สมาชิกปกติและสมบัติบีคิวของกึ่งกรุปการแปลงและริงของการแปลงเชิงเส้น

นางสาวศันสนีย์ เณรเทียน

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REGULAR ELEMENTS AND THE $\mathcal{BQ}$-PROPERTY OF TRANSFORMATION SEMIGROUPS AND RINGS OF LINEAR TRANSFORMATIONS

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ศันสนีย์ เฌรเทียน : สมาชิกปกติและสมบัติบีคิวของกึ่งกรุปการแปลงและริงของการแปลงเชิงเส้น (REGULAR ELEMENTS AND THE ***BQ***-PROPERTY OF TRANSFORMATION SEMIGROUPS AND RINGS OF LINEAR TRANSFORMATIONS) อ.ที่ปรึกษา : ศ. ดร. ยุพาภรณ์ เข็มประสิทธิ์, 64 หน้า ISBN 974-14-2046-3

เราเรียกสมาชิก x ของกึ่งกรุป [ริง] A ว่าเป็น*สมาชิกปกติ* ถ้ามีสมาชิก y ของ A ซึ่ง $x = xyx$ และเรียก A ว่า*กึ่งกรุปปกติ* [*ริงปกติ (แบบวอนนอยแมน)*] ถ้าทุกๆ สมาชิกของ A เป็นสมาชิกปกติ *ควอซีไอดีล*ของกึ่งกรุป [ริง] A คือกึ่งกรุปย่อย [ริงย่อย] Q ของ A ซึ่ง $AQ \cap QA \subseteq Q$ และ*ไบไอดีล*ของ A คือกึ่งกรุปย่อย [ริงย่อย] B ของ A ซึ่ง $BAB \subseteq B$ ทบทวนว่าสำหรับเซตย่อยไม่ว่าง X และ Y ของริง A XY แทนเซตของผลบวกจำกัดทั้งหมดในรูปแบบ $\sum x_i y_i$ เมื่อ $x_i \in X$ และ $y_i \in Y$ เรากล่าวว่ากึ่งกรุปหรือริงมี*สมบัติบีคิว* ถ้าควอซีไอดีลและไบไอดีลของกึ่งกรุปหรือริงนี้เป็นสิ่งเดียวกัน เป็นที่รู้กันแล้วว่ากึ่งกรุปปกติและริงปกติมีสมบัติบีคิว

สำหรับเซตไม่ว่าง X ให้ $T(X)$ แทนกึ่งกรุปการแปลงเต็มบน X และสำหรับ $\varnothing \neq Y \subseteq X$ ให้ $T(X, Y)$ และ $\overline{T}(X, Y)$ เป็นกึ่งกรุปย่อยของ $T(X)$ ที่กำหนดโดย

$$T(X, Y) = \{ \alpha \in T(X) \mid \operatorname{ran} \alpha \subseteq Y \} \text{ และ } \overline{T}(X, Y) = \{ \alpha \in T(X) \mid Y\alpha \subseteq Y \}$$

ไซมอนส์และมากิลล์แนะนำและศึกษา $T(X, Y)$ และ $\overline{T}(X, Y)$ ในปี 1975 และ 1966 ตามลำดับ

ถ้า V เป็นปริภูมิเวกเตอร์บนฟิลด์ F ให้ $L_F(V)$ เป็นเซตของการแปลงเชิงเส้น $\alpha : V \to V$ ทั้งหมด สำหรับปริภูมิย่อย W ของ V นิยาม $L_F(V, W)$ และ $\overline{L}_F(V,W)$ ในทำนองเดียวกันดังนี้

$$L_F(V, W) = \{ \alpha \in L_F(V) \mid \operatorname{ran} \alpha \subseteq W \} \text{ และ } \overline{L}_F(V,W) = \{ \alpha \in L_F(V) \mid W\alpha \subseteq W \}$$

และนอกจากนี้เราจะพิจารณา

$$K_F(V, W) = \{ \alpha \in L_F(V) \mid W \subseteq \ker \alpha \}$$

ดังนั้น $L_F(V, W)$, $\overline{L}_F(V,W)$ และ $K_F(V, W)$ เป็นกึ่งกรุปย่อยของ $(L_F(V), \circ)$ และริงย่อยของ $(L_F(V), +, \circ)$ โดยที่ $\circ$ และ $+$ เป็นการประกอบและการบวกปกติของการแปลงเชิงเส้น ตามลำดับ

การวิจัยนี้ประกอบด้วยส่วนหลักสองส่วน ในส่วนแรกเราให้เงื่อนไขที่จำเป็นและเพียงพอสำหรับสมาชิกของกึ่งกรุปเหล่านี้ที่จะเป็นสมาชิกปกติ สิ่งที่ตามมาเราให้ลักษณะที่จะบอกว่าเมื่อใดกึ่งกรุปเหล่านี้เป็นกึ่งกรุปปกติ ในส่วนที่สองเราให้เงื่อนไขที่จำเป็นและเพียงพอสำหรับกึ่งกรุปและริงเหล่านี้ที่จะมีสมบัติบีคิว

ภาควิชา ...คณิตศาสตร์... ลายมือชื่อนิสิต ศันสนีย์ เฌรเทียน

สาขาวิชา ...คณิตศาสตร์... ลายมือชื่ออาจารย์ที่ปรึกษา ยุพาภรณ์ เข็มประสิทธิ์

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An element x of a semigroup [ring] A is said to *regular* if there is an element y of A such that $x = xyx$, and A is called a *regular semigroup* [(*Von Neumann*) *regular ring*] if every element of A is regular. A *quasi-ideal* of a semigroup [ring] A is a subsemigroup [subring] Q of A such that $AQ \cap QA \subseteq Q$, and a *bi-ideal* of A is a subsemigroup [subring] B of A such that $BAB \subseteq B$. Recall that for nonempty subsets X and Y of a ring A, XY denotes the set of all finite sums of the form $\sum x_iy_i$ where $x_i \in X$ and $y_i \in Y$. We say that a semigroup or a ring has the ***BQ-property*** if its quasi-ideals and bi-ideals coincide. It is known that every regular semigroup and every regular ring has the ***BQ***-property.

For a nonempty set X, let $T(X)$ denote the full transformation semigroup on X, and for $\varnothing \neq Y \subseteq X$, let $T(X, Y)$ and $\overline{T}(X, Y)$ be the subsemigroups of $T(X)$ defined by

$$T(X, Y) = \{ \alpha \in T(X) \mid \operatorname{ran} \alpha \subseteq Y \} \text{ and } \overline{T}(X, Y) = \{ \alpha \in T(X) \mid Y\alpha \subseteq Y \}.$$

Symons and Magill introduced and studied $T(X, Y)$ and $\overline{T}(X, Y)$ in 1975 and 1966, respectively.

If V is a vector space over a field F, let $L_F(V)$ be the set of all linear transformations $\alpha : V \rightarrow V$. For a subspace W of V, define $L_F(V, W)$ and $\overline{L}_F(V,W)$ analogously as follows :

$$L_F(V, W) = \{ \alpha \in L_F(V) \mid \operatorname{ran} \alpha \subseteq W \} \text{ and } \overline{L}_F(V,W) = \{ \alpha \in L_F(V) \mid W\alpha \subseteq W \},$$

and we also consider

$$K_F(V, W) = \{ \alpha \in L_F(V) \mid W \subseteq \ker \alpha \}.$$

Then $L_F(V, W)$, $\overline{L}_F(V,W)$ and $K_F(V, W)$ are subsemigroups of $(L_F(V), \circ)$ and subrings of $(L_F(V), +, \circ)$ where $\circ$ and $+$ are the composition and usual addition of linear transformations, respectively.

This research consists of two major parts. In the first part, we give necessary and sufficient conditions for the elements of these semigroups to be regular. As a consequence, characterizations determining when these semigroups are regular are given. In the second part, we provide necessary and sufficient conditions for these semigroups and rings to have the ***BQ***-property.

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Academic Year ....2006..............

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CONTENTS

INTRODUCTION

In both semigroups and rings, quasi-ideals are a generalization of one-sided ideals and bi-ideals generalize quasi-ideals. The notion of quasi-ideal was introduced by Steinfeld ([19], [18]) in 1953 and 1956 for rings and semigroups, respectively. The notion of bi-ideal for semigroups was introduced in 1952 by Good and Hughes [4] while the notion of bi-ideal for rings was given much later. It was introduced by Lajos and Szász [14] in 1971.

Kapp [9] used $\mathcal{BQ}$ to denote the class of all semigroups whose bi-ideals and quasi-ideals coincide and Mielke [16] called a semigroup in the class $\mathcal{BQ}$ a $\mathcal{BQ}$-semigroup. The following semigroups were known to be in the class $\mathcal{BQ}$: regular semigroups (Lajos [13]), left [right] simple semigroups (Kapp [9]) and left [right] 0-simple semigroups (Kapp [9]). In fact, Calais [2] proved that a semigroup S belongs to $\mathcal{BQ}$ if and only if the bi-ideal and the quasi-ideal of S generated by any $x, y \in S$ are identical.

This research deals with both semigroups and rings whose their bi-ideals and quasi-ideals are identical. Then we shall say that a semigroup or a ring has the *$\mathcal{BQ}$-property* if its bi-ideals and quasi-ideals coincide, or equivalently, its bi-ideals are quasi-ideals. In fact, from [10], we have that every (Von Neumann) regular ring has the $\mathcal{BQ}$-property. Hence we deduce that in both semigroups and rings, the regularity implies the $\mathcal{BQ}$-property. However, the converse is not generally true. By the definition, a ring $(R, +, \cdot)$ is regular if and only if $(R, \cdot)$ is a regular semigroup. However, this is not true for the $\mathcal{BQ}$-property. It is not difficult to see that for a ring $(R, +, \cdot)$, if the semigroup $(R, \cdot)$ has the $\mathcal{BQ}$-property, then the ring $(R, +, \cdot)$ has the $\mathcal{BQ}$-property. The converse is not true in general. This can be seen in this work.

Some transformation semigroups having the $\mathcal{BQ}$-property have been studied in [11]. In [12] and [17], the authors characterized when their target semigroups of linear tranformations have the $\mathcal{BQ}$-property.

We denote by $T(X)$ the full transformation semigroup on a nonempty set X. It is well-known that $T(X)$ is a regular semigroup. For a nonempty subset Y of X, let

$$T(X,Y) = \{\alpha \in T(X) \mid \text{ran } \alpha \subseteq Y\},$$
$$\overline{T}(X,Y) = \{\alpha \in T(X) \mid Y\alpha \subseteq Y\}.$$

Then $T(X,Y) \subseteq \overline{T}(X,Y)$ and both are subsemigroups of $T(X)$. The semigroup $T(X,Y)$ was introduced and studied by Symons [21] in 1975 while Magill [15] introduced and studied the semigroup $\overline{T}(X,Y)$ in 1966.

The semigroup, under composition, of all linear transformations from a vector space V over a field F into itself is denoted by $L_F(V)$. It is also known that $L_F(V)$ is a regular semigroup. For a subspace W of V, $L_F(V,W)$ and $\overline{L}_F(V,W)$ are defined analogously, that is,

$$L_F(V,W) = \{\alpha \in L_F(V) \mid \text{ran } \alpha \subseteq W\},$$
$$\overline{L}_F(V,W) = \{\alpha \in L_F(V) \mid W\alpha \subseteq W\}.$$

The semigroup $L_F(V,W)$ motivates us to consider the subsemigroup

$$K_F(V,W) = \{\alpha \in L_F(V) \mid W \subseteq \ker \alpha\}$$

of $L_F(V)$. In fact, $(L_F(V), +, \circ)$ is a ring where $+$ and $\circ$ are the usual addition and composition of linear transformations and $(L_F(V), +, \circ)$ has $L_F(V,W)$, $\overline{L}_F(V,W)$ and $K_F(V,W)$ as subrings. Observe that the semigroups $L_F(V)$, $L_F(V,W)$, $\overline{L}_F(V,W)$ and $K_F(V,W)$ mean $(L_F(V), \circ)$, $(L_F(V,W), \circ)$, $(\overline{L}_F(V,W), \circ)$ and $(K_F(V,W), \circ)$, respectively.

In this research, we determine the regular elements of all the semigroups defined above and characterize when these semigroups are regular and when they have the $\mathcal{BQ}$-property. Moreover, we give characterizations determining when the ring $(L_F(V,W), +, \circ)$, $(\overline{L}_F(V,W),\ +, \circ)$ and $(K_F(V,W), +, \circ)$ have the $\mathcal{BQ}$-property.

This research is organized as follows :

Chapter I contains definitions and quoted results which will be used for this research. For better understanding, some examples are also provided.

In Chapter II, we give necessary and sufficient conditions for the elements of the semigroups $T(X,Y)$ and $\overline{T}(X,Y)$ to be regular. In addition, the numbers of regular elements of $T(X,Y)$ and $\overline{T}(X,Y)$ are counted in terms of the Stirling number of the second kind when X is finite.

In Chapter III, necessary and sufficient conditions for the elements of the semigroups $L_F(V,W), \overline{L}_F(V,W)$ and $K_F(V,W)$ to be regular are provided. The conditions for the regularity of the elements of $L_F(V,W)$ and $\overline{L}_F(V,W)$ are the same as those for $T(X,Y)$ and $\overline{T}(X,Y)$ in Chapter II. We also apply the characterizations of the regular elements of $L_F(V,W)$ and $K_F(V,W)$ to determine the regular elements of some matrix semigroups over F.

Chapter IV deals with the $\mathcal{BQ}$-property of the semigroups $T(X,Y)$ and $\overline{T}(X,Y)$. It is shown that $T(X,Y)$ always has the $\mathcal{BQ}$-property. The semigroup $\overline{T}(X,Y)$ has the $\mathcal{BQ}$-property if and only if $Y = X, |Y| = 1$ or $|X| \leq 3$. Calais's theorem mentioned previously is useful for this work.

In Chapter V, we have similarly that the semigroups $L_F(V,W)$ and $K_F(V,W)$ always have the $\mathcal{BQ}$-property. However, it is shown that $\overline{L}_F(V,W)$ has the $\mathcal{BQ}$-property if and only if one of the following conditions holds.

(i) $W = V$.

(ii) $W = \{0\}$.

(iii) $F = \mathbb{Z}_2, \dim_F W = 1$ and $\dim_F V = 2$.

Calais's theorem is also referred for this characterization.

