# CHAPTER II

# REGRESSIVE TRANSFORMATION SEMIGROUPS

Throughout this chapter, let $X$ denote a partially ordered set.

## 2.1 Regular Elements of Regressive Transformation Semigroups

In this section, we show that every regular element of a regressive transformation semigroup $S$ on $X$ is an idempotent of $S$ and then a necessary and sufficient condition for an element of $S$ to be an idempotent is given. The regularity of each of $PT_{RE}(X)$, $T_{RE}(X)$, $I_{RE}(X)$, $U_{RE}(X)$, $V_{RE}(X)$ and $W_{RE}(X)$ is also characterized in term of $X$.

**Theorem 2.1.1.** Let $S$ be a regressive transformation semigroup on $X$. Then every regular element of $S$ is an idempotent.

**Proof.** Let $\alpha \in S$ be a regular element of $S$. Then there exists an element $\beta \in S$ such that $\alpha = \alpha\beta\alpha$. Let $x \in \Delta\alpha$. Since $\alpha$ and $\beta$ are regressive, $x\alpha = x\alpha\beta\alpha \le x\alpha\beta \le x\alpha$. This implies that $x\alpha = x\alpha\beta$, so $x\alpha = x\alpha\beta\alpha = (x\alpha\beta)\alpha$ $= (x\alpha)\alpha = x\alpha^2$. This proves that $\Delta\alpha \subseteq \Delta\alpha^2$ and $x\alpha = x\alpha^2$ for all $x \in \Delta\alpha$. But $\Delta\alpha^2 \subseteq \Delta\alpha$, so $\alpha = \alpha^2$. Hence $\alpha$ is an idempotent of $S$. $\square$

**Theorem 2.1.2.** Let $S$ be a regressive transformation semigroup on $X$ and $\alpha \in S$. Then $\alpha$ is an idempotent of $S$ if and only if for every $a \in \nabla\alpha$, $a = \min(a\alpha^{-1})$.

**Proof.** Assume that $\alpha$ is an idempotent of $S$. Then $\nabla\alpha \subseteq \Delta\alpha$ and $x\alpha = x$ for all $x \in \nabla\alpha$. Let $a \in \nabla\alpha$. Then $a\alpha = a$, so $a \in a\alpha^{-1}$. If $x \in a\alpha^{-1}$, then $a = x\alpha$, so $a = x\alpha \le x$ since $\alpha$ is regressive. This proves that $a$ is the minimum element of $a\alpha^{-1}$.

Conversely, assume that $a=\min\left(a\alpha^{-1}\right)$ for all $a\in\nabla\alpha$. Then for every $a\in\nabla\alpha$, $a\in a\alpha^{-1}\subseteq\Delta\alpha$. Thus $\nabla\alpha\subseteq\Delta\alpha$. Since for every $a\in\nabla\alpha$, $a\in a\alpha^{-1}$, we have that $a\alpha=a$ for all $a\in\nabla\alpha$. Hence $\alpha$ is an idempotent of $S$, as required. □

**Corollary 2.1.3.** Let $S$ be a regressive transformation semigroup on $X$ and $\alpha\in S$. Then $\alpha$ is a regular element of $S$ if and only if for every $a\in\nabla\alpha$, $a=\min\left(a\alpha^{-1}\right)$.

**Proof.** It follows from Theorem 2.1.1 and Theorem 2.1.2. □

**Lemma 2.1.4.** Let $S$ be a regressive transformation semigroup on $X$ and $\alpha\in S$. If $x\in\Delta\alpha$ is a minimal element of $X$, then $x\alpha=x$.

**Proof.** Since $\alpha$ is regressive, $x\alpha\leq x$. But $x$ is a minimal element of $X$, so $x\alpha=x$. □

**Lemma 2.1.5.** Let $S$ be a regressive transformation semigroup on $X$. If $X$ is isolated, then for every $\alpha\in S$, $\alpha=1_{\Delta\alpha}$.

Hence if $X$ is isolated, then $S$ is a regular semigroup.

**Proof.** Let $\alpha\in S$. Since $X$ is isolated, every element of $X$ is a minimal element of $X$. By Lemma 2.1.4, $x\alpha=x$ for all $x\in\Delta\alpha$. Then $\alpha=1_{\Delta\alpha}$. □

**Theorem 2.1.6.** Let $S$ be $PT_{RE}(X)$, $I_{RE}(X)$, $U_{RE}(X)$ or $W_{RE}(X)$. Then $S$ is a regular semigroup if and only if $X$ is isolated.

