APPENDIX

We now generalize the last theorem from the theory of skewsemifields to skewring.

Definition 1. Let $(S, +, \bullet)$ be a semiring with a multiplicative zero 0. S is called a skewring if $(S, +)$ is a group.

Definition 2. Let R be an ordered skewring. Then the set $P = \{ x \in R \,/\, x \geq 0 \}$ is called the positive cone of R.

Definition 3. Let R be an ordered skewring and $x \in R$. Then x is dense if there for every $y \in R$ such that $x < y$, there exists a $z \in R$ such that $x < z < y$.

If x is not dense then we shall call x discrete.

Definition 4. Let R and M be skewrings. A function $f : R \rightarrow M$ is called an order homomorphism if and only if f is an isotone homomorphism of skewrings.

The definitions of order monomorphisms, order epimorphisms and order isomorphisms are defined as one would expect. If there exists an order ismorphism from R onto M, we denote this by $R \cong_o M$.

Definition 5. A totally ordered skewring R is called an Archemedean skewring if and only if for all $x, y \in R$ if $0 < x < y$ then there exists an $n \in \mathbf{Z}^+$ such that $y < nx$.

Theorem 6. A totally ordered skewring R can be embedded into a complete totally ordered skewring if and only if it is Archimedean.

Proof Let $|R| > 1$.

1. 0 is dense. Let $\mathbf{R}$ be the set of all subsets D of R having the property that :

1) $\varnothing \neq D \neq R$.

2) For all $x, y \in R$, $x \in D$ and $y < x$ imply that $y \in D$.

3) For every $x \in D$, there exists a $y \in D$ such that $x < y$.

Define + on $\mathbf{R}$ by $A + B = \{ a + b \ / \ a \in A \text{ and } b \in B \}$ for all $A, B \in \mathbf{R}$. Clearly, $A + B \neq \varnothing$.

Claim 1, there exists a $p \in R \setminus A$ such $p > 0$. Since $A \neq R$, there exists an $x \in R \setminus A$.

Case 1 : $x > 0$. Let $p = x$.

Case 2 : $x = 0$. Then there exists a $p \in R$ such that $p > 0$, so $p \notin A$.

Case 3 : $x < 0$. Then $-x > 0$ and $-x \notin A$. Let $p = -x$, so we have claim 1.

Similarly, there exists a $q \in R \setminus B$ such that $q > 0$. Let $a \in A$ and $b \in B$. Then $a < p$ and $b < q$, so $a + b < p + q$. Then $p + q \notin A + B$, so $A + B \neq R$. Next, let $x, y \in R$ be such that $y < x$ and $x \in A + B$. Then there exist $a \in A$ and $b \in B$ such that $x = a + b$, so $y - b < a$. Then $y - b \in A$, so $y = (y - b) + b \in A + B$. Next, let $x \in A + B$. Then there exist $a \in A$ and $b \in B$ such that $x = a + b$. Since $A, B \in \mathbf{R}$, there exist $m \in A$ and $n \in B$ such that $a < m$ and $b < n$, so $x = a + b < m + n$ and $m + n \in A + B$. Hence + is well-defined. Clearly, the associative law holds.

Let $D_0 = \{ x \in R \ / \ x < 0 \}$. Then $D_0 \in \mathbf{R}$. Let $A \in \mathbf{R}$. Let $a \in A$ and $y \in D_0$. Then $y < 0$, so $a + y < a$. By 2), $a + y \in A$, so $A + D_0 \subseteq A$. Next, let $a \in A$. Then there exists an $x \in A$ such that $a < x$, so $a - x < 0$. Then $a = a + (a - y) \in A + D_0$, so $A \subseteq A + D_0$. Hence $A = A + D_0$.

