CHAPTER I

PRELIMINARIES

In this chapter we shall give some notations, definitions and theorems used in this thesis. Our notations are :

$\mathbf{Z}$ is the set of all integers,

$\mathbf{Z}^+$ is the set of all positive integers,

$\mathbf{Q}$ is the set of all rational numbers,

$\mathbf{Q}^+$ is the set of all positive rational numbers,

$\mathbf{Q}_0^+ = \mathbf{Q}^+ \cup \{0\}$,

$\mathbf{R}$ is the set of all real numbers

$\mathbf{R}^+$ is the set of all positive real numbers and

$\mathbf{R}_0^+ = \mathbf{R}^+ \cup \{0\}$.

In this thesis, if we do not give the definitions of a binary operation or order on any subset of $\mathbf{R}$ then we shall mean the usual binary operation and order on it.

As usual one may write $y \geq x$ for $x \leq y$ and $x < y$ or $y > x$ to mean that $x \leq y$ and $x \neq y$. If neither $x \leq y$ nor $y \leq x$ then x and y are said to be <u>incomparable</u> and this is denoted by $x \| y$.

<u>Definition 1.1.</u> For a subset B of a partially ordered set P. The set of all <u>upper</u> [<u>lower</u>] <u>bounds</u> of B will be denoted by U(B) [L(B)]. If B is the empty set then U(B) = L(B) = P, while if B has no upper bound in P then U(B) = ∅. Similarly, if B has no lower bound in P then L(B) = ∅.

<u>Remark 1.2.</u> Let $(P, \leq)$ be a partially ordered set. Then the following statements clearly hold : for all subsets A, B of P,

1) $A \subseteq B$ implies that $U(A) \supseteq U(B)$ and $L(A) \supseteq L(B)$,

2) $L(U(B)) \supseteq B$ and $U(L(B)) \supseteq B$,

3) $U(L(U(B))) = B$ and $L(U(L(B))) = B$.

Definition 1.3. Let $(P, \leq)$ be a partially ordered set. P is said to be complete if and only if every nonempty subset of P which has a lower bound has an infimum.

The same proof given in [6], pp. 5 shows that a partially ordered set is complete if and only if every nonempty subset of P which has an upper bound has a supremum.

Definition 1.4. Let $(P, \leq)$ be a partially ordered set. P is a lower [upper] semilattice if and only if $\inf\{x, y\}$ [$\sup\{x, y\}$] exists for all $x, y \in P$ and we denote $\inf\{x, y\}$ [$\sup\{x, y\}$] by $x \wedge y$ [$x \vee y$]. P is said to be a lattice if and only if P is both a lower and upper semilattice.

Definition 1.5. Let $(P, \leq)$ be a partially ordered set. A nonempty subset S of P is called dense in P if and only if for all $x, y \in P$, $x < y$ implies that there exists a $z \in S$ such that $x < z < y$.

Definition 1.6. Let $(S, +)$ be a semigroup. S is said to be a band if and only if for every $x \in S$, $x + x = x$.

Let $(L, \leq)$ be an upper [lower] semilattice. Define a binary operation $+_{\leq}$ on L by $x +_{\leq} y = x \vee y$ [$x \wedge y$] for all $x, y \in L$. Then we have that $(L, +_{\leq})$ is a commutative band.

Let $(L, +)$ be a commutative band. Define a binary operation $\leq_{+}$ on L by $x \leq_{+} y$ if and only if $x + y = y$ [$x + y = x$] for all $x, y \in L$. Then we have that

$(L, \leq_+)$ is an upper [lower] semilattice such that $x \vee y = x + y$ [$x \wedge y = x + y$] for all $x, y \in L$.

<u>Proposition 1.7.</u> ([3]) Let L be a nonempty set. Let $\mathcal{S}$ be the set of all semilattice structures on L and $\mathcal{C}$ the set of all commutative band structures on L. Then there exists a bijection between $\mathcal{S}$ and $\mathcal{C}$.

<u>Definition 1.8.</u> Let L be a nonempty set and $\vee, \wedge$ be binary operations on L such that

1) $(L, \wedge)$ and $(L, \vee)$ are commutative bands and
2) for all $x, y \in L$, $x \vee (x \wedge y) = x$ and $x \wedge (x \vee y) = x$.

Then $(L, \wedge, \vee)$ is called a <u>lattice algebra</u>.

Let $(L, \wedge, \vee)$ be a lattice algebra. Define $\leq_{\wedge\vee}$ on L by $x \leq_{\wedge\vee} y$ if and only if $x \wedge y = x$ for all $x, y \in L$. Then we have that $(L, \leq_{\wedge\vee})$ is a lower semilattice.

Note that for all $x, y \in L$, we define $x \leq_{\wedge\vee} y$ if and only if $x \wedge y = x$ is equivalent to $x \vee y = y$. Hence $(L, \leq_{\wedge\vee})$ is a lattice.

Let $(L, \leq)$ be a lattice. Then we have that $(L, \wedge_\leq, \vee_\leq)$ is a lattice algebra where $x \wedge_\leq y = \inf\{x, y\}$ and $x \vee_\leq y = \sup\{x, y\}$ for all $x, y \in L$.

<u>Proposition 1.9.</u> ([3]) Let L be a nonempty set. Let $\mathcal{A}$ be the set of all lattice algebra structures on L and $\mathcal{B}$ the set of all lattice structures on L. Then there exists a bijection between $\mathcal{A}$ and $\mathcal{B}$.

<u>Definition 1.10.</u> Let L be a lattice algebra. L is said to be a <u>distributive lattice algebra</u> if and only if for all $x, y, z \in L$, $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

<u>Remark 1.11.</u> 1) Let L be a lattice algebra. Then L is a distributive lattice algebra if and only if for all $x, y, z \in L$, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

2) Let L be a distributive lattice algebra. Then for all $x, y, z \in L$, $(x \vee y) \wedge [(x \wedge y) \vee z] = (x \wedge y) \vee [(x \vee y) \wedge z]$.

Proof 1) See [3], pp. 5.

2) Let $x, y, z \in L$. Then $(x \vee y) \wedge [(x \wedge y) \vee z]$ $= [(x \vee y) \wedge (x \wedge y)] \vee [(x \vee y) \wedge z] = (x \wedge y) \vee [(x \vee y) \wedge z]$. #

Definition 1.12. Let L be a lattice. L is said to be a distributive lattice if and only if for all $x, y, z \in L$, $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

Proposition 1.13. ([3]) Let L be a nonempty set, $\mathbb{A}$ the set of all distributive lattice algebra structures on L and $\mathbb{B}$ the set of all distributive lattice structures on L. Then there exists a bijection between $\mathbb{A}$ and $\mathbb{B}$.

Definition 1.14. Let L be lattice. L is said to be a modular lattice if and only if for all $x, y, z \in L$, $x \leq z$ implies that $x \vee (y \wedge z) = (x \vee y) \wedge z$

Note that every distributive lattice is a modular lattice, but the converse is not true.

Definition 1.15. Let $(P, \leq)$ and $(P', \leq')$ be partially ordered sets. A function $f : P \to P'$ is said to be isotone if and only if $x \leq y$ implies that $f(x) \leq' f(y)$ for all $x, y \in P$, f is said to be an order isomorphism if and only if f is a bijection and both f and f^{-1} are isotone. In this case, P and P′ are called order isomorphic.

Definition 1.16. Let P and P′ be lattices. A function $f : P \to P'$ is said to be a lattice homomorphism if and only if for all $x, y \in P$, $f(x \vee y) = f(x) \vee f(y)$ and $f(x \wedge y) = f(x) \wedge f(y)$.

Remark 1.17. Let P and P′ be lattices and $f: P \to P'$ a function. Then the following statements clearly hold :

1) If f is a lattice homomorphism then f is isotone.
2) If f is an order isomorphism then f is a lattice homomorphism.

Definition 1.18. A subset C of a partially ordered set P is to be an ordered convex subset if and only if for all $x, y \in C$ and $z \in P$, the inequalities $x \le z \le y$ imply that $z \in C$.

From now we shall call an ordered convex subset an o-convex subset.

Examples 1.19. 1) Let P be a partially ordered set and $x \in P$. Then $\{x\}$ is an o-convex subset of P.

2) Every interval of **R** is an o-convex subset of **R**.

3) In $\mathbf{R} \times \mathbf{R}$, $\{(x,y) \mid x^2 + y^2 \le 4\}$ is an o-convex subset of $\mathbf{R} \times \mathbf{R}$ where $(x,y) \le (z,w)$ if and only if $x \le z$ and $y \le w$ for all $x, y, z, w \in \mathbf{R}$.

Remark 1.20. ([3]) 1) The intersection of a family of o-convex subsets of a partially ordered set is an o-convex set. Also the union of an increasing chain of o-convex subsets is an o-convex set.

2) If $f: P \to P'$ is an isotone map and C′ an o-convex subset of P′ then $f^{-1}(C')$ is an o-convex subset of P.

Definition 1.21. A triple $(S, +, \bullet)$ is a semiring if and only if

1) $(S, +)$ and $(S, \bullet)$ are semigroups and
2) for all $x, y, z \in S$, $x(y+z) = xy + xz$ and $(y+z)x = yx + zx$.

Definition 1.22. Let $(S, +, \bullet)$ be a semiring with multiplicative zero 0. S is said to be a 0-skewsemifield if and only if $(S^*, \bullet)$ is a group and for every $x \in S$,

$x + 0 = x = 0 + x$ where $S^* = S\setminus\{0\}$. A subset H of a 0-skewsemifield K is called a subskewsemifield of K if and only if H is a 0-skewsemifield under the same operation. A subset S of K^* is said to be conic if and only if $S \cap S^{-1} = \{1\}$ where $S^{-1} = \{x^{-1} / x \in S\}$.

Remark 1.23. ([4]) The intersection of subskewsemifields of a 0-skewsemifield is a subskewsemifield. Hence the intersection of all subskewsemifields is the smallest subskewsemifield of a 0-skewsemifield and will be called the prime skewsemifield.

