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A FIXED PATH THEORY FOR LEVEL-CONTRACTION MAPS OF THE CYLINDER OF A COMPACT METRIC SPACE

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ในวิทยานิพนธ์นี้เราแสดงว่าสำหรับปริภูมิอิงระยะทางที่กระชับ X ใดๆ ถ้า $f : X \times I \to X \times I$ เป็นการส่งแบบหดตัวเชิงระดับ แล้วจะมีวิถี $\rho : I \to X \times I$ เพียงหนึ่งเดียว ที่ถูกตรึงด้วย f นั่นคือ $f \circ \rho = \rho$

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In this thesis, we show that for any compact metric space X ,if $f : X \times I \to X \times I$ is a level-contraction map then there exists a unique path $\rho : I \to X \times I$ which is fixed by f ; i.e., $f \circ \rho = \rho$.

Department **Mathematics** Student's signature......................................

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CONTENTS

CHAPTER I

Introduction

A considerable amount of study in fixed point theory has been done on the existence of a fixed point of various kinds of maps. In this thesis, we add one more dimension to the problem by considering a certain kind of maps between the cylinder of a compact metric space and try to prove the existence of a path fixed by such a map.

More precisely, let X be a compact metric space and f a level-contraction map of the cylinder $X \times I$; i.e., a continuous map $f : X \times I \to X \times I$ such that $f(X \times \{t\}) \subseteq X \times \{t\}$ and $f_{|X \times \{t\}}$ is a contraction map for each $t \in I$. Then we try to prove that there exists a unique path $\rho : I \to X \times I$ such that $\rho(t) \in X \times \{t\}$ for each $t \in I$ which is fixed by f; i.e., $f \circ \rho = \rho$.

The thesis is organized as follows. In Chapter 2, we give some notations, definitions and basic theorems that will be used throughout. In Chapter 3, we prove the special case for α-level-contraction maps. In Chapter 4, we prove the general case for level-contraction maps.

CHAPTER II

Preliminalies

Notations.

$\mathbb{R}$ = The set of all real numbers.

$\mathbb{N}$ = The set of all positive integers.

$I = [0, 1]$.

$Y^X = \{f \mid f : X \to Y\}$.

$C(X, Y) = \{f \in Y^X \mid f \text{ is continuous}\}$.

$C_l(X \times Y, Z \times Y) = \{f \in C(X \times Y, Z \times Y) \mid f(X \times \{y\}) \subseteq Z \times \{y\} \quad , \forall y \in Y\}$.

When X is a one-point space, we will identify $C_l(X \times Y, Z \times Y)$ with

$$C_l(Y, Z \times Y) = \{f \in C(Y, Z \times Y) \mid f(y) \in Z \times Y \quad , \forall y \in Y\}.$$

For a map $f : X \to X$ and $n \geq 1$, we will use f^n to denote $f \circ f \circ \ldots \circ f$ (n times) and f^0 denotes the identity map.

Definition 2.1. For a topological space X, the *cylinder* of X is defined to be $X \times I$ and denoted by CX.

Definition 2.2. A *metric* on a set X is a map $d : X \times X \to \mathbb{R}$ satisfying the following conditions for $x, y, z \in X$,

1. $d(x, y) \geq 0$.

2. $d(x, y) = 0$ if and only if $x = y$.

3. $d(x,y) = d(y,x)$.

4. $d(x,z) \leq d(x,y) + d(y,z)$.

The pair (X,d) is called a metric space.

Example 2.3.

1. The *usual metric* on $\mathbb{R}$ is defined by $d(x,y) = |x-y|$ for $x,y \in \mathbb{R}$.

2. For any space X, taken with the *discrete metric* defined by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

for $x,y \in X$.

Let (X,d_X) and (Y,d_Y) be metric spaces. Define $d_{X\times Y} : (X\times Y)\times(X\times Y) \to \mathbb{R}$ by

$$d_{X\times Y}((x_1,y_1),(x_2,y_2)) = \sqrt{d_X(x_1,x_2)^2 + d_Y(y_1,y_2)^2}$$

for $x_1, x_2 \in X$ and $y_1, y_2 \in Y$.

Then, it is not hard to show that $(X \times Y, d_{X\times Y})$ is a metric space. We will call $d_{X\times Y}$ the *product metric* on $X \times Y$ because of the following theorem.

Theorem 2.4. The metric $d_{X\times Y}$ induces the product topology on $X \times Y$.

Proof. Let (X,d_X) and (Y,d_Y) be metric spaces and define $d_{X\times Y}$ as above.