Let $D \in \mathbf{R} \setminus \{ D_0 \}$. Let $-D = \{ p \in R \ / \text{ there exists an } q \in R \setminus D \text{ such that } p < -q \}$. By claim 1, there exists a $p \in R \setminus D$ such that $p > 0$. Let $x = p + p$. Then $x > p$, so $-x < -p$. Hence $-x \in -D$, so $-D \neq \varnothing$. Since $D \neq \varnothing$, there exists a $d \in D$. Let $x \in R \setminus D$. Then $d < x$, so $-x < -d$. Hence $-d \notin -D$. Next, let $x, y \in R$ such that $y < x$ and $x \in -D$. Then there exists a $q \in R \setminus D$ such that $y < x < -q$, so $y \in -D$. Next, let $p \in -D$. Then there exists a $q \in R\backslash D$ such that $x < -q$, so there exists an $r \in R$ such that $p < r < -q$. Then $r \in -D$, hence $-D \in \mathbf{R}$.

To show that $D + (-D) \subseteq D_0$, let $x \in D$ and $y \in -D$. Then there exists a $q \in R\backslash D$ such that $y < -q$, so $x < q$. Then $x + y < q + y < q + (-q) = 0$, so $x + y \in D_0$. Hence $D + (-D) \subseteq D_0$.

Claim 2, for all $A \in \mathbf{R}$ and $x \in R$, if $x > 0$ and $0 \in A$ imply that there exists a $q \in R\backslash A$ such that $q - x, -x + q \in A$.

Let $0 \in A$ and $x > 0$. Suppose that $nx \in D$ for all $n \in \mathbf{Z}^+$. Let $p \in R$.

Case 1 : $p \leq x$. Then $p \in A$.

Case 2 : $p > x$. By the Archemedean property, there exists an $N \in \mathbf{Z}^+$ such that $p < Nx$, so $p \in A$. Then $A = \mathbf{R}$ which is a contradiction, so there exists an $N_0 \in \mathbf{Z}^+$ such that $N_0 x \notin D$.

Let $n_0 = \min \{ n \in \mathbf{Z}^+ \ / \ nx \notin D \}$. Then $n_0 \geq 1$.

Case 1 : $n_0 = 1$. Then $(1 - n_0)x = (n_0 - 1)x = 0 \in A$, so let $q = x$.

Case 2 : $n_0 > 1$. Then $n_0 - 1 \in \mathbf{Z}^+$, so $(n_0 - 1)x = (n_0 - 1)x \in A$. Let $q = n_0 x$, so we have claim 2.

To show that $D_0 \subseteq D + (-D)$, let $x \in D_0$. Then $x < 0$.

Case 1 : $0 \in D$. Then $x \in D$. Since 0 is dense, there exists a $d \in R$ such that $x < d < 0$, so $d \in D$. Then $0 < d - x$. By claim 2, there exists a $t \in R\backslash D$ such that $[-(d - x) + t] \in D$. Since $d < 0$, $-t + d < -t$, so $(-t + d) \in -D$. Then $x = [-(d - x) + t] + (-t + d) \in D + (-D)$.

Case 2 : $0 \notin D$. Then for every $y \in D$, $y < 0$, hence $D \subseteq D_0$. Then $D \subset D_0$, so there exists a $q \in D_0\backslash D$. Then $q < 0$, so $0 < -q$. Hence $0 \in -D$. By definition, $-(-D) = \{ m \in R \ / \text{ there exists an } n \in R\backslash(-D) \text{ such that } m < -n \}$. To show that $-(-D) \subseteq D$, let $z \in -(-D)$. Then there exists an $n \in R\backslash(-D)$ such that $z < -n$. Suppose that $z \notin D$. If $n \leq -z$ then $n \in -D$ which is a contradiction. Then $n > -z$, so $-n < z$ which is a contradiction. Then $z \in D$, so $-(-D) \subseteq D$. Since 0 is dense, there exists a $d \in R$ such that $x < d < 0$, so $0 < -x + d$. By claim 2, there exists a $t \in R \setminus D$ such that $t-(-x+d) \in D$. Since $d<0$, $d-t<-t$, so $d-t \in -(-D)$. Then $x=(d-t)+[t-(-x+d)] \in D+(-D)$. Thus $D+(-D)=D_0$, hence $\mathbf{R}$ is a group.