Proposition 1.24. ([4]) If K is a 0-skewsemifield then the prime skewsemifield of K is either isomorphic to $\mathbf{Q}_0^+$ or $\mathbf{Z}_p$ where p is a prime number or the skewsemifield $\{0, 1\}$ with $1 + 1 = 1$.

Proposition 1.25. ([4]) Let K be a 0-skewsemifield. If there exists an $x \in K^*$ such that x has a right [left] additive inverse. Then every element in K has an additive inverse and hence K is a skewfield.

In our thesis, we shall study only 0-skewsemifields which are not skewfields. So from now on we shall use the word skewsemifield for 0-skewsemifield.

Examples 1.26. 1) $\mathbf{Q}_0^+, \mathbf{R}_0^+$ are skewsemifields.

2) Let G be a group with multiplicative zero 0. Then we can define a binary operation $\oplus$ on G by $x \oplus y = \begin{cases} x & \text{if } x \neq 0 \\ y & \text{if } x = 0 \end{cases}$, for all $x, y \in G$.

Then G is a skewsemifield.

3) Let $n \in \mathbf{Z}^+$ be such that $n \geq 2$. Let $K_n = \{0\} \cup \{A \in M_n(\mathbf{R})\ [M_n(\mathbf{Q})] / A_{ij} > 0 \text{ if } i = j \text{ and } A_{ij} = 0 \text{ if } i > j\}$. Then K_n with the usual binary operation is a skewsemifield.

4) Let G be a lattice group. Let a be an element not representing in G. Then we define a binary operation + on G by $x + y = x \vee y$ and $x + a = x = a + x$ for all $x, y \in G$. Define $ax = a = xa$ for every $x \in G$. Then $G \cup \{a\}$ is a skewsemifeild.

Definition 1.27. A semiring $(S, +, \bullet)$ is said to be left [right] additively cancellative if and only if $x + z = y + z$ implies that $x = y$ [$z + x = z + y$ implies that $x = y$] for all $x, y, z \in S$, additively cancellative (A.C.) if it is both left additively cancellative and right additively cancellative, left [right] multiplicatively cancellative if and only if $zx = zy$ and $z \neq 0$ imply that $x = y$ [$xz = yz$ and $z \neq 0$ imply that $x = y$] for all $x, y, z \in S$, multiplicatively cancellative (M.C.) if it is both left multiplicatively cancellative and right multiplicatively cancellative where 0 denotes the multiplicative zero 0 of S if it exists, and cancellative if S is both A.C. and M.C.

Definition 1.28. A semigroup $(S, \bullet)$ is said to satisfy the right [left] Ore condition if and only if for all $a, b \in S \setminus \{0\}$ there exist $x, y \in S \setminus \{0\}$ such that $ax = by$ [$xa = yb$] where 0 denotes the multiplicative zero of S if it exists.

Definition 1.29. Let S and M be semirings. A function $f : S \to M$ is called a homomorphism of S into M if and only if for all $x, y \in S$,

1) $f(0) = 0$ if 0 exists,
2) $f(x + y) = f(x) + f(y)$ and
3) $f(xy) = f(x)f(y)$.

And the multiplicatively kernel of f is the set $\{x \in S \mid f(x) = 1\}$, denoted by m-ker f.

A homomorphism $f : S \to M$ is called a monomorphism if and only if f is an injection, an epimorphism if f is onto and an isomorphism if f is a bijection. S and M are said to be isomorphic if there exists an isomorphism S onto M and we denote this by $S \cong M$. Note that if $f : S \to M$ is an isomorphism then f^{-1} is also an isomorphism.

Definition 1.29. Let S be a semiring with a multiplicative zero 0 such that $|S| > 1$. Then a skewsemifield K is said to be a skewsemifield of right [left] quotients of S if and only if there exists a monomorphism $i : S \to K$ such that for every $x \in K$, there exist $a \in S, b \in S \setminus \{0\}$ such that $x = i(a)i(b)^{-1}$ [$x = i(b)^{-1}i(a)$]. A monomorphism i satisfying the above property is said to be a right [left] quotients embedding of S into K.

Example 1.30. Let $S = \{ \begin{bmatrix} x & z \\ 0 & y \end{bmatrix} / x, y \in \mathbf{Z}^+ \text{ and } z \in \mathbf{Z} \} \cup \{0\}$ and $K = \{ \begin{bmatrix} a & c \\ 0 & b \end{bmatrix} / a, b \in \mathbf{Q}^+ \text{ and } c \in \mathbf{Q} \} \cup \{0\}$. Then S and K with the usual addition and multiplication are a semiring with a multiplicative zero 0 and skewsemifield, respectively. In [5], pp. 17 it was shown that K is a skewsemifield of right quotient of S.

Theorem 1.31. ([5]) Let S be a semiring with multiplicative zero 0. Then S can be embedded into a skewsemifield if and only if

1) S is multiplicatively cancellative and
2) $(S, \bullet)$ satisfies the right [left] Ore condition.

Proof We shall now give the construction of the skewsemifield of quotients of a semiring S which appears in [5], pp. 18 – 23.

Assume that S is M.C. and $(S, \bullet)$ satisfies the right [left] Ore Condition. Define a relation $\sim$ on $S \times (S \setminus \{0\})$, by $(x,y) \sim (z,w)$ if and only if there exist $a, b \in S \setminus \{0\}$ such that $xa = zb$ and $ya = wb$ for all $(x,y), (z,w) \in S \times (S \setminus \{0\})$. In [5], pp. 18 it was shown that $\sim$ is an equivalent relation.

Let $K = S \times (S\setminus\{0\})/_{\sim}$.

Let $\alpha, \beta \in K$. Define $\bullet$ on K in the following way: Choose $(a,b) \in \alpha$ and $(c,d) \in \beta$. Since $b \in S\setminus\{0\}$ and $c \in S$, there exist $x \in S$ and $y \in S\setminus\{0\}$ such that $bx = cy$. Define $\alpha\beta = [(ax,dy)]$.

To show that $\bullet$ is well-defined, let $(a,b), (a',b'), (c,d), (c',d') \in K$ be such that $(a,b) \sim (a',b')$ and $(c,d) \sim (c',d')$. Then there exist $x, x', y, y' \in S\setminus\{0\}$ such that $bx = dy$ and $b'x' = d'y'$. We must show that $(ax,dy) \sim (a'x',d'y')$. Since $(a,b) \sim (a',b')$, there exist $u, v \in S\setminus\{0\}$ such that $au = a'v$ and $bu = b'v$. Since $(c,d) \sim (c',d')$, there exist $u', v' \in S\setminus\{0\}$ such that $cu' = c'v'$ and $du' = d'v'$. Since $dy, d'y' \in S\setminus\{0\}$, there exist $p, q \in S\setminus\{0\}$ such that $dyp = d'y'q$. We must show that $axp = a'x'q$. Since $y'q, v' \in S\setminus\{0\}$, there exist $g, h \in S\setminus\{0\}$ such that $y'qg = v'h$. Since $v \in S\setminus\{0\}$ and $x'q \in S$, there exist $l \in S$ and $k \in S\setminus\{0\}$ such that $vl = x'qk$. Then $du'h = d'v'h = d'y'qg = dypg$. Since $d \neq 0$, $u'h = ypg$, so $bxpg = cypg = cu'h = c'v'h = c'y'qg = b'x'qg$. Since $g \neq 0$, $bxp = b'x'q$, so $bxpk = b'x'qk = b'vl = bul$. Since $b \neq 0$, $x'pk = ul$. Then $ax'pk = aul = a'vl = a'x'qk$. Since $k \neq 0$, $axp = a'x'q$, so $\bullet$ is well-defined.

In [5], pp. 20 it was shown that $(K^*, \bullet)$ is a group with $[(a,a)]$ and $[(0,a)]$ as the identity and multiplicative zero respectively, and $[(b,a)]$ as the inverse of $[(a,b)]$ for all $a, b \in S\setminus\{0\}$.

Let $\alpha, \beta \in K$. Define $+$ on K in the following way: $(a,b) \in \alpha$ and $(c,d) \in \beta$. There exist $x, y \in S\setminus\{0\}$ such that $bx = dy$. Define $\alpha + \beta = [(ax + cy, bx)]$.

To show that $+$ is well-defined, let $(a,b), (a',b'), (c,d), (c',d') \in K$ be such that $(a,b) \sim (a',b')$ and $(c,d) \sim (c',d')$. Then there exist $x, x', y, y' \in S\setminus\{0\}$ such that $bx = dy$ and $b'x' = d'y'$. We must show that $(ax + cy, bx) \sim (a'x' + c'y', b'x')$. Since $(a,b) \sim (a',b')$, there exist $u, v \in S\setminus\{0\}$ such that $au = a'v$ and $bu = b'v$. By $(c,d) \sim (c',d')$, there exist $u', v' \in S\setminus\{0\}$ such that $cu' = c'v'$ and $du' = d'v'$. Since $bx, b'x' \in S\setminus\{0\}$, there exist $p, q \in S\setminus\{0\}$ such that $bxp = b'x'q$. We must show that $(ax + cy)p = (a'x' + c'y')q$. Since $x'q, v \in S\setminus\{0\}$, there exist $g, h \in S\setminus\{0\}$

such that $x'qg = vh$. Since $y'q, v'p \in S\backslash\{0\}$, there exist $k, l \in S\backslash\{0\}$ such that $y'qk = v'pl$. Therefore $buh = b'vh = b'x'qg = bxpg$. Since $b \neq 0$, $uh = xpg$, so $axpg = auh = a'vh = a'x'qg$. Since $g \neq 0$, $axp = a'x'q$. Since $dy = bx$, $dyp = bxp = b'x'q = d'y'q$. Therefore $dypk = d'y'qk = d'v'pl = du'pl$. Since $d \neq 0$, $ypk = u'pl'$, so $cypk = cu'pl = c'v'pl = c'y'qk$. Since $k \neq 0$, $cyp = c'y'q$, so $(ax + cy)p = (a'x' + c'y')q$. Therefore $+$ is well-defined.

In [5], pp. 21. it was shown that $+$ is associative and $\bullet$ is distributive over $+$ in K, hence K is a skewsemifield.