We are concerned with the $\mathcal{BQ}$-property of the rings $(L_F(V,W), +, \circ)$, $(K_F(V,W), +, \circ)$ and $(\overline{L}_F(V,W), +, \circ)$ in the last chapter. We have that the rings $(L_F(V,W), +, \circ)$ and $(K_F(V,W), +, \circ)$ have the $\mathcal{BQ}$-property since the semigroups $(L_F(V,W), \circ)$ and $(K_F(V,W), \circ)$ have the $\mathcal{BQ}$-property. The conditions for the ring $(\overline{L}_F(V,W), +, \circ)$

to have the $\mathcal{BQ}$-property are much wider than those for the semigroup $(\overline{L}_F(V,W),\circ)$. It is shown that the ring $(\overline{L}_F(V,W),+,\circ)$ has the $\mathcal{BQ}$-property if and only if one of the following conditions holds.

(i) $W = V$.

(ii) $W = \{0\}$.

(iii) $F = \mathbb{Z}_p$ for some prime p and $\dim_F W = 1$.

(iv) $F = \mathbb{Z}_p$ for some prime p and $\dim_F (V/W) = 1$.

CHAPTER I

PRELIMINARIES

Let $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ denote respectively the set of natural numbers (positive intergers), the set of integers and the set of real numbers. For $n \in \mathbb{N}$, $\mathbb{Z}_n$ denotes the set of integers modulo n.

For $n, r \in \mathbb{N}$ with $n \geq r$, the number of partitions $\{1, \ldots, n\}$ into r blocks is denoted by $S(n, r)$ and is called a *Stirling number* of the second kind. It is known that

$$S(n,r) = \frac{1}{r!}\sum_{i=0}^{r}(-1)^i\binom{r}{i}(r-i)^n$$

([1], page 12). Hence the number of maps from $\{1, 2, \ldots, n\}$ onto $\{1, 2, \ldots, r\}$ is $S(n, r)r!$.

The cardinality of a set X is denoted by $|X|$.

For a semigroup S, let $S^1 = S$ if S has an identity, otherwise, let S^1 be the semigroup S with an identity 1 adjoined.

An element a of a semigroup S is said to be *regular* if $a = axa$ for some $x \in S$, and S is called a *regular semigroup* if every element of S is regular. The set of all regular elements of a semigroup S is denoted by Reg (S). *Regular elements* of a ring $R = (R, +, \cdot)$ are regular elements of $(R, \cdot)$, and we call R a (*Von Neumann*) *regular ring* if every element of R is regular. The set of all regular elements of the ring R is also denoted by Reg (R).

In this research, the value of a map α at x in the domain of α is denoted by $x\alpha$ and the range of α is denoted by ran α.

For a nonempty set X, let $T(X)$ be the full transformation semigroup on X, that is, the semigroup, under composition, of all mappings from X into itself.

It is known that $T(X)$ is a regular semigroup ([6], page 4). The *kernel* of $\alpha \in T(X), \ker\alpha$, is the equivalence relation $\alpha \circ \alpha^{-1}$ on X, that is,

$$\ker\alpha = \{(x, y) \in X \times X \mid x\alpha = y\alpha\}.$$

Then $x\ker\alpha = (x\alpha)\alpha^{-1}$ for all $x \in X$, in particular, if $x \in \operatorname{ran}\alpha, x\alpha^{-1}$ is a $\ker\alpha$-class. Also, the mapping $x\ker\alpha \mapsto x\alpha$ is a bijection of $X/\ker\alpha$ onto $\operatorname{ran}\alpha$. Hence for any $\alpha \in T(X)$, the set of equivalence classes of $\ker\alpha$ and $\operatorname{ran}\alpha$ have the same cardinality.

For a vector space V over a field F, let $L_F(V)$ denote the semigroup, under composition, of all linear transformations from V into itself. Denote by $M_n(F)$ the multiplicative semigroup of all $n \times n$ matrices over a field F. We have that $(L_F(V), +, \circ)$ is a ring where $+$ and $\circ$ are the usual addition and composition of linear transformations, respectively. It is well-known that $M_n(F) \cong L_F(V)$ if $\dim_F(V) = n$ ([8], page 330), and $L_F(V)$ is a regular semigroup ([7], page 63). Hence $M_n(F)$ is a regular semigroup. Recall that for $\alpha \in L_F(V)$,

$$\ker\alpha = \{v \in V \mid v\alpha = 0\}.$$

The entry of $A \in M_n(F)$ in the $i^{\underline{th}}$ row and $j^{\underline{th}}$ column will be denoted by A_{ij}.

A *quasi-ideal* of a semigroup S is a subsemigroup Q of S such that $SQ \cap QS \subseteq Q$, and a *bi-ideal* of S is a subsemigroup B of S such that $BSB \subseteq B$.

For nonempty subsets X and Y of a ring R, XY denotes the set of all finite sums of the form $\sum x_i y_i$ where $x_i \in X$ and $y_i \in Y$. Also, for a nonempty subset X of a ring R, $\mathbb{Z}X$ denotes the set of all finite sums of the form $\sum k_i x_i$ where $k_i \in \mathbb{Z}$ and $x_i \in X$. Quasi-ideals and bi-ideals of rings are defined analogously. That is, a *quasi-ideal* of R is a subring Q of R such that $RQ \cap QR \subseteq Q$, and a *bi-ideal* of R is a subring B of R such that $BRB \subseteq B$.

In both semigroups and rings, every left ideal and every right ideal is clearly a quasi-ideal and every quasi-ideal is a bi-ideal. The following example shows that the converse is not generally true.

Example 1.1. Let F be a field and $n \in \mathbb{N}$.

(1) For $k, l \in \{1, 2, \ldots, n\}$, let $Q_n^{kl}(F)$ consist of all matrices $C \in M_n(F)$ such that

$$C_{ij} = 0 \quad \text{if} \quad i \neq k \quad \text{or} \quad j \neq l.$$

Then for $k, l \in \{1, 2, \ldots, n\}$, $Q_n^{kl}(F)$ is a subsemigroup [subring] of the semigroup [ring] $M_n(F)$,

$$M_n(F)Q_n^{kl}(F) = \left\{ \begin{array}{c} l \\ \downarrow \\ \begin{bmatrix} 0 & \ldots & 0 & x_1 & 0 & \ldots & 0 \\ 0 & \ldots & 0 & x_2 & 0 & \ldots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \ldots & 0 & x_n & 0 & \ldots & 0 \end{bmatrix} \end{array} \;\middle|\; x_1, x_2, \ldots, x_n \in F \right\}$$

and

$$Q_n^{kl}(F)M_n(F) = \left\{ \begin{array}{c} \\ \\ \\ k \rightarrow \\ \\ \\ \\ \end{array} \begin{bmatrix} 0 & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 0 \\ x_1 & x_2 & \ldots & x_n \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 0 \end{bmatrix} \;\middle|\; x_1, x_2, \ldots, x_n \in F \right\}$$

which imply that $M_n(F)Q_n^{kl}(F) \cap Q_n^{kl}(F)M_n(F) = Q_n^{kl}(F)$, so $Q_n^{kl}(F)$ is a quasi-ideal of the semigroup [ring] $M_n(F)$. Moreover, if $n > 1$, then for all $k, l \in \{1, 2, \ldots, n\}$, $Q_n^{kl}(F)$ is neither a left ideal nor a right ideal of the semigroup [ring] $M_n(F)$.

(2) For $n \geq 4$, let $SU_n(F)$ be the subsemigroup [subring] of the semigroup [ring] $M_n(F)$ consisting of all strictly upper triangular matrices over F. Let

$$B = \left\{ \begin{bmatrix} 0 & \dots & 0 & x & 0 \\ 0 & \dots & 0 & 0 & y \\ 0 & \dots & 0 & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & 0 \end{bmatrix} \middle| \; x, \; y \in F \right\}.$$

Then $B^2 = \{0\}$, so B is a subsemigroup [subring] of the semigroup [ring] $SU_n(F)$. Moreover, $BSU_n(F)B = \{0\} \subseteq B$. But

$$\begin{aligned} \begin{bmatrix} 0 & \dots & 0 & 1 \\ 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & 0 \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} 0 & \dots & 0 & 1 & 0 \\ 0 & \dots & 0 & 0 & 1 \\ 0 & \dots & 0 & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & \dots & 0 & 1 & 0 \\ 0 & \dots & 0 & 0 & 1 \\ 0 & \dots & 0 & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \dots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & 1 \\ 0 & \dots & 0 & 0 \end{bmatrix} \\ &\in (SU_n(F)B \cap BSU_n(F)) \smallsetminus B, \end{aligned}$$

so B is a bi-ideal but not a quasi-ideal of the semigroup [ring] $SU_n(F)$.

Example 1.1 shows that quasi-ideals generalize left ideals and right ideals and bi-ideals generalize quasi-ideals.

For a subset A of a semigroup S [ring R], let $(A)_q$ and $(A)_b$ denote respectively the quasi-ideal and the bi-ideal of S [R] generated by A, that is, $(A)_q$ is the intersection of all quasi-ideals of S [R] containing A and $(A)_b$ is the intersection of all bi-ideals of S [R] containing A (see [20], page 10 and 12). Observe that $(A)_b \subseteq (A)_q$ since every quasi-ideal is a bi-ideal.

Proposition 1.2. ([3], page 84-85) *For a nonempty subset A of a semigroup S,*
(i) $(A)_q = S^1A \cap AS^1$ *and*
(ii) $(A)_b = AS^1A \cup A$.

Proposition 1.3. ([22]) *For a nonempty subset A of a ring R,*

$$(A)_q = \mathbb{Z}A + (RA \cap AR).$$

Proposition 1.4. ([14]) *For a nonempty subset A of a ring R,*

$$(A)_b = \mathbb{Z}A + \mathbb{Z}A^2 + ARA.$$

In particular, if R has an identity, then $(A)_b = \mathbb{Z}A + ARA$.

Let $\mathcal{BQ}$ be the class of all semigroups whose bi-ideals and quasi-ideals coincide and an elements in $\mathcal{BQ}$ are called *$\mathcal{BQ}$-semigroups*. Important $\mathcal{BQ}$-semigroups are the following ones.

Proposition 1.5. ([13]) *Every regular semigroup is a $\mathcal{BQ}$-semigroup.*

Proposition 1.6. ([9]) *Every left [right] simple semigroup and left [right] 0-simple semigroup is a $\mathcal{BQ}$-semigroup.*

Recall that a semigroup S is *left [right] simple* if S has no proper left [right] ideal, and a semigroup S with zero 0 is called *left [right] 0-simple* if $S^2 \neq \{0\}$ and S has no proper nonzero left [right] ideal.

Some examples of $\mathcal{BQ}$-semigroups which are neither regular nor left [right] simple are as follows.

Example 1.7. ([11]) Let X be an infinite set and $S(X)$ the subsemigroup of $T(X)$ defined by

$$S(X) = \{\alpha \in T(X) \mid X \smallsetminus \operatorname{ran} \alpha \text{ is infinite}\}.$$

Then $S(X)$ is a $\mathcal{BQ}$-semigroup but it is neither regular nor left [right] simple semigroup.

Example 1.8. ([12]) For an infinite dimensional vector space V over a field F, define the subsemigroup $S(V)$ of $L_F(V)$ by

$$S(V) = \{\alpha \in L_F(V) \mid \alpha \text{ is 1-1 and } \dim_F(V/\text{ran } \alpha) \text{ is infinite}\}.$$

Then $S(V)$ is not regular and $S(V)$ is a $\mathcal{BQ}$-semigroup if and only if $\dim_F(V) = \mathcal{N}_0$.

In fact, $\mathcal{BQ}$-semigroups have been characterized by Calais [2] as follows:

Proposition 1.9. ([2]) *A semigroup S is a $\mathcal{BQ}$-semigroup if and only if $(x, y)_b = (x, y)_q$ for all $x, y \in S$.*

A $\mathcal{BQ}$*-ring* is defined similarly to a $\mathcal{BQ}$-semigroup, that is, a $\mathcal{BQ}$-ring is a ring whose bi-ideals are quasi-ideals. Kapp [10] provided a sufficient condition for a bi-ideal of a ring R to be a quasi-ideal of R as follows: If B is a bi-ideal of a ring R such that every element of B is regular in R, then B is a quasi-ideal of R. Then we have the following proposition as its direct consequence.

Proposition 1.10. *Every regular ring is a $\mathcal{BQ}$-ring.*

This research is concerned with both semigroups and rings whose bi-ideals and quasi-ideals coincide. Then we shall say that a semigroup or a ring has the $\mathcal{BQ}$-*property* if its quasi-ideals and bi-ideals are identical. Then every regular semigroup and every regular ring has the $\mathcal{BQ}$-property.

For a nonempty subset Y of a nonempty set X, let

$$T(X, Y) = \{\alpha \in T(X) \mid \text{ran } \alpha \subseteq Y\},$$
$$\overline{T}(X, Y) = \{\alpha \in T(X) \mid Y\alpha \subseteq Y\}.$$

Then $T(X, Y) \subseteq \overline{T}(X, Y)$ and both are subsemigroups of $T(X)$. Note that 1_X, the identity map on X, belongs to $\overline{T}(X, Y)$ and if $Y \neq X$, then $1_X \notin T(X, Y)$. The semigroup $T(X, Y)$ was introduced and studied by Symons [21] in 1975 while Magill [15] introduced and studied the semigroup $\overline{T}(X, Y)$ in 1966. Observe that these two types of transformation semigroups are generalizations of full transformation semigroups.