For $r, r' > 0, x \in X$ and $y \in Y$, the open ball centered at $(x,y) \in X \times Y$ of radius r is given by

$$B_{X\times Y}((x,y);r) = \{(x',y') \in X\times Y \mid d_{X\times Y}((x,y),(x'y')) < r\}.$$

Let $(x,y) \in X\times Y$ and $r, r' > 0$. We need to show that there exist open balls in X and Y whose product is a subset of $B_{X\times Y}((x,y);r)$ and conversely, there exists an open ball in $X\times Y$ with respect to the metric $d_{X\times Y}$ which is a subset of $B_X(x;r) \times B_Y(y;r')$.

Claim that $B_X(x;\frac{r}{2}) \times B_Y(y;\frac{r}{2}) \subseteq B_{X\times Y}((x,y);r)$.

Let $(x',y') \in B_X(x;\frac{r}{2}) \times B_Y(y;\frac{r}{2})$. Then, $d_X(x,x') < \frac{r}{2}$ and $d_Y(y,y') < \frac{r}{2}$.

Hence,

$$\begin{aligned} d_{X\times Y}((x,y),(x',y')) &= \sqrt{d_X(x,x')^2 + d_Y(y,y')^2} \\ &< \sqrt{\frac{r^2}{4} + \frac{r^2}{4}} \\ &= \frac{1}{2}\sqrt{2r^2} \\ &< \frac{1}{2}\sqrt{4r^2} \\ &= r. \end{aligned}$$

Thus, $B_X(x;\frac{r}{2}) \times B_Y(y;\frac{r}{2}) \subseteq B_{X\times Y}((x,y);r)$.

Conversely, let $r_0 = \min\{r,r'\}$.

Claim that $B_{X\times Y}((x,y);r_0) \subseteq B_X(x;r) \times B_Y(y;r')$.

Let $(x'',y'') \in B_{X\times Y}((x,y);r_0)$. Then, $d_{X\times Y}((x,y),(x'',y'')) < r_0$ and hence,

$$d_X(x,x'') \leq \sqrt{d_X(x,x'')^2 + d_Y(y,y'')^2} = d_{X\times Y}((x,y),(x'',y'')) < r_0 \leq r.$$

Similarly, we have $d_Y(y,y'') < r'$.

Hence, $(x'';y'') \in B_X(x;r) \times B_Y(y;r')$.

Thus, $B_{X\times Y}((x,y);r_0) \subseteq B_X(x;r) \times B_Y(y;r')$.

Therefore, the metric $d_{X\times Y}$ induces the product topology as required.

□

Definition 2.5. A subset Y of a metric space (X, d_X) is said to be *bounded* if there is some positive number M such that $d_X(y_1, y_2) \leq M$, for each $y_1, y_2 \in Y$.

Definition 2.6. A *sequence* in a metric space (X, d_X) is a map $x : \mathbb{N} \to X$. We will denote its value at $n \in \mathbb{N}$ by x_n instead of $x(n)$. And we denote x itself by the symbol $(x_1, x_2, x_3, ...)$ or (x_n).

We say that (x_n) *converges to* x_0, denoted by $(x_n) \to x_0$, if for each $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $d_X(x_n, x_0) < \epsilon$ for all $n \geq N$.

Definition 2.7. Let (X, d_X) be a metric space. A sequence (x_n) in X is said to be a *Cauchy sequence* in X if for each $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $d_X(x_n, x_m) < \epsilon$ for all $n, m \geq N$.

Definition 2.8. A metric space (X, d_X) is said to be *complete* if every Cauchy sequence in X converges.

Definition 2.9. Let (X, d_X) and (Y, d_Y) be metric spaces and (f_n) a sequence of maps from X to Y; i.e., $f_n \in Y^X$ for each $n \in \mathbb{N}$.

We say that (f_n) *converges pointwise* to $f \in Y^X$ if for each $x \in X$ and $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $d_Y(f_n(x), f(x)) < \epsilon$ for all $n \geq N$.

And we say that (f_n) *converges uniformly* to $f \in Y^X$ if for each $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for each $x \in X$, $d_Y(f_n(x), f(x)) < \epsilon$ for all $n \geq N$.

Theorem 2.10. Let (X, d_X) and (Y, d_Y) be metric spaces and (f_n) a sequence of continuous maps from X to Y. If (f_n) converges uniformly to f, then f is continuous.

Proof. The proof can be found in [2]. □

Definition 2.11. A collection $\mathcal{A}$ of subsets of a space X is said to *cover* X, or to be a *covering* of X, if $\bigcup \mathcal{A} = X$.

A covering $\mathcal{A}$ of X is called an *open covering* of X if each element of $\mathcal{A}$ is open in X.

Definition 2.12. A space X is said to be *compact* if every open covering $\mathcal{A}$ of X contains a finite subcollection that also covers X.