**Proof.** If $X$ is isolated, then by Lemma 2.1.5, S is a regular semigroup.

Suppose that $X$ is not isolated. Then there exist $a,b\in X$ such that $a<b$. Define the partial transformation $\alpha$ of $X$ by $\Delta\alpha=\{b\}$ and $\nabla\alpha=\{a\}$. Then $\alpha\in S$. Since $\nabla\alpha\nsubseteq\Delta\alpha$, $\alpha$ is not an idempotent of $S$. By Theorem2.1.1, $\alpha$ is not a regular element of $S$. Hence $S$ is not a regular semigroup. □

**Theorem 2.1.7.** Let $S$ be $T_{RE}(X)$ or $V_{RE}(X)$. Then $S$ is a regular semigroup if and only if for every chain $C$ of $X$, $|C| \le 2$.

**Proof.** Assume that $X$ contains a chain of three elements. Then $X$ has elements $a,b$ and $c$ such that $a < b < c$. Define $\alpha : X \to X$ by $a\alpha = b\alpha = a$, $c\alpha = b$ and $x\alpha = x$ for all $x \in X \setminus \{a,b,c\}$. Then $\alpha \in S$. Since $b \in \nabla\alpha$ and $b\alpha = a \ne b$, it follows that $\alpha$ is not an idempotent of $S$. By Theorem 2.1.1, $\alpha$ is not a regular element of $S$. Therefore $S$ is not a regular semigroup.

Conversely, assume that every chain $C$ of $X$, $|C| \le 2$. Then for $a,b,c \in X$, $a \le b \le c$ implies that $a = b$ or $b = c$. Let $\alpha \in S$ and $x \in X$. Since $\alpha$ is regressive, $x\alpha^2 \le x\alpha \le x$. Then $x\alpha^2 = x\alpha$ or $x\alpha = x$ which implies that $x\alpha^2 = x\alpha$. This proves that $\alpha^2 = \alpha$, so $\alpha$ is regular. Hence $S$ is a regular semigroup. □

## 2.2 Eventual Regularity of $PT_{RE}(X)$, $T_{RE}(X)$ and $I_{RE}(X)$

In this section, we prove that the condition of having a positive integer $n$ such that $|C| \le n$ for every chain $C$ of $X$ is a necessary and sufficient condition for $S$ to be eventually regular where $S$ is $PT_{RE}(X)$, $T_{RE}(X)$ or $I_{RE}(X)$.

**Theorem 2.2.1.** If $X$ contains a sequence of disjoint finite chains $C_1, C_2, C_3, \ldots$ such that $|C_1| < |C_2| < |C_3| < \ldots$, then $PT_{RE}(X)$, $T_{RE}(X)$ and $I_{RE}(X)$ are not eventually regular.

**Proof.** For each $i \in \mathbf{N}$, let

$$C_i = \{x_{i1}, x_{i2}, \ldots, x_{ik_i}\} \text{ where } x_{i1} < x_{i2} < \ldots < x_{ik_i}.$$

Without loss of generality, we may assume that $|C_1| > 1$, otherwise we consider the sequence $C_2, C_3, C_4, \ldots$ instead. Define the partial transformation $\alpha$ of $X$ by

$$x_{ij}\alpha = x_{i,j-1} \text{ for all } i \in \mathbf{N} \text{ and } j \in \{2,3,\ldots,k_i\}.$$

Then $\alpha$ is one-to-one and regressive, so $\alpha \in PT_{RE}(X)$ and $I_{RE}(X)$. We also have that

for all $n, i \in \mathbf{N}$, $n < k_i$ implies that $x_{ik_i}\alpha^n = x_{i,k_i-n}$ .....(*)

Define $\overline{\alpha} : X \to X$ by

$$x\overline{\alpha} = \begin{cases} x\alpha & \text{if} \quad x = x_{ij} \quad \text{for some } i \in N \text{ and } j \in \{2,3,\dots,k_i\}, \\ x & \text{otherwise.} \end{cases}$$

Since $\alpha$ is regressive, $\overline{\alpha}$ is regressive. Thus $\overline{\alpha} \in T_{RE}(X)$.