Define $\le$ on $\mathbf{R}$ by $D \le C$ if and only if $D \subseteq C$, for all $C, D \in \mathbf{R}$. Clearly, $\mathbf{R}$ is an ordered group. To show that $\le$ is a total order, let $C, D \in \mathbf{R}$. Suppose that $C \not\subset D$ and $D \not\subset C$. Then there exist $c \in C \setminus D$ and $d \in D \setminus C$. Thus $c<d$, so $c \in D$ which is a contradiction. Then $C \subseteq D$ or $D \subseteq C$, so $C \le D$ or $D \le C$.

To show that $\mathbf{R}$ is complete, let $\{ D_i \,/\, i \in I \}$ be a family in $\mathbf{R}$ such that there exists a $C \in \mathbf{R}$ with the property that $D_i \le C$ for all $i \in I$. Let $D = \cup D_i$. Since $I \ne \emptyset$, there exists an $i_o \in I$ such that $\emptyset \ne D_{i_o} \subseteq D$. Since $C \ne \mathbf{R}$, there exists an $a \in R \setminus C$, so $a \notin D_i$ for all $i \in I$. Then $a \notin D$. Next, let $p, q \in R$ be such that $p<q$ and $q \in D$. Then there exists an $i_0 \in I$ such that $q \in D_{i_0}$, so $p \in D_{i_0} \subseteq D$. Next, let $x \in D$. Then there exists an $i_0 \in I$ such that $x \in D_{i_0}$, so there exists a $y \in D_{i_0} \subseteq D$ with the property that $x<y$. Then $D \in \mathbf{R}$. Clearly, D is a least upper bound of $\{ D_i \,/\, i \in I \}$, so $\mathbf{R}$ is complete.

Let $A, B \in \mathbf{R}$ be such that $A, B \ge D_0$. Define $AB = \{ z \in R \,/$ there exist $a \in A \setminus D_0$ and $b \in B \setminus D_0$ such that $z<ab \} \cup D_0$. Then $AB \ne \emptyset$. Since $A, B \ne \emptyset$, there exist $x \in R \setminus A$ and $y \in R \setminus B$, so $x, y \ge 0$. Then $xy \ge 0$, so $xy \notin D_0$. If $A=D_0$ or $B=D_0$ then $xy \notin D_0 = AB$. Suppose that $A \ne D_0$ and $B \ne D_0$. Let $a \in A \setminus D_0$ and $b \in B \setminus D_0$. Then $x>a$ and $y>b$, so $xy \ge ab$. Then $xy \notin AB$. Clearly, for all x and $y \in R$, $x \in AB$ and $y<x$ imply that $y \in AB$. Next, let $x \in AB$.

Case 1 : $x \in D_0$. Then there exists a $p \in D_0 \subseteq AB$ such that $x<p$.

Case 2 : $x \notin D_0$. Then there exist $a \in A \setminus D_0$ and $b \in B \setminus D_0$ such that $x<ab$. Since 0 is dense, there exists a $p \in R$ such that $x<p<ab$, so $p \in AB$. Hence $AB \in \mathbf{R}$.

$$\text{Define } \bullet \text{ on } \mathbf{R} \text{ by } A \bullet B = \begin{cases} AB \text{ if } A \ge D_0 \text{ and } B \ge D_0 , \\ -(A(-B)) \text{ if } A \ge D_0 \text{ and } B<D_0 , \\ -((-A)B) \text{ if } A<D_0 \text{ and } B \ge D_0 , \\ (-A)(-B) \text{ if } A<D_0 \text{ and } B<D_0 . \end{cases}$$

Claim 3, for all $A, B, C \in \mathbf{R}$, $A, B, C \geq D_0$ imply that $A(BC) = (AB)C$.

Let $A, B, C \in \mathbf{R}$ be such that $A, B, C \geq D_0$.

Case 1 : there exist $a, b > 0$ such that $ab = 0$. Let $X, Y \in \mathbf{R}$ be such that X and $Y > D_0$. Next, Let $x \in X \backslash D_0$ and $y \in Y \backslash D_0$. If $x = 0$ or $y = 0$ then $xy = 0$. Suppose that $x, y > 0$.