Definition 2.13. Let (X, d_X) and (Y, d_Y) be metric spaces and $f \in Y^X$. We say that f is *uniformly continuous* on X if for each $\epsilon > 0$, there exists $\delta > 0$ such that for $x, x' \in X$ with $d_X(x, x') < \delta$, we have $d_Y(f(x), f(x')) < \epsilon$.

Theorem 2.14. Let (X, d_X) and (Y, d_Y) be metric spaces and $f \in C(X, Y)$. If X is compact, then f is uniformly continuous.

Proof. The proof can be found in [2]. □

Definition 2.15. A metric space X is said to be *totally bounded* if for every $\epsilon > 0$, there is a finite covering of X by ϵ-balls.

Theorem 2.16. A totally bounded metric space is always bounded.

Proof. The proof can be found in [1]. □

Theorem 2.17. A metric space X is compact if and only if it is complete and totally bounded.

Proof. The proof can be found in [2]. □

Definition 2.18. A point $x \in X$ is said to be a *fixed point* of the map $f : X \to X$ if $f(x) = x$.

Example 2.19.

1. Let $f : \mathbb{R} \to \mathbb{R}$ be the map defined by $f(x) = x$. Then the fixed point set of f is $\mathbb{R}$.

2. Let $f : I \to I$ be the map defined by $f(x) = x^2$. Then the fixed point set of f is $\{ 0,1 \}$.

3. Let $f : [-\pi, \pi] \to \mathbb{R}$ be the map defined by $f(x) = \sin(x)$. Then the fixed point set of f is $\{ 0 \}$.

Definition 2.20. Let (X, d) be a metric space. A map $f : X \to X$ is said to be a *contraction map* if there is a constant $0 \leq \alpha < 1$ such that for all $x, y \in X$,

$$d(f(x), f(y)) \leq \alpha d(x, y).$$

The constant α in the above inequality is called a *contraction factor* of f, and we will call a contraction map whose contraction factor is α an *α-contraction map*. The smallest α for which the above inequality holds is called the *Lipschitz constant* of f.

Theorem 2.21. Every contraction map is uniformly continuous.

Proof. Let (X, d) be a metric space and $f : X \to X$ an α-contraction map.

If $\alpha = 0$, then f is a constant map. Clearly, f is uniformly continuous.

If $\alpha \neq 0$, then let $\epsilon > 0$ and choose $\delta = \dfrac{\epsilon}{\alpha}$. Hence, for $x, y \in X$ with $d(x, y) < \delta$, we have

$$d(f(x), f(y)) \leq \alpha d(x, y) < \epsilon.$$

Thus, f is uniformly continuous.

□

Theorem 2.22. Let (X, d_X) be a metric space and $f : X \to X$ a map. The map f is a contraction map whose Lipschitz constant is α if and only if $\alpha = \sup\{\frac{d_X(f(x), f(y))}{d_X(x, y)} \mid x, y \in X \text{ and } x \neq y\} \in [0, 1)$.

Proof. ($\Leftarrow$) Clearly, by assumption, we have for each $x, y \in X$,

$$d_X(f(x), f(y)) \leq \alpha d_X(x, y).$$

Hence, f is a contraction map.

Let $A = \{0 \leq \beta < 1 \mid d_X(f(x), f(y)) \leq \beta d_X(x, y) \text{ for all } x, y \in X\} \neq \varnothing$.

Since A is bounded below by 0, let $\hat{\alpha} = \inf A$. Clearly, $\hat{\alpha}$ is the Lipschitz constant and hence, $\hat{\alpha} \leq \alpha$.

For each $\beta \in A$, we have for each $x, y \in X$ and $x \neq y$,

$$\frac{d_X(f(x), f(y))}{d_X(x, y)} \leq \beta.$$

Hence, β is an upper bound of $\{\frac{d_X(f(x), f(y))}{d_X(x, y)} \mid x, y \in X \text{ and } x \neq y\}$.

By the definition of α, we have $\alpha \leq \beta$ for each $\beta \in A$. Therefore, α is a lower bound of A.

By the definition of $\hat{\alpha}$, we have $\alpha \leq \hat{\alpha}$.

Thus, $\alpha = \hat{\alpha}$; i.e., α is the Lipschitz constant of f.

($\Rightarrow$) Assume that f is a contraction whose Lipschitz constant is α . Then, for each $x, y \in X$, $d_X(f(x), f(y)) \leq \alpha d_X(x, y)$.

Let $B = \{\frac{d_X(f(x), f(y))}{d_X(x, y)} \mid x, y \in X \text{ and } x \neq y\}$.

Since B is bounded above by α, $\sup B$ exists and $0 \leq \sup B \leq \alpha < 1$.