Let $c \in S\backslash\{0\}$, define $i : S \to K$ by $i(x) = [(xc,c)]$, for every $x \in S$.

In [5], pp. 23. it was shown that i is a right quotients embedding of S into K. Therefore K is a skewsemifield of a right quotients of S. $_{\#}$

Proposition 1.32. ([5]) Let S be an M.C. semiring with multiplicative zero 0 satisfying right [left] Ore condition. Then $S \times (S\backslash\{0\})/_{\sim}$ is the smallest skewsemifield containing S up to isomorphism where $\sim$ is the equivalence relation given in the proof of Theorem 1.31.

Proof See [5], pp. 26. $_{\#}$

Definition 1.33. Let K be a skewsemifield. A subset C of K is called a normal subset of K if and only if for every $x \in K$, $xC = Cx$.

Remark 1.34. Let K be a skewsemifield and C a normal subset of K. Then the following statements are equivalent :

1) for all $a, b \in C$ and $\alpha, \beta \in K, \alpha + \beta = 1$ implies that $\alpha a + \beta b \in C$.
2) for all $a, b \in C$ and $\alpha, \beta \in K, \alpha + \beta = 1$ implies that $\alpha a + b\beta \in C$.
3) for all $a, b \in C$ and $\alpha, \beta \in K, \alpha + \beta = 1$ implies that $a\alpha + \beta b \in C$.
4) for all $a, b \in C$ and $\alpha, \beta \in K, \alpha + \beta = 1$ implies that $a\alpha + b\beta \in C$.

Definition 1.35. Let K be a skewsemifield. A subset C of K is called an algebraically convex subset of K if and only if for all $x, y \in C$ and $a, b \in K$ such that $a + b = 1$, $ax + by \in C$.

From now on we shall call an algebraically convex subset an a-convex subset.

Proposition 1.36. Let K be a skewsemifield and C a subset of K. Then C is an a-convex subset of K if and only if for all $a, b \in K$, $aC + bC = (a+b)C$.

Proof Assume that C is an a-convex set of K. Let $a, b \in K$. If $a = 0$ then done. So suppose that $a \neq 0$. Let $c, c' \in C$. Since $(a+b)^{-1}a + (a+b)^{-1}b$ $= (a+b)^{-1}(a+b) = 1$ and by assumption, $(a+b)^{-1}ac + (a+b)^{-1}ac' \in C$, so there exists a $c'' \in C$ such that $(a+b)^{-1}ac + (a+b)^{-1}ac' = c''$ Therefore $ac + bc' = (a+b)c'' \in (a+b)C$. Thus $aC + bC \subseteq (a+b)C$.
Clearly, $(a+b)C \subseteq aC + bC$. Hence $aC + bC = (a+b)C$.

Conversely, assume that for all $\alpha, \beta \in K$, $\alpha C + \beta C = (\alpha + \beta)C$. Let $x, y \in C$ and $a, b \in K$ be such that $a + b = 1$. Then $ax + by \in aC + bC = (a+b)C = C$. Hence C is an a-convex subset of K. #

Remark 1.37. Let K be a skewsemifield.

1) The intersection of a family of a-convex subsets of K is an a-convex subset of K and the union of an increasing chain of a-convex subsets of K is an a-convex subset of K.

2) If A, B are a-convex subsets of K then AB is an a-convex subset of K where $AB = \{ ab \mid a \in A \text{ and } b \in B \}$..

3) Let C be an a-convex subset of K. Then for all $n \in \mathbf{Z}^+$, $a_1, \ldots, a_n \in K$, $x_1, \ldots, x_n \in C$, $\sum_{i=1}^{n} a_i = 1$ implies that $\sum_{i=1}^{n} a_i x_i \in C$.

4) Let C be a subset of K, the smallest a-convex normal subset of K

containing C is $\{ \sum_{i=1}^{n} a_i[x_i c_i (x_i)^{-1}] \mathbin{/} n \in \mathbf{Z}^+,\ c_i \in C,\ a_i,\ x_i \in K^*$ such that $\sum_{i=1}^{n} a_i = 1$ for every $i \in \{1,\ldots,n\} \}$.

Proof 1) Clear.

2) Let $k, k' \in K$ and $x \in kAB + k'AB$. Then there exist $a, a' \in A$ and $b, b' \in B$ such that $x = kab + k'a'b' \in kaB + k'a'B$. By Proposition 1.36., $kaB + k'a'B = (ka + k'a')B$, so there exits a $b'' \in B$ such that $x = (ka + k'a')b''$. By Proposition 1.36., $ka + k'a' \in kA + k'A = (k + k')A$, so there exits an $a'' \in A$ such that $ka + k'a' = (k + k')a''$. Then $x = (k + k')a''b'' \in (k + k')AB$. Therefore $kAB + k'AB = (k + k')AB$. Hence AB is an a-convex set.

3) If $n = 2$ then done. Let $n \in \mathbf{Z}^+$ be such that $n > 2$. Suppose that 3) is true for the case $n - 1$. Let $x_1,\ldots,x_n \in C$ and $a_1,\ldots,a_n \in K$ be such that $\sum_{i=1}^{n} a_i = 1$. Let $a'_{n-1} = a_{n-1} + a_n$. By Proposition 1.36., $(a_{n-1})(x_{n-1}) + (a_n)(x_n) \in (a_{n-1})C + (a_n)C = (a_{n-1} + a_n)C$, so there exists an $x \in C$ such that $(a_{n-1})(x_{n-1}) + (a_n)(x_n) = (a_{n-1} + a_n)x = (a'_{n-1})x$. Then $\sum_{i=1}^{n} a_i x_i = (\sum_{i=1}^{n-2} a_i x_i + (a'_{n-1})x) \in C$.

4) Let $B = \{ \sum_{i=1}^{n} a_i[x_i c_i (x_i)^{-1}] \mathbin{/} n \in \mathbf{Z}^+,\ c_i \in C,\ a_i,\ x_i \in K^*$ such that $\sum_{i=1}^{n} a_i = 1$, for all $i \in \{1,\ldots,n\} \}$. To show that B is a normal subset of K, let $b = \sum_{i=1}^{n} a_i[x_i c_i (x_i)^{-1}] \in B$ and $t \in K^*$. Then $tbt^{-1} = t(\sum_{i=1}^{n} a_i[x_i c_i (x_i)^{-1}])t^{-1}$ $= \sum_{i=1}^{n} t(a_i[x_i c_i (x_i)^{-1}])t^{-1} = \sum_{i=1}^{n} ta_i t^{-1}(t[x_i c_i (x_i)^{-1}]t^{-1}) = \sum_{i=1}^{n} ta_i t^{-1}[(tx_i)c_i(tx_i)^{-1}]$. Since $\sum_{i=1}^{n} t(a_i)t^{-1} = t(\sum_{i=1}^{n} a_i)t^{-1} = 1$, $tbt^{-1} \in B$. Next, to show the a-convexity of B, let $\sum_{i=1}^{n} a_i[x_i c_i (x_i)^{-1}]$, $\sum_{i=1}^{n} b_i[y_i k_i (y_i)^{-1}] \in B$ and let $\alpha, \beta \in K$ be such that $\alpha + \beta = 1$. Since $\sum_{i=1}^{n} \alpha a_i + \sum_{i=1}^{n} \beta b_i = \alpha(\sum_{i=1}^{n} a_i) + \beta(\sum_{i=1}^{n} b_i) = \alpha + \beta = 1$,

$\alpha(\sum_{i=1}^{n} a_i[x_i c_i (x_i)^{-1}]) + \beta(\sum_{i=1}^{n} b_i[y_i k_i (y_i)^{-1}]) = \sum_{i=1}^{n} \alpha a_i[x_i c_i (x_i)^{-1}] + \sum_{i=1}^{n} \beta b_i[y_i k_i (y_i)^{-1}] \in B$.

Clearly, B contains C, so B is an a-convex normal subset of K containing C. #

<u>Definition 1.38.</u> A subset C of a skewsemifield K is called a <u>normal subgroup</u> of K if and only if C is a multiplicative normal subgroup of K^*.

<u>Remark 1.39.</u> Let K be a skewsemifield.

1) $\{1\}$ and K^* are trivial a-convex normal subgroups.

2) The intersection of a family of a-convex normal subgroups of K is an a-convex normal subgroup. Also the union of an increasing chain of a-convex subgroups is an a-convex normal subgroup.

3) If A and B are a-convex normal subgroups of K then AB is an a-convex normal subgroup of K.

<u>Proposition 1.40.</u> Let K be a skewsemifield and C a multiplicative normal subgroup of K. Then the following statements are equivalent.

1) C is an a-convex set.

2) For all $x, y \in C$ and $a \in K$, $(x+a)^{-1}(y+a) \in C$.

3) For all $x, y \in C$ and $a \in K$, $(x+a)(y+a)^{-1} \in C$.

4) For all $x \in C$ and $a, b \in K$ such that $a+b=1$, $ax+b \in C$.

5) For all $x, y \in C$ and $a, b \in K$ such that $a+b \in C$, $ax+by \in C$.

6) For all $x \in C$ and $a, b \in K$ such that $a+b \in C$, $ax+b \in C$.

<u>Proof</u> 4) $\Rightarrow$ 2) Let $x, y \in C$ Then $xy^{-1} \in C$. Let $a \in K$. Then $(x+a)^{-1}(y+a) = (x+a)^{-1}y + (x+a)^{-1}a = (x+a)^{-1}x(x^{-1}y) + (x+a)^{-1}a \in C$.

2) $\Rightarrow$ 1) Let $x, y \in C$. Then $y^{-1}x \in C$. Let $a, b \in K$ be such that $a+b=1$. If $a=0$ then done. So suppose that $a \neq 0$. Then $y^{-1}(ax+by)$ $= (y^{-1}a)[a^{-1}(ax+by)] = (a^{-1}y)^{-1}[a^{-1}(ax+by)] = [y+a^{-1}by][x+a^{-1}by] \in C$.

