INTRODUCTION

The following transformation semigroups on any set X are well-known :

$PT(X)$: the partial transformation semigroup on X,

$T(X)$: the full transformation semigroup on X,

$I(X)$: the 1-1 partial transformation semigroup on X (the symmetric inverse semigroup on X),

$U(X)$: the semigroup of all almost identical partial transformations of X,

$V(X)$: the semigroup of all almost identical transformations of X and

$W(X)$: the semigroup of all almost identical 1-1 partial transformations of X.

All the above transformation semigroups are regular for every set X. Since every regular semigroup is eventually regular, these semigroups are eventually regular.

Transformation semigroups on partially ordered sets have long been studied. For a partially ordered set X, let

$PT_{RE}(X)$ = the regressive partial transformation semigroup on X,

$T_{RE}(X)$ = the full regressive transformation semigroup on X,

$I_{RE}(X)$ = the regressive 1-1 partial transformation semigroup on X,

$U_{RE}(X)$ = the semigroup of all regressive almost identical partial transformations of X,

$V_{RE}(X)$ = the semigroup of all regressive almost identical transformations of X and

$W_{RE}(X)$ = the semigroup of all regressive almost identical 1-1 partial transformations of X.

For a partially ordered set X, $PT_{RE}(X)$, $T_{RE}(X)$, $I_{RE}(X)$, $U_{RE}(X)$, $V_{RE}(X)$ and $W_{RE}(X)$ are subsemigroups of $PT(X)$, $T(X)$, $I(X)$, $U(X)$, $V(X)$ and $W(X)$, respectively. However, for a partially ordered set X, $PT_{RE}(X)$, $T_{RE}(X)$, $I_{RE}(X)$, $U_{RE}(X)$, $V_{RE}(X)$ and $W_{RE}(X)$ need not be regular. Since every finite semigroup is eventually regular, it follows that if X is a finite partially ordered set, then they are all eventually regular.

It was proved by A.Umar in [2] that if X is a finite chain, then the semigroup

$$S = \{\alpha \in T_{RE}(X) / |im(\alpha)| < |X|\}$$

is generated by idempotents and S is not regular if $|X| \geq 3$. A.Umar also proved in [3] that for chains X and Y, $T_{RE}(X)$ and $T_{RE}(Y)$ are isomorphic if and only if X and Y are order-isomorphic. In [1], T.Saito, K.Aoki and K.Kajitori gave necessary and sufficient conditions on partially ordered sets X and Y for $T_{RE}(X)$ and $T_{RE}(Y)$ to be isomorphic, and the result of A.Umar in [3] mentioned above is a corollary of this result.

The researches on regressive transformation semigroups which we have studied seldom related to their regularity. We can easily prove that every regular element of a regressive transformation semigroup is an idempotent and a characterization of its idempotents can be given. These results are a part of Chapter II. Eventually regular semigroups are a generalization of regular semigroups and periodic semigroups. The first purpose of this research is to give necessary and sufficient conditions for a partially ordered set X such that S is eventually regular where S is $PT_{RE}(X)$, $T_{RE}(X)$ or $I_{RE}(X)$ and to show that $U_{RE}(X)$, $V_{RE}(X)$ and $W_{RE}(X)$ are always eventually regular for any partially ordered set X. These results are given in Chapter II.

If S is a transformation semigroup on a set and $\theta \in S$, let (S,θ) denote the semigroup S with the product $*$ defined by $\alpha * \beta = \alpha\theta\beta$ for all $\alpha, \beta \in S$. The second purpose of this research is to introduce necessary and sufficient conditions of a partially ordered set X and θ such that (S,θ) is eventually regular where S is $PT_{RE}(X)$, $T_{RE}(X)$ or $I_{RE}(X)$ and $\theta \in S$ and to show that if X is a partially ordered set and S is $U_{RE}(X)$, $V_{RE}(X)$ or $W_{RE}(X)$, then for any $\theta \in S$, (S,θ) is eventually regular. This study is given in Chapter III.