$\square$

Example 2.23.

1. Let $f : I \to I$ be the map defined by $f(x) = \frac{x}{3}$ for $x \in I$. Clearly, f is a contraction map whose Lipschitz constant is $\frac{1}{3}$.

2. Let $f : [1,\infty) \to [1,\infty)$ be the map defined by $f(x) = \frac{x}{2} + \frac{1}{x}$ for $x \in [1,\infty)$. Then, f is a contraction map whose Lipschitz constant is $\frac{1}{2}$. This is because for $x, y \in [1,\infty)$, we have

$$\begin{aligned} |\frac{x}{2} + \frac{1}{x} - (\frac{y}{2} + \frac{1}{y})| &= |\frac{x-y}{2} + \frac{y-x}{xy}| \\ &= |\frac{1}{2} - \frac{1}{xy}||x-y| \end{aligned}$$

and $\sup\{ |\frac{1}{2} - \frac{1}{xy}| \mid x, y \in [1,\infty) \text{ and } x \neq y\} = \frac{1}{2}$.

Then, by the previous theorem, f is a contraction map whose Lipschitz constant is $\frac{1}{2}$.

Let $f : X \times I \to X \times I$ be a map such that for each $t \in I$, $f(X \times \{t\}) \subseteq X \times \{t\}$ and $f_{|X\times\{t\}}$ is a contraction map. We will see from Example 2.24.1 that f may not be continuous. Even though the continuity of f is also assumed, we will see from Example 2.24.2 that f can not be a contraction map.

Example 2.24.

1. Let $f : I \times I \to I \times I$ be defined by

$$f(s,t) = \begin{cases} (0,0) & \text{if } t = 0 \\ (\frac{s}{2}, t) & \text{if } t \neq 0 \end{cases}$$

Then, $f_{|I\times\{t\}}$ is a contraction map for all $t \in I$, but f is not continuous.

2. Let $f : I \times I \to I \times I$ be defined by $f(s,t) = (\frac{s}{2}, t)$ for $(s,t) \in I \times I$.

Then, f is continuous and $f_{|I\times\{t\}}$ is a contraction map for each $t \in I$.

However, f is not a contraction map because

$$||f(1,1) - f(1,0)|| = ||(\frac{1}{2}, 1) - (\frac{1}{2}, 0)|| = 1 = ||(1,1) - (1,0)||.$$

Definition 2.25. Let X be a metric space. A map $f \in C_l(X \times I, X \times I)$ is said to be a *level-contraction map* if $f_{|X \times \{t\}}$ is a contraction map for each $t \in I$.

Definition 2.26. Let X be a metric space and $\alpha \in [0, 1)$. A map $f \in C_l(X \times I, X \times I)$ is said to be an *α-level-contraction map* if $f_{|X \times \{t\}}$ is an α-contraction map for each $t \in I$.

Theorem 2.27. (*Banach Fixed Point Theorem*): Consider a nonempty metric space (X, d). Suppose that X is complete and $f : X \to X$ is a contraction map. Then, f has precisely one fixed point x_∞. Moreover, for each $x \in X$, $\lim_{n \to \infty} f^n(x) = x_\infty$.

Proof. The proof can be found in [3]. □

Example 2.28.

1. Let $f : I \to I$ be defined by $f(x) = \dfrac{x}{2}$. Then f is a contraction map. Since I is a complete metric space, then *Banach Fixed Point Theorem* implies that f has a unique fixed point. By direct calculation, it is not difficult to see that the fixed point of f is 0.

2. Let $f : (0, 1) \to (0, 1)$ be defined by $f(x) = \dfrac{x}{3}$. Then f is a contraction map. But $(0, 1)$ is not a complete metric space, so it does not guarantee that f has a fixed point. In fact, f does not have a fixed point in this case.

CHAPTER III

The Fixed Path of an α-Level-Contraction Map

In this chapter, we will prove that every α-level-contraction map of the cylinder of a compact metric space has precisely one fixed path.

First, we will prove that the sequence of the α-level-contraction maps (f^n) converges uniformly. And then, we will construct a unique path $\rho \in C_l(I, X \times I)$ which is fixed by f; i.e., $f \circ \rho = \rho$.

Throughout the rest of this thesis, we will assume that (X, d_X) is a nonempty metric space.

Lemma 3.1. Let (X, d_X) be a bounded complete metric space, $\alpha \in [0, 1)$ and $f \in C_l(X \times I, X \times I)$ an α-level-contraction map, then the sequence (f^n) converges uniformly to the map $f^\infty : X \times I \to X \times I$ defined by

$$f^\infty(x, t) = \lim_{n\to\infty} f^n(x, t) = (\hat{x}_t, t)$$

where $\hat{x}_t$ is the fixed point of $f_{|X\times\{t\}}$.