1) $\Rightarrow$ 5) Let $x, y \in C$ and $a, b \in K$ such that $a+b \in C$. Then there exists a $c \in C$ such that $a+b=c$. Then $c^{-1}a + c^{-1}b = c^{-1}(a+b) = 1$. By 1), $c^{-1}ax + c^{-1}by \in C$. Then $ax+by = c(c^{-1}ax + c^{-1}by) \in C$.

6) $\Rightarrow$ 1) Let $x, y \in C$ and $a, b \in K$ such that $a+b \in C$. Then

$xy^{-1} \in C$. By 6), $axy^{-1} + b \in C$. Then $ax + by = (axy^{-1} + b)y \in C$.

3) $\Rightarrow$ 2) Let $x, y \in C$ and $a \in K$. By 3), $(y + a)(x + a)^{-1} \in P$. Since C is a normal set, $(x + a)^{-1}(y + a) = (x + a)^{-1}(y + a)(x + a)^{-1}(x + a) \in C$.

2) $\Rightarrow$ 3) Dually, 3) $\Rightarrow$ 2).

The remaining cases are clearly seen to be true. #

Let $\mathcal{C}$ be the set of all a-convex normal subgroups of a skewsemifield K Let $C, C' \in \mathcal{C}$. Then $C \vee C' = CC'$ and $C \wedge C' = C \cap C'$, so $\mathcal{C}$ is a lattice. Morever, $\mathcal{C}$ is modular.

To prove this, let $C, C', C'' \in \mathcal{C}$ be such that $C \subseteq C''$. Let $x \in CC' \cap C''$. Then there exist $c \in C, c' \in C'$ such that $x = cc'$ and $x \in C''$. Then $c' = c^{-1}x \in C''$, so $c' \in C' \cap C''$. Therefore $x = cc' \in C(C' \cap C'')$. Then $CC' \cap C'' \subseteq C(C' \cap C'')$. Since $C \subseteq C''$, $C(C' \cap C'') \subseteq CC' \cap C''$, so $C \vee (C' \wedge C'') = C(C' \cap C'') = CC' \cap C''$ $= (C \vee C') \wedge C''$.

Remark 1.41. Let $f : K \to M$ be a nonzero homomorphism of skewsemifields. Then the following statements hold :

1) $f(0) = 0$ if and only if $x = 0$ for every $x \in K$.

2) $f(x^{-1}) = (f(x))^{-1}$ for every $x \in K^*$.

3) m-ker f is an a-convex normal subgroup of K.

4) If C' is an a-convex normal subgroup of M then $f^{-1}(C')$ is an a-convex normal subgroup of K containing m-ker f.

5) If f is onto and C an a-convex normal subgroup of K then f(C) is an a-convex normal subgroup of M.

In our thesis, we shall study only nonzero homomorphism. So from now on we shall use the word homomorphism for nonzero homomorphism.

We shall now give an example of a-convex normal subgroup of a skewsemifield.

Example 1.42. Let K be a skewsemifield. Then $K^* \times K^* \cup \{ (0,0) \}$ is a skewsemifield. Define $f : K^* \times K^* \cup \{ (0,0) \} \to K$ by $f(x,y) = x$ for every $(x,y) \in K^* \times K^* \cup \{ (0,0) \}$. It easy to show that f is a homomorphism and $\text{m-ker} f = \{ (1,x) \ / \ x \in K^* \}$. By remark 1.41., 2), $\{ (1,x) \ / \ x \in K^* \}$ is an a-convex normal subgroup of $K^* \times K^* \cup \{ (0,0) \}$.

Proposition 1.43. Let $f : K \to M$ be an epimorphism of skewsemifields. Let $\mathbb{A}$ be the set of all a-convex normal subgroups of K containing ker f and $\mathbb{B}$ the set of all a-convex normal subgroups of M. Then there exists an order isomorphism from $\mathbb{A}$ onto $\mathbb{B}$.

Proof Define $\varphi : \mathbb{A} \to \mathbb{B}$ by $\varphi(A) = f(A)$ for all $A \in \mathbb{A}$ and $\psi : \mathbb{B} \to \mathbb{A}$ by $\psi(B) = f^{-1}(B)$ for all $B \in \mathbb{B}$. To show that $\varphi \circ \psi = Id_{\mathbb{B}}$, let $B \in \mathbb{B}$. Then $\varphi \circ \psi(B) = \varphi(\psi(B)) = \varphi(f^{-1}(B)) = f(f^{-1}(B))$. Since f is onto, $f(f^{-1}(B)) = B$, so $\varphi \circ \psi(B) = B$. Therefore $\varphi \circ \psi = Id_{\mathbb{B}}$. To show that $\psi \circ \varphi = Id_{\mathbb{A}}$, let $A \in \mathbb{A}$. Then $\psi \circ \varphi(A) = \psi(\varphi(A)) = \psi(f(A)) = f^{-1}(f(A))$. We must to show that $f^{-1}(f(A)) = A$. Clearly, $A \subseteq f^{-1}(f(A))$. Let $x \in f^{-1}(f(A))$. Then $f(x) \in f(A)$, so there exists an $a \in A$ such that $f(x) = f(a)$. Therefore $f(xa^{-1}) = f(x)(f(a))^{-1} = 1$, so $xa^{-1} \in \ker f \subseteq A$. Then $x = (xa^{-1})a \in A$, so $f^{-1}(f(A)) \subseteq A$. Therefore $\psi \circ \varphi(A) = A$, so φ is a bijection. Clearly, φ and ψ are isotone. Hence φ is an order isomorphism from $\mathbb{A}$ onto $\mathbb{B}$.

Definition 1.44. Let K be a skewsemifield and ρ an equivalence relation on K. ρ is called a congruence on K if for all $x, y, z \in K$,

1) $x \rho 0$ if and only if $x = 0$,

2) $x \rho y$ implies that $(xz) \rho (yz)$ and $(zx) \rho (zy)$, and

3) $x \rho y$ implies that $(x + z) \rho (y + z)$ and $(z + x) \rho (z + y)$.

<u>Remark 1.45.</u> 1) The intersection of a family of congruences on a skewsemifield K is a congruence on K.

2) $x \rho y$ implies that $x^{-1} \rho y^{-1}$ for all $x, y \in K^*$.

Let ρ be a congruence on a skewsemifield K. We shall show that $[1]_\rho = \{ x \in K \ / \ x \rho 1 \}$ is an a-convex normal subgroup of K.

Since $1 \rho 1$, $1 \in [1]_\rho$, so $[1]_\rho \neq \varnothing$. Let $x, y \in [1]_\rho$. Then $x \rho 1$ and $1 \rho y$. Therefore $x \rho y$. Thus $xy^{-1} \rho 1$, so $xy^{-1} \in [1]_\rho$. Hence $[1]_\rho$ is a multiplicative subgroup of K. Next, to show that $[1]_\rho$ is an a-convex normal set, let $x \in [1]_\rho$. Then $x \rho 1$. Let $y \in K^*$. Then $xy^{-1} \rho y^{-1}$, so $yxy^{-1} \rho 1$. Then $yxy^{-1} \in [1]_\rho$. Next, let $a, b \in K^*$ be such that $a + b = 1$. Since $x \rho 1$, $ax \rho a$, so $(ax + b) \rho (a + b)$. Therefore $ax + b \in [1]_\rho$. Hence $[1]_\rho$ is an a-convex normal subgroup of K.

Let $\mathcal{C}$ be the set of all congruence on a skewsemifield K. Let $\rho, \rho' \in \mathcal{C}$. Clearly, $\rho \wedge \rho' = \rho \cap \rho'$.

Define $x \rho^* y$ if and only if there exists a $u \in [1]_\rho$ such that $x \rho' uy$, for all $x, y \in K$. To show that ρ^* is an equivalent relation, let $x \in K$. Let $u = 1$. Then $u \in [1]_\rho$ and $x = ux$. Since ρ' is reflexive, $x \rho' ux$, so $x \rho^* x$. Then ρ^* is reflexive. Let $x, y \in K$ be such that $x \rho^* y$. Then there exists a $u \in [1]_\rho$ such that $x \rho' y$. Thus $u^{-1}x \rho' y$ and so $y \rho' u^{-1}x$. Then $y \rho^* x$, hence ρ^* is anti-symmetric. Let $x, y, z \in K$ be such that $x \rho^* y$ and $y \rho^* z$. Then there exist $u, v \in [1]_\rho$ such that $x \rho' uy$ and $y \rho' vz$, so $uy \rho^* uvz$. Therefore $x \rho' uvz$, so $x \rho^* z$ and hence ρ^* is transitive.

Next, to show that ρ^* is a congruence, let $x, y \in K$ be such that $x \rho^* y$. Then there exists a $u \in [1]_\rho$ such that $x \rho' uy$. Let $z \in K$.

Case 1, $z = 0$. Then $x + z = x = z + y$, $y + z = y = z + y$, $zx = 0 = zy$ and $xz = 0 = yz$. Therefore $zx \rho^* zy$, $xz \rho^* yz$, $(x + z) \rho^* (y + z)$ and $(z + x) \rho^* (z + y)$.

Case 2, $z \neq 0$. Then $xz \rho' uyz$ and thus $xz \rho^* yz$. Since $x \rho' uy$, $zx \rho' zuy$, so $zx \rho' (zuz^{-1})zy$. Since $zuz^{-1} \in [1]_\rho$, $zx \rho^* zy$. By $x \rho' uy$, $(x+ z) \rho' (uy + z)$. By Proposition 1.36., $uy + z \in ([1]_\rho)y + ([1]_\rho)z = ([1]_\rho)(y + z)$, so there exists a $u' \in [1]_\rho$

such that $uy + z = u'(y + z)$. Then $(x + z)\, \rho'\, u'(y + z)$, so $(x + z)\, \rho^*\, (y + z)$. We can prove similarly that $(z + x)\, \rho^*\, (z + y)$. Hence $\rho^* \in \mathcal{C}$.