Preliminaries for this research are given in Chapter I.

CHAPTER I

PRELIMINARIES

Let S be a semigroup. For $a \in S$, if $a^2 = a$, a is said to be an *idempotent* of S. If $a \in S$ and i is a positive integer such that $a^i = a^{i+1}$, then $a^i = a^{2i}$, so a^i is an idempotent of S. An element a of S is said to be *regular* if $a = aba$ for some $b \in S$ and we call S a *regular semigroup* if every element of S is regular. Then every idempotent of S is a regular element of S. An element a of S is said to be *eventually regular* if there exists a positive integer n such that a^n is a regular element of S. If every element of S is eventually regular, then S is said to be an *eventually regular semigroup.*

For $a \in S$, let $\langle a \rangle$ denote the subsemigroup of S generated by a. Then for $a \in S$,

$$\langle a \rangle = \{a, a^2, a^3, \ldots\}.$$

We call S a *periodic semigroup* if for every $a \in S$, $\langle a \rangle$ is finite. It is known that if $a \in S$ and $a^i = a^j$ for some distinct positive integers i and j, then a^k is an idempotent of S for some positive integer k. Therefore if S is a periodic semigroup, then for every element a in S, there exists a positive integer k such that a^k is an idempotent of S.

Hence every regular semigroup and every periodic semigroup is an eventually regular semigroup. Since every finite semigroup is periodic, we have that every finite semigroup is eventually regular.

Let X be a set. For any set A, let 1_A denote the identity map on A. A *partial transformation* of X is a map from a subset of X into X. The *empty transformation* of X is the partial transformation with empty domain and it is denoted by 0. For a partial transformation α of X, let $\Delta\alpha$ and $\nabla\alpha$ denote the domain and the range(image) of α, respectively. Let $PT(X)$ be the set of all partial transformations of X. For $\alpha, \beta \in PT(X)$, define the product $\alpha\beta$ as follows:

If $\nabla\alpha \cap \Delta\beta = \varnothing$, let $\alpha\beta = 0$. If $\nabla\alpha \cap \Delta\beta \neq \varnothing$, let $\alpha\beta$ be the composition of the maps $\alpha|_{(\nabla\alpha\cap\Delta\beta)\alpha^{-1}}$ and $\beta|_{\nabla\alpha\cap\Delta\beta}$. Then $PT(X)$ is a semigroup having 0 and 1_X as its zero and identity, respectively and for $\alpha, \beta \in PT(X)$, $\Delta\alpha\beta = (\nabla\alpha \cap \Delta\beta)\alpha^{-1} \subseteq \Delta\alpha$, $\nabla\alpha\beta = (\nabla\alpha \cap \Delta\beta)\beta \subseteq \nabla\beta$. The semigroup $PT(X)$ is called the *partial transformation semigroup* on X.

By a *transformation semigroup* on X, we mean a subsemigroup of $PT(X)$.

By a transformation of X, we mean a map of X into itself. Let $T(X)$ be a set of all transformations of X, that is,

$$T(X) = \{\alpha \in PT(X) / \Delta\alpha = X\}.$$

Then $T(X)$ is a subsemigroup of $PT(X)$ containing 1_X and it is called the *full transformation semigroup* on X.

Let $I(X)$ be the set of all 1-1 partial transformations of X, that is,

$$I(X) = \{\alpha \in PT(X) / \alpha \ \textit{is} \ 1-1.\}.$$

Then $I(X)$ is a subsemigroup of $PT(X)$ containing 0 and 1_X, and it is called the *1-1 partial transformation semigroup* or the *symmetric inverse semigroup* on X.

It is well-known that $PT(X)$, $T(X)$ and $I(X)$ are all regular. Also, for $\alpha \in PT(X)$, $\alpha^2 = \alpha$ if and only if $\nabla\alpha \subseteq \Delta\alpha$ and $x\alpha = x$ for all $x \in \nabla\alpha$.