We introduce the subsemigroups $L_F(V,W)$ and $\overline{L}_F(V,W)$ analogously where W is a subspace of V, that is,

$$L_F(V,W) = \{\alpha \in L_F(V) \mid \text{ran } \alpha \subseteq W\},$$
$$\overline{L}_F(V,W) = \{\alpha \in L_F(V) \mid W\alpha \subseteq W\}.$$

Then $L_F(V,W) \subseteq \overline{L}_F(V,W)$. Clearly, 0 (the zero map on V) belongs to $L_F(V,W)$ and $\overline{L}_F(V,W)$, and $1_V \in \overline{L}_F(V,W)$ while $1_V \notin L_F(V,W)$ if $W \neq V$. We also consider the subsemigroup $K_F(V,W)$ of the semigroup $L_F(V)$ defined by

$$K_F(V,W) = \{\alpha \in L_F(V) \mid W \subseteq \ker \alpha\}.$$

Hence

$$K_F(V,W) = \{\alpha \in L_F(V) \mid W\alpha = \{0\}\}.$$

Then $K_F(V,W) \subseteq \overline{L}_F(V,W)$. Notice that $0 \in K_F(V,W)$, $L_F(V,V) = L_F(V) = K_F(V,\{0\})$, $L_F(V,\{0\}) = \{0\} = K_F(V,V)$ and $\overline{L}(V,V) = L_F(V) = \overline{L}(V,\{0\})$. Thus if $W = \{0\} \neq V$ or $W = V \neq \{0\}$, then $L_F(V,W) \neq K_F(V,W)$. Moreover, if $\{0\} \neq W \subsetneq V$, then $L_F(V,W)$ and $K_F(V,W)$ are not subsets of each other. To see this, assume that $\{0\} \neq W \subsetneq V$. Let B_1 be a basis of W and B a basis of V containing B_1. Define $\alpha, \beta \in L_F(V)$ on B by bracket notation as follows:

$$\alpha = \begin{bmatrix} v & \mathsf{B} \smallsetminus \mathsf{B}_1 \\ v & 0 \end{bmatrix}_{v \,\in\, \mathsf{B}_1}, \quad \beta = \begin{bmatrix} \mathsf{B}_1 & v \\ 0 & v \end{bmatrix}_{v \,\in\, \mathsf{B} \smallsetminus \mathsf{B}_1}.$$

Then ran $\alpha = \langle \mathsf{B}_1 \rangle = \ker \beta$, $\ker \alpha = \langle \mathsf{B} \smallsetminus \mathsf{B}_1 \rangle = \text{ran } \beta$. Therefore we deduce that $\alpha \in L_F(V,W) \smallsetminus K_F(V,W)$ and $\beta \in K_F(V,W) \smallsetminus L_F(V,W)$. We can see that $L_F(V,W)$, $\overline{L}_F(V,W)$ and $K_F(V,W)$ are subrings of the ring $(L_F(V), +, \circ)$ by the following facts:

$$\text{for } \alpha,\beta \in L_F(V,W),\ \text{ran } (\alpha+\beta) = V(\alpha+\beta) \subseteq V\alpha + V\beta \subseteq W + W = W,$$
$$\text{ran } (-\alpha) = \text{ran } \alpha \subseteq W,$$
$$\text{for } \alpha,\beta \in \overline{L}_F(V,W),\ W(\alpha+\beta) \subseteq W\alpha + W\beta \subseteq W + W = W,$$
$$W(-\alpha) = W\alpha \subseteq W,$$
$$\text{and } \quad \text{for } \alpha,\beta \in K_F(V,W),\ W(\alpha+\beta) \subseteq W\alpha + W\beta \subseteq \{0\} + \{0\} = \{0\},$$
$$W(-\alpha) = W\alpha = \{0\}.$$

For $1 \le k \le n$, let $C_n(F,k)$ and $R_n(F,k)$ be the matrix semigroups defined by

$$C_n(F,k) = \{A \in M_n(F) \mid A_{ij} = 0 \text{ for all } i,j \in \{1,\dots,n\} \text{ and } j > k\},$$
$$R_n(F,k) = \{A \in M_n(F) \mid A_{ij} = 0 \text{ for all } i,j \in \{1,\dots,n\} \text{ and } i > k\}.$$

In other words, $C_n(F,k)$ consists of all matrices in $M_n(F)$ of the form

$$\begin{bmatrix} a_{11} & \cdots & a_{1k} & 0 & \cdots & 0 \\ a_{21} & \cdots & a_{2k} & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nk} & 0 & \cdots & 0 \end{bmatrix}$$

and $R_n(F,k)$ consists of all matrices in $M_n(F)$ of the form

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kn} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Observe that $R_n(F,n) = M_n(F) = C_n(F,n)$. It is clearly seen that if $t_1,\dots,t_k \in \{1,\dots,n\}$ with $t_1 < t_2 < \cdots < t_k$, then S_1 and S_2 defined by

$$S_1 = \{A \in M_n(F) \mid A_{ij} = 0 \text{ for all } i,j \in \{1,\dots,n\} \text{ and } j \notin \{t_1,\dots,t_k\}\},$$
$$S_2 = \{A \in M_n(F) \mid A_{ij} = 0 \text{ for all } i,j \in \{1,\dots,n\} \text{ and } i \notin \{t_1,\dots,t_k\}\}$$

are subsemigroups of $M_n(F)$ which are clearly isomorphic to $C_n(F,k)$ and $R_n(F,k)$, respectively. Notice that $C_n(F,k)$ and $R_n(F,k)$ are also subrings of the ring $(M_n(F),+,\cdot)$ where $+$ and $\cdot$ are the usual addition and multiplication of matrices.

We recall the following basic facts of vector spaces and linear transformations which will be used.

(1) If $\alpha \in L_F(V)$, B_1 is a basis of $\ker\alpha$, B_2 is a basis of $\operatorname{ran}\alpha$ and for each $u \in \mathsf{B}_2$, choose an element $u' \in u\alpha^{-1}$, then $\mathsf{B}_1 \cup \{u' \mid u \in \mathsf{B}_2\}$ is a basis of V.

(2) If U_1 and U_2 are subspaces of V, B_1 is a basis of the subspace $U_1 \cap U_2$, $\mathsf{B}_2 \subseteq U_1 \smallsetminus \mathsf{B}_1$ and $\mathsf{B}_3 \subseteq U_2 \smallsetminus \mathsf{B}_1$ are such that $\mathsf{B}_1 \cup \mathsf{B}_2$ and $\mathsf{B}_1 \cup \mathsf{B}_3$ are bases of U_1 and U_2, respectively, then $\mathsf{B}_1 \cup \mathsf{B}_2 \cup \mathsf{B}_3$ is a basis of the subspace $U_1 + U_2$ of V. In particular, if $U_1 \cap U_2 = \{0\}$, then $\mathsf{B}_2 \cup \mathsf{B}_3$ is a basis of $U_1 + U_2$.

(3) If W is a subspace of V such that $\dim_F(V/W) = 1$ and B is a basis of W, then for every $u \in V \smallsetminus W$, $\mathsf{B} \cup \{u\}$ is a basis of V.

(4) If B_1 is a basis of W and B is a basis of V containing B_1, then $\{v + W \mid v \in \mathsf{B} \smallsetminus \mathsf{B}_1\}$ is the basis of the quotient space V/W and $v_1 + W \neq v_2 + W$ for all distinct $v_1, v_2 \in \mathsf{B} \smallsetminus \mathsf{B}_1$. Hence $\dim_F(V/W) = |\mathsf{B} \smallsetminus \mathsf{B}_1|$.

CHAPTER II

REGULAR ELEMENTS OF SEMIGROUPS OF TRANSFORMATIONS OF SETS

In this chapter, the regular elements of the semigroups $T(X,Y)$ and $\overline{T}(X,Y)$ are characterized. Some remarkable relationships of Reg $(T(X,Y))$ and Reg $(\overline{T}(X,Y))$ are also given. In addition, Reg $(T(X,Y))$ and Reg $(\overline{T}(X,Y))$ are counted in terms of $|X|, |Y|$, and their Stirling numbers of the second kind when X is finite.

Throughout this chapter, X denotes a nonempty set and $\varnothing \neq Y \subseteq X$. First, we recall that

$$T(X,Y) = \{\alpha \in T(X) \mid \text{ran } \alpha \subseteq Y\},$$
$$\overline{T}(X,Y) = \{\alpha \in T(X) \mid Y\alpha \subseteq Y\}.$$

For $n, r \in \mathbb{N}$ with $n \geq r$, the number of all mappings from $\{1, 2, \ldots, n\}$ onto $\{1, 2, \ldots, r\}$ is $r!S(n,r)$ where

$$S(n,r) = \frac{1}{r!}\sum_{i=0}^{r}(-1)^i\binom{r}{i}(r-i)^n$$

(a Stirling number of the second kind).

Theorem 2.1. *For $\alpha \in T(X,Y)$, the following statements are equivalent.*

(i) $\alpha \in \mathit{Reg}\ (T(X,Y))$.

(ii) $\mathit{ran}\ \alpha = Y\alpha$.

(iii) $x\mathit{ker}\ \alpha \cap Y \neq \varnothing$ *for every* $x \in X$.

(iv) $x\alpha^{-1} \cap Y \neq \varnothing$ *for every* $x \in \mathit{ran}\ \alpha$.

Proof. (i) $\Rightarrow$ (ii). Let $\beta \in T(X,Y)$ be such that $\alpha = \alpha\beta\alpha$. Then $X\alpha\beta \subseteq Y$, and so ran $\alpha = X\alpha = X\alpha\beta\alpha = (X\alpha\beta)\alpha \subseteq Y\alpha \subseteq X\alpha =$ ran α. Hence (ii) holds.

(ii) $\Rightarrow$ (iii). For any $x \in X$, $x\alpha \in$ ran $\alpha = Y\alpha$, so $x\alpha = y\alpha$ for some $y \in Y$ which implies that $y \in (x\alpha)\alpha^{-1} = x$ker α.

(iii) $\Rightarrow$ (iv). This is trivial since for every $x \in$ ran $\alpha, x\alpha^{-1}$ is a ker α-class.

(iv) $\Rightarrow$ (i). For each $x \in$ ran α, choose an element $x' \in x\alpha^{-1} \cap Y$. Then $x'\alpha = x$ for every $x \in$ ran α. Let a be a fixed element of Y. Define $\beta : X \to X$ by bracket notation as follows:

$$\beta = \begin{bmatrix} x & X \setminus \text{ran } \alpha \\ x' & a \end{bmatrix}_{x \, \in \, \text{ran } \alpha},$$

that is, $x\beta = x'$ for all $x \in$ ran α and $x\beta = a$ for all $x \in X \setminus$ran α. Then ran $\beta \subseteq Y$ and for every $x \in X$, $x\alpha\beta\alpha = (x\alpha)\beta\alpha = (x\alpha)'\alpha = x\alpha$. Hence $\beta \in T(X, Y)$ and $\alpha = \alpha\beta\alpha$. $\square$

As a consequence of Theorem 2.1, a necessary and sufficient condition for $T(X, Y)$ to be a regular semigroup can be given as follows:

Corollary 2.2. *The semigroup $T(X, Y)$ is regular if and only if either $X = Y$ or $|Y| = 1$.*

Proof. Suppose that $Y \subsetneq X$ and $|Y| > 1$. Let a and b be two distinct elements of Y. Define $\alpha : X \to X$ by

$$\alpha = \begin{bmatrix} Y & X \setminus Y \\ a & b \end{bmatrix}.$$

Then ran $\alpha = \{a, b\} \subseteq Y$ and $b\alpha^{-1} \cap Y = (X \setminus Y) \cap Y = \varnothing$. Hence $\alpha \in T(X, Y)$ and by Theorem 2.1, α is not a regular element of $T(X, Y)$. This proves that if $T(X, Y)$ is a regular semigroup, then $Y = X$ or $|Y| = 1$.

Since $T(X, Y) = T(X)$ if $Y = X$ and $|T(X, Y)| = 1$ if $|Y| = 1$, the converse holds. $\square$

Theorem 2.3. *For $\alpha \in \overline{T}(X, Y)$, the following statements are equivalent.*

(i) $\alpha \in$ *Reg* $(\overline{T}(X, Y))$.

(ii) *ran* $\alpha \cap Y = Y\alpha$.

(iii) $x\ker\alpha \cap Y \neq \varnothing$ *for every* $x \in X$ *with* $x\alpha \in Y$*, that is,* $x \in Y\alpha^{-1}$.

(iv) $x\alpha^{-1} \cap Y \neq \varnothing$ *for every* $x \in \operatorname{ran}\alpha \cap Y$.

Proof. (i) $\Rightarrow$ (ii). Let $\beta \in \overline{T}(X,Y)$ be such that $\alpha = \alpha\beta\alpha$. Then $Y\alpha \subseteq X\alpha \cap Y = \operatorname{ran}\alpha \cap Y$. If $x \in \operatorname{ran}\alpha \cap Y$, then $x \in Y$ and $x = a\alpha$ for some $a \in X$. Consequently, $x = a\alpha = a\alpha\beta\alpha = x\beta\alpha \in Y\beta\alpha \subseteq Y\alpha$. Hence (ii) holds.

(ii) $\Rightarrow$ (iii). Let $x \in X$ be such that $x\alpha \in Y$. Then $x\alpha \in \operatorname{ran}\alpha \cap Y = Y\alpha$, so $x\alpha = y\alpha$ for some $y \in Y$. This implies that $y \in (x\alpha)\alpha^{-1} = x\ker\alpha$. Hence $y \in x\ker\alpha \cap Y$.

(iii) $\Rightarrow$ (iv). If $x \in \operatorname{ran}\alpha \cap Y$, then $x = a\alpha$ for some $a \in X$, so $a \in x\alpha^{-1} \subseteq Y\alpha^{-1}$. By (iii), $a\ker\alpha \cap Y \neq \varnothing$. But $a\ker\alpha = (a\alpha)\alpha^{-1} = x\alpha^{-1}$, so $x\alpha^{-1} \cap Y \neq \varnothing$.

(iv) $\Rightarrow$ (i). For each $x \in \operatorname{ran}\alpha \cap Y$, choose an element $x' \in x\alpha^{-1} \cap Y$. Also, for $x \in \operatorname{ran}\alpha \setminus Y$, choose an element $\overline{x} \in x\alpha^{-1}$. Then $x'\alpha = x$ for every $x \in \operatorname{ran}\alpha \cap Y$ and $\overline{x}\alpha = x$ for all $x \in \operatorname{ran}\alpha \setminus Y$. Let a be a fixed element of Y. Define $\beta : X \to X$ by

$$\beta = \begin{bmatrix} x & t & X \setminus \operatorname{ran}\alpha \\ x' & \overline{t} & a \end{bmatrix}_{\substack{x \,\in\, \operatorname{ran}\,\alpha\cap Y \\ t \,\in\, \operatorname{ran}\,\alpha\setminus Y}}.$$

Then $Y\beta \subseteq \{x' \mid x \in \operatorname{ran}\alpha \cap Y\} \cup \{a\} \subseteq Y$ and for $x \in X$,

$$x\alpha\beta\alpha = (x\alpha)\beta\alpha = \begin{cases} (x\alpha)'\alpha = x\alpha & \text{if } x\alpha \in \operatorname{ran}\alpha \cap Y \\ \overline{(x\alpha)}\alpha = x\alpha & \text{if } x\alpha \in \operatorname{ran}\alpha \setminus Y. \end{cases}$$

Hence $\beta \in \overline{T}(X,Y)$ and $\alpha = \alpha\beta\alpha$. $\square$

We also have the following corollary which characterizes when $\overline{T}(X,Y)$ is a regular semigroup.

Corollary 2.4. *The semigroup* $\overline{T}(X,Y)$ *is regular if and only if either* $X = Y$ *or* $|Y| = 1$.

Proof. Suppose that $Y \subsetneq X$ and $|Y| > 1$. Let $a, b \in Y$ and α be as in the proof of Corollary 2.2. Then $Y\alpha = \{a\} \subseteq Y$, so $\alpha \in \overline{T}(X,Y)$. Since $b \in \operatorname{ran}\alpha \cap Y$

and $b\alpha^{-1} \cap Y = (X \smallsetminus Y) \cap Y = \varnothing$, by Theorem 2.3, α is not a regular element of $\overline{T}(X,Y)$.