Let $S$ be $PT_{RE}(X)$, $T_{RE}(X)$ or $I_{RE}(X)$ and let

$$\beta = \begin{cases} \alpha & \text{if } S = PT_{RE}(X) \text{ or } I_{RE}(X), \\ \overline{\alpha} & \text{if } S = T_{RE}(X). \end{cases}$$

Let $m \in \mathbf{N}$. Since $(k_1, k_2, k_3, \dots)$ is a strictly increasing sequence of positive integers, $k_j > 2m$ for some $j \in \mathbf{N}$. By (*) and the definition of $\beta$, we have that $x_{jk_j}\beta^m = x_{jk_j}\alpha^m = x_{j,k_j-m}$ and $x_{jk_j}\beta^{2m} = x_{jk_j}\alpha^{2m} = x_{j,k_j-2m}$. Since $k_j - m \neq k_j - 2m$, $x_{j,k_j-m} \neq x_{j,k_j-2m}$. Then $\beta^m \neq \beta^{2m}$. This proves that $\beta^n$ is not an idempotent of $S$ for every $n \in \mathbf{N}$. By Theorem 2.1.1, $\beta^n$ is not regular in $S$ for every $n \in \mathbf{N}$. Hence $\beta$ is not eventually regular in $S$, and so $S$ is not an eventually regular semigroup. $\square$

**Lemma 2.2.2.** If $X$ contains an infinite chain, then there exists a sequence of disjoint finite chains $C_1, C_2, C_3, \dots$ of $X$ such that $|C_1| < |C_2| < |C_3| < \dots$.

**Proof.** Let $Y$ be an infinite chain of $X$. Let $x_{11} \in Y$ and $C_1 = \{x_{11}\}$. Since $Y$ is infinite, $Y \setminus C_1$ is infinite. Then there exist $x_{21}$ and $x_{22}$ in $Y \setminus C_1$ such that $x_{21} \neq x_{22}$. Since $Y$ is a chain, we may assume that $x_{21} < x_{22}$. Let $C_2 = \{x_{21}, x_{22}\}$. Then $C_1 \cap C_2 = \varnothing$ and $|C_1| < |C_2|$. Again, since $Y$ is infinite, $Y \setminus (C_1 \cup C_2)$ is infinite. Then $Y \setminus (C_1 \cup C_2)$ contains distinct elements $x_{31}, x_{32}$ and $x_{33}$. We may assume that $x_{31} < x_{32} < x_{33}$ since $Y$ is a chain. Therefore we have the disjoint finite chains $C_1, C_2$ and $C_3$ such that $|C_1| < |C_2| < |C_3|$. By this process, we obtain a sequence of disjoint finite chains $C_1, C_2, C_3, \ldots$ such that $|C_1| < |C_2| < |C_3| < \ldots$, as required. □

**Theorem 2.2.3.** If $X$ contains an infinite chain, then $PT_{RE}(X)$, $T_{RE}(X)$ and $I_{RE}(X)$ are not eventually regular.

**Proof.** It follows from Theorem 2.2.1 and Lemma 2.2.2. □

**Theorem 2.2.4.** Let $S$ be $PT_{RE}(X)$, $T_{RE}(X)$ or $I_{RE}(X)$. Then $S$ is eventually regular if and only if there exists a positive integer $n$ such that $|C| \le n$ for every chain $C$ of $X$.

**Proof.** First, assume that there exists a positive integer $n$ such that $|C| \le n$ for every chain $C$ of $X$. To show that $S$ is eventually regular, let $\alpha \in S$. Let $x \in \Delta\alpha^n$. Then $x \in \Delta\alpha^i$ for all $i \in \{1,2,\ldots,n\}$. Since $\alpha$ is regressive, $x \ge x\alpha \ge x\alpha^2 \ge \ldots \ge x\alpha^n$. Then $\{x, x\alpha, \ldots, x\alpha^n\}$ is a chain of $X$. By assumption, $|\{x, x\alpha, \ldots, x\alpha^n\}| \le n$. Then $x\alpha^j = x\alpha^{j+1}$ for some $j \in \{0,1,\ldots,n-1\}$ where $x\alpha^0 = x$. This implies that $x\alpha^{n-1} = x\alpha^n$. Since $x \in \Delta\alpha^n$, $x\alpha^{n-1} \in \Delta\alpha$, so $x\alpha^n \in \Delta\alpha$. Then $(x\alpha^{n-1})\alpha = (x\alpha^n)\alpha$. Therefore $x\alpha^n = x\alpha^{n+1}$, so we have that $x \in \Delta\alpha^{n+1}$. But $\Delta\alpha^{n+1} \subseteq \Delta\alpha^n$, so $\Delta\alpha^n = \Delta\alpha^{n+1}$. This proves that $\Delta\alpha^n = \Delta\alpha^{n+1}$ and $x\alpha^n = x\alpha^{n+1}$ for all $x \in \Delta\alpha^n$. Hence $\alpha^n = \alpha^{n+1}$. Consequently, $\alpha^n$ is an idempotent of $S$, so $\alpha$ is an eventually regular element of $S$. Therefore $S$ is an eventually regular semigroup.