Subcase 1.1 : $0 < x \leq a$ and $0 < y \leq b$. Then $0 \leq xy \leq ab = 0$, so $xy = 0$.

Subcase 1.2 : $0 < a < x$ and $0 < y \leq b$. By the Archemedean property, there exists an $n \in \mathbf{Z}^+$ such that $x < na$. Then $0 \leq xy \leq (na)b \leq n(ab) = 0$, so $xy = 0$.

Subcase 1.3 : $0 < x \leq a$ and $0 < b < y$. The proof is similar to the proof of subcase 1.2.

Subcase 1.4 : $0 < a < x$ and $0 < b < y$. By the Archemedean property, there exist $n, m \in \mathbf{Z}^+$ such that $x < na$ and $y < mb$. Then $0 \leq xy \leq (na)(mb) = (nm)(ab) = 0$, so $xy = 0$. Hence $XY = D_0$, so $A(BC) = D_0 = (AB)C$.

Case 2 : for all $a, b > 0$, $ab > 0$. Then for all $a, b, c \in R$, $a < b$ and $0 < c$ imply that $ac < bc$ and $ca < cb$. To show that $(AB)C \subseteq A(BC)$, Let $x \in (AB)C$.

Subcase 2.1 : $x \in D_0$. Then $x \in A(BC)$.

Subcase 2.2 : there exist $a \in A \backslash D_0$ and $p \in BC \backslash D_0$ such that $x < ap$. Then there exist $b \in B \backslash D_0$ and $c \in C \backslash D_0$ such that $p < bc$, so $x < ap \leq a(bc) = (ab)c$. Since a and $b \geq 0$, $ab \geq 0$, so $ab \notin D_0$. There exist $k \in A$ and $l \in B$ such that $a < k$ and $b < l$, so $k > 0$ and $l > 0$. Then $ab \leq kb < kl$, so $ab \in AB \backslash D_0$. Hence $x \in (AB)C$, so $A(BC) \subseteq (AB)C$. Similarly, $(AB)C \subseteq A(BC)$. Therefore $(AB)C = A(BC)$, so we have claim 3.

To show $\bullet$ is associative, let $A, B, C \in \mathbf{R}$.

Case 1: $A, B, C \geq D_0$. Then done.

Case 2 : $A, B, \geq D_0$ and $C < D_0$. Then $A(BC) = A[-(B(-C))] = -[A(B(-C))]$ $= -[(AB)(-C)] = (AB)C = (AB)C$.

Case 3 : $A, C \geq D_0$ and $B < D_0$. Then $A(BC) = A[-((-B)C)] = -[A((-B)C)]$ $= -[(A(-B))C] = [-(A(-B))]C = (AB)C$.

Case 4 : $A \geq D_0$ and $B, C < D_0$. Then $A(BC) = A[(-B)(-C)] = A[(-B)(-C)]$

$= [A(-B)](-C)]= (-(-[(A(-B)])) (-C) = [-(A(-B))]C = (AB)C.$

Case 5 : $A < D_0$ and $B, C \geq D_0$. Then $A(BC) = A(BC) = -[(-A)(BC)]$

$= -[((-A)B)C] = -[-(-[(-A)B])] = (-[(-A)B])C = (AB)C.$

Case 6 : $A, C < D_0$ and $B \geq D_0$. Then $A(BC) = A[-(B(-C))] = (-A)[B(-C)]$

$= [(-A)B](-C) = [-(-[(-A)B])] C = -[(-A)B]C = (AB)C.$

Case 7 : $A, B < D_0$ and $C \geq D_0$. Then $A(BC) = A[-((-B)C)] = (-A)[(-B)C]$

$= [(-A)(-B)]C) = [(-A)(-B)] C = (AB)C.$

Case 8 : $A, B, C < D_0$. Then $A(BC) = A[(-B)(-C)] = -[(-A)((-B)(-C))]$

$= -[((-A)(-B))(-C)] = [(-A)(-B)]C = (AB)C.$

Claim 4, for all $A, B, C \in \mathbf{R}$, $A, B, C \geq D_0$ imply that $A(B + C) = AB + AC$.