Proof. Since X is a complete metric space and f is an α-level-contraction map, it is clear that f^∞ as above is well-defined.

Since X is bounded, there exists $M > 0$ such that $d_X(x, x') \leq M$ for each $x, x' \in X$. In particular, $d_{X\times I}((x, t), (\hat{x}_t, t)) = d_X(x, \hat{x}_t) \leq M$ for each $x \in X$ and $t \in I$.

Let $\epsilon > 0$. Since $0 \leq \alpha < 1$, there is $N \in \mathbb{N}$ such that $\alpha^n < \dfrac{\epsilon}{M}$ for all $n \geq N$. Then, for $n \geq N$ and $(x,t) \in X \times I$, we have

$$\begin{aligned} d_{X\times I}(f^n(x,t), f^\infty(x,t)) &= d_{X\times I}(f^n(x,t), (\hat{x}_t, t)) \\ &= d_{X\times I}(f^n(x,t), f^n(\hat{x}_t, t)) \\ &\leq \alpha^n d_{X\times I}((x,t), (\hat{x}_t, t)) \\ &< \frac{\epsilon}{M}(M) = \epsilon. \end{aligned}$$

Hence, the convergence is uniform.

□

Theorem 3.2. Let X be a bounded complete metric space, $\alpha \in [0,1)$ and $f \in C_l(X \times I, X \times I)$ an α-level-contraction map, then there is a unique path $\rho \in C_l(I, X \times I)$ which is fixed by f.

Proof. Choose any $x_0 \in X$ and let $\rho = f^\infty_{|\{x_0\}\times I} : I \to X \times I$ where f^∞ is the limit map defined in the previous lemma.

Clearly, $\rho(t) = f^\infty(x_0, t) \in X \times \{t\}$ and

$$f(\rho(t)) = f(f^\infty(x_0,t)) = f(\lim_{n\to\infty} f^n(x_0,t)) = \lim_{n\to\infty} f^{n+1}(x_0,t) = f^\infty(x_0,t) = \rho(t)$$

for all $t \in I$; i.e., ρ is fixed by f.

By the previous lemma, the sequence (f^n) converges uniformly to f^∞. Since f^n is continuous for each $n \in \mathbb{N}$, then f^∞ is continuous by Theorem 2.10 and hence, so is ρ.

The uniqueness of ρ follows directly from the uniqueness of the fixed point of a contraction map. □

Corollary 3.3. Let X be a compact metric space, $\alpha \in [0,1)$ and $f \in C_l(X \times I, X \times I)$ an α-level-contraction map, then there is a unique path $\rho \in C_l(I, X \times I)$ which is fixed by f.

Proof. Since X is compact, X is complete and bounded. The result then follows directly from Theorem 3.2.

□

Example 3.4. Let $f : I \times I \to I \times I$ be defined by $f(s,t) = (\frac{st}{2} + \frac{1}{4}, t)$ for $s, t \in I$.

Then, we have for each $t \in I$ and $s_1, s_2 \in I$,

$$\begin{aligned}||f(s_1,t) - f(s_2,t)|| &= ||(\frac{s_1 t}{2} + \frac{1}{4}, t) - (\frac{s_2 t}{2} + \frac{1}{4}, t)|| \\ &= \frac{t}{2}||(s_1,t) - (s_2,t)|| \\ &\leq \frac{1}{2}||(s_1,t) - (s_2,t)||.\end{aligned}$$

By Corollary 3.3, there is a unique path $\rho \in C_l(I, I \times I)$ fixed by f. Also, by direct calculation, we have $\rho(t) = (\frac{-1}{2(t-2)}, t)$ for $t \in I$.

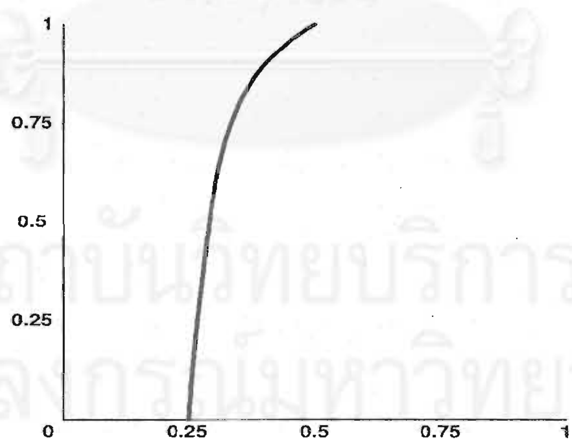

Figure 3.1: The image of ρ in $I \times I$.