Conversely, suppose that there is no positive integer $n$ such that $|C| \le n$ for every chain $C$ of $X$. Then for each $n \in \mathrm{N}$, there exists a chain $C$ of $X$ such that $|C| > n$. Let $C_1$ be a finite chain of $X$. If $X \setminus C_1$ does not contain a finite chain $C$ of $X$ such that $|C| > |C_1|$, then for every finite chain $C$ of $X$, $|C| \le 2|C_1|$ which is a contradiction. Let $C_2 \subseteq X \setminus C_1$ be a finite chain of $X$ and $|C_2| > |C_1|$. Then $C_1$ and $C_2$ are disjoint. If $X \setminus (C_1 \cup C_2)$ does not contain a finite chain $C$ of $X$ such that $|C| > |C_2|$, then for every chain $C$ of $X$, $|C| < 3|C_2|$, a contradiction. Hence by this inductive construction, we have a sequence of disjoint finite chains $C_1, C_2, C_3, \ldots$ of $X$ such that $|C_1| < |C_2| < |C_3| < \ldots$. By Theorem 2.2.1, $S$ is not eventually regular. □

**Example.** Let $X$ be a nonempty subset of R. If $X$ is finite, then $PT_{RE}(X)$, $T_{RE}(X)$ and $I_{RE}(X)$ are finite, so they are all eventually regular. If $X$ is infinite, by Theorem 2.2.3, each of $PT_{RE}(X)$, $T_{RE}(X)$ and $I_{RE}(X)$ is not eventually regular.

**Example.** Let $A$ be a nonempty subset of R. Let $X = A \times A$ and let $\leq$ be the dictionary partial order on $X$, that is, for $a,b,c,d \in A$,

$$(a,b) \leq (c,d) \text{ if and only if } (i)\ a < c \text{ or } (ii)\ a = c \text{ and } b \leq d.$$

Then $(X,\leq)$ is a chain. Hence $PT_{RE}(X)$, $T_{RE}(X)$ and $I_{RE}(X)$ are eventually regular if and only if $A$ is finite by Theorem 2.2.3 and the fact that every finite semigroup is eventually regular.

**Example.** Let $X$ be the set of all sequences $(x_n)_{n \in N}$ of positive integers. Then $X$ is infinite. Define a partial order $\leq$ on $X$ by

$$(x_n) \leq (y_n) \text{ if and only if } x_n \leq y_n \text{ for all } n \in \mathbf{N}.$$

Then $(X,\leq)$ is an infinite partially ordered set. The elements $(1,2,2,2,\ldots)$ and $(2,1,1,1,\ldots)$ belong to $X$ which are not comparable. Then $(X,\leq)$ is not a chain. For each $i \in \mathbf{N}$, let $a^{(i)}$ be the element $(x_n)_{n \in N}$ of $X$ defined by

$$x_n = \begin{cases} 2 & \text{if } n \in \{1,2,\ldots,i\}, \\ 1 & \text{if } n \in \{i+1, i+2, \ldots\}. \end{cases}$$

Then $a^{(1)} < a^{(2)} < a^{(3)} < \dots$, so $\{a^{(1)}, a^{(2)}, a^{(3)}, \dots\}$ is an infinite chain of $X$. By Theorem 2.2.3, each of $PT_{RE}(X)$, $T_{RE}(X)$ and $I_{RE}(X)$ is not eventually regular.