Let $A, B, C \in \mathbf{R}$ be such that $A, B, C \geq D_0$.

If there exist $a, b > 0$ such that $ab = 0$ then $A(B + C) = D_0 = D_0 + D_0 = AB + AC$, so done. Suppose that for all $a, b > 0$, $ab > 0$. Let $x \in A(B + C)$.

Case 1 : $x \in D_0$. Then $x \in AB + AC$.

Case 2 : there exist $a \in A \backslash D_0$ and $p \in (B + C) \backslash D_0$ such that $0 \leq x < ap$. Then there exist $b \in B$ and $c \in C$ such that $p = b + c$, so $x < ap = a(b + c) = ab + ac$. Since $a \in A$, there exists a $z \in A \backslash D_0$ such that $a < z$, so there exist $k \in B \backslash D_0$ and $l \in C \backslash D_0$ such that $b < k$ and $c < l$, so $ab \leq ak < zk$ and $ac \leq al < zl$. Then $ab \in AB$ and $ac \in AC$. Then $x \in AB + AC$, so $A(B + C) \subseteq AB + AC$.

Next, let $x \in AB + AC$. Then there exist $y \in AB$ and $z \in AC$ such that $x = y + z$.

Case 1 : $y, z \in D_0$. Then $x = y + z < z$, so $x \in AB + AC$.

Case 2 : $y \in D_0$ and there exist $a \in A \backslash D_0$ and $c \in C \backslash D_0$ such that $0 \leq z < ac$. Then $x = y + z < z < ac = a(0 + c) \in A(B + C)$.

Case 3 : there exist $a \in A \backslash D_0$ and $b \in B \backslash D_0$ such that $0 \leq y < ab$ and $z \in D_0$. The proof is similar to the proof of case 2.

Case 4 : there exist $a_1, a_2 \in A \backslash D_0$, $b \in C \backslash D_0$ and $c \in C \backslash D_0$ such that $0 \leq y < a_1 b$ and $0 \leq z < a_2 c$. WLOG, suppose that $a_1 \leq a_2$. Then $x = y + z < a_1 b + a_2 c \leq a_2 b + a_2 c$

$= a_2(b + c) \in A(B + C)$, so $AB + AC \in A(B + C)$, Then $A(B + C) = AB + AC$, so we have claim 4.

To show $\bullet$ is distributive over $+$ in $\mathbf{R}$, let $X, Y, Z \in \mathbf{R}$.

Case 1 : $X, Y, Z \geq D_0$. Then done.

Case 2 : $X, Y, \geq D_0$ and $Z < D_0$.

Subcase 2.1 : $Y + Z \geq D_0$. Then $X(Y + Z) - (XZ) = X(Y + Z) - (-[X(-Z)])$ $= X(Y + Z) + X(-Z) = X[(Y + Z) + (-Z)] = XY$, so $X(Y + Z) = (XY) + (XZ)$.

Subcase 2.2 : $Y + Z < D_0$. Then $-(XY) + [X(Y + Z)] = -(XY) + (-[X(-(Y + Z))])$ $= -[XY + X(-Y-Z)] = -[X(Y + (-Y - Z))] = -[X(-Z)] = XZ$, so $X(Y + Z)$ $= (XY) + (XZ)$.

Case 3 : $X, Z \geq D_0$ and $Y < D_0$. The proof is similar to the proof of case 2.

Case 4 : $X \geq D_0$ and $Y, Z < D_0$. Then $Y + Z < D_0$, so $X(Y + Z) = -[X(-(Y + Z))]$ $= -[X(-Y - Z)] = -[X(-Y) + X(-Z)] = -[X(-Y)] + (-[X(-Z)]) = (XY) + (XZ)$.

Case 5 : $X < D_0$ and $Y, Z \geq D_0$. Then $X(Y + Z) = -[(-X)(Y + Z)]$ $= -[((-X)Y) + ((-X)Z)] = -[(-X)Y] + [(-X)Z] = (XY) + (XZ)$.