If $Y = X$, then $\overline{T}(X,Y) = T(X)$ which is regular. Next, assume that $Y = \{c\}$. Then $c\alpha = c$ for all $\alpha \in \overline{T}(X,Y)$. To show that $\overline{T}(X,Y)$ is regular, let $\alpha \in \overline{T}(X,Y)$. For each $x \in \text{ran } \alpha \smallsetminus \{c\}$, choose an element $x' \in x\alpha^{-1}$. Then $x'\alpha = x$ for all $x \in \text{ran } \alpha \smallsetminus \{c\}$. Let $c' = c$ and define $\beta \in T(X)$ by

$$\beta = \begin{bmatrix} x & X \smallsetminus \text{ran } \alpha \\ x' & c \end{bmatrix}_{x \, \in \, \text{ran } \alpha}.$$

Then $Y\beta = \{c\}\beta = \{c'\} = \{c\} = Y$ and for $x \in X, x\alpha\beta\alpha = (x\alpha)'\alpha = x\alpha$. This proves that if $|Y| = 1$, then $\overline{T}(X,Y)$ is a regular semigroup, as required. □

The following result which is obtained from Theorem 2.1 and Theorem 2.3 shows that any nonregular element of $T(X,Y)$ cannot be regular in $\overline{T}(X,Y)$.

Corollary 2.5. *Reg* $(\overline{T}(X,Y)) \subseteq$ *Reg* $(T(X,Y)) \cup (\overline{T}(X,Y) \smallsetminus T(X,Y))$, *or equivalently,*

$$T(X,Y) \smallsetminus \mathit{Reg}\ (T(X,Y)) \subseteq \overline{T}(X,Y) \smallsetminus \mathit{Reg}\ (\overline{T}(X,Y)).$$

Proof. Let $\alpha \in \text{Reg } (\overline{T}(X,Y))$ and assume that $\alpha \in T(X,Y)$. Then $\text{ran } \alpha \cap Y = Y\alpha$ by Theorem 2.3 and $\text{ran } \alpha \subseteq Y$. These imply that $\text{ran } \alpha = Y\alpha$, so $\alpha \in \text{Reg } (T(X,Y))$ by Theorem 2.1. □

Next, the cardinalities of the regular elements in the semigroups $T(X,Y)$ and $\overline{T}(X,Y)$ are investigated when X is finite. First, we note that if $|X| = n$ and $|Y| = m$, then

$$|T(X)| = n^n,$$

$$|T(X,Y)| = m^n,$$

$$|\overline{T}(X,Y)| = m^m \times n^{n-m}.$$

Theorem 2.6. *If $|X| = n$ and $|Y| = m$, then*

$$|Reg\ (T(X,Y))| = \sum_{r=1}^{m} \binom{m}{r} r!S(m,r)r^{n-m}.$$

Proof. Let $\varnothing \neq Y' \subseteq Y$ and $|Y'| = r$. Then the number of maps from Y onto Y' is $r!S(m,r)$. Consequently, the number of maps α from X onto Y' such that $Y\alpha = Y'$ is $r!S(m,r)r^{n-m}$. Hence

$$|\{\alpha \in T(X,Y) \mid \text{ran } \alpha = Y' = Y\alpha\}| = r!S(m,r)r^{n-m}.$$

But we have from Theorem 2.1((i) $\Leftrightarrow$ (ii)) that

$$\{\alpha \in T(X,Y) \mid \text{ran } \alpha = Y' = Y\alpha\} = \{\alpha \in \text{Reg } (T(X,Y)) \mid \text{ran } \alpha = Y'\},$$

so

$$|\{\alpha \in \text{Reg } (T(X,Y)) \mid \text{ran } \alpha = Y'\}| = r!S(m,r)r^{n-m}.$$

This implies that for $1 \leq r \leq m$,

$$|\{\alpha \in \text{Reg } (T(X,Y)) \mid |\text{ran } \alpha| = r\}| = \binom{m}{r} r!S(m,r)r^{n-m}.$$

Therefore it follows that

$$|\text{Reg } (T(X,Y))| = \sum_{r=1}^{m} \binom{m}{r} r!S(m,r)r^{n-m},$$

as desired. $\square$

Theorem 2.7. *If $|X| = n$ and $|Y| = m$, then*

$$|Reg\ (\overline{T}(X,Y))| = \sum_{r=1}^{m} \binom{m}{r} r!S(m,r)(n-m+r)^{n-m}.$$

Proof. Let $\varnothing \neq Y' \subseteq Y$ and $|Y'| = r$. Then the number of maps from Y onto Y' is $r!S(m,r)$. Therefore it follows that the number of maps $\alpha : X \to X$ such that $Y\alpha = Y'$ and ran $\alpha \cap Y = Y'$ is $r!S(m,r)(n-m+r)^{n-m}$ since $|(X \smallsetminus Y) \cup Y'| = |X \smallsetminus Y| + |Y'| = n - m + r$. Hence

$$|\{\alpha \in \overline{T}(X,Y) \mid \text{ran } \alpha \cap Y = Y' = Y\alpha\}| = r!S(m,r)(n-m+r)^{n-m}.$$

We have from Theorem 2.3((i) $\Leftrightarrow$ (ii)) that

$$\{\alpha \in \overline{T}(X,Y) \mid \text{ran } \alpha \cap Y = Y' = Y\alpha\} = \{\alpha \in \text{Reg } (\overline{T}(X,Y)) \mid \text{ran } \alpha \cap Y = Y'\}$$

which implies that

$$|\{\alpha \in \text{Reg } (\overline{T}(X,Y)) \mid \text{ran } \alpha \cap Y = Y'\}| = r!S(m,r)(n-m+r)^{n-m}.$$

Consequently, for $1 \leq r \leq m$,

$$|\{\alpha \in \text{Reg } (\overline{T}(X,Y)) \mid |\text{ran } \alpha \cap Y| = r\}| = \binom{m}{r} r!S(m,r)(n-m+r)^{n-m},$$

whence

$$|\text{Reg } (\overline{T}(X,Y))| = \sum_{r=1}^{m} \binom{m}{r} r!S(m,r)(n-m+r)^{n-m}.$$

□

Example 2.8. Since $S(n,r) = \dfrac{1}{r!}\displaystyle\sum_{i=0}^{r}(-1)^i\binom{r}{i}(r-i)^n$, we have $S(3,1) = 1$, $S(3,2)$ $= 3$ and $S(3,3) = 1$.

(1) Let $|X| = 4$ and $|Y| = 3$. By Theorem 2.6 and Theorem 2.7, we have respectively that

$$\begin{aligned}
|\text{Reg } (T(X,Y))| &= \sum_{r=1}^{3}\binom{3}{r} r!S(3,r)r \\
&= (3 \times 1! \times 1 \times 1) + (3 \times 2! \times 3 \times 2) + (1 \times 3! \times 1 \times 3) \\
&= 3 + 36 + 18 = 57, \\
|\text{Reg } (\overline{T}(X,Y))| &= \sum_{r=1}^{3}\binom{3}{r} r!S(3,r)(1+r) \\
&= (3 \times 1! \times 1 \times 2) + (3 \times 2! \times 3 \times 3) + (1 \times 3! \times 1 \times 4) \\
&= 6 + 54 + 24 = 84.
\end{aligned}$$

Hence

$$|T(X,Y) \setminus \text{Reg } (T(X,Y))| = 3^4 - 57 = 81 - 57 = 24,$$

$$|\overline{T}(X,Y) \setminus \text{Reg } (\overline{T}(X,Y))| = (3^3 \times 4^1) - 84 = 108 - 84 = 24,$$

and so by Corollary 2.5, $T(X,Y) \smallsetminus \text{Reg}\ (T(X,Y)) = \overline{T}(X,Y) \smallsetminus \text{Reg}\ (\overline{T}(X,Y))$. Since $|\overline{T}(X,Y) \smallsetminus T(X,Y)| = 108 - 81 = 27$, we deduce that $|\text{Reg}\ (T(X,Y)) \cup (\overline{T}(X,Y) \smallsetminus T(X,Y))| = 57 + 27 = 84 = |\text{Reg}\ (\overline{T}(X,Y))|$, so by Corollary 2.5, we have that $\text{Reg}\ (\overline{T}(X,Y)) = \text{Reg}\ (T(X,Y)) \cup (\overline{T}(X,Y) \smallsetminus T(X,Y))$. Therefore every element in $\overline{T}(X,Y) \smallsetminus T(X,Y)$ is regular in $\overline{T}(X,Y)$.

(2) Assume that $|X| = 5$ and $|Y| = 3$. Then

$$\begin{aligned}
|\text{Reg}\ (T(X,Y))| &= \sum_{r=1}^{3} \binom{3}{r} r! S(3,r) r^2 \\
&= (3 \times 1! \times 1 \times 1^2) + (3 \times 2! \times 3 \times 2^2) + (1 \times 3! \times 1 \times 3^2) \\
&= 3 + 72 + 54 = 129, \\
|\text{Reg}\ (\overline{T}(X,Y))| &= \sum_{r=1}^{3} \binom{3}{r} r! S(3,r)(2+r)^2 \\
&= (3 \times 1! \times 1 \times 3^2) + (3 \times 2! \times 3 \times 4^2) + (1 \times 3! \times 1 \times 5^2) \\
&= 27 + 288 + 150 = 465.
\end{aligned}$$

Hence

$$\begin{aligned}
|T(X,Y) \smallsetminus \text{Reg}\ (T(X,Y))| &= 3^5 - 129 \\
&= 243 - 129 = 114, \\
|\overline{T}(X,Y) \smallsetminus \text{Reg}\ (\overline{T}(X,Y))| &= (3^3 \times 5^2) - 465 \\
&= 675 - 465 = 210, \\
|\overline{T}(X,Y) \smallsetminus T(X,Y)| &= 675 - 243 = 432, \\
|\text{Reg}\ (T(X,Y)) \cup (\overline{T}(X,Y) \smallsetminus T(X,Y))| &= 129 + (675 - 243) \\
&= 129 + 432 = 561.
\end{aligned}$$

It follows from Corollary 2.5 that $T(X,Y) \smallsetminus \text{Reg}\ (T(X,Y)) \subsetneq \overline{T}(X,Y) \smallsetminus \text{Reg}\ (\overline{T}(X,Y))$ and $\text{Reg}\ (\overline{T}(X,Y)) \subsetneq \text{Reg}\ (T(X,Y)) \cup (\overline{T}(X,Y) \smallsetminus T(X,Y))$. Since $\text{Reg}\ (T(X,Y)) \subseteq \text{Reg}\ (\overline{T}(X,Y))$, we deduce that there is an element of $\overline{T}(X,Y) \smallsetminus T(X,Y)$ which is not regular in $\overline{T}(X,Y)$.

From Example 2.8(1), it is natural to ask whether it is true that for a set X and $\varnothing \neq Y \subseteq X$, if $|X \smallsetminus Y| = 1$, then Reg $(\overline{T}(X,Y)) =$ Reg $(T(X,Y)) \cup (\overline{T}(X,Y) \smallsetminus T(X,Y))$. Also, does the converse hold if $Y \neq X$ and $|Y| > 1$? The later question is motivated by Example 2.8(2). The following theorem shows that these are true in general. Note that by Corollary 2.2 and Corollary 2.4, if $X = Y$ or $|Y| = 1$, then both $T(X,Y)$ and $\overline{T}(X,Y)$ are regular which implies that Reg $(\overline{T}(X,Y)) =$ Reg $(T(X,Y)) \cup (\overline{T}(X,Y) \smallsetminus T(X,Y))$.

Theorem 2.9. *If* $|X \smallsetminus Y| = 1$, *then Reg* $(\overline{T}(X,Y)) =$ *Reg* $(T(X,Y)) \cup (\overline{T}(X,Y) \smallsetminus T(X,Y))$, *and the converse holds if* $Y \subsetneq X$ *and* $|Y| > 1$.

Proof. Assume that $X \smallsetminus Y = \{c\}$ and let $\alpha \in \overline{T}(X,Y) \smallsetminus T(X,Y)$ be given. Then $Y\alpha \subseteq Y$ and $X\alpha \nsubseteq Y$. But $X = Y \cup \{c\}$, so $c\alpha = c$. Hence ran $\alpha \cap Y = (Y \cup \{c\})\alpha \cap Y = (Y\alpha \cup \{c\}) \cap Y = Y\alpha \cap Y = Y\alpha$. By Theorem 2.3, $\alpha \in$ Reg $(\overline{T}(X,Y))$. Hence Reg $(T(X,Y)) \cup (\overline{T}(X,Y) \smallsetminus T(X,Y)) \subseteq$ Reg $(\overline{T}(X,Y))$. The reverse inclusion is obtained from Corollary 2.5.

Conversely, let $Y \subsetneq X$ and $|Y| > 1$ and assume that $|X \smallsetminus Y| > 1$. Let $a, b \in X \smallsetminus Y$ be distinct and c and d be distinct elements of Y. Define $\alpha : X \to X$ by

$$\alpha = \begin{bmatrix} a & b & X \smallsetminus \{a,b\} \\ c & b & d \end{bmatrix}.$$

Since $Y \subseteq X \smallsetminus \{a,b\}$, $Y\alpha = \{d\} \subseteq Y$ and ran $\alpha = \{c,b,d\} \nsubseteq Y$, we have that $\alpha \in \overline{T}(X,Y) \smallsetminus T(X,Y)$. Also, ran $\alpha \cap Y = \{c,d\} \neq \{d\} = Y\alpha$. By Theorem 2.3, $\alpha \notin$ Reg $(\overline{T}(X,Y))$.

Hence the proof is complete. □

Remark 2.10. Let X be infinite. We shall give some remarks relating to the cardinalities of Reg $(T(X,Y))$ and Reg $(\overline{T}(X,Y))$. First, we note that if $|Y| = 1$, then $|$Reg $(T(X,Y))| = |T(X,Y)| = 1$. The following three facts are provided.