**Example.** Let $X = \{(x,y) \in \mathrm{N} \times \mathrm{N} / x \le y\}$. Define a partially order $\le$ on $X$ by

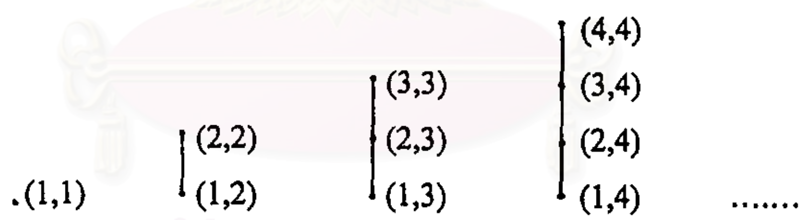
$$(a,b) \le (c,d) \text{ if and only if } a \le c \text{ and } b = d . \quad \dots\dots (*)$$

Since $(1,2),(2,3) \in X$, $(1,2) \not\le (2,3)$ and $(2,3) \not\le (1,2)$, $X$ is an infinite partially ordered set which is not a chain. Let $C$ be a chain of $X$. By $(*)$, there exists a positive integer $m$ such that for $(a,b) \in C$, $b = m$. Again by $(*)$, $C \subsetneq \{(1,m),(2,m),\dots,(m,m)\}$. Hence $C$ is finite. This proves that $X$ does not contain an infinite chain. For $n \in \mathrm{N}$, let

$$C_n = \{(1,n),(2,n),\dots,(n,n)\}.$$

Then for every $n \in \mathrm{N}$, $C_n$ is a chain of $X$ and $|C_n| = n$. Thus for every $n \in \mathrm{N}$, $C_{n+1}$ is a chain of $X$ such that $|C_{n+1}| = n+1 > n$. By Theorem 2.2.4, all of $PT_{RE}(X)$, $T_{RE}(X)$ and $I_{RE}(X)$ are not eventually regular.

In fact, the partially ordered set $(X,\le)$ can be shown by the following diagram:




**Example.** Let $m \in \mathrm{N} \setminus \{1\}$ and $X$ the set of all sequences $(x_n)_{n\in N}$ such that $x_n \in \{1,2,\dots,m\}$ for all $n \in \mathrm{N}$. Define a partial order $\le$ on $X$ by

$$(x_n)_{n\in N} \le (y_n)_{n\in N} \text{ if and only if (i) } x_n \le y_n \text{ for all } n \in \mathrm{N} \text{ and (ii) } x_n = y_n \text{ for all } n \ge m \quad \dots\dots (*)$$

Since $(1,1,1,\dots)$, $(2,2,2,\dots) \in X$ and they are not comparable, $(X,\le)$ is an infinite partially ordered set which is not a chain.

Let $\overline{m} = m^{m-1}$. Claim that for every chain $C$ of $X$, $|C| \le \overline{m}$. Let $C$ be a chain of $X$ and let $(a_n)_{n\in N} \in C$. By $(*)$, for every $(x_n)_{n\in N} \in C$, $x_n = a_n$ for all

$n \geq m$. Then $C$ is a subset of $\{(x_n)_{n\in N} \in X / x_n = a_n$ for all $n \geq m\}$. But $|\{(x_n)_{n\in N} \in X \mid x_n = a_n \text{ for all } n \geq m\}| = m^{m-1} = \overline{m}$, so $|C| \leq \overline{m}$. Hence by Theorem 2.2.4, all of $PT_{RE}(X)$; $T_{RE}(X)$ and $I_{RE}(X)$ are eventually regular semigroups.

## 2.3 Eventual Regularity of $U_{RE}(X)$, $V_{RE}(X)$ and $W_{RE}(X)$

We show in this section that a regressive transformation semigroup on $X$ in which each element is almost identical is eventually regular, and then we have that all of $U_{RE}(X)$, $V_{RE}(X)$ and $W_{RE}(X)$ are always eventually regular.

**Lemma 2.3.1.** Let $S$ be a regressive transformation semigroup on $X$ such that for every $\alpha \in S$, $\alpha$ is almost identical. Then $S$ is eventually regular.