The *shift* of $\alpha \in PT(X)$ is defined to be the set $\{x \in \Delta\alpha / x\alpha \neq x\}$ and it is denoted by $s(\alpha)$. For $\alpha \in PT(X)$, α is said to be *almost identical* if $s(\alpha)$ is finite. Let

$U(X)$ = the set of all almost identical partial transformations of X,

$V(X)$ = the set of all almost identical transformations of X

and

$W(X)$ = the set of all almost identical 1-1 partial transformations of X.

Then

$$U(X) = \{\alpha \in PT(X) / s(\alpha) \text{ is finite.}\},$$

$$V(X) = \{\alpha \in T(X) / s(\alpha) \text{ is finite.}\}$$

and

$$W(X) = \{\alpha \in I(X) / s(\alpha) \text{ is finite.}\}.$$

We have that for $\alpha, \beta \in PT(X)$, $s(\alpha\beta) \subseteq s(\alpha) \cup s(\beta)$. Then $U(X)$, $V(X)$ and $W(X)$ are subsemigroups of $PT(X)$, $T(X)$ and $I(X)$, respectively, $0, 1_X \in U(X), 1_X \in V(X)$ and $0, 1_X \in W(X)$. In fact, $U(X)$, $V(X)$ and $W(X)$ are regular semigroups.

The following notations will be used.

$\mathbf{N}$ = the set of positive integers,

$\mathbf{Z}^-$ = the set of negative integers,

$\mathbf{Z}$ = the set of integers

and

$\mathbf{R}$ = the set of real numbers.

In this research, the partial order on any subset of **R** always means the natural partial order of real numbers.

Let S be a transformation semigroup on X and $\theta \in S$. Let (S, θ) denote the semigroup S with the product $*$ defined by $\alpha * \beta = \alpha\theta\beta$ for all $\alpha, \beta \in S$. We call the semigroup (S, θ) a *generalized transformation semigroup* on X.

Let X be a partially ordered set. An element a of X is said to be an *isolated point* if for every $x \in X$, $x \leq a$ or $x \geq a$ implies $x = a$. X is said to be *isolated* if every point of X is isolated. For $A \subseteq X$, let max(A) and min(A) denote the maximum element and the minimum element of A, respectively if they exist. By a *chain* of X we mean a chain Y such that $Y \subseteq X$ and the partial order of Y is the partial order of X restricted to Y.

For $\alpha \in PT(X)$, α is said to be *regressive* if $x\alpha \leq x$ for all $x \in \Delta\alpha$. A transformation semigroup on X is said to be *regressive* if all of its elements are regressive. For a transformation semigroup $S(X)$ on X, let

$$S_{RE}(X) = \{\alpha \in S(X) / \alpha \text{ is regressive.}\}$$

which is a subsemigroup of $S(X)$ if it is nonempty. Note that 1_X belongs to $PT_{RE}(X)$, $T_{RE}(X)$, $I_{RE}(X)$, $U_{RE}(X)$, $V_{RE}(X)$ and $W_{RE}(X)$, all of $PT_{RE}(X)$, $I_{RE}(X)$, $U_{RE}(X)$ and $W_{RE}(X)$ contain 0,

$$U_{RE}(X)=\{\alpha \in PT_{RE}(X)/\, s(\alpha) \text{ is finite.}\},$$

$$V_{RE}(X)=\{\alpha \in T_{RE}(X)/\, s(\alpha) \text{ is finite.}\}$$

and

$$W_{RE}(X)=\{\alpha \in I_{RE}(X)/\, s(\alpha) \text{ is finite.}\}.$$

For a set X, let $M(X)$ and $E(X)$ denote the semigroup of all 1-1 transformations of X and the semigroup of all onto transformations of X, respectively, so we have that $1_X \in M(X)$ and $1_X \in E(X)$. If X is a partially ordered set, we have that both $M_{RE}(X)$ and $E_{RE}(X)$ contain 1_X.

For any set A, let $|A|$ denote the cardinality of A.