For the case where f is a level-contraction map, if we know that the Lipschitz constant $\tilde{\alpha}_t$ of $f_{|X\times\{t\}}$ depends continuously on t, we can simply use the compactness of I to make f is an $\tilde{\alpha}$-level-contraction map by letting

$$\tilde{\alpha} = \max\{\tilde{\alpha}_t | t \in I\} \in [0,1).$$

Then, Corollary 3.3 implies that there is a path $\rho \in C_l(I, X \times I)$ which is fixed by f. Unfortunately, this situation fails in general as we will see in the following example.

Example 3.5. Let $f : I \times I \to I \times I$ be defined by

$$f(s,t) = \begin{cases} (\frac{t}{2}\sin(\frac{s}{t}), t) & \text{if} \quad t \neq 0 \\ (0,t) & \text{if} \quad t = 0 \end{cases}$$

Then, f is a level-contraction because it is clearly continuous and for each $t \in (0,1]$ and $s, s' \in I$, we have

$$\begin{aligned} ||f(s,t) - f(s',t)|| &= ||(\frac{t}{2}\sin\frac{s}{t}, t) - (\frac{t}{2}\sin\frac{s'}{t}, t)|| \\ &= \frac{t}{2}||(\sin\frac{s}{t}, t) - (\sin\frac{s'}{t}, t)|| \\ &\leq \frac{t}{2}||(\frac{s}{t}, t) - (\frac{s'}{t}, t)|| \\ &= \frac{t}{2t}||(s,t) - (s',t)|| \\ &= \frac{1}{2}||(s,t) - (s',t)||. \end{aligned}$$

Clearly, $\tilde{\alpha}_t = \frac{1}{2}$ for all $t \in (0,1]$ and $\tilde{\alpha}_0 = 0$.

Thus, the map $t \mapsto \tilde{\alpha}_t$ is not continuous at 0.

Therefore, if we assume the continuity of the map $t \mapsto \tilde{\alpha}_t$, we will get the following results:

Theorem 3.6. Let X be a compact metric space and $f \in C_l(X \times I, X \times I)$ a level-contraction map where the Lipschitz constant of $f_{|X \times \{t\}}$ is $\tilde{\alpha}_t$. If the map $t \mapsto \tilde{\alpha}_t$ is continuous, then there is a unique path $\rho \in C_l(I, X \times I)$ which is fixed by f.

Proof. As described before Example 3.5.

□

Corollary 3.7. Let X be a compact metric space and $f \in C_l(X \times I, X \times I)$. If $\sup\{\frac{d_{X\times I}(f(x,t), f(y,t))}{d_{X\times I}((x,t),(y,t))} \mid x, y \in X \text{ and } x \neq y\} \in [0,1)$ and continuous on t, then there is a unique path $\rho \in C_l(I, X \times I)$ which is fixed by f.

Proof. By Theorem 2.22 , f is a level-contraction map where the Lipschitz constant of $f_{|X \times \{t\}}$ is

$$\tilde{\alpha}_t = \sup\{\frac{d_{X\times I}(f(x,t), f(y,t))}{d_{X\times I}((x,t),(y,t))} \mid x, y \in X \text{ and } x \neq y\}.$$

From the continuity of the map $t \mapsto \tilde{\alpha}_t$ and Theorem 3.6, there is a unique path $\rho \in C_l(I, X \times I)$ which is fixed by f as desired. □

Without the continuity of $t \mapsto \tilde{\alpha}_t$, we hope to show directly that $\sup\{\tilde{\alpha}_t | t \in I\} < 1$, but it still fails when considering the following example.

Example 3.8. Let $g : I \times I \to I \times I$ be defined by

$$g(s,t) = \begin{cases} (t(1-t) - st, t) & \text{if} \quad 0 \leq s \leq 1-t \\ (0,t) & \text{if} \quad 1-t \leq s \leq 1 \end{cases}$$

Then, $g \in C_l(I \times I, I \times I)$ and g is a level-contraction. This is because for each $t \in (0,1)$ and $s, s' \in I$.

If $0 \leq s \leq 1-t$ and $0 \leq s' \leq 1-t$, then we have

$$||g(s,t) - g(s',t)|| = ||(t(1-t) - st, t) - (t(1-t) - s't, t)|| = t||(s,t) - (s',t)||.$$

If $1-t \leq s \leq 1$ and $0 \leq s' \leq 1-t$, then we have

$$\begin{aligned} ||g(s,t) - g(s',t)|| &= ||(0,t) - (t(1-t) - s't, t)|| \\ &= \sqrt{t^2(1-t-s')^2} \\ &\leq t\sqrt{(s-s')^2} \\ &= t||(s,t) - (s',t)||. \end{aligned}$$

If $1-t \leq s \leq 1$ and $1-t \leq s' \leq 1$, then we have

$$||g(s,t) - g(s',t)|| = ||(0,t) - (0,t)|| = 0 \leq t||(s,t) - (s',t)||.$$

So, $\tilde{\alpha}_t = t$ for each $t \in (0,1)$ and $\tilde{\alpha}_0 = 0 = \tilde{\alpha}_1$.