Case 6 : $X, Z, < D_0$ and $Y \geq D_0$.

Subcase 6.1 : $Y + Z \geq D_0$. Then $X(Y + Z) - (XZ) = -[(-X)(Y + Z)] - [(-X)(-Z)]$ $= -[(-X)(Y + Z) + (-X)(-Z)] = -[(-X)(Y + Z - Z)] = -[(-X)Y] = XY$, so $X(Y + Z)$ $= (XY) + (XZ)$.

Subcase 6.2 : $Y + Z < D_0$. Then $-(XY) + [X(Y + Z)]$ $= -[-(-X)Y] + [(-X)(-(Y + Z))] = (-X)Y + (-X)(-Y-Z) = (-X)[Y + (-Y - Z)]$ $= (-X)(-Z) = XZ$, so $X(Y + Z) = (XY) + (XZ)$.

Case 7 : $X, Y \geq D_0$ and $Z \geq D_0$. The proof is similar to the proof of case 6.

Case 8 : $X, Y, Z < D_0$. Then $Y + Z < D_0$, so $X(Y + Z) = (-X)(-(Y + Z)) = (-X)(-Y - Z)$ $= (-X)[(-Y) + (-Z)] = (-X)(-Y) + (-X)(-Z) = (XY) + (XZ)$.

Hence $\mathbf{R}$ is a skewring, so $\mathbf{R}$ is a complete totally ordered skewring.

Let $x \in R$. Let $D_x = \{ y \in R \ / \ y < x \}$. Clearly, $D_x \in \mathbf{R}$. Define $i : R \to \mathbf{R}$ by $i(x) = D_x$ for every $x \in R$. To show that i is injective, let $x, y \in R$ be such that $i(x) = i(y)$. If $x \neq y$ then $D_x \neq D_y$ which is a contradiction. Then $x = y$, so i is injective.

Let $x, y \in R$. To show that $D_x + D_y \subseteq D_{x+y}$, let $a \in D_x$ and $b \in D_y$. Then $a < x$ and $b < y$, so $a + b < x + y$. Then $a + b \in D_{x+y}$, so $D_x + D_y \subseteq D_{x+y}$. Next, let $c \in D_{x+y}$. Then $c < x + y$, so $c - y < x$. Hence there exists an $r \in R$ such that $c - y < r < x$, so $r \in D_x$. Since $c - y < r$, $-r + c < y$, we get that $-r + c \in D_y$. Then $c = r + (-r + c) \in D_x + D_y$, so $D_{x+y} \subseteq D_x + D_y$. Thus $i(x)+i(y) = D_x + D_y = D_{x+y} = i(x+y)$.

Claim 5, for all $x, y \in R$, $x,y \geq 0$ imply that $D_xD_y = D_{xy}$.

Let $x, y \in R$ be such that $x,y \geq 0$. If $x = 0$ or $y = 0$ then done. So suppose that $x,y > 0$. Let $z \in D_xD_y$.

Case 1 : $z \in D_0$. Then $z \in D_{xy}$.

Case 2 : there exist $a \in D_x \backslash D_0$ and $b \in D_y \backslash D_0$ such that $0 \leq z < ab$. Then $z < ab \leq xy$, so $z \in D_{xy}$. Hence $D_xD_y \subseteq D_{xy}$.

To show that $D_{xy} \subseteq D_xD_y$., let $c \in D_{xy}$. Then $c < xy$. If there exist $a, b > 0$ such that $ab = 0$ then $c \in D_0 = D_xD_y$. So suppose that for all $a, b > 0$, $ab > 0$. Since $c < xy$, $xy - c > 0$. Let $z = xy - c$. Then $c = xy - z$ and $z > 0$. Suppose that for all $p \in D_x \backslash D_0$ and for all $q \in D_y \backslash D_0$, $pq \leq c$.

Claim (*), for all $0 < r_x \leq x$ and $0 < r_y \leq y$, $z < xr_y + r_xy$.