Next, to show that $\rho^* \subseteq \rho' \circ \rho$, let $x, y \in K$ be such that $x\, \rho^* y$. Then there exists a $u \in [1]_\rho$ such that $x\, \rho'\, uy$. Then $uy\, \rho\, y$, so $x\, (\rho' \circ \rho)\, y$ and thus $\rho^* \subseteq \rho' \circ \rho$. Next, let $x, y \in K$ be such that $x\, (\rho' \circ \rho)\, y$. Then there exists a $z \in K$ such that $x\, \rho'\, z$ and $z\, \rho\, y$. If $z = 0$ then $x = 0 = y$, so $x\, \rho\, y$. Suppose that $z \neq 0$. Then $yz^{-1}\, \rho\, 1$, so $yz^{-1} \in [1]_\rho$. Thus $zy^{-1} = (yz^{-1})^{-1} \in [1]_\rho$ and $z\, \rho'\, (zy^{-1})y$, so $x\, \rho'\, (zy^{-1})y$ and therefore $x\, \rho^*\, y$, so $\rho' \circ \rho \subseteq \rho^*$. Hence $\rho^* = \rho' \circ \rho$, so for all $\rho, \rho' \in \mathcal{C}$, $\rho' \circ \rho = \rho \vee \rho' = \rho' \vee \rho = \rho \circ \rho'$. Hence $(\mathcal{C}, \circ)$ is a commutative semigroup.

Next, to show that $\rho \vee \rho' = \rho^*$, let $x, y \in K$ be such that $x\, \rho\, y$. If $y = 0$ then $x = 0$. So suppose that $y \neq 0$. Then $xy^{-1}\, \rho\, 1$ and $x = (xy^{-1})y$. Then $x\, \rho'\, (xy^{-1})y$, so $x\, \rho^*\, y$. Therefore $\rho \subseteq \rho^*$. Clearly $\rho' \subseteq \rho^*$.

Let $\rho \subseteq \rho''$ and $\rho' \subseteq \rho''$. Let $x, y \in K$ be such that $x\, \rho^* y$. Then there exists a $u \in [1]_\rho$ such that $x\, \rho'\, uy$. If $y = 0$ then $x = 0$. Suppose that $y \neq 0$. Then $u\, \rho\, 1$ and $xy^{-1}\, \rho' u$. Since $\rho \subseteq \rho''$ and $\rho' \subseteq \rho''$, $u\, \rho''\, 1$ and $xy^{-1}\, \rho''\, 1$, so $xy^{-1}\, \rho''\, 1$. Therefore $x\, \rho''\, y$. Hence $\rho^* \subseteq \rho''$, so $\rho \vee \rho' = \rho^* = \rho' \circ \rho$. Then $\mathcal{C}$ is a lattice. Morever, we shall show that $\mathcal{C}$ is modular.

To prove this, let $\rho, \rho', \rho^* \in \mathcal{C}$ be such that $\rho \subseteq \rho^*$. Let $x, y \in K$ be such that $x\, [(\rho^* \cap \rho') \circ \rho] y$. Then there exists a $z \in K$ such that $x\, (\rho' \cap \rho^*)\, z$ and $z\, \rho\, y$, so $x\, \rho'\, z$ and $x\, \rho^*\, z$. Then $x\, (\rho' \circ \rho)\, y$. Since $\rho \subseteq \rho^*$, $z\, \rho\, y$, so $x\, \rho^*\, y$. Therefore $x\, [(\rho' \circ \rho) \cap \rho^*]\, y$, hence $(\rho' \cap \rho^*) \circ \rho \subseteq (\rho' \circ \rho) \cap \rho^*$. Next, let $x, y \in K$ be such that $x\, [(\rho' \circ \rho) \cap \rho^*]\, y$. Then $x\, (\rho' \circ \rho) y$ and $x\, \rho^*\, y$. Therefore there exists a $z \in K$ such that $x\, \rho'\, z$ and $z\, \rho\, y$. Since $\rho \subseteq \rho^*$, $z\, \rho^*\, y$, so $y\, \rho^*\, z$. Then $x\, (\rho' \cap \rho)\, y$, so $x\, [(\rho' \cap \rho^*) \circ \rho]\, y$. Therefore $(\rho' \circ \rho) \cap \rho^* \subseteq (\rho' \cap \rho^*) \circ \rho$, so $\rho \vee (\rho' \wedge \rho^*)$ $= (\rho' \cap \rho^*) \circ \rho = (\rho' \circ \rho) \cap \rho^* = (\rho \vee \rho') \wedge \rho^*$.

Let C be an a-convex normal subgroup a skewsemifield K. The relation ρ_c on K given by $x\, \rho_c\, y$ if and only if $xy^{-1} \in C$ or $x = y = 0$, for all $x, y \in K$, clearly

ρ_c is a congruence on K and $[x]_{\rho c} = xC$.

Let $K/_{\rho c}$ be the set of all equivalence classes of K with respect to ρ_c, we shall use the notation K/C instead of $K/_{\rho c}$.

Define + and $\bullet$ on K/C as follows: let $xC, yC \in K/C$, let $xC + yC = (x + y)C$ and $(xC)(yC) = (xy)C$.

To show that + and $\bullet$ are well-defined, let $xC, yC \in K/C$. Choose $a \in xC$ and $b \in yC$. Then $xa^{-1} \in C$ or $x = a = 0$ and $yb^{-1} \in C$ or $y = b = 0$. If $x = a = 0$ or $y = b = 0$ then $ab = 0 = xy$ and $a + b = x + y$, so $(ab)C = (xy)C$ and $(a + b)C =(x + y)C$. Suppose that $a, b \neq 0$. Then $xa^{-1}, yb^{-1} \in C$. Since C is a normal subset of K, $a(yb^{-1})a^{-1} \in C$, so $(xy)(ab)^{-1} = xyb^{-1}a^{-1}$ $= (xa^{-1})(ayb^{-1}a^{-1}) \in C$. Therefore $(xy)C = (ab)C$, so $\bullet$ is well-defined. By the a-convexity of C, $(x + y)(a + b)^{-1}= (xa^{-1})[a(a + b)^{-1}]+ (yb^{-1})[b(a + b)^{-1}] \in C$. Then $(x + y)C= (a + b)C$, hence + is well-defined. Then we have that $(K/C, +, \bullet)$ is a skewsemifield.

<u>Proposition 1.46.</u> Let K be a skewsemifiled, $\mathbb{A}$ the set of all congruences on K and $\mathbb{B}$ the set of all a-convex normal subgroups of K. Then there exists an order isomorphism from $\mathbb{A}$ onto $\mathbb{B}$.

<u>Proof</u> Define $\varphi : \mathbb{A} \to \mathbb{B}$ by $\varphi(\rho) = [1]_\rho$ for all $\rho \in \mathbb{A}$. To show that φ is an injection, let $\rho, \rho' \in \mathbb{A}$ be such that $[1]_\rho = \varphi(\rho) = \varphi(\rho') = [1]_{\rho'}$. Let $x, y \in K$ be such that $x \rho y$. If $y = 0$ then $x = 0$, so $x \rho' y$. Suppose that $y \neq 0$. Then $xy^{-1} \rho 1$, so $xy^{-1} \in [1]\rho = [1]\rho'$. Therefore $xy^{-1} \rho' 1$, so $x \rho' y$. Thus $\rho \subseteq \rho'$ Similarly $\rho' \subseteq \rho$.Then $\rho = \rho'$, so φ is an injection. Next, to show that φ is onto, let $C \in \mathbb{B}$. Then $\rho_c \in A$. Let $x \in [1]_{\rho c}$. Then $x \rho_c 1$, so $x \in C$. Then $[1]\rho_c \subseteq C$. Let $x \in C$. Then $x \rho_c 1$, so $x \in [1]_{\rho c}$. Then $C \subseteq [1]\rho_c$, so $\varphi(\rho_c) = [1]_{\rho c} = C$. Therefore φ is a bijection. Clearly, φ is isotone and for all $C, C' \in \mathbb{B}$, $C \subseteq C'$ implies that $\varphi^{-1}(C) = \rho_c \subseteq \rho_{c'} = \varphi^{-1}(C')$. Then φ^{-1} is isotone, so φ is an order isomorphism

from $\mathbb{A}$ onto $\mathbb{B}$. #

Corollary 1.47. Let K be a skewsemifield and C an a-convex normal subgroup of K. Let $\mathbb{A}$ be the set of all a-convex normal subgroups of $K_{/C}$ except { C } and $\mathbb{B}$ the set of all a-convex subgroups of K such that strictly contain C. Then there exists an order isomorphism from $\mathbb{A}$ onto $\mathbb{B}$.

Proof Claim that for every $D \in \mathbb{A}$, $\cup D$ is an a-convex subgroup of K which strictly contains C. To prove this, let $x, y \in \cup D$. Then there exist $\alpha, \beta \in D$ such that $x \in \alpha$ and $y \in \beta$. Thus $xy \in \alpha\beta$ and $\alpha\beta \in D$, so $xy \in \cup D$. Since D is a subgroup of $K_{/C}$, there exists an $\alpha^{-1} \in D$ such that $\alpha^{-1}\alpha = C$. Since $1 \in C$, there exist $u \in \alpha^{-1}$ and $v \in \alpha$ such that $uv = 1$. Since $x, v \in \alpha$, $xC = \alpha = vC$, so $vx^{-1} \in C$. Then $x^{-1} = u(vx^{-1}) \in uC = \alpha^{-1} \subseteq \cup D$. Next, let $z \in K^*$. Then $zxz^{-1} \in (zxz^{-1})C$. Since $(zxz^{-1})C = (zC)\alpha(zC)^{-1} \in D$, we get that $zxz^{-1} \in \cup D$. Let $a, b \in K$ be such that $a + b = 1$. Then $aC + bC = (a + b)C = C$, so $(ax + by)C = (aC)(xC) + (bC)(yC) \in D$ and thus $ax + by \in \cup D$. Since $D \neq \{ C \}$, there exists an $\gamma \in D$ such that $\gamma \neq C$, and so there exists an $x \in \gamma$ such that $x \notin C$. Hence $\cup D$ is an a-convex subgroup of K which strictly contains C, so we have claim.