**Proof.** Let $\alpha \in S$. Since $\alpha$ is almost identical, $s(\alpha)$ is finite. Let $n = |s(\alpha)|$. Let $x \in \Delta\alpha^{n+1}$. Then $x\alpha^i \in \Delta\alpha^{n+1-i}$ for all $i \in \{0,1,\ldots,n\}$ where $x\alpha^0 = x$. Since $\Delta\alpha^{n+1-i} \subseteq \Delta\alpha$ for all $i \in \{0,1,\ldots,n\}$, we have that $x, x\alpha, x\alpha^2, \ldots, x\alpha^n \in \Delta\alpha$. Since $\alpha$ is regressive, $x \geq x\alpha \geq x\alpha^2 \geq \ldots \geq x\alpha^n \geq x\alpha^{n+1}$. If $x > x\alpha > x\alpha^2 > \ldots > x\alpha^n > x\alpha^{n+1}$, then $|\{x, x\alpha, \ldots, x\alpha^n\}| = n+1$ and $\{x, x\alpha, \ldots, x\alpha^n\} \subseteq s(\alpha)$ which is a contradiction since $|s(\alpha)| = n$. Then $x\alpha^i = x\alpha^{i+1}$ for some $i \in \{1,2,\ldots,n\}$ which implies that $x\alpha^n = x\alpha^{n+1}$. Since $x\alpha^n \in \Delta\alpha$, $x\alpha^{n+1} \in \Delta\alpha$ and so $x\alpha^{n+1} = x\alpha^{n+2}$. This proves that $\Delta\alpha^{n+1} \subseteq \Delta\alpha^{n+2}$ and $x\alpha^{n+1} = x\alpha^{n+2}$ for all $x \in \Delta\alpha^{n+1}$. But $\Delta\alpha^{n+2} \subseteq \Delta\alpha^{n+1}$, so $\alpha^{n+1} = \alpha^{n+2}$. Hence $\alpha^{n+1}$ is an idempotent of $S$. Thus $\alpha$ is eventually regular. Therefore $S$ is an eventually regular semigroup. □

**Theorem 2.3.2.** The regressive transformation semigroups $U_{RE}(X)$, $V_{RE}(X)$ and $W_{RE}(X)$ are eventually regular.

**Proof.** It follows directly from Lemma 2.3.1. □

## 2.4 Eventual Regularity of $M_{RE}(X)$ and $E_{RE}(X)$

In this section, the condition that every chain of $X$ has a minimum element is a necessary and sufficient condition for $M_{RE}(X)$ to be eventually regular and it implies that $M_{RE}(X)=\{1_X\}$. Also, we show that a necessary and sufficient condition for $E_{RE}(X)$ to be eventually regular is that every chain of $X$ has a maximum element and this forces $E_{RE}(X)$ to be trivial.

**Lemma 2.4.1.** (1) If $X$ is a chain without a minimum element, then $X$ has a chain of the form

$$\{x_n / n \in \mathbf{Z}^-\} \text{ where } x_i > x_{i-1} \text{ for all } i \in \mathbf{Z}^-.$$

(2) If $X$ is a chain without maximum element, then $X$ has a chain of the form

$$\{x_n / n \in \mathbf{N}\} \text{ where } x_{i+1} > x_i \text{ for all } i \in \mathbf{N}.$$

**Proof.** (1) Assume that $X$ is a chain and $X$ has no minimum element. Let $x_{-1} \in X$. Then $x_{-1}$ is not a minimum element of $X$, so there exists an element $x_{-2} \in X \setminus \{x_{-1}\}$ such that $x_{-1} \nless x_{-2}$. Since $X$ is a chain, $x_{-2} < x_{-1}$. Again, by assumption, $x_{-2}$ is not a minimum element of $X$, so there exists an element $x_{-3} \in X \setminus \{x_{-2}\}$ such that $x_{-2} \nless x_{-3}$. Since $X$ is a chain, $x_{-3} < x_{-2}$. Now, we have $x_{-3} < x_{-2} < x_{-1}$. By continuing this process inductively, we obtain a chain $\{x_n / n \in \mathbf{Z}^-\}$ of $X$ where $x_i > x_{i-1}$ for all $i \in \mathbf{Z}^-$, as required.

(2) The proof of (2) can be given similarly to that of (1). $\square$

**Lemma 2.4.2.** (1) If $X = \{x_n / n \in \mathbf{Z}^-\}$ where $x_i > x_{i-1}$ for all $i \in \mathbf{Z}^-$ and $\alpha : X \to X$ is defined by $x_i\alpha = x_{i-1}$ for all $i \in \mathbf{Z}^-$, then $\alpha \in M_{RE}(X)$ and $\alpha^n \neq \alpha^{2n}$ for all $n \in \mathbf{N}$.

(2) If $X = \{x_n / n \in \mathbf{N}\}$ where $x_{i+1} > x_i$ for all $i \in \mathbf{N}$ and $\alpha : X \to X$ is defined by $x_1\alpha = x_1$ and $x_{i+1}\alpha = x_i$ for all $i \in \mathbf{N}$, then $\alpha \in E_{RE}(X)$ and $\alpha^n \neq \alpha^{2n}$ for all $n \in \mathbf{N}$.