(i) If $|Y| > 1$, then $|$Reg $(T(X,Y))| \geq 2^{|X|}$. To see this, let a and b be distinct elements of Y. For any $A \in P(X \setminus \{a, b\})$ (the power set of $X \setminus \{a, b\}$), define $\alpha_A : X \to X$ by

$$\alpha_A = \begin{bmatrix} A \cup \{a\} & X \setminus (A \cup \{a\}) \\ a & b \end{bmatrix}.$$

Then ran $\alpha_A = \{a, b\} = (\{a, b\})\alpha_A = Y\alpha_A$ for every $A \in P(X \setminus \{a, b\})$, so $\{\alpha_A \mid A \in P(X \setminus \{a, b\})\} \subseteq \text{Reg } (T(X, Y))$ by Theorem 2.1. Since for distinct $A, B \in P(X \setminus \{a, b\}), \alpha_A \neq \alpha_B$, it follows that $|P(X \setminus \{a, b\})| \leq |\text{Reg } (T(X, Y))|$. However, $|X| = |X \setminus \{a, b\}|$, so $|P(X)| = |P(X \setminus \{a, b\})|$. Therefore it follows that

$$|\text{Reg } (T(X, Y))| \geq |P(X)| = 2^{|X|}.$$

(ii) If $|Y| = |X|$, then $|\text{Reg } (T(X, Y))| = |T(X)|$. To prove this, assume that $|Y| = |X|$. Then $|T(Y)| = |T(X)|$ through a map $\alpha \mapsto \varphi^{-1}\alpha\varphi$ where $\varphi : X \to Y$ is a bijection. For $\alpha \in T(Y)$, define a map $\alpha' : X \to X$ by $\alpha'|_Y = \alpha$ and $(X \setminus Y)\alpha' \subseteq$ ran α. Hence for every $\alpha \in T(Y), \alpha' \in T(X, Y)$ and ran $\alpha' =$ ran $\alpha = Y\alpha'$, so $\alpha' \in \text{Reg } (T(X, Y))$ for all $\alpha \in T(Y)$ by Theorem 2.1. Moreover, $\alpha \mapsto \alpha'$ is an injective map from $T(Y)$ into Reg $(T(X, Y))$, so

$$|T(X)| \geq |\text{Reg } (T(X, Y))| \geq |\{\alpha' \mid \alpha \in T(Y)\}| = |T(Y)| = |T(X)|,$$

and the required result is obtained.

(iii) $|\text{Reg } (\overline{T}(X, Y))| = |T(X)|$. If $|Y| = |X|$, then by (ii), $|\text{Reg } (T(X, Y))| = |T(X)|$. Since Reg $(T(X, Y)) \subseteq$ Reg $(\overline{T}(X, Y)) \subseteq \overline{T}(X, Y) \subseteq T(X)$, we have that $|\text{Reg } (\overline{T}(X, Y))| = |T(X)|$ when $|Y| = |X|$. Next, assume that $|Y| < |X|$. Then $|X| = |X \setminus Y| + |Y| = |X \setminus Y|$ since X is infinite and $|Y| < |X|$, and hence $|T(X \setminus Y)| = |T(X)|$. For $\alpha \in T(X \setminus Y)$, define a map $\overline{\alpha} : X \to X$ by $\overline{\alpha}|_{X \setminus Y} = \alpha$ and $Y\overline{\alpha} \subseteq Y$. Thus for every $\alpha \in T(X \setminus Y)$, $\overline{\alpha} \in \overline{T}(X, Y)$ and ran $\overline{\alpha} \cap Y = ($ran $\alpha \cup Y\overline{\alpha}) \cap Y = Y\overline{\alpha}$. It follows from Theorem 2.3 that $\{\overline{\alpha} \mid \alpha \in T(X \setminus Y)\} \subseteq \text{Reg } (\overline{T}(X, Y))$. Since $\alpha \mapsto \overline{\alpha}$ is an injective map from

$T(X \smallsetminus Y)$ into Reg $(\overline{T}(X,Y))$, we have

$$|T(X)| \geq |\text{Reg } (\overline{T}(X,Y))| \geq |\{\overline{\alpha} \mid \alpha \in T(X \smallsetminus Y)\}| = |T(X \smallsetminus Y)| = |T(X)|,$$

and thus $|\text{Reg } (\overline{T}(X,Y))| = |T(X)|$.

CHAPTER III

REGULAR ELEMENTS OF SEMIGROUPS OF LINEAR TRANSFORMATIONS

In this chapter, we consider the subsemigroups $L_F(V,W)$ and $\overline{L}_F(V,W)$ of $L_F(V)$ analogous to the subsemigroups $T(X,Y)$ and $\overline{T}(X,Y)$ of $T(X)$, respectively. Also, the subsemigroup $K_F(V,W)$ of $L_F(V)$ is considered. The regular elements of these three semigroups are characterized. Such results for $L_F(V,W)$ and $K_F(V,W)$ are then applied to determine the regular elements of the matrix semigroups $C_n(F,k)$ and $R_n(F,k)$, respectively.

First, we recall the semigroups $L_F(V,W), \overline{L}_F(V,W), K_F(V,W)$, $C_n(F,k)$ and $R_n(F,k)$, where W is a subspace of a vector space V over a field F, $n,k \in \mathbb{N}$ and $k \leq n$, as follows:

$$L_F(V,W) = \{\alpha \in L_F(V) \mid \operatorname{ran} \alpha \subseteq W\},$$

$$\overline{L}_F(V,W) = \{\alpha \in L_F(V) \mid W\alpha \subseteq W\},$$

$$K_F(V,W) = \{\alpha \in L_F(V) \mid W \subseteq \ker \alpha\},$$

$$C_n(F,k) = \{A \in M_n(F) \mid A_{ij} = 0 \text{ for all } i,j \in \{1,\dots,n\} \text{ and } j > k\},$$

$$R_n(F,k) = \{A \in M_n(F) \mid A_{ij} = 0 \text{ for all } i,j \in \{1,\dots,n\} \text{ and } i > k\}.$$

In other words, $C_n(F,k)$ consists of all matrices in $M_n(F)$ of the form

$$\begin{bmatrix} a_{11} & \cdots & a_{1k} & 0 & \cdots & 0 \\ a_{21} & \cdots & a_{2k} & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nk} & 0 & \cdots & 0 \end{bmatrix}$$

and $R_n(F,k)$ consists of all matrices in $M_n(F)$ of the form

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kn} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Observe that $R_n(F,n) = M_n(F) = C_n(F,n)$.

Throughout this chapter, let W be a subspace of a vector space V over a field F, $n \in \mathbb{N}$ and $k \in \{1,\ldots,n\}$.

Theorem 3.1. *For $\alpha \in L_F(V,W), \alpha \in Reg\,(L_F(V,W))$ if and only if ran $\alpha = W\alpha$.*

Proof. If $\alpha = \alpha\beta\alpha$ for some $\beta \in L_F(V,W)$, then $W\alpha \subseteq V\alpha = V\alpha\beta\alpha = (V\alpha\beta)\alpha \subseteq W\alpha$, so ran $\alpha = W\alpha$.

For the converse, assume that ran $\alpha = W\alpha$. Let B_1 be a basis of $\ker\alpha$, B_2 a basis of ran α and B_3 a basis of V containing B_2. Since ran $\alpha = W\alpha$, for each element $u \in \mathsf{B}_2$, there is an element $u' \in W$ such that $u'\alpha = u$. Then $\mathsf{B}_1 \cup \{u' \mid u \in \mathsf{B}_2\}$ is a basis of V. Define $\beta \in L_F(V)$ on the basis B_3 of V by

$$\beta = \begin{bmatrix} u & \mathsf{B}_3 \smallsetminus \mathsf{B}_2 \\ u' & 0 \end{bmatrix}_{u \in \mathsf{B}_2}.$$

Then ran $\beta = \langle\{u' \mid u \in \mathsf{B}_2\}\rangle \subseteq W$, so $\beta \in L_F(V,W)$. Since $\mathsf{B}_1\alpha\beta\alpha = \{0\} = \mathsf{B}_1\alpha$ and $u'\alpha\beta\alpha = u\beta\alpha = u'\alpha$ for all $u \in \mathsf{B}_2$, we have that $\alpha = \alpha\beta\alpha$. Hence α is a regular element of $L_F(V,W)$, as desired. □

Corollary 3.2. *The semigroup $L_F(V,W)$ is regular if and only if either $W = V$ or $W = \{0\}$.*

Proof. Assume that $\{0\} \subsetneq W \subsetneq V$. Let B_1 be a basis of W and B a basis of V containing B_1. Let $w \in \mathsf{B}_1$ and define $\alpha \in L_F(V)$ by

$$\alpha = \begin{bmatrix} \mathsf{B}_1 & \mathsf{B} \smallsetminus \mathsf{B}_1 \\ 0 & w \end{bmatrix}.$$

Then ran $\alpha = \langle w \rangle \subseteq W$ and $W\alpha = \langle \mathsf{B}_1 \rangle \alpha = \{0\}$, thus ran $\alpha \neq W\alpha$. Hence $\alpha \in L_F(V, W)$ and by Theorem 3.1, α is not a regular element of $L_F(V, W)$.

Since $L_F(V, V) = L_F(V)$ and $L_F(V, \{0\}) = \{0\}$, the converse holds. $\square$

Theorem 3.3. *For $\alpha \in \overline{L}_F(V, W)$, $\alpha \in Reg\ (\overline{L}_F(V, W))$ if and only if ran $\alpha \cap W = W\alpha$.*

Proof. Since $W\alpha \subseteq W$, we have $W\alpha \subseteq$ ran $\alpha \cap W$. Assume that $\alpha = \alpha\beta\alpha$ for some $\beta \in \overline{L}_F(V, W)$. If $v \in$ ran $\alpha \cap W$, then $v \in W$ and $v = u\alpha$ for some $u \in V$ which imply that

$$v = u\alpha = u\alpha\beta\alpha = v\beta\alpha \in W\beta\alpha \subseteq W\alpha.$$

Hence ran $\alpha \cap W = W\alpha$.

Conversely, assume that ran $\alpha \cap W = W\alpha$. Let B_1 be a basis of ran $\alpha \cap W$, $\mathsf{B}_2 \subseteq$ ran $\alpha \smallsetminus \mathsf{B}_1$ and $\mathsf{B}_3 \subseteq W \smallsetminus \mathsf{B}_1$ such that $\mathsf{B}_1 \cup \mathsf{B}_2$ and $\mathsf{B}_1 \cup \mathsf{B}_3$ are bases of ran α and W, respectively. Then $\mathsf{B}_1 \cup \mathsf{B}_2 \cup \mathsf{B}_3$ is a basis of ran $\alpha + W$. Let $\mathsf{B}_4 \subseteq V \smallsetminus (\mathsf{B}_1 \cup \mathsf{B}_2 \cup \mathsf{B}_3)$ be such that $\mathsf{B}_1 \cup \mathsf{B}_2 \cup \mathsf{B}_3 \cup \mathsf{B}_4$ is a basis of V. Since $\mathsf{B}_1 \subseteq$ ran $\alpha \cap W = W\alpha$, for each $u \in \mathsf{B}_1$, there is an element $u' \in W$ such that $u'\alpha = u$. Since $\mathsf{B}_2 \subseteq$ ran α, for each $v \in \mathsf{B}_2$, there is an element $\overline{v} \in v\alpha^{-1}$ such that $\overline{v}\alpha = v$. Define $\beta \in L_F(V)$ on the basis $\mathsf{B}_1 \cup \mathsf{B}_2 \cup \mathsf{B}_3 \cup \mathsf{B}_4$ by

$$\beta = \begin{bmatrix} u & v & \mathsf{B}_3 \cup \mathsf{B}_4 \\ u' & \overline{v} & 0 \end{bmatrix}_{\substack{u \in \mathsf{B}_1 \\ v \in \mathsf{B}_2}}.$$

It follows that $W\beta = \langle \mathsf{B}_1 \cup \mathsf{B}_3 \rangle \beta = \langle \{u' \mid u \in \mathsf{B}_1\} \rangle \subseteq W$, so $\beta \in \overline{L}_F(V, W)$. Let

B_0 be a basis of $\ker\alpha$. Then $\mathsf{B}_0 \cup \{u' \mid u \in \mathsf{B}_1\} \cup \{\overline{v} \mid v \in \mathsf{B}_2\}$ is a basis of V. Since

$$\mathsf{B}_0\alpha\beta\alpha = \{0\} = \mathsf{B}_0\alpha,\ u'\alpha\beta\alpha = u\beta\alpha = u'\alpha \text{ for all } u \in \mathsf{B}_1,$$
$$\overline{v}\alpha\beta\alpha = v\beta\alpha = \overline{v}\alpha \text{ for all } v \in \mathsf{B}_2,$$

we have $\alpha = \alpha\beta\alpha$, so $\alpha \in \mathrm{Reg}\ (\overline{L}_F(V,W))$, as desired. □

Corollary 3.4. *The semigroup $\overline{L}_F(V,W)$ is regular if and only if either $W = V$ or $W = \{0\}$.*

Proof. Assume that $\{0\} \neq W \subsetneq V$. Let B_1 be a basis of W and B a basis of V containing B_1. Then $\mathsf{B}_1 \neq \varnothing \neq \mathsf{B} \smallsetminus \mathsf{B}_1$. Let $w \in \mathsf{B}_1$ and $u \in \mathsf{B} \smallsetminus \mathsf{B}_1$. Define $\alpha \in L_F(V)$ by

$$\alpha = \begin{bmatrix} u & \mathsf{B} \smallsetminus \{u\} \\ w & 0 \end{bmatrix}.$$

Then $W\alpha = \langle \mathsf{B}_1 \rangle \alpha \subseteq \langle \mathsf{B} \smallsetminus \{u\} \rangle \alpha = \{0\}$, so $\alpha \in \overline{L}_F(V,W)$. Since $\mathrm{ran}\ \alpha \cap W = \langle w \rangle \neq \{0\} = W\alpha$, by Theorem 3.3, we deduce that α is not a regular element of $\overline{L}_F(V,W)$. Hence $\overline{L}_F(V,W)$ is not a regular semigroup.

Since $\overline{L}_F(V,V) = L_F(V) = \overline{L}_F(V,\{0\})$, the converse holds. □

Theorem 3.5. *For $\alpha \in K_F(V,W), \alpha \in Reg\ (K_F(V,W))$ if and only if $ran\ \alpha \cap W = \{0\}$.*

Proof. Assume that $\alpha = \alpha\beta\alpha$ for some $\beta \in K_F(V,W)$. If $v \in \mathrm{ran}\ \alpha \cap W$, then $v \in W$ and $v = u\alpha$ for some $u \in V$, and hence $v = u\alpha = u\alpha\beta\alpha = v\beta\alpha \in W\beta\alpha = \{0\}$. This shows that $\mathrm{ran}\ \alpha \cap W = \{0\}$.

