambda = 2k$.

Case 1. $k = 2$. Then $\lambda = 4 = 2^2$, so $\lambda_1 = 1$ and $i = 2$. Since $\lambda_1 + 1 = 1 + 1 = 2$, $\lambda_2 = 1$ and $r = 1$. For $t = 2$, we have $2 \nmid \Big(\lambda_2\lambda_1^l + \sum\limits_{0\leqslant\nu\leqslant l-1} \lambda_1^\nu\Big)$ for all $l \in \mathbb{N}(t-2)$ and $\lambda_2\lambda_1^{t-1} + \sum\limits_{0\leqslant\nu\leqslant t-2} \lambda_1^\nu = \lambda_2\lambda_1 + 1 = 1\cdot 1 + 1 = 2$. By Theorem 3.13(ii), the trajectory $\langle \alpha_0, \alpha_1, \alpha_2, \ldots\rangle$ is cyclic.

Case 2. $k = 3$. Then $\lambda = 3 \cdot 2$, so $\lambda_1 = 2$ and $i = 1$. Since $\lambda_1 + 1 = 2 + 1 = 3$, $\lambda_2 = 1$ and $r = 1$. For $t = 2$, we have $3 \nmid \Big(\lambda_2\lambda_1^l + \sum\limits_{0\leqslant\nu\leqslant l-1} \lambda_1^\nu\Big)$ for all $l \in \mathbb{N}(t-2)$ and $\lambda_2\lambda_1^{t-1} + \sum\limits_{0\leqslant\nu\leqslant t-2} \lambda_1^\nu = \lambda_2\lambda_1 + 1 = 1\cdot 2 + 1 = 3$. By Theorem 3.13(ii), the trajectory $\langle \alpha_0, \alpha_1, \alpha_2, \ldots\rangle$ is cyclic. □

Proposition 4.2.3. *Let $A = \left[\begin{smallmatrix}k & 1\\ k^{2n} & k\end{smallmatrix}\right]$ and $\beta = kd\left[\begin{smallmatrix}1\\k^n\end{smallmatrix}\right]$ for some $n, d \in \mathbb{N}$ with $k \nmid d$ and $n \geq 2$; and let $\alpha_0 \in \mathbb{Z}^2 - (k\mathbb{Z})^2$ be such that $\alpha_m = a\left[\begin{smallmatrix}1\\k^n\end{smallmatrix}\right]$ for some $a, m \in \mathbb{Z}_*$, where $k \nmid a$, $a \neq d$ and $a(1 + k^{n-1}) + d = a \cdot k^s$ for some $s \in \mathbb{N}$. Then A satisfies the hypotheses in Theorem 3.10(v.1). Let λ, λ_1, λ_2, i and r be as in Theorem 3.14. Then $\lambda = k(1+k^{n-1})$, $\lambda_1 = 1 + k^{n-1}$ and $i = 1$, and the trajectory $\langle \alpha_0, \alpha_1, \alpha_2, \ldots\rangle$ is cyclic.*

Proof. As usual, it is easy to check that A satisfies the hypotheses in Theorem 3.10(v.1). Let $\vec{e} = \left[\begin{smallmatrix}1\\k^n\end{smallmatrix}\right]$. Since $A\vec{e} = k(1 + k^{n-1})\left[\begin{smallmatrix}1\\k^n\end{smallmatrix}\right]$, $\lambda = k(1 + k^{n-1})$, and since $n \geq 2$, $k \nmid (1 + k^{n-1})$, so $\lambda_1 = 1 + k^{n-1}$ and $i = 1$. Since $a\lambda_1 + d = a(1 + k^{n-1}) + d$ $= a \cdot k^s$, $\lambda_2 = a$ and $r = s$. For $t = 2$, we have

$$k \nmid \Big(\lambda_2\lambda_1^l + \big(\sum_{0\leqslant\nu\leqslant l-1} \lambda_1^\nu\big)d\Big)$$

for all $l \in \mathbb{N}(t-2)$ and

$$\lambda_2\lambda_1^{t-1} + \Big(\sum_{0\leqslant\nu\leqslant t-2} \lambda_1^\nu\Big)d = \lambda_2\lambda_1 + d = a\lambda_1 + d = a \cdot k^s.$$

By Theorem 3.14(ii) the trajectory $\langle \alpha_0, \alpha_1, \alpha_2, \ldots\rangle$ is cyclic. □

Proposition 4.2.4. *Let* $A = \left[\begin{smallmatrix} n & k-n \\ n & k-n \end{smallmatrix}\right]$ *and* $\beta = ka\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$ *for some* $n \in \{1, 2, \ldots, k-1\}$ *and* $a \in \mathbb{N}$ *with* $k \nmid a$*; and let* $\alpha_0 \in \mathbb{Z}^2 - (k\mathbb{Z})^2$ *be such that* $\alpha_m = a\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$ *for some* $m \in \mathbb{Z}_*$*. Then* A *satisfies the hypotheses in Theorem 3.10(v.1). Let* λ*,* λ_1*,* λ_2*,* i *and* r *be as in Theorem 3.13. Then* $\lambda = k$*,* $\lambda_1 = 1$*,* $i = 1$*, and the trajectory* $\langle \alpha_0, \alpha_1, \alpha_2, \ldots \rangle$ *is cyclic.*

Proof. Again, it is easy to check that A satisfies the hypotheses in Theorem 3.10(v.1). Let $\vec{e} = \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$. Since $A\vec{e} = k\vec{e}$, $\lambda = k$, so $\lambda_1 = 1$ and $i = 1$.