Let $0 < r_x \leq x$ and $0 < r_y \leq y$. Then $0 \leq x - r_x$ and $0 \leq y - r_y$, so $x - r_x \in D_x \backslash D_0$ and $y - ry \in D_y \backslash D_0$. By hypothesis, $xy - z = c \geq (x - r_x)(y - r_y) = xy - xr_y - r_xy + r_xr_y$, so $-z \geq - xr_y - r_xy + r_xr_y > -xr_y - r_xy$. Hence $z < xr_y + r_xy$, so we have claim (*).

Since $z > 0$, there exist $p, q > 0$ such that $z = p + q$. Let $C = \{ D_r \;/\; 0 < r \leq y \}$. Then $\inf(C) = D_0$.

Claim (**), for all $D > D_0$, $\inf(DC) = D(\inf(C)) = D_0$.

Let $D > D_0$ and $D_r \in C$. Then $D_r \geq \inf(C)$, so $DD_r \geq D(\inf(C))$. Then $\inf(DC)$ exists, say B and $B \geq D_0$. Let $0 < r \leq y$. Then there exists an $r_1 \in R$ such that $0 < r_1 < r \leq y$. Then $0 < r - r_1 \leq y - r_1 < y$. Let $r_2 = r - r_1$. Then $r = r_1 + r_2$, so $D_r = D_{r1+r2} = D_{r1} + D_{r2}$. Since $D_{r1}, D_{r2} \in C$, $DD_r = D(D_{r1} + D_{r2}) = DD_{r1} + DD_{r2} \geq \inf(DC) + \inf(DC) = B + B$. Then $B = \inf(DC) \geq B + B$, so $D_0 \geq B$. Hence $\inf(DC) = D_0$, so we have claim (**).

Since $|R| > 1$, there exists a $t \in R$ such that $t > 0$, $x + t > x > 0$. Then $D_{x+t} > D_0$.

By claim (**), $\inf(D_{x+t}C) = D_0$. Since $p > 0$, $D_p > D_0$, so there exists a $0 < d \le y$ such that $D_{x+t}D_d < D_p$. Then there exists an $r_y \in R$ such that $0 < r_y < d$, so $xr_y < xd$. Then $xr_y \in D_{x+t}D_d < D_p$, so $xr_y < p$.

Similarly, there exists an $0 < r_x \le x$ such that $r_xy < q$. Then $xr_y + r_xy < p + q = z$ which is contradicts to claim (*). Hence there exist $a \in D_x \backslash D_0$ and $b \in D_y \backslash D_0$ such that $c < ab$, so $c \in D_xD_y$. Thus $D_{xy} \subseteq D_xD_y$ and hence $D_xD_y = Dxy$, so we have claim 5.

Let $p, q \in R$.

Case 1 : $p, q \ge 0$. Then done.

Case 2 : $p < 0$ and $q \ge 0$. Then $-p > 0$. Since $-[(-p)q] = pq$, $i(pq)$ $= i(-[(-p)(q)]) = D_{-[(-p)q]} = -D_{(-p)q} = -[D_{(-p)}D_q] = D_pD_q = i(p)i(q)$.

Case 3 : $p \ge 0$ and $q < 0$. The proof is similar to the proof of case 2.

Case 4 : $p, q < 0$. Then $-p, -q > 0$. Since $pq = (-p)(-q)$, $i(pq) = i((-p)(-q))$ $= D_{(-p)(-q)} = (D_{(-p)})(D_{(-q)}) = (-D_p)(-D_q) = D_pD_q = i(p)i(q)$, so i is a monomorphism. Clearly, i is isotone, so $i(P_R) \subseteq P_{i(R)}$. To show that $P_{i(R)} \subseteq i(P_R)$, Let $D_x \in P_{i(R)}$. Then $D_x \ge D_0$. If $x < 0$ then $D_x < D_0$ which is a contradiction. Then $x \ge 0$, so $i(P_R) = P_{i(R)}$. Thus i is an order monomorphism, so $R \cong_o i(R)$. Hence R can be embedded into a complete totally ordered skewring.