Conversely, assume that $\mathrm{ran}\ \alpha \cap W = \{0\}$. Let B_1 be a basis of $\ker\alpha$, B_2 a basis of $\mathrm{ran}\ \alpha$ and B_3 a basis of W. Since $\mathrm{ran}\ \alpha \cap W = \{0\}$, we have that $\mathsf{B}_2 \cup \mathsf{B}_3$ is a basis of $\mathrm{ran}\ \alpha + W$. Let B_4 be a basis of V containing $\mathsf{B}_2 \cup \mathsf{B}_3$. For each element $u \in \mathsf{B}_2$, let $u' \in V$ be such that $u'\alpha = u$. Then $\mathsf{B}_1 \cup \{u' \mid u \in \mathsf{B}_2\}$ is a basis of V. Define $\beta \in L_F(V)$ by

$$\beta = \begin{bmatrix} u & \mathsf{B}_4 \smallsetminus \mathsf{B}_2 \\ u' & 0 \end{bmatrix}_{u \in \mathsf{B}_2}.$$

Since $\mathsf{B}_3 \subseteq \mathsf{B}_4 \setminus \mathsf{B}_2$, it follows that $W\beta = \langle \mathsf{B}_3 \rangle \beta = \{0\}$, so $\beta \in K_F(V, W)$. Moreover, $\mathsf{B}_1\alpha\beta\alpha = \{0\} = \mathsf{B}_1\alpha$ and $u'\alpha\beta\alpha = u\beta\alpha = u'\alpha$ for all $u \in \mathsf{B}_2$. Hence we have $\alpha = \alpha\beta\alpha$, so α is a regular element of $K_F(V, W)$. □

Corollary 3.6. *The semigroup $K_F(V, W)$ is regular if and only if either $W = V$ or $W = \{0\}$.*

Proof. Assume that $\{0\} \subsetneq W \subsetneq V$. Let $\mathsf{B}_1, \mathsf{B}, w, u$ and $\alpha \in L_F(V)$ be as in Corollary 3.4. Since $W\alpha = \{0\}$, we have $\alpha \in K_F(V, W)$. Also, $\operatorname{ran} \alpha \cap W = \langle w \rangle \cap W = \langle w \rangle \neq \{0\}$. By Theorem 3.5, α is not a regular element of $K_F(V, W)$.

The converse holds since $K_F(V, V) = \{0\}$ and $K_F(V, \{0\}) = L_F(V)$. □

To characterize the regular elements of $C_n(F, k)$ and $R_n(F, k)$ by Theorem 3.1 and Theorem 3.5, respectively, some lemmas are needed.

Let V^* and V^{**} be the dual space and the double dual space of V, respectively. For $A \subseteq V$, the annihilator of A is denoted by A^0, that is,

$$A^0 = \{f \in V^* \mid f(v) = 0 \text{ for all } v \in A\}$$

and let $A^{00} = (A^0)^0$, that is,

$$A^{00} = \{T \in V^{**} \mid T(f) = 0 \text{ for all } f \in A^0\}.$$

For $(x_1, \ldots, x_n) \in F^n$, define $h_{(x_1,\ldots,x_n)} : F^n \to F$ by

$$h_{(x_1,\ldots,x_n)}(t_1, \ldots, t_n) = t_1x_1 + \cdots + t_nx_n \text{ for all } (t_1, \ldots, t_n) \in F^n.$$

Then we have

$$(F^n)^* = \left\{h_{(x_1,\ldots,x_n)} \mid (x_1, \ldots, x_n) \in F^n\right\} \qquad \text{(I)}$$

([5], page 149). For $x \in F^n$, define $T_x : (F^n)^* \to F$ by

$$T_x(f) = f(x) \text{ for all } x \in F^n.$$

Then

$$(F^n)^{**} = \{T_x \mid x \in F^n\} \quad \text{and}$$
$$T_x \neq T_y \ \text{ for all distinct } x, y \in F^n \tag{II}$$

([5], page 147). If U is a subspace of F^n, then

$$U^{00} = \{T_u \mid u \in U\} \tag{III}$$

([5], page 148–149). Note that if A_1 and A_2 are subsets of F^n such that $A_1 \subseteq A_2$, then $A_1^0 \supseteq A_2^0$ and $A_1^{00} \subseteq A_2^{00}$.

Lemma 3.7. *Let $(a_{11}, \ldots, a_{1n}), \ldots, (a_{m1}, \ldots, a_{mn}), (b_1, \ldots, b_n)$ be elements of F^n. Then the following two conditions are equivalent.*

(i) $(b_1, \ldots, b_n) \in \langle (a_{11}, \ldots, a_{1n}), \ldots, (a_{m1}, \ldots, a_{mn}) \rangle$.

(ii) *For every $(x_1, \ldots, x_n) \in F^n$, $a_{i1}x_1 + \cdots + a_{in}x_n = 0$ for all $i \in \{1, \ldots, m\}$, then $b_1x_1 + \cdots + b_nx_n = 0$.*

Proof. Let $U_1 = \langle (a_{11}, \ldots, a_{1n}), \ldots, (a_{m1}, \ldots, a_{mn}) \rangle$ and $U_2 = \langle (b_1, \ldots, b_n) \rangle$.

Assume that (i) holds. Then $U_2 \subseteq U_1$ which implies that $U_2^0 \supseteq U_1^0$. Let $(x_1, \ldots, x_n) \in F^n$ be such that $a_{i1}x_1 + \cdots + a_{in}x_n = 0$ for all $i \in \{1, \ldots, m\}$. Then

$$h_{(x_1,\ldots,x_n)}(a_{i1}, \ldots, a_{in}) = 0 \text{ for all } i \in \{1, \ldots, m\}.$$

It follows that $h_{(x_1,\ldots,x_n)} \in U_1^0$. But $U_1^0 \subseteq U_2^0$, so $h_{(x_1,\ldots,x_n)}(b_1, \ldots, b_n) = 0$, that is, $b_1x_1 + \cdots + b_nx_n = 0$. Hence (ii) holds.

To show that (ii) implies (i), assume that (ii) holds. Then we have that for every $(x_1, \ldots, x_n) \in F^n, h_{(x_1,\ldots,x_n)} \in \langle \{(a_{11}, \ldots, a_{1n}), \ldots, (a_{m1}, \ldots, a_{mn})\} \rangle^0$ implies that $h_{(x_1,\ldots,x_n)} \in \langle \{(b_1, \ldots, b_n)\} \rangle^0$. It follows from (I) that $U_1^0 \subseteq U_2^0$. Then $U_2^{00} \subseteq U_1^{00}$. Hence by (III),

$$\{T_x \mid x \in U_2\} = U_2^{00} \subseteq U_1^{00} = \{T_x \mid x \in U_1\}.$$

By (II), we deduce that $U_2 \subseteq U_1$, so (i) holds. □

For a matrix $A \in M_n(F)$, define $g_A : F^n \to F^n$ by

$$Xg_A = XA \ \text{ for all } X \in F^n.$$

Clearly, $g_A \in L_F(F^n)$ for all $A \in M_n(F)$. Let $\{e_1, \dots, e_n\}$ be the standard basis of F^n over F. Therefore we have

$$e_i g_A = (A_{i1}, \dots, A_{in}) \text{ for all } i \in \{1, \dots, n\} \text{ and } A \in M_n(F). \qquad \text{(IV)}$$

Lemma 3.8. *The mapping $\varphi : M_n(F) \to L_F(F^n)$ defined by $A\varphi = g_A$ for all $A \in M_n(F)$ is an isomorphism from $M_n(F)$ onto $L_F(F^n)$.*

Proof. It is clear that φ is a homomorphism. It follows from (IV) that φ is 1-1. If $\alpha \in L_F(F^n)$, then define $A \in M_n(F)$ by

$$(A_{i1}, \dots, A_{in}) = e_i\alpha \text{ for all } i \in \{1, \dots, n\}.$$

Then by (IV), $e_i g_A = e_i\alpha$ for all $i \in \{1, \dots, n\}$, and thus $A\varphi = g_A = \alpha$. Hence the lemma is proved. □

Lemma 3.9. *Let U_1 and U_2 be subspaces of F^n spanned by $\{e_1, \dots, e_k\}$ and $\{e_{k+1}, \dots, e_n\}$, respectively. Then*

(i) $L_F(F^n, U_1) = \{g_A \mid A \in C_n(F, k)\}$ *and*

(ii) $K_F(F^n, U_2) = \{g_A \mid A \in R_n(F, k)\}$.

Proof. We have from the definitions of U_1 and U_2 that

$$U_1 = \{(x_1, \dots, x_k, 0, \dots, 0) \mid x_1, \dots, x_k \in F\}$$

and

$$U_2 = \begin{cases} \{(0, \dots, 0)\} & \text{if } k = n, \\ \{(0, \dots, 0, x_{k+1}, \dots, x_n) \mid x_{k+1}, \dots, x_n \in F\} & \text{if } k < n. \end{cases}$$

(i) For $A \in M_n(F)$,

$$\begin{aligned} g_A \in L_F(F^n, U_1) &\Leftrightarrow \text{ran } g_A \subseteq U_1 \\ &\Leftrightarrow (A_{i1}, \dots, A_{in}) \in U_1 \text{ for all } i \in \{1, \dots, n\} \quad \text{from (IV)} \\ &\Leftrightarrow A_{ij} = 0 \text{ for all } i, j \in \{1, \dots, n\} \text{ with } j > k \\ &\Leftrightarrow A \in C_n(F, k). \end{aligned}$$

Hence by Lemma 3.8, (i) holds.

(ii) If $k = n$, then $K_F(F^n, U_2) = L_F(F^n)$ and $R_n(F, k) = M_n(F)$, so (ii) holds by Lemma 3.8. Next, assume that $k < n$. Then for $A \in M_n(F)$,

$$\begin{aligned}
g_A \in K_F(F^n, U_2) &\Leftrightarrow U_2 \subseteq \ker g_A \\
&\Leftrightarrow U_2 g_A = \{(0, \ldots, 0)\} \\
&\Leftrightarrow e_i g_A = (0, \ldots, 0) \ \text{ for all } i \in \{k+1, \ldots, n\} \\
&\Leftrightarrow (A_{i1}, \ldots, A_{in}) = (0, \ldots, 0) \\
&\qquad\qquad \text{for all } i \in \{k+1, \ldots, n\} \quad \text{from (IV)} \\
&\Leftrightarrow A \in R_n(F, k).
\end{aligned}$$

Hence (ii) holds by Lemma 3.8. □

Theorem 3.10. *For $A \in C_n(F, k)$, A is regular in $C_n(F, k)$ if and only if for any $x_1, \ldots, x_k \in F$,*

$$\begin{aligned}
&A_{i1}x_1 + \cdots + A_{ik}x_k = 0 \textit{ for all } i \in \{1, \ldots, k\} \\
&\qquad\qquad \Rightarrow A_{i1}x_1 + \cdots + A_{ik}x_k = 0 \textit{ for all } i \in \{k+1, \ldots, n\},
\end{aligned} \tag{1}$$

that is, for any $(x_1, \ldots, x_k) \in F^k$,

$$\begin{bmatrix} A_{11} & \cdots & A_{1k} \\ \vdots & \ddots & \vdots \\ A_{k1} & \cdots & A_{kk} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} A_{k+1,1} & \cdots & A_{k+1,k} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nk} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Proof. Let U be the subspace of F^n spanned by $\{e_1, \ldots, e_k\}$. Then by Lemma 3.8 and Lemma 3.9(i), $C_n(F, k) \cong L_F(F^n, U)$ through the mapping $A \mapsto g_A$.

Let $A \in C_n(F, k)$. Since $A_{ij} = 0$ for all $i, j \in \{1, \ldots, n\}$ with $j > k$, by (IV), we have

$$\begin{aligned}
\text{ran } g_A &= \langle (A_{11}, \ldots, A_{1k}, 0, \ldots, 0), \ldots, (A_{n1}, \ldots, A_{nk}, 0, \ldots, 0) \rangle, \\
U g_A &= \langle (A_{11}, \ldots, A_{1k}, 0, \ldots, 0), \ldots, (A_{k1}, \ldots, A_{kk}, 0, \ldots, 0) \rangle.
\end{aligned} \tag{2}$$

Hence

$$\begin{aligned}
A \in \text{Reg }(C_n(F,k)) &\Leftrightarrow g_A \in \text{Reg }(L_F(F^n,U)) \\
&\Leftrightarrow \text{ran } g_A = U g_A \text{ from Theorem 3.1} \\
&\Leftrightarrow (A_{i1},\ldots,A_{ik},0,\ldots,0) \\
&\qquad \in \langle (A_{11},\ldots,A_{1k},0,\ldots,0),\ldots,(A_{k1},\ldots,A_{kk},0,\ldots,0)\rangle \\
&\qquad\quad \text{for all } i \in \{k+1,\ldots,n\} \text{ from (2)} \\
&\Leftrightarrow (A_{i1},\ldots,A_{ik}) \in \langle (A_{11},\ldots,A_{1k}),\ldots,(A_{k1},\ldots,A_{kk})\rangle \text{ in } F^k \\
&\qquad\qquad\qquad \text{for all } i \in \{k+1,\ldots,n\} \\
&\Leftrightarrow \text{(1) holds from Lemma 3.7.}
\end{aligned}$$

Therefore the theorem is proved. □

The following two corollaries are direct consequences of Theorem 3.10.

Corollary 3.11. *If $A \in C_n(F,k)$ is of the form*

$$\begin{bmatrix} a_{11} & \cdots & a_{1k} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{k1} & \cdots & a_{kk} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix},$$

then A is regular in $C_n(F,k)$.

We note here that if S consists of all matrices $A \in M_n(F)$ of the form given in Corollary 3.11, then S is a subsemigroup of $M_n(F)$ contained in $C_n(F,k)$ and $S \cong M_k(F)$. This implies that S is a regular subsemigroup of $C_n(F,k)$.

Corollary 3.12. *Let $k < n$ and $A \in C_n(F,k)$ be of the form*

$$\begin{bmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ a_{k+1,1} & \cdots & a_{k+1,k} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nk} & 0 & \cdots & 0 \end{bmatrix}.$$

Then A is regular in $C_n(F,k)$ if and only if A is a zero matrix.

Also, as a consequence of Theorem 3.12, $C_n(F,k)$ is a regular semigroup only in the case that $k = n$, or equivalently, $C_n(F,k) = M_n(F)$.

Corollary 3.13. *The semigroup $C_n(F,k)$ is a regular semigroup if and only if $k = n$.*

Proof. Assume that $k < n$. Define $A \in M_n(F)$ by

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{bmatrix}.$$

Then $A \in C_n(F,k)$. Since $k < n$, by Corollary 3.12, A is not regular in $C_n(F,k)$.