Case 1. $k = 2$. Since $\lambda_1 + 1 = 1 + 1 = 2$, $\lambda_2 = 1$ and $r = 1$. For $t = 2$ we have $2 \nmid \Big(\lambda_2\lambda_1^l + \sum\limits_{0 \leqslant \nu \leqslant l-1} \lambda_1^\nu\Big)$ for all $l \in \mathbb{N}(t-2)$ and $\lambda_2\lambda_1^{t-1} + \sum\limits_{0 \leqslant \nu \leqslant t-2} \lambda_1^\nu = \lambda_2\lambda_1 + 1$ $= 1 \cdot 1 + 1 = 2$. By Theorem 3.13(ii), the trajectory $\langle \alpha_0, \alpha_1, \alpha_2, \ldots \rangle$ is cyclic.

Case 2. $k \neq 2$. Then $k \geq 3$ and $k - 1 \in \mathbb{N}$. Since $\lambda_1 + 1 = 1 + 1 = 2$, $\lambda_2 = 2$ and $r = 0$. For $t = k - 1$ we have $\lambda_2\lambda_1^{t-1} + \sum\limits_{0 \leqslant \nu \leqslant t-2} \lambda_1^\nu = \lambda_2\lambda_1^{k-2} + \sum\limits_{0 \leqslant \nu \leqslant k-3} \lambda_1^\nu$ $= 2 \cdot (1)^{k-1} + \sum\limits_{0 \leqslant \nu \leqslant k-3} 1^\nu = 2 + k - 2 = k$ and $k \nmid \Big(\lambda_2\lambda_1^l + \sum\limits_{0 \leqslant \nu \leqslant l-1} \lambda_1^\nu\Big)$ for all $l \in \mathbb{N}(t-2)$. By Theorem 3.13(ii) the trajectory $\langle \alpha_0, \alpha_1, \alpha_2, \ldots \rangle$ is cyclic. □

Proposition 4.2.5. *Let* $A = \left[\begin{smallmatrix} n & k-n \\ n & k-n \end{smallmatrix}\right]$ *and* $\beta = kd\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$ *for some* $n \in \{1, 2, \ldots, k-1\}$ *and* $d \in \mathbb{N}$ *with* $k \nmid d$*; and let* $\alpha_0 \in \mathbb{Z}^2 - (k\mathbb{Z})^2$ *be such that* $\alpha_m = a\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$ *for some* $a, m \in \mathbb{Z}_*$*, where* $k \nmid a$*,* $a \neq d$ *and* $a + d = a \cdot k^s$ *for some* $s \in \mathbb{N}$*. Then* A *satisfies the hypotheses in Theorem 3.10(v.1). Let* λ*,* λ_1*,* λ_2*,* i *and* r *be as in Theorem 3.14. Then* $\lambda = k$*,* $\lambda_1 = 1$ *and* $i = 1$*, and the trajectory* $\langle \alpha_0, \alpha_1, \alpha_2, \ldots \rangle$ *is cyclic.*

Proof. It is easy to check that A satisfies the hypotheses in Theorem 3.10(v.1). Let $\vec{e} = \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$. Since $A\vec{e} = k\vec{e}$, $\lambda = k$, so $\lambda_1 = 1$ and $i = 1$. Since $a\lambda_1 + d$ $= a + d = a \cdot k^s$, $\lambda_2 = a$ and $r = s$. For $t = 2$ we have

$$k \nmid \Big(\lambda_2\lambda_1^l + (\sum_{0 \leqslant \nu \leqslant l-1} \lambda_1^\nu)d\Big)$$

for all $l \in \mathbb{N}(t-2)$ and

$$\lambda_2\lambda_1^{t-1} + \Big(\sum_{0\leqslant\nu\leqslant t-2} \lambda_1^{\nu}\Big)d = \lambda_2\lambda_1 + d = a + d = a \cdot k^s.$$

By Theorem 3.14(ii), the trajectory $\langle \alpha_0, \alpha_1, \alpha_2, \ldots \rangle$ is cyclic. □

4.3 Conclusion

Theorem 3.10 gives some information on the situations in which the pair (A, β) satisfies the condition $(*)$. In the situations described in parts (iii) and (v.1) of this theorem, we proviode some explicit examples in which the trajectory $\langle \alpha, T(\alpha), T^2(\alpha), \ldots \rangle$ is cyclic. The situations described in parts (ii),(iv) and (v.3) are more complicated, and await further analysis. Deeper insight may be needed to construct some clearer conditions ensuring that the trajectory will be cyclic.

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VITA

Miss Umarin Pintoptang was born in June 6, 1974 at Phitsanulok, Thailand. She graduated Bachelor of Science(Mathematics) from Naresuan University, NU., Phitsanulok in May 1997. After that, she had been an instructor in the department of Mathematics, Faculty of Science, NU. for 3 semesters. During her study for Master degree at Chulalongkorn university, from June 1998 to March 2000, she got a scholarship from The Ministry of University Affairs.