Define $\varphi : \mathbb{A} \to \mathbb{B}$ by $\varphi(D) = \cup D$ for all $D \in \mathbb{A}$ and $\Psi : \mathbb{B} \to \mathbb{A}$ by $\Psi(D) = \Pi(D)$ for all $D \in \mathbb{B}$ where Π is the projection map of K onto $K_{/C}$. To show that $\varphi \circ \psi = Id_B$, let $B \in \mathbb{B}$. Then $\varphi \circ \psi(B) = \varphi(\psi(B)) = \varphi(\Pi(B)) = \cup(\Pi(B))$. To show that $\cup(\Pi B) = B$, let $b \in B$. Since $b \in bC = \Pi(d)$, so $d \in \cup(\Pi(B))$ and thus $B \subseteq \cup(\Pi B)$. Next, let $x \in \cup(\Pi(B))$. Then there exists a $b \in B$ such that $x \in \Pi(b) = bC$, so $xb^{-1} \in C \subseteq B$. Thus $x = (xb^{-1})b \in B$, so $\cup(\Pi B) \subseteq B$ and therefore $\cup(\Pi B) = B$, so $\varphi \circ \psi = Id_B$.

To show that $\psi \circ \varphi = Id_A$, let $D \in \mathbb{A}$. Then $\psi \circ \varphi(D) = \psi(\varphi(D)) = \psi(\cup D) = \Pi(\cup D)$. We must to show that $\Pi(\cup D) = D$. Let $\alpha \in D$ and let $x \in \alpha$. Then $\alpha = xC = \Pi(x) \in \Pi(\cup D)$, so $D \subseteq \Pi(\cup D)$. Next, let $\beta \in \Pi(\cup D)$. Then there exists an $x \in \cup D$ such that $\Pi(x) = \beta$, so there exits an $\alpha \in D$ such that $x \in \alpha$.

Therefore $\beta = \Pi(x) = xC = \alpha \in \mathcal{D}$, so $\Pi(\cup\mathcal{D}) \subseteq \mathcal{D}$. Hence $\Pi(\cup\mathcal{D}) = \mathcal{D}$, so $\psi \circ \varphi = Id_{\mathbb{A}}$. Thus φ is a bijection. Clearly, φ and φ^{-1} are isotone, hence φ is an order isomorphism from $\mathbb{A}$ onto $\mathbb{B}$. #

Proposition 1.48. Let K be a skewsemifield and $C \subseteq K^*$. Then C is an a-convex normal subgroup of K if and only if C is the multiplicatively kernel of some epimorphism.

Proof Assume that C is an a-convex normal subgroup of K. Define $\Pi : K \to K/_C$ by $\Pi(x) = xC$, for every $x \in K$. Then Π is an epimorphism and $m\text{-ker}\,\Pi = C$.

The converse follows from Remark 1.41., 2). #

The map $\Pi : K \to K/_C$ given in Proposition 1.48. is called the canonical projection of K onto $K/_C$.

Proposition 1.49. Let C be an a-convex normal subgroup of a skewsemifield K. Then $K/_C$ is a right [left] additively cancellative skewsemifield if and only if for all $x, \alpha, \beta \in K^*$, $\alpha + \beta = 1$ and $\alpha x + \beta \in C$ imply that $x \in C$ [$\alpha + \beta = 1$ and $\alpha + \beta x \in C$ imply that $x \in C$].

Proof Let $x, \alpha, \beta \in K^*$ be such that $\alpha + \beta = 1$ and $\alpha x + \beta \in C$. Then $\alpha xC + \beta C = (\alpha x + \beta)C = C = (\alpha + \beta)C = \alpha C + \beta C$. Therefore $\alpha xC = \alpha C$, so $x = \alpha^{-1}(\alpha x) \in C$.

Conversely, assume that for all $x, \alpha, \beta \in K^*$, $\alpha + \beta = 1$ and $\alpha x + \beta \in C$ imply that $x \in C$. Let $x, y, z \in K$ be such that $xC + zC = yC + zC$.

Case 1 : $x = 0$. Then $zC = (y + z)C$. Suppose that $y \neq 0$. Then $(1 + y^{-1}z)^{-1}y^{-1}(0) + (1 + y^{-1}z)^{-1}y^{-1}z = (1 + y^{-1}z)^{-1}y^{-1}z = [y(1 + y^{-1}z)]^{-1}z$ $= (y + z)^{-1}z \in C$. By assumption, $0 = y^{-1}(0) \in C$ which is a contradiction. Then

$y = 0$, so $xC = yC$.

Case 2 : $x \neq 0$. Then $(x + z)C=(y + z)C$, so $(1 + x^{-1}z)^{-1}x^{-1}y + (1 + x^{-1}z)^{-1}x^{-1}z$

$(1 + x^{-1}z)^{-1}(x^{-1}y + x^{-1}z) = (1 + y^{-1}z)^{-1}x^{-1}(y + z) = (x + z)^{-1}(y + z) \in C$. By assumption, $x^{-1}y \in C$, so $xC = yC$.

Hence $K/_C$ is a right additively cancellative skewsemifield. #

Theorem 1.51. (First Isomorphism Theorem).

Let $f : K \to M$ be a homomorphism of skewsemifields. Then $K/_{m\text{-}\ker f} \cong \operatorname{Im} f$. Hence if f is onto then $K/_{m\text{-}\ker f} \cong M$.

Proof By Remark 1.41., 2), m-ker f is an a-convex normal subgroup of K. Define $\varphi : K/_{m\text{-}\ker f} \to \operatorname{Im} f$ as follows : let $\alpha \in K/_{m\text{-}\ker f}$. Let $x \in \alpha$, define $\varphi(x) = f(x)$. To show that φ is well-defined, let $x, y \in K$ be such that $x(m\text{-}\ker f) = y(m\text{-}\ker f)$. Then there exists an $a \in m\text{-}\ker f$ such that $x = ya$, so $\varphi(x(m\text{-}\ker f)) = f(x) = f(ya) = f(y)f(a) = f(y) = \varphi(y(m\text{-}\ker f))$. Then φ is well-defined and $\varphi(0) = f(0) = 0$. Clearly, φ is a bijection and a homomorphism. Hence $K/_{m\text{-}\ker f} \cong \operatorname{Im} f$. #

Lemma 1.52. Let H be a subskewsemifield of a skewsemifield K and C an a-convex normal subgroup of K. Then $H \cap C$ is an a-convex normal subgroup of H and HC is a subskewsemifield of K.

Proof Clearly, $H \cap C$ is a multiplicative normal subgroup of H. Let $x \in H \cap C$ and $a, b \in H$ be such that $a + b = 1$. Then $ax + b \in H$. Since C is an a-convex normal set of K, $ax + b \in C$, so $ax + b \in H \cap C$. Threfore $H \cap C$ is an a-convex normal subgroup of H.

Since $1 \in H$ and $1 \in C$, $1 \in HC$, so $HC \neq \varnothing$. Let $a, b \in (HC)^*$. Then there exist $u, v \in H^*$ and $x, y \in C$ such that $a = ux$ and $b = vy$. Since C is a normal set, $ab^{-1} = (ux)(vy)^{-1} = uxy^{-1}v^{-1} = (uv^{-1})[v(xy^{-1})v^{-1}] \in HC$. By Proposition 1.36.,

$a + b = ux + vy \in uC + vC = (u + v)C \subseteq HC$. Hence HC is a subskewsemifield. #

Theorem 1.53. (Second Isomorphism Theorem).

Let H be a subskewsemifield of a skewsemifield K and C an a-convex normal subgroup of K. Then $H/_{H \cap C} \cong HC/_C$.

Proof Define $\varphi : H \to HC/_C$ by $\varphi(x) = xC$, for every $x \in H$. Then φ is an epimorphism. Since for every $x \in H$, $xC = \varphi(x) = C$ if and only if $x \in C$, m-ker $\varphi = H \cap C$. Then $H/_{H \cap C} \cong HC/_C$. #

Lemma 1.54. Let D and H be a-convex normal subgroups of a skewsemifield K such that $H \subseteq D$. Then $D/_H$ is an a-convex normal subgroup of $K/_H$.

Proof Clearly, $D/_H$ is a multiplicative normal subgroup of $K/_H$. To show the a-convexity, let $x, y \in D$ and $\alpha H, \beta H \in K/_H$ be such that $(\alpha + \beta)H = \alpha H + \beta H = H$. Then $\alpha + \beta \in H$, so there exist $a \in \alpha$, $b \in \beta$ and $h \in H$ such that $a + b = h$. Thus $ah^{-1} + bh^{-1} = (a + b)h^{-1} = 1$. By the a-convexity of D, $(ah^{-1})x + (bh^{-1})y \in D$, so $(\alpha H)(xH) + (\beta H)(yH) = (ah^{-1}x + bh^{-1}y)H \in K/_H$. Therefore $D/_H$ is an a-convex normal subgroup of $K/_H$. #

Theorem 1.55. (Third Isomorphism Theorem).

Let K be a skewsemifield, D and H a-convex normal subgroups of K such $H \subseteq D$. Then $K/_H/_{D/_H} \cong K/_D$.

Proof Define $\varphi : K/_H \to K/_D$ by $\varphi(xH) = xD$, for every $x \in K$. Then φ is an epimorphism. Since for every $a \in K$, $aD = \varphi(aH) = D$ if and only if $a \in D$, m-ker $\varphi = D/_H$. Then $K/_H/_{D/_H} \cong K/_D$. #

Proposition 1.56. Let $f : K \to M$ be an epimorphism of skewsemifields. If C' is an a-convex normal subgroup of M then $K/f^{-1}(C') \cong M/C'$.

Proof By Remark 1.41., 3), $f^{-1}(C')$ is an a-convex normal subgroup of M. Define $\varphi : K \to M/C'$ by $\varphi(x) = f(x)C'$, for every $x \in K$. Then φ is an epimorphism. Let $x \in$ m-ker φ, then $f(x)C' = \varphi(x) = C'$, so $f(x) \in C'$. Therefore $x \in f^{-1}(C')$. Hence m-ker $\varphi \subseteq f^{-1}(C')$. Similarly, $f^{-1}(C') \subseteq$ m-ker φ, so $f^{-1}(C') =$ m-ker φ. By Theorem 1.51., $K/f^{-1}(C') \cong M/C'$.#

Lemma 1.57 Let A and B be subskewsemifields of a skewsemifield K, A_1 and B_1 a-convex normal subgroups of A and B, respectively. Then $(A_1 \cap B)(A \cap B_1)$ is an a-convex normal subgroup of $A \cap B$ and $(A \cap B)A_1$ and $(A \cap B)B_1$ are subskewsemifields of K.