To show that i(R) is dense, let $A, B \in \mathbf{R}$ be such that $A < B$. Then there exists an $x \in B \backslash A$, so there exists a $y \in B$ such that $x < y$. Clearly, $A \le i(y) \le B$. Since $y \in B$ and $y \notin i(y)$, $i(y) < B$. Since $x \notin A$ and $x \in i(y)$, we get that $A < i(y)$.

2. 0 is discrete. Then there exists an $a \in R$ such that $a > 0$ and there does not exist $z \in R$ such that $a > z > 0$. Claim 6, $R = \{ na \mid n \in \mathbf{Z}^+ \} =: \langle a \rangle$.

Let $x \in R$.

Case 1 : $x = a$ or $x = 0$. Then done.

Case 2 : $x > a$. Let $A = \{ n \in \mathbf{Z}^+ \mid x < na \}$. By the Archimedean property, there exists a $m \in \mathbf{Z}^+$ such that $x < ma$, so $A \ne \varnothing$. Let $N = \min A$. Then $N > 1$, so $N - 1 \in \mathbf{Z}^+$, so $x \ge (N-1)a$. Suppose that $x > (N-1)a$. Then $Na > x > (N-1)a$, so $0 > x - Na > -a$. Then $0 < -x + Na < a$ which is a contradiction. Then

$x = (N-1)a \in$ <a>.

Case 3 : $x < a$. Then $x < 0$, so $-x > 0$. Then $-x \geq a$. By case 1 and case 2, there exists an $n \in \mathbf{Z}$ such that $-x = na$, $x = (-n)a \in$ <a>. Hence R = <a>, so we have claim 6.

Claim 7, for $m, n \in \mathbf{Z}$, $m < n$ implies that $ma < na$.

Let $m, n \in \mathbf{Z}$ be such that $m < n$. Then $n - m \in \mathbf{Z}^+$. Since $a > 0$, $na - ma = (n-m)a > 0$. Then $ma < na$, so we have claim 7.

By claim 6 and claim 7, we have that for every $r \in R$, there exists a unique $n \in \mathbf{Z}$ such that $r = na$. Since $a^2 \in R$ and $a^2 > 0$, there exists a unique $n_0 \in \mathbf{Z}^+$ such that $a^2 = n_0a$.

Define $\bullet$ on $\mathbf{Z}$ by $m \bullet n = mnn_0$ for all $m, n \in \mathbf{Z}$. Let $m_1, m_2, m_3 \in \mathbf{Z}$. Then $m_1(m_2m_3) = m_1[(m_2m_3)n_0] = [m_1(m_2m_3)n_0]n_0 = [((m_1m_2)n_0)m_3)]n_0 = [(m_1m_2)n_0]m_3$ $= (m_1m_2)m_3$ and $m_1(m_2 + m_3) = [m_1(m_2 + m_3)]n_0 = m_1m_2n_0 + m_1m_3n_0$ $= (m_1m_2) + (m_1m_3)$. Hence $(\mathbf{Z}, +, \bullet)$ is a skewring.

Clearly, $(\mathbf{Z}, +, \bullet, \leq)$ is a complete totally ordered commutative ring.

Define $i : R \rightarrow \mathbf{Z}$ as follows : let $r \in R$. Then there exists a unique $n \in \mathbf{Z}$ such that $r = na$. Let $i(r) = n$. Clearly, i is a bijection, i and i^{-1} are isotone.

Let $x, y \in R$. Then there exist $m, n \in \mathbf{Z}$ such that $x = ma$ and $y = na$. Then $i(x + y) = i(ma + na) = i([m + n]a) = m + n = i(x) + i(y)$ and $i(xy) = i((ma)(na))$ $= i((mn)a^2) = i((mnn_0)a) = mnn_0 = mn = i(x)i(y)$. Thus i is an order isomorphism, so $R \cong_o \mathbf{Z}$

<u>Corollary 6.</u> An Archimedean totally ordered skewring is a commutative ring.

<u>Proof</u> In [6], pp. 130 – 136 it was shown that all complete ordered skewrings were classified and were shown to be both multiplicatively and additively commutative.

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