Hence, $\sup\{\tilde{\alpha}_t \mid t \in I\} = 1$.

However, there is still a fixed path $\rho : I \to I \times I$ defined by $\rho(t) = (\dfrac{t(1-t)}{1+t}, t)$ for $t \in I$.

Figure 3.2: The image of ρ in $I \times I$.

CHAPTER IV

The Fixed Path of a Level-Contraction Map

In this chapter we will prove that every level-contraction map of the cylinder of a compact metric space has precisely one fixed path.

Lemma 4.1. Let (X, d_X) be a metric space and $f \in C_l(X \times I, X \times I)$. If f is uniformly continuous, then for each $\epsilon > 0$, there exists $\delta > 0$ such that for $t, t' \in I$ with $d_I(t, t') < \delta$, we have $d_{X\times I}(f(x,t), f(x,t')) < \epsilon$, for all $x \in X$.

Proof. Let $\epsilon > 0$. By the uniform continuity of f, there exists $\delta > 0$ such that for any $(x,t),(x,t') \in X \times I$ with $d_I(t,t') = d_{X\times I}((x,t),(x,t')) < \delta$, we have $d_{X\times I}(f(x,t), f(x,t')) < \epsilon$. Hence, for any $t, t' \in I$ with $d_I(t,t') < \delta$, we have $d_{X\times I}(f(x,t), f(x,t')) < \epsilon$ for all $x \in X$.

□

Theorem 4.2. Let (X, d_X) be a complete metric space and $f \in C_l(X \times I, X \times I)$ a level-contraction map. If f is uniformly continuous, then there is a unique path $\rho \in C_l(I, X \times I)$ which is fixed by f.

Proof. Since X is a complete metric space and f is a level-contraction map, then for each $t \in I$, $f_{|X\times\{t\}}$ has a unique fixed point, says $\hat{x}_t$.

Define the map $\rho : I \to X \times I$ by $\rho(t) = (\hat{x}_t, t)$ for $x \in X$ and $t \in I$. Clearly, ρ is fixed by f. So, it remains to show that ρ is continuous.

Suppose that ρ is not continuous at some $t_0 \in I$.

Then there exists $\epsilon_0 > 0$ such that for each $\delta > 0$, there exists $t_\delta \in I$ with

$$d_I(t_\delta, t_0) < \delta \text{ and } d_{X\times I}((\hat{x}_{t_\delta}, t_\delta), (\hat{x}_{t_0}, t_0)) \geq \epsilon_0. \quad \textbf{(1)}$$

Since $f_{|X\times\{t_0\}}$ is a contraction map, let $\alpha_{t_0} \in [0, 1)$ be its contraction factor.

Since $(1 - \alpha_{t_0})\epsilon_0 > 0$, then by Lemma 4.1, there exists $\delta_0 > 0$ such that for $t \in I$ with $d_I(t, t_0) < \delta_0$, we have for each $x \in X$,

$$d_{X\times I}(f(x,t), f(x, t_0)) < (1 - \alpha_{t_0})\epsilon_0. \quad \textbf{(2)}$$

By **(1)**, there exists $t_{\delta_0} \in I$ such that

$$d_I(t_{\delta_0}, t_0) < \delta_0 \text{ and } d_{X\times I}((\hat{x}_{t_{\delta_0}}, t_{\delta_0}), (\hat{x}_{t_0}, t_0)) \geq \epsilon_0,$$

and hence, by **(2)**, we obtain for each $x \in X$,

$$d_{X\times I}(f(x, t_{\delta_0}), f(x, t_0)) < (1 - \alpha_{t_0})\epsilon_0 \leq (1 - \alpha_{t_0})d_{X\times I}((\hat{x}_{t_{\delta_0}}, t_{\delta_0}), (\hat{x}_{t_0}, t_0)). \quad \textbf{(3)}$$

Note that if $\hat{x}_{t_{\delta_0}} = \hat{x}_{t_0}$, then by **(3)**, we will have

$$\begin{aligned} d_{X\times I}((\hat{x}_{t_{\delta_0}}, t_{\delta_0}), (\hat{x}_{t_0}, t_0)) &< (1 - \alpha_{t_0})d_{X\times I}((\hat{x}_{t_{\delta_0}}, t_{\delta_0}), (\hat{x}_{t_0}, t_0)) \\ &< d_{X\times I}((\hat{x}_{t_{\delta_0}}, t_{\delta_0}), (\hat{x}_{t_0}, t_0)) \end{aligned}$$

which is a contradiction. Hence, $\hat{x}_{t_{\delta_0}} \neq \hat{x}_{t_0}$. In particular, $t_{\delta_0} \neq t_0$.