Proof Clearly, $A \cap B$ is a subskewsemifield of A. Since A_1 is an a-convex normal subgroup of A and (by Lemma 1.53.), $(A \cap B)A_1$ is a subskewsemifield of A, so $(A \cap B)A_1$ is also a subskewsemifield of K. Similarly, $(A \cap B)B_1$ is a subskewsemifield of K. By Lemma 1.52., $(A_1 \cap B) = A_1 \cap (A \cap B)$ which is an a-convex normal subgroup of $(A \cap B)$. Similarly, $A \cap B_1$ is an a-convex normal subgroup of $A \cap B$. By Remark 1.39., 3), $(A_1 \cap B)(A \cap B_1)$ is an a-convex normal subgroup of $(A \cap B)$. #

Proposition 1.58. Let A and B be subskewsemifields of a skewsemifield K, A_1 and B_1 a-convex normal subgroups of A and B, respectively. Then $(A \cap B)A_1/(A \cap B_1)A_1 \cong (A \cap B)B_1/(A_1 \cap B)B_1$.

Proof Define $f : (A \cap B)A_1 \to (A \cap B)/(A_1 \cap B)(A \cap B_1)$ as follows : let $c \in A \cap B$, $a_1 \in A_1$, define $f(ca_1) = c[(A_1 \cap B)(A \cap B_1)]$. To show that f is well-defined, let $c_1, c_2 \in (A \cap B)$ and $a_{11}, a_{12} \in A_1$ be such that $c_1a_{11} = c_2a_{12}$. Then $(c_2)^{-1}c_1 = a_{11}(a_{12})^{-1} \in (A \cap B) \cap A_1 = A_1 \cap B \subseteq (A_1 \cap B)(A \cap B_1)$. Therefore $f(c_1a_{11}) = c_1[(A_1 \cap B)(A \cap B_1)] = c_2[(A_1 \cap B)(A \cap B_1)] = f(c_2a_{12})$. Therefore f is well-defined. Clearly, f is an epimorphism.

To show that $(A \cap B_1)A_1 = \text{m-ker} f$, let $c \in A \cap B_1$ and $a \in A_1$. Then $(A_1 \cap B)(A \cap B_1) = f(a) = c[(A_1 \cap B)(A \cap B_1)]$. Therefore $(A \cap B_1)A_1 \subseteq \text{m-ker} f$.

Next, let $c \in A \cap B$ and $a \in A_1$ be such that $c[(A_1 \cap B)(A \cap B_1)] = f(ca) = (A_1 \cap B)(A \cap B_1)$. Then $c \in (A_1 \cap B)(A \cap B_1)$, so there exist $x \in A_1 \cap B$ and $y \in A \cap B_1$ such that $c = xy$. Then $ca = xya$. Since A_1 is a normal set of A, there exists a $z \in A_1$ such that $xy = yz$, so $ca = xya = yza \in (A \cap B)A_1$. Then $\text{m-ker} f \subseteq (A \cap B_1)A_1$ and hence $(A \cap B_1)A_1 = \text{m-ker} f$. By Theorem 1.51., we get that $(A \cap B)A_1/(A \cap B_1)A_1 \cong (A \cap B)/(A_1 \cap B)(A \cap B_1)$. Similarly, we get that $(A \cap B)B_1/(A_1 \cap B)B_1 \cong (A \cap B)/(A_1 \cap B)(A \cap B_1)$. Hence $(A \cap B)A_1/(A \cap B_1)A_1 \cong (A \cap B)B_1/(A_1 \cap B)B_1$. #

<u>Definition 1.59.</u> Let $\{ K_i \mid i \in I \}$ be a family of skewsemifields. The <u>direct product</u> of the family $\{ K_i \mid i \in I \}$, denoted by $\prod_{i \in I} K_i$, is the set of all elements $(x_i)_{i \in I}$ in the cartesian product of the family $\{ K_i^* \mid i \in I \} \cup \{ 0 \}$ where $0 = (0_i)_{i \in I}$ together with operations + and $\bullet$ defined as usual, that is for all $(x_i)_{i \in I}, (y_i)_{i \in I} \in \prod_{i \in I} K_i$,

$$(x_i)_{i \in I} + (y_i)_{i \in I} = (x_i + y_i)_{i \in I} \text{ and}$$

$$(x_i)_{i \in I} \bullet (y_i)_{i \in I} = (x_iy_i)_{i \in I}.$$

Then we have that $\prod_{i \in I} K$ is a skewsemifield.

<u>Proposition 1.60.</u> Let $\{ K_i \mid i \in I \}$ be a family of skewsemifields. Then the following statements hold :

1) for each $i \in I$, the canonical projection $\Pi_k : \prod_{i \in I} K_i \to K_k$ given by $\Pi_k((x_i)_{i \in I}) = x_k$ is an epimorphism.

2) if $1_i + 1_i = 1_i$ for every $i \in I$ then for each $k \in I$ the canonical injection

$I_k : K_k \to \prod_{i \in I} K$ given by $I_k(x_k) = (x_i)_{i \in I}$ where $x_i = 1_i$ for $i \neq k$, and $I_k(0) = 0$ is a monomorphism of skewsemifields.

<u>Proof</u> Obvious. #

<u>Proposition 1.61.</u> Let $\{ K_i \ / \ i \in I \}$ be a family of skewsemifields and C_i an a-convex normal subgroup of K_i for every $i \in I$. Then $\prod_{i \in I} C_i$ is an a-convex normal subgroup of $\prod_{i \in I} K_i$ and $\prod_{i \in I} K_i / \prod_{i \in I} C_i \cong \prod_{i \in I} (K_i/C_i)$.

<u>Proof</u> Define $\varphi : \prod_{i \in I} K_i \to \prod_{i \in I} (K_i/C_i)$ by $\varphi((x_i)_{i \in I}) = ((x_iC_i)_{i \in I})$, for every $(xi)_{i \in I} \in \prod_{i \in I} K_i$. Then φ is an epimorphism.

To show that $\text{m-ker}\,\varphi = \prod_{i \in I} C_i$, let $(x_i)_{i \in I} \in \text{m-ker}\,\varphi$. Then $((x_iC_i)_{i \in I} = \varphi(x_i)_{i \in I} = (C_i)_{i \in I}$, so $x_i = C_i$ for all $i \in I$. Therefore $x_i \in C_i$ for all $i \in I$, so $(x_i)_{i \in I} \in \prod_{i \in I} C_i$. and hence $\text{m-ker}\,\varphi \subseteq \prod_{i \in I} C_i$. Clearly, $\prod_{i \in I} C_i \subseteq \text{m-ker}\,\varphi$, so $\text{m-ker}\,\varphi = \prod_{i \in I} C_i$. By Remark 1.42., we get that $\prod_{i \in I} C_i$ is an a-convex normal subgroup of $\prod_{i \in I} K_i$ and by Theorem 1.51., $\prod_{i \in I} K_i / \prod_{i \in I} C_i \cong \prod_{i \in I} (K_i/C_i)$. #

<u>Definition 1.62.</u> Let L be a subskewsemifield of a direct product of a family skewsemifields $\{ K_i \ / \ i \in I \}$. L is said to be a <u>subdirect</u> <u>product</u> of $\{ K_i \ / \ i \in I \}$ if and only if for every $k \in I$, $\Pi_k(L) = K_k$ where Π_k is the projection map.

<u>Definition 1.63.</u> Let $\{ K_i \ / \ i \in I \}$ be a family of skewsemifields and L a skewsemifield. Let $g : L \to \prod_{i \in I} K_i$ be a homomorphism. Then g is said to be a <u>representation</u> of L as a subdirect product of $\{ K_i \ / \ i \in I \}$ if and only if $\text{Im}\, g$ is a subdirect product of $\{ K_i \ / \ i \in I \}$.

Definition 1.64. Let K be a skewsemifield. K is said to be subdirectly irreducible if and only if for every family $\{ K_i \,/\, i \in I \}$ of skewsemifields and for every injective representation $f : K \to \prod_{i \in I} K_i$, there exists a $k \in I$ such that $\Pi_k \circ f$ is an isomorphism.

If a skewsemifield K is not subdirectly irreducible, we shall say that K is subdirectly reducible.

Theorem 1.65. Let $g : L \to \prod_{i \in I} K_i$ be a representation of L as a subdirect product of $\{ K_i \,/\, i \in I \}$. Then $\text{Im } g \cong L/_{\bigcap_{k \in I} \text{m-ker}\, \Pi_k \circ g}$.

Proof Define $\varphi : L \to \text{Im } g$ by $\varphi(x) = g(x)$ for every $x \in L$. Then φ is an epimorphism. To show that $\text{m-ker}\, \varphi = \bigcap_{k \in I} \text{m-ker}\, \Pi_k \circ g$, let $x \in L$ be such that $g(x) = \varphi(x) = (1_i)_{i \in I}$. Then $(\Pi_k \circ g)(x) = 1_k$ for all $k \in I$, so $x \in \text{m-ker}\, \Pi_k \circ g$ for all $k \in I$. Thus $x \in \bigcap_{k \in I} \text{m-ker}\, \Pi_k \circ g$, so $\text{m-ker}\, \varphi \subseteq \bigcap_{k \in I} \text{m-ker}\, \Pi_k \circ g$.