Since $C_n(F,n) = M_n(F)$, the converse holds. □

Theorem 3.14. *For $A \in R_n(F,k), A$ is regular in $R_n(F,k)$ if and only if for any $x_1, \ldots, x_k \in F$,*

$$\begin{aligned} &A_{1j}x_1 + \cdots + A_{kj}x_k = 0 \text{ for all } j \in \{1, \ldots, k\} \\ &\qquad \Rightarrow A_{1j}x_1 + \cdots + A_{kj}x_k = 0 \text{ for all } j \in \{k+1, \ldots, n\}, \end{aligned} \tag{1}$$

that is, for any $(x_1, \ldots, x_k) \in F^k$,

$$\begin{bmatrix} x_1 & \cdots & x_k \end{bmatrix} \begin{bmatrix} A_{11} & \cdots & A_{1k} \\ \vdots & \ddots & \vdots \\ A_{k1} & \cdots & A_{kk} \end{bmatrix} = \begin{bmatrix} 0 & \cdots & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 & \cdots & x_k \end{bmatrix} \begin{bmatrix} A_{1,k+1} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{k,k+1} & \cdots & A_{kn} \end{bmatrix} = \begin{bmatrix} 0 & \cdots & 0 \end{bmatrix}.$$

Proof. This is true if $k = n$ since $R_n(F, n) = M_n(F)$. Assume that $k < n$ and U is a subspace of F^n spanned by $\{e_{k+1}, \ldots, e_n\}$. By Lemma 3.8 and Lemma 3.9(ii), $R_n(F, k) \cong K_F(F^n, U)$ by $A \mapsto g_A$. Note that

$$U = \{(0, \ldots, 0, x_{k+1}, \ldots, x_n) \mid x_{k+1}, \ldots, x_n \in F\}. \tag{2}$$

Let $A \in R_n(F, k)$. Then $A_{ij} = 0$ for all $i, j \in \{1, \ldots, n\}$ with $i > k$ and

$$\begin{aligned} A \in \text{Reg } (R_n(F, k)) &\Leftrightarrow g_A \in \text{Reg } (K_F(F^n, U)) \\ &\Leftrightarrow \text{ran } g_A \cap U = \{(0, \ldots, 0)\} \text{ from Theorem 3.5,} \end{aligned}$$

Thus to prove the theorem, it suffices to show that ran $g_A \cap U = \{(0, \ldots, 0)\}$ if and only if (1) holds. First, assume that ran $g_A \cap U = \{(0, \ldots, 0)\}$ and let $x_1, \ldots, x_k \in F$ be such that $A_{1j}x_1 + \cdots + A_{kj}x_k = 0$ for all $j \in \{1, \ldots, k\}$. Then

$$\begin{aligned} &(x_1, \ldots, x_k, 0, \ldots, 0)g_A \\ &\quad = (x_1, \ldots, x_k, 0, \ldots, 0)A \\ &\quad = (A_{11}x_1 + \cdots + A_{k1}x_k, \ldots, A_{1n}x_1 + \cdots + A_{kn}x_k) \\ &\quad = (0, \ldots, 0, A_{1,k+1}x_1 + \cdots + A_{k,k+1}x_k, \ldots, A_{1n}x_1 + \cdots + A_{kn}x_k) \\ &\quad \in \text{ran } g_A \cap U = \{(0, \ldots, 0)\} \text{ from (2).} \end{aligned}$$

This implies that $A_{1j}x_1 + \cdots + A_{kj}x_k = 0$ for all $j \in \{k+1, \ldots, n\}$.

Conversely, assume that (1) holds. Let $(y_1, \ldots, y_n) \in$ ran $g_A \cap U$. Then $y_j = 0$ for all $j \in \{1, \ldots, k\}$ by (2) and $(y_1, \ldots, y_n) = (a_1, \ldots, a_n)g_A$ for some

$(a_1,\dots,a_n)\in F^n$. It follows that $A_{1j}a_1+\cdots+A_{kj}a_k=y_j$ for all $j\in\{1,\dots,n\}$. Then $A_{1j}a_1+\cdots+A_{kj}a_k=0$ for all $j\in\{1,\dots,k\}$. By (1), $A_{1j}a_1+\cdots+A_{kj}a_k=0$ for all $j\in\{k+1,\dots,n\}$. Thus $(y_1,\dots,y_n)=(0,\dots,0)$. This shows that ran $g_A\cap U=\{(0,\dots,0)\}$.

Therefore the proof is complete. □

From Theorem 3.14, we clearly have the next two corollaries.

Corollary 3.15. *If $A\in R_n(F,k)$ is of the form*

$$\begin{bmatrix} a_{11} & \cdots & a_{1k} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{k1} & \cdots & a_{kk} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix},$$

then A is regular in $R_n(F,k)$.

Corollary 3.16. *Let $k<n$ and $A\in R_n(F,k)$ be of the form*

$$\begin{bmatrix} 0 & \cdots & 0 & a_{1,k+1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{k,k+1} & \cdots & a_{kn} \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Then A is regular in $R_n(F,k)$ if and only if A is a zero matrix.

Corollary 3.17. *The semigroup $R_n(F,k)$ is a regular semigroup if and only if $k=n$.*

Proof. If $k < n$, then by Corollary 3.16,

$$A = \begin{bmatrix} 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

is a nonregular element of $R_n(F, k)$.

If $k = n$, then $R_n(F, k) = M_n(F)$. Therefore the converse holds. □

Remark 3.18. In our presentation, we applied Theorem 3.1 and Theorem 3.5 to obtain Theorem 3.10 and Theorem 3.14, respectively. In fact, Theorem 3.10 implies Theorem 3.14 and the converse is also true. It follows from the following facts:

(i) If the semigroups S_1 and S_2 are anti-isomorphic, that is, there is a bijection $\varphi : S_1 \to S_2$ such that $(xy)\varphi = (y\varphi)(x\varphi)$ for all $x, y \in S_1$, it is clearly that $\text{Reg }(S_2) = (\text{Reg }(S_1))\,\varphi$.

(ii) The mapping $A \mapsto A^t$, the transpose of A, from $C_n(F, k)$ $[R_n(F, k)]$ into $R_n(F, k)$ $[C_n(F, k)]$ is clearly an anti-isomorphism.

Example 3.19. Consider the matrices A and B over $\mathbb{R}$ defined by

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 3 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

If we consider $A \in C_3(\mathbb{R}, 2)$, then A is not a regular element of $C_3(\mathbb{R}, 2)$ by Theorem 3.10 since $A_{11}(1) + A_{12}(-1) = 0 = A_{21}(1) + A_{22}(-1)$ and $A_{31}(1) + A_{32}(-1) = -2 \neq 0$. Consider B as an element of $C_4(\mathbb{R}, 3)$ and $R_4(\mathbb{R}, 2)$. By Corollary 3.11, $B \in \text{Reg }(C_4(\mathbb{R}, 3))$. To show that $B \in \text{Reg }(R_4(\mathbb{R}, 2))$ by Theorem 3.14, let $x_1, x_2 \in \mathbb{R}$ be such that $B_{11}x_1 + B_{21}x_2 = 0 = B_{12}x_1 + B_{22}x_2$. Then $3x_2 = 0 = x_1 + 2x_2$ which implies that $x_1 = x_2 = 0$, so $B_{13}x_1 + B_{23}x_2 = 0 = B_{14}x_1 + B_{24}x_2$.

CHAPTER IV

THE $\mathcal{BQ}$-PROPERTY OF SEMIGROUPS OF TRANSFORMATIONS OF SETS

The $\mathcal{BQ}$-property of the semigroups of $T(X,Y)$ and $\overline{T}(X,Y)$ are considered in this chapter. The characterizations of $T(X,Y)$ and $\overline{T}(X,Y)$ to have the $\mathcal{BQ}$-property will provide some examples of $\mathcal{BQ}$-semigroups which are not regular.

Recall that the semigroups $T(X,Y)$ and $\overline{T}(X,Y)$, where Y is a nonempty subset of a set X, are defined as follows:

$$T(X,Y) = \{\alpha \in T(X) \mid \text{ran } \alpha \subseteq Y\},$$
$$\overline{T}(X,Y) = \{\alpha \in T(X) \mid Y\alpha \subseteq Y\}.$$

Throughout this chapter, let X be a nonempty set and $\varnothing \neq Y \subseteq X$.

We first show that the semigroup $T(X,Y)$ always has the $\mathcal{BQ}$-property.

Lemma 4.1. *If B is a bi-ideal of a regular semigroup S, then B has the $\mathcal{BQ}$-property.*

Proof. Since B is a bi-ideal of S, we have $BSB \subseteq B$. Let A be a bi-deal of B. Then $ABA \subseteq A$. To show that A is a quasi-ideal of B, let $x \in AB \cap BA$. Since S is regular, $x = xyx$ for some $y \in S$. These imply that

$$x = xyx \in ABSBA \subseteq ABA \subseteq A.$$

Hence $AB \cap BA \subseteq A$. This proves that every bi-ideal of B is a quasi-ideal of B. Hence B has the $\mathcal{BQ}$-property. □

Lemma 4.2. *The semigroup $T(X,Y)$ is a left ideal of $T(X)$.*

Proof. Since ran $(\beta\alpha) \subseteq$ ran α for all $\alpha, \beta \in T(X)$, it follows that $T(X)T(X,Y) \subseteq T(X,Y)$. Hence $T(X,Y)$ is a left ideal of $T(X)$. □

Theorem 4.3. *The semigroup $T(X,Y)$ always has the $\mathcal{BQ}$-property.*

Proof. Since $T(X,Y)$ is a left ideal of $T(X)$ by Lemma 4.2, $T(X,Y)$ is a bi-ideal of $T(X)$. But since $T(X)$ is a regular semigroup, by Lemma 4.1, $T(X,Y)$ has the $\mathcal{BQ}$-property. □

To characterize when $\overline{T}(X,Y)$ is a $\mathcal{BQ}$-semigroup, Proposition 1.2, Proposition 1.5, Proposition 1.9 and Corollary 2.4 and the following three lemmas are needed.

Lemma 4.4. *Let S be a semigroup. If $\varnothing \neq A \subseteq$ Reg (S), then $(A)_b = (A)_q$.*

Proof. We know that $(A)_b \subseteq (A)_q$. Let $x \in (A)_q$. By Proposition 1.2(i), $x = sa = bt$ for some $s, t \in S^1$ and $a, b \in A$. Since $a \in$ Reg (S), $a = aa'a$ for some $a' \in S$. Then

$$x = sa = saa'a = bta'a \in ASA \subseteq (A)_b$$

by Proposition 1.2(ii). Hence we have $(A)_b = (A)_q$, as desired. □

Lemma 4.5. *Let S be a semigroup, $\varnothing \neq A \subseteq S$ and $B \subseteq$ Reg (S). If $(A)_b = (A)_q$, then $(A \cup B)_b = (A \cup B)_q$.*

Proof. We first show that $S^1A \cap BS^1$ and $S^1B \cap AS^1$ are subsets of $(A \cup B)_b$. Let $x \in S^1A \cap BS^1$. Then $x = sa = bt$ for some $s, t \in S^1$, $a \in A$ and $b \in B$. Since $b \in$ Reg (S), $b = bb'b$ for some $b' \in S$. It follows that

$$x = bt = bb'bt = bb'sa \in BSA \subseteq (A \cup B)S(A \cup B) \subseteq (A \cup B)_b.$$

This shows that $S^1A \cap BS^1 \subseteq (A \cup B)_b$. It can be shown similarly that $S^1B \cap AS^1 \subseteq (A \cup B)_b$. Consequently,

$$\begin{aligned}(A \cup B)_q &= S^1(A \cup B) \cap (A \cup B)S^1 \\ &= (S^1A \cup S^1B) \cap (AS^1 \cup BS^1) \\ &= (S^1A \cap AS^1) \cup (S^1A \cap BS^1) \cup (S^1B \cap AS^1) \cup (S^1B \cap BS^1)\end{aligned}$$

$$
\begin{aligned}
&= (A)_q \cup (S^1A \cap BS^1) \cup (S^1B \cap AS^1) \cup (B)_q \\
&= (A)_b \cup (S^1A \cap BS^1) \cup (S^1B \cap AS^1) \cup (B)_b \\
&\qquad\qquad\qquad \text{from the assumption and Lemma 4.4} \\
&\subseteq (A)_b \cup (A \cup B)_b \cup (A \cup B)_b \cup (B)_b \\
&= (A \cup B)_b.
\end{aligned}
$$

But $(A \cup B)_b \subseteq (A \cup B)_q$, so $(A \cup B)_b = (A \cup B)_q$. □

Lemma 4.6. *If* $|X| = 3$ *and* $|Y| = 2$, *then for all* $\alpha, \beta \in \overline{T}(X, Y)$, $(\alpha, \beta)_b = (\alpha, \beta)_q$ *in* $\overline{T}(X, Y)$.

Proof. For convenience, let X_a denote the constant map whose domain and range are X and $\{a\}$, respectively.