**Proof.** (1) Assume that $X$ and $\alpha$ satisfy the assumption of (1). It is clear that $\alpha \in M_{RE}(X)$. We have that for $n \in \mathbf{N}$ and $i \in \mathbf{Z}^-$, $x_i\alpha^n = x_{i-n}$. Since $x_{i-n} \neq x_{i-2n}$ for all $i \in \mathbf{Z}^-$ and $n \in \mathbf{N}$, it follows that $\alpha^n \neq \alpha^{2n}$ for all $n \in \mathbf{N}$.

(2) Assume that $X$ and $\alpha$ satisfy the assumption of (2). Then $\alpha$ belongs to $E_{RE}(X)$ and $x_i\alpha = x_{i-1}$ for all $i \in \mathbf{N} \setminus \{1\}$. Then $x_i\alpha^n = x_{i-n}$ for all $i \in \{n+1, n+2, n+3, \ldots\}$ and $n \in \mathbf{N}$. It follows that $\alpha^n \neq \alpha^{2n}$ for all $n \in \mathbf{N}$. □

**Theorem 2.4.3.** The following statements are equivalent.

(1) Every chain of $X$ has a minimum element.

(2) $M_{RE}(X) = \{1_X\}$.

(3) $M_{RE}(X)$ is regular.

(4) $M_{RE}(X)$ is eventually regular.

**Proof.** (1)$\Rightarrow$(2). Assume that (1) holds. Let $\alpha \in M_{RE}(X)$ and $x \in X$. Since $\alpha$ is regressive, $x \geq x\alpha \geq x\alpha^2 \geq \ldots$. Then $\{x\alpha^n / n \in \mathbf{N}\}$ is a chain of $X$. By assumption, there exists a positive integer $k$ such that $x\alpha^k = \min\{x\alpha^n / n \in \mathbf{N}\}$. Since $x\alpha^k \geq x\alpha^{k+1}$, $x\alpha^k = x\alpha^{k+1} = (x\alpha)\alpha^k$. But $\alpha^k$ is one-to-one, so we have that $x\alpha = x$. This proves that $\alpha = 1_X$. Then (2) is obtained.

(2)$\Rightarrow$(3). Trivial.

(3)$\Rightarrow$(4). Trivial.

(4)$\Rightarrow$(1). Suppose that (1) is not true. Then there exists a chain $Y$ of $X$ such that $Y$ has no minimum element. By Lemma 2.4.1.(1), there exists a chain

$C=\{x_n / n\in \mathbf{Z}^-\}$ of $Y$ with $x_i > x_{i-1}$ for all $i\in \mathbf{Z}^-$. Define $\alpha : C\to C$ by $x_i\alpha = x_{i-1}$ for all $i\in\mathbf{Z}^-$. By Lemma 2.4.2.(1), $\alpha\in M_{RE}(C)$ and $\alpha^n \neq \alpha^{2n}$ on $C$ for all $n\in\mathbf{N}$. Extend $\alpha$ to $\beta : X\to X$ by defining $\beta$ as follows :

$$x\beta=\begin{cases} x\alpha & if \quad x\in C,\\ x & if \; x\in X\setminus C.\end{cases}$$

Since $\alpha$ is one-to-one and regressive, $\beta$ is also one-to-one and regressive. Then $\beta\in M_{RE}(X)$. Since $\alpha^n\neq\alpha^{2n}$ on $C$ for every $n\in\mathbf{N}$ and $\alpha$ is the restriction of $\beta$ to $C$, we have that $\beta^n\neq\beta^{2n}$ on $X$ for all $n\in\mathbf{N}$. Hence $\beta^n$ is not an idempotent for every $n\in\mathbf{N}$. By Theorem 2.1.1, $\beta^n$ is not regular for all $n\in\mathbf{N}$. Hence $\beta$ is not an eventually regular element of $M_{RE}(X)$, so (4) is not true. $\square$

**Theorem 2.4.4.** The following statements are equivalent.

(1) Every chain of $X$ has a maximum element.

(2) $E_{RE}(X)=\{1_X\}$.

(3) $E_{RE}(X)$ is regular.

(4) $E_{RE}(X)$ is eventually regular.