Next, let $x \in \bigcap_{k \in I} \text{m-ker}\, \Pi_k \circ g$. Then $(\Pi_k \circ g)(x) = 1_k$ for all $k \in I$. Therefore $g(x) = (1_i)_{i \in I}$. Since $\varphi(x) = g(x) = 1$, $x \in \text{m-ker}\, \varphi$. Hence $\bigcap_{k \in I} \text{m-ker}\, \Pi_k \circ g \subseteq \text{m-ker}\, \varphi$, so $\text{m-ker}\, \varphi = \bigcap_{k \in I} \text{m-ker}\, \Pi_k \circ g$. By Theorem 1.51., $\text{Im } g \cong L/_{\bigcap \text{m-ker}\, \Pi_k \circ g}$. #

Corollary 1.66. Let $g : L \to \prod_{i \in I} K_i$ be an injective representation of L as a subdirect product of $\{ K_i \,/\, i \in I \}$. Then $\bigcap_{k \in I} \text{m-ker}\, \Pi_k \circ g = \{ 1 \}$, hence $\text{Im } g \cong L$.

Proof To show that $\bigcap_{k \in I} \text{m-ker}\, \Pi_k \circ g = \{ 1 \}$, let $x \in \bigcap_{k \in I} \text{m-ker}\, \Pi_k \circ g$. Then $(\Pi_k \circ g)(x) = 1_k$ for all $k \in I$, so $g(x) = (1_i)_{i \in I}$. Since g is a monomorphism, $x = 1$. Therefore $\bigcap_{k \in I} \text{m-ker}\, \Pi_k \circ g = \{ 1 \}$. #

Proposition 1.67. Let L be a skewsemifield and $\mathcal{C} = \{ C_i \ / \ C_i$ is an a-convex normal subgroup of L for all $i \in I \}$. Define $f_{\mathcal{C}} : L \to \prod_{i \in I} (L/C_i)$ by $f_{\mathcal{C}}(x) = (xC_i)_{i \in I}$, for all $x \in L$. Then $f_{\mathcal{C}}$ is a representation of L as a subdirect product of $\{ L/C_i \ / \ i \in I\}$. Furthermore, if $\bigcap_{i \in I} C_i = \{ 1 \}$ then $f_{\mathcal{C}}$ is an injective representation of L.

Proof Clearly, $f_{\mathcal{C}}$ is a homomorphism of L. To show that $\operatorname{Im} f_{\mathcal{C}}$ is a subdirect product, let $k \in I$ and $x \in L$. Then $\Pi_k \circ f_{\mathcal{C}}(x) = \Pi_k((f_{\mathcal{C}}(x))) = \Pi_k((x_i)_{i \in I}) = xC_k \in L/C_k$, so $\Pi_k(\operatorname{Im} f_{\mathcal{C}}) \subseteq L/C_k$. Next, let $x \in L$. Then $xC_k \in L/C_k$, so $f_{\mathcal{C}}(x) \in \prod_{i \in I} (L/C_i)$ and $\Pi_k((f_{\mathcal{C}}(x)) = xC_k \in \Pi_k(\operatorname{Im} f_{\mathcal{C}})$. Therefore $L/C_k \subseteq \Pi_k(\operatorname{Im} f_{\mathcal{C}})$, so $(\Pi_k \circ f_{\mathcal{C}})(L) = \Pi_k(\operatorname{Im} f_{\mathcal{C}}) = L/C_k$. Hence $f_{\mathcal{C}}$ is a representation of L as a subdirect product of $\{ L/C_i \ / \ i \in I\}$.

Assume that $\bigcap_{i \in I} C_i = \{ 1 \}$. To show that $f_{\mathcal{C}}$ is an injection, let $x \in L$ be such that $(x_i)_{i \in I} = f_{\mathcal{C}}(x) = (C_i)_{i \in I}$. Hence $x \in C_i$ for all $i \in I$. By assumption, $x = 1$, so $f_{\mathcal{C}}$ is an injection. Hence $f_{\mathcal{C}}$ is an injective representation of L.$_{\#}$

Proposition 1.68. Let K be a skewsemifield and $\mathcal{C}$ the set of all a-convex normal subgroups of K except $\{ 1 \}$. Then K is a subdirectly irreducible skewsemifields if and only if $\mathcal{C}$ has a minimum element.

Proof Assume that K is a subdirectly irreducible skewsemifield. Suppose that $\mathcal{C}$ has no minimum element. Then $\cap\mathcal{C} = \{ 1 \}$. By Proposition 1.68., $f_{\mathcal{C}} : K \to \prod_{i \in I} (K/C)$ defined by $f_{\mathcal{C}}(x) = (xC)_{C \in \mathcal{C}}$ is an injective representation of L as a subdirect product of $\{ K/C \ / \ C \in \mathcal{C} \}$. By assumption, there exists a $C' \in \mathcal{C}$ such that $\Pi_{C'} \circ f_{\mathcal{C}}$ is an isomorphism of L. To show that $C' \subseteq \{ 1 \}$, let $x \in C'$. Then $\Pi_{C'} \circ f_{\mathcal{C}}(x) = \Pi_{C'}((f_{\mathcal{C}}(x))) = \Pi_{C'}((xC)_{C \in \mathcal{C}}) = xC'$ and $x \in \Pi_{C'} \circ f_{\mathcal{C}}$. Since $\Pi_{C'} \circ f_{\mathcal{C}}$ is an injection, $x = 1$, so $C' \subseteq \{ 1 \}$ which is a contradiction since $C' \in \mathcal{C}$. Therefore $\mathcal{C}$ has a minimum element.

Conversely, assume that $\mathcal{C}$ has a minimum element, say C_m. Let $\{K_i \,/\, i \in I\}$ be a family of skewsemifields and $f : L \to \prod_{i \in I} K_i$ an injective representation of K as a subdirect product of $\{K_i \,/\, i \in I\}$. By Remark 1.42., 2) $\{\text{m-ker}\,\Pi_i \circ f \,/\, i \in I\}$ is a set of a-convex normal subgroup of K. Since f is an injection, $\bigcap_{i \in I} \text{m-ker}\,\Pi_i \circ g = \{1\}$. Suppose that for every $i \in I$, $\text{m-ker}\,\Pi_i \circ f \neq \{1\}$. Then $\{\text{m-ker}\,\Pi_i \circ f \,/\, i \in I\} \subseteq \mathcal{C}$, so $C_m \subseteq \bigcap_{i \in I} \text{m-ker}\,\Pi_i \circ g = \{1\}$ which is a contradiction. Hence there exists a $k \in I$ such that $\text{m-ker}\,\Pi_k \circ f = \{1\}$, so $\Pi_k \circ f$ is an injection. Therefore $\Pi_k \circ f$ is an isomorphism, so K is a subdirectly irreducible skewsemifield. #

Next, we want to show that every skewsemifield is a subdirect product of subdirectly irreducible skewsemifields. First, we need three lemmas.

<u>Lemma 1.69.</u> Let K be a skewsemifield and $x, y \in K^*$ distinct. Let $\mathcal{C} = \{C \,/\, C$ is an a-convex normal subgroup of K and $xy^{-1} \notin C\}$. Then $\mathcal{C}$ has a maximal element.

<u>Proof</u> Since $\{1\} \in \mathcal{C}$, $\mathcal{C} \neq \varnothing$. Let $\mathcal{D}$ be a nonempty chain of $\mathcal{C}$. Then $\cup\mathcal{D}$ is an upper bound of $\mathcal{D}$ and $\cup\mathcal{D} \in \mathcal{C}$. By Zorn's Lemma, $\mathcal{C}$ has a maximal element. #

<u>Lemma 1.70.</u> Using the same assumptions of Lemma 1.69., let M be a maximal element in $\mathcal{C}$. Let $\mathbb{A} = \{C \,/\, C$ is an a-convex normal subgroup of K and $M \subset C\}$. Then $\mathbb{A}$ has a minimum element.

<u>Proof</u> Since $K^* \in \mathbb{A}$, $\mathbb{A} \neq \varnothing$. If there exists a $C \in \mathbb{A}$ such that $xy^{-1} \notin C$ then it contradicts to the maximality of M. Then for every $C \in \mathbb{A}$, $xy^{-1} \in C$, so $\cap\mathbb{A}$ is an a-convex normal subgroup of K which is the minimum element of $\mathbb{A}$. #

Lemma 1.71. Using the same assumptions of Lemma 1.69., let M be a maximal element in $\mathcal{C}$. $K_{/M}$ is a subdirectly irreducible skewsemifield.

Proof Let $\mathcal{D}$ be the set of all a-convex normal subgroups of $K_{/M}$ except { M }. By Corollary 1.47., $\mathcal{D}$ is isomorphic to the set for all a-convex normal subgroups of K strictly containing M. By Lemma 1.70., $\mathcal{D}$ has a minimum element. By Proposition 1.68., $K_{/M}$ is a subdirectly irreducible skewsemifield. #

Theorem 1.72. Let K be a skewsemifield. Then K is a subdirect product of subdirectly irreducible skewsemifields.

Proof If $|K| = 2$ then done. Suppose that $|K| > 2$. By Lemma 1.69., there exists an a-convex normal subgroup C_{xy} of K such that $xy^{-1} \notin C_{xy}$ for all $x, y \in K^*$ where $x \neq y$. By Lemma 1.71., $K_{/C_{xy}}$ is a subdirectly irreducible skewsemifield for all $x, y \in K^*$ such that $x \neq y$.

Let $\mathcal{C} = \{ C_{xy} \ / \ x, y \in K^* \text{ and } x \neq y \}$. Let $x \in \cap\mathcal{C}$. If $x \neq 1$ then $x \notin C_{x1}$ which is a contradiction since $x \in \cap\mathcal{C}$. Hence $\cap\mathcal{C} = \{ 1 \}$. By Proposition 1.67., $f_{\mathcal{C}} : K \rightarrow \prod_{C \in \mathcal{C}} K_{/C}$ is an injective representation of K as a subdircet product of $\{ K_{/C} \ / \ C \in \mathcal{C} \}$. Therefore $f_{\mathcal{C}}(K)$ is a subdirect product of $\{ K_{/C} \ / \ C \in \mathcal{C} \}$. By $K \cong f_{\mathcal{C}}(K)$, K is a subdirect product of subdirectly irreducible skewsemifields. #

We cannot generalize the last theorem to positively ordered skewsemifields. It has been done for semifeilds (i.e. both multiplication and addition are assumed to be commutative) in [3]. We proved it here because we felt that it is interesting in the theory of skewsemifields.