Now, for any $n \in \mathbb{N}$, we have

$$\begin{aligned} d_{X\times I}((\hat{x}_{t_{\delta_0}}, t_{\delta_0}), (\hat{x}_{t_0}, t_0)) &= d_{X\times I}(f^n(\hat{x}_{t_{\delta_0}}, t_{\delta_0}), f^n(\hat{x}_{t_0}, t_0)) \\ &\leq d_{X\times I}(f^n(\hat{x}_{t_{\delta_0}}, t_{\delta_0}), f^n(\hat{x}_{t_{\delta_0}}, t_0)) + d_{X\times I}(f^n(\hat{x}_{t_{\delta_0}}, t_0), f^n(\hat{x}_{t_0}, t_0)) \\ &\leq d_{X\times I}(f^n(\hat{x}_{t_{\delta_0}}, t_{\delta_0}), f^n(\hat{x}_{t_{\delta_0}}, t_0)) + \alpha_{t_0}^n d_{X\times I}((\hat{x}_{t_{\delta_0}}, t_0), (\hat{x}_{t_0}, t_0)) \\ &\leq d_{X\times I}(f^n(\hat{x}_{t_{\delta_0}}, t_{\delta_0}), f(\hat{x}_{t_{\delta_0}}, t_0)) + d_{X\times I}(f(\hat{x}_{t_{\delta_0}}, t_0), f^n(\hat{x}_{t_{\delta_0}}, t_0)) \\ &\quad + \alpha_{t_0}^n d_{X\times I}((\hat{x}_{t_{\delta_0}}, t_0), (\hat{x}_{t_0}, t_0)) \\ &\leq d_{X\times I}(f(\hat{x}_{t_{\delta_0}}, t_{\delta_0}), f(\hat{x}_{t_{\delta_0}}, t_0)) + \alpha_{t_0} d_{X\times I}((\hat{x}_{t_{\delta_0}}, t_0), f^{n-1}(\hat{x}_{t_{\delta_0}}, t_0)) \\ &\quad + \alpha_{t_0}^n d_{X\times I}((\hat{x}_{t_{\delta_0}}, t_0), (\hat{x}_{t_0}, t_0)). \end{aligned}$$

By letting $n \to \infty$, we have $\alpha_{t_0}^n \to 0$ and $f^{n-1}(\hat{x}_{t_{\delta_0}}, t_0) \to (\hat{x}_{t_0}, t_0)$. Hence,

$$d_{X\times I}((\hat{x}_{t_{\delta_0}}, t_{\delta_0}), (\hat{x}_{t_0}, t_0)) \leq d_{X\times I}(f(\hat{x}_{t_{\delta_0}}, t_{\delta_0}), f(\hat{x}_{t_{\delta_0}}, t_0)) + \alpha_{t_0} d_{X\times I}((\hat{x}_{t_{\delta_0}}, t_0), (\hat{x}_{t_0}, t_0)).$$

Since $d_{X\times I}((\hat{x}_{t_{\delta_0}}, t_0), (\hat{x}_{t_0}, t_0)) < d_{X\times I}((\hat{x}_{t_{\delta_0}}, t_{\delta_0}), (\hat{x}_{t_0}, t_0))$, it follows from **(3)** that

$$\begin{aligned} d_{X\times I}((\hat{x}_{t_{\delta_0}}, t_{\delta_0}), (\hat{x}_{t_0}, t_0)) &< (1-\alpha_{t_0}) d_{X\times I}((\hat{x}_{t_{\delta_0}}, t_{\delta_0}), (\hat{x}_{t_0}, t_0)) + \alpha_{t_0} d_{X\times I}((\hat{x}_{t_{\delta_0}}, t_{\delta_0}), (\hat{x}_{t_0}, t_0)) \\ &= d_{X\times I}((\hat{x}_{t_{\delta_0}}, t_{\delta_0}), (\hat{x}_{t_0}, t_0)) \end{aligned}$$

which is a contradiction.

Therefore, ρ must be continuous.

□

Corollary 4.3. Let X be a compact metric space and $f \in C_l(X \times I, X \times I)$ a level-contraction map. Then there is a unique path $\rho \in C_l(I, X \times I)$ which is fixed by f.

Proof. Since f is continuous and $X \times I$ is compact, f is uniformly continuous by Theorem 2.14. Then, the result follows from the previous theorem.

□

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