Assume that $X = \{a, b, c\}$ and $Y = \{a, b\}$. Clearly,

$$
\begin{aligned}
\overline{T}(X, Y) = \Big\{ 1_X, X_a, X_b, &\begin{bmatrix} a & b & c \\ a & a & b \end{bmatrix}, \begin{bmatrix} a & b & c \\ a & a & c \end{bmatrix}, \begin{bmatrix} a & b & c \\ b & b & a \end{bmatrix}, \begin{bmatrix} a & b & c \\ b & b & c \end{bmatrix}, \\
&\begin{bmatrix} a & b & c \\ a & b & a \end{bmatrix}, \begin{bmatrix} a & b & c \\ a & b & b \end{bmatrix}, \begin{bmatrix} a & b & c \\ b & a & a \end{bmatrix}, \begin{bmatrix} a & b & c \\ b & a & b \end{bmatrix}, \begin{bmatrix} a & b & c \\ b & a & c \end{bmatrix} \Big\}.
\end{aligned}
$$

By Theorem 2.3((i)$\Leftrightarrow$ (ii)),

$$
\overline{T}(X, Y) \smallsetminus \operatorname{Reg}\,(\overline{T}(X, Y)) = \left\{ \begin{bmatrix} a & b & c \\ a & a & b \end{bmatrix}, \begin{bmatrix} a & b & c \\ b & b & a \end{bmatrix} \right\}.
$$

Let

$$
\lambda = \begin{bmatrix} a & b & c \\ a & a & b \end{bmatrix} \quad \text{and} \quad \eta = \begin{bmatrix} a & b & c \\ b & b & a \end{bmatrix}.
$$

Note that $\lambda^2 = X_a = \eta\lambda$ and $\eta^2 = X_b = \lambda\eta$. To show that $(\alpha, \beta)_b = (\alpha, \beta)_q$ for all $\alpha, \beta \in \overline{T}(X, Y)$, by Lemma 4.5, it suffices to show that $(\lambda)_b = (\lambda)_q$, $(\eta)_b = (\eta)_q$

and $(\lambda,\eta)_b = (\lambda,\eta)_q$. By direct multiplication, we have

$$\overline{T}(X,Y)\lambda = \{\lambda, X_a\}, \lambda\overline{T}(X,Y) = \{\lambda, X_a, X_b, \eta\}, \lambda\overline{T}(X,Y)\lambda = \{X_a\},$$
$$\overline{T}(X,Y)\eta = \{\eta, X_b\}, \eta\overline{T}(X,Y) = \{\eta, X_a, X_b, \lambda\}, \eta\overline{T}(X,Y)\eta = \{X_b\},$$
$$\lambda\overline{T}(X,Y)\eta = \{X_b\}, \eta\overline{T}(X,Y)\lambda = \{X_a\}.$$

Hence

$$\begin{aligned}
(\lambda)_b &= \lambda\overline{T}(X,Y)\lambda \cup \{\lambda\} = \{X_a, \lambda\} = \overline{T}(X,Y)\lambda \cap \lambda\overline{T}(X,Y) = (\lambda)_q,\\
(\eta)_b &= \eta\overline{T}(X,Y)\eta \cup \{\eta\} = \{X_b, \eta\} = \overline{T}(X,Y)\eta \cap \eta\overline{T}(X,Y) = (\eta)_q,\\
(\lambda,\eta)_b &= \{\lambda,\eta\}\overline{T}(X,Y)\{\lambda,\eta\} \cup \{\lambda,\eta\}\\
&= \lambda\overline{T}(X,Y)\lambda \cup \lambda\overline{T}(X,Y)\eta \cup \eta\overline{T}(X,Y)\lambda \cup \eta\overline{T}(X,Y)\eta \cup \{\lambda,\eta\}\\
&= \{X_a, X_b, \lambda, \eta\},\\
(\lambda,\eta)_q &= \overline{T}(X,Y)\{\lambda,\eta\} \cap \{\lambda,\eta\}\overline{T}(X,Y)\\
&= (\overline{T}(X,Y)\lambda \cup \overline{T}(X,Y)\eta) \cap (\lambda\overline{T}(X,Y) \cup \eta\overline{T}(X,Y))\\
&= \{\lambda, X_a, \eta, X_b\} = (\lambda,\eta)_b.
\end{aligned}$$

□

Theorem 4.7. *The semigroup $\overline{T}(X,Y)$ has the $\mathcal{BQ}$-property if and only if one of the following statements holds.*

(i) $Y = X$.

(ii) $|Y| = 1$.

(iii) $|X| \leq 3$.

Proof. Assume that (i), (ii) and (iii) are false. Then $X \smallsetminus Y \neq \varnothing, |Y| > 1$ and $|X| > 3$.

Case 1 : $|Y| = 2$. Let $Y = \{a, b\}$. Since $|X| > 3$, $|X \smallsetminus Y| > 1$. Let $c \in X \smallsetminus Y$.

Then $X \setminus \{a,b,c\} \neq \varnothing$. Define $\alpha, \beta, \gamma \in \overline{T}(X,Y)$ by

$$\alpha = \begin{bmatrix} a & b & c & X \setminus \{a,b,c\} \\ b & b & a & c \end{bmatrix}, \quad \beta = \begin{bmatrix} c & x \\ a & x \end{bmatrix}_{x \in X \setminus \{c\}},$$
$$\gamma = \begin{bmatrix} a & b & X \setminus \{a,b\} \\ b & b & c \end{bmatrix}.$$

Then

$$a\alpha\beta = b = a\gamma\alpha, \quad b\alpha\beta = b = b\gamma\alpha, \quad c\alpha\beta = a = c\gamma\alpha$$

and

$$(X \setminus \{a,b,c\})\alpha\beta = \{a\} = (X \setminus \{a,b,c\})\gamma\alpha \neq (X \setminus \{a,b,c\})\alpha,$$

so $\alpha \neq \alpha\beta = \gamma\alpha \in (\alpha)_q$ by Proposition 1.2(i). If $\alpha\beta \in (\alpha)_b$, then by Proposition 1.2(ii), $\alpha\beta = \alpha\eta\alpha$ for some $\eta \in \overline{T}(X,Y)$. Hence we have

$$a = c\alpha\beta = c\alpha\eta\alpha = (a\eta)\alpha.$$

This implies that $a\eta = c$ which is contrary to $a \in Y$ and $c \in X \setminus Y$. Thus $(\alpha)_b \neq (\alpha)_q$, so by Proposition 1.9, $\overline{T}(X,Y)$ does not have the $\mathcal{BQ}$-property.

Case 2 : $|Y| > 2$. Let a, b, c be distinct elements of Y. Let $\alpha, \beta, \gamma \in \overline{T}(X,Y)$ be defined by

$$\alpha = \begin{bmatrix} a & Y \setminus \{a\} & X \setminus Y \\ b & a & c \end{bmatrix}, \quad \beta = \begin{bmatrix} a & b & x \\ b & a & x \end{bmatrix}_{x \in X \setminus \{a,b\}},$$
$$\gamma = \begin{bmatrix} a & Y \setminus \{a\} & x \\ c & a & x \end{bmatrix}_{x \in X \setminus Y}.$$

Then

$$a\alpha\beta = a = a\gamma\alpha \neq a\alpha, \ (Y \setminus \{a\})\alpha\beta = \{b\} = (Y \setminus \{a\})\gamma\alpha$$

and

$$(X \setminus Y)\alpha\beta = \{c\} = (X \setminus Y)\gamma\alpha.$$

Thus $\alpha \neq \alpha\beta = \gamma\alpha \in (\alpha)_q$. If $\alpha\beta \in (\alpha)_b$, then $\alpha\beta = \alpha\eta\alpha$ for some $\eta \in \overline{T}(X, Y)$. Therefore we have that for every $x \in X \smallsetminus Y$,

$$c = x\alpha\beta = x\alpha\eta\alpha = (c\eta)\alpha$$

which implies that $c\eta \in X \smallsetminus Y$. It is a contradiction since $c \in Y$. Hence $(\alpha)_b \neq (\alpha)_q$, and so by Proposition 1.9, $\overline{T}(X, Y)$ does not have the $\mathcal{BQ}$-property.

If $Y = X$ or $|Y| = 1$, then $\overline{T}(X, Y)$ is regular by Corollary 2.4 which implies by Proposition 1.5 that $\overline{T}(X, Y)$ has the $\mathcal{BQ}$-property. If $|X| = 3$ and $|Y| = 2$, then by Lemma 4.6 and Proposition 1.9, $\overline{T}(X, Y)$ has the $\mathcal{BQ}$-property.

Hence the theorem is proved. □

Two direct consequences of Proposition 1.5, Corollary 2.4, Theorem 4.7 and the proof of Lemma 4.6 are as follows :

Corollary 4.8. *If $|X| \neq 3$, then the following statements are equivalent.*

(i) $\overline{T}(X, Y)$ *is a $\mathcal{BQ}$-semigroup.*

(ii) $Y = X$ *or* $|Y| = 1$.

(iii) $\overline{T}(X, Y)$ *is a regular semigroup.*

Corollary 4.9. *The semigroup $\overline{T}(X, Y)$ is a nonregular $\mathcal{BQ}$-semigroup if and only if $|X| = 3$ and $|Y| = 2$. Hence for each set X with $|X| = 3$, there are exactly 3 semigroups $\overline{T}(X, Y)$ which are nonregular $\mathcal{BQ}$-semigroups, and each of such $\overline{T}(X, Y)$ contains 12 elements.*

Remark 4.10. (i) From Corollary 2.2 and Theorem 4.3, we have that for $|Y| > 1$ and $Y \subsetneq X$, $T(X, Y)$ is a $\mathcal{BQ}$-semigroup but not a regular semigroup.

(ii) By Lemma 4.2, $T(X, Y)$ is a left ideal of $T(X)$. But for $\alpha \in T(X, Y)$ and $\beta \in \overline{T}(X, Y)$, $X\alpha\beta \subseteq Y\beta \subseteq Y$, so $T(X, Y)$ is an ideal of $\overline{T}(X, Y)$. We have that $1_X \in \overline{T}(X, Y) \smallsetminus T(X, Y)$ if $Y \neq X$. Hence if $Y \neq X$, then $\overline{T}(X, Y)$ is neither left nor right simple. Therefore we deduce from Corollary 4.9 that if $|X| = 3$ and $|Y| = 2$, then $\overline{T}(X, Y)$ is an example of $\mathcal{BQ}$-semigroup which is neither regular nor left [right] simple (see Proposition 1.5 and Proposition 1.6).

CHAPTER V

THE $\mathcal{BQ}$-PROPERTY OF SEMIGROUPS OF LINEAR TRANSFORMATIONS

In this chapter, the semigroups $L_F(V,W)$, $\overline{L}_F(V,W)$ and $K_F(V,W)$ are studied. We have similarly to $T(X,Y)$ that $L_F(V,W)$ always has the $\mathcal{BQ}$-property. Moreover, $K_F(V,W)$ has also the $\mathcal{BQ}$-property. However, the characterization of $\overline{L}_F(V,W)$ to have the $\mathcal{BQ}$-property also depends on the field F.

Throughout this chapter, let V be a vector space over a field F and W a subspace of V. Recall that

$$L_F(V,W) = \{\alpha \in L_F(V) \mid \operatorname{ran} \alpha \subseteq W\},$$
$$\overline{L}_F(V,W) = \{\alpha \in L_F(V) \mid W\alpha \subseteq W\},$$
$$K_F(V,W) = \{\alpha \in L_F(V) \mid W \subseteq \ker \alpha\}.$$

By the same proof given for Lemma 4.2, we have

Lemma 5.1. *The semigroup $L_F(V,W)$ is a left ideal of $L_F(V)$.*

Lemma 5.2. *The semigroup $K_F(V,W)$ is a right ideal of $L_F(V)$.*

Proof. Since $W \subseteq \ker \alpha \subseteq \ker \alpha\beta$ for all $\alpha \in K_F(V,W)$ and $\beta \in L_F(V)$, it follows that $K_F(V,W)L_F(V) \subseteq K_F(V,W)$. Hence $K_F(V,W)$ is a right ideal of $L_F(V)$. □

Hence Lemma 4.1, Lemma 5.1 and Lemma 5.2 yield the following results.

Theorem 5.3. *The semigroup $L_F(V,W)$ always has the $\mathcal{BQ}$-property.*

Theorem 5.4. *The semigroup $K_F(V,W)$ always has the $\mathcal{BQ}$-property.*

Let $n \in \mathbb{N}$, $\{e_1, \ldots, e_n\}$ be the standard basis of F^n over F, U_1 and U_2 subspaces of F^n spanned by $\{e_1, \ldots, e_k\}$ and $\{e_{k+1}, \ldots, e_n\}$, respectively. By Lemma 3.8 and Lemma 3.9, we have

$$C_n(F,k) \cong L_F(F^n, U_1) \text{ and } R_n(F,k) \cong K_F(F^n, U_2)$$

where for $k \in \mathbb{N}$ and $k \leq n$,

$$C_n(F,k) = \{A \in M_n(F) \mid A_{ij} = 0 \text{ for all } i,j \in \{1,\ldots,n\} \text{ and } j > k\},$$
$$R_n(F,k) = \{A \in M_n(F) \mid A_{ij} = 0 \text{ for all } i,j \in \{1,\ldots,n\} \text{ and } i > k\}.$$

From these facts, Theorem 5.3 and Theorem 5.4, we obtain the following corollary.

Corollary 5.5. *For $n, k \in \mathbb{N}$ with $k \leq n$, the semigroups $C_n(F,k)$ and $R_n(F,k)$ have the $\mathcal{BQ}$-property.*

To prove the main theorem, the following lemma is also needed. Lemma 4.5 and Theorem 3.3 are used in the course of its proof.

Lemma 5.6. *If $F = \mathbb{Z}_2, \dim_F V = 2$ and $\dim_F W = 1$, then for all $\alpha, \beta \in \overline{L}_F(V,W), (\alpha,\beta)_b = (\alpha,\beta)_q$ in $\overline{L}_F(V,W)$.*

Proof. Let $\{w\}$ be a basis of W and $\{w, u\}$ a basis of V. Since $F = \mathbb{Z}_2$, it follows that $W = \{0, w\}$ and $V = \{0, w, u, u+w\}$. Clearly, both $\{u, u+w\}$ and $\{w, u+w\}$ are also bases of V. Thus $\langle w \rangle \cap \langle u \rangle = \langle w \rangle \cap \langle u+w \rangle = \langle u \rangle \cap \langle u+w \rangle = \{0\}$. All the elements of $\overline{L}_F(V,W)$ defined on the basis $\{w, u\}$ of V can be given as follows:

$$\overline{L}_F(V,W) = \Big\{0, 1_V, \begin{bmatrix} w & u \\ 0 & w \end{bmatrix}, \begin{bmatrix} w & u \\ 0 & u \end{bmatrix}, \begin{bmatrix} w & u \\ 0 & w+u \end{bmatrix}, \begin{bmatrix} w & u \\ w & 0 \end{bmatrix}, \begin{bmatrix} w & u \\ w & w \end{bmatrix}, \begin{bmatrix} w & u \\ w & w+u \end{bmatrix}\Big\}$$

By Theorem 3.3,

$$\overline{L}_F(V,W) \smallsetminus \text{Reg}\,(\overline{L}_F(V,W)) = \Big\{ \begin{bmatrix} w & u \\ 0 & w \end{bmatrix} \Big\}.$$

Let $\lambda = \begin{bmatrix} w & u \\ 0 & w \end{bmatrix}$. Note that $\lambda^2 = 0$. To prove the lemma, by Lemma 4.5, it suffices to show that $(\lambda)_b = (\lambda)_q$. By direct multiplication, we have

$$\overline{L}_F(V,W)\lambda = \{0,\lambda\},\ \lambda\overline{L}_F(V,W) = \{0,\lambda\},\ \lambda\overline{L}_F(V,W)\lambda = \{0\}.$$

Consequently,

$$(\lambda)_b = \lambda\overline{L}_F(V,W)\lambda \cup \{\lambda\} = \{0,\lambda\} = \overline{L}_F(V,W)\lambda \ \cap\ \lambda\overline{L}_F(V,W) = (\lambda)_q.$$

□

Theorem 5.7. *The semigroup $\overline{L}_F(V,W)$ has the $\mathcal{BQ}$-property if and only if one of the following statements holds.*

(i) $W = V$.

(ii) $W = \{0\}$.

(iii) $F = \mathbb{Z}_2, \dim_F V = 2$ *and* $\dim_F W = 1$.

Proof. Assume that (i), (ii) and (iii) are false. Then (1) $\{0\} \neq W \subsetneq V$ and (