**Proof.** (1)$\Rightarrow$(2). Assume that every chain of $X$ has a maximum element. To show that $E_{RE}(X)=\{1_X\}$, let $\alpha\in E_{RE}(X)$ and $x\in X$. Let $x_1=x$. Since $\alpha$ is onto, there exists an element $x_2\in X$ such that $x_2\alpha=x_1$. Since $\alpha$ is regressive, $x_1=x_2\alpha\le x_2$. Since $\alpha$ is onto, there exists an element $x_3\in X$ such that $x_3\alpha=x_2$. Since $\alpha$ is regressive, $x_2=x_3\alpha\le x_3$. Then we have that $x_1\le x_2\le x_3$, $x_2\alpha=x_1$ and $x_3\alpha=x_2$. By this inductive process, we obtain a sequence $x_1,x_2,x_3,...$ of $X$ such that $x_1\le x_2\le x_3\le ...$ and $x_{n+1}\alpha=x_n$ for all $n\in\mathbf{N}$. It follows by induction that

$$x_{n+1}\alpha^n=x_1 \text{ for every } n\in\mathbf{N} \qquad .....(*)$$

Since $\{x_n / n\in\mathbf{N}\}$ is a chain of $X$, by assumption, there exists a positive integer $m$ such that $x_n\le x_m$ for all $n\in\mathbf{N}$. Then $x_n=x_m$ for all $n\ge m$. In particular, $x_{m+1}=x_m$. Therefore $x_m=x_{m+1}\alpha=x_m\alpha$ which implies by induction that

$$x_m\alpha^n = x_m \text{ for all } n \in \mathbf{N} \quad \ldots\ldots(**)$$

If $m=1$, then $x_1\alpha = x_1$. If $m>1$, then by $(*)$ and by $(**)$, $x_1 = x_m\alpha^{m-1} = x_m$, so $x_1\alpha = x_1$ since $x_m\alpha = x_m$. Hence $x\alpha = x$. This proves that $\alpha = 1_X$.

(2)$\Rightarrow$(3). Trivial.

(3)$\Rightarrow$(4). Trivial.

(4)$\Rightarrow$(1). Assume that (1) is not true. Then there exists a chain $Y$ of $X$ such that $Y$ has no maximum element. By Lemma 2.4.1.(2), there exists a chain $C = \{x_n / n \in \mathbf{N}\}$ of $Y$ with $x_i < x_{i+1}$ for all $i \in \mathbf{N}$. Define $\alpha : C \to C$ by $x_1\alpha = x_1$ and $x_{i+1}\alpha = x_i$ for all $i \in \mathbf{N}$. By Lemma 2.4.2.(2), $\alpha \in E_{RE}(C)$ and $\alpha^n \neq \alpha^{2n}$ on $C$ for all $n \in \mathbf{N}$. Extend $\alpha$ to $\beta : X \to X$ by

$$x\beta = \begin{cases} x\alpha & \text{if} \quad x \in C, \\ x & \text{if } x \in X \setminus C. \end{cases}$$

Since $\alpha$ is regressive and $\nabla\alpha = C$, $\nabla\beta = X$ and $\beta$ is regressive. Then $\beta \in E_{RE}(X)$. Since $\alpha^n \neq \alpha^{2n}$ on $C$ for every $n \in \mathbf{N}$ and $\alpha$ is the restriction of $\beta$ to $C$, we have that $\beta^n \neq \beta^{2n}$ on $X$ for all $n \in \mathbf{N}$. Hence $\beta^n$ is not an idempotent for every $n \in \mathbf{N}$. By Theorem 2.1.1, $\beta^n$ is not a regular element of $E_{RE}(X)$ for every $n \in \mathbf{N}$. Hence $\beta$ is not eventually regular. Therefore (4) is not true. □

**Example.** By Theorem 2.4.3, $M_{RE}(\mathbf{N}) = \{1_N\}$ but $M_{RE}(\mathbf{Z}^-)$ and $M_{RE}(\mathbf{Z})$ are not eventually regular.

By Theorem 2.4.4, $E_{RE}(\mathbf{Z}^-) = \{1_{Z^-}\}$ but $E_{RE}(\mathbf{N})$ and $E_{RE}(\mathbf{Z})$ are not eventually regular.

**Example.** Let $a, b \in \mathbf{R}$ be such that $a < b$. Let $I$ be the interval $(a,b)$, $[a,b)$, $(a,b]$ or $[a,b]$. Then $(a,b)$ is a chain of $I$ and $(a,b)$ has neither a maximum element nor a minimum element. Then by Theorem 2.4.3 and Theorem 2.4.4, both $M_{RE}(I)$ and $E_{RE}(I)$ are not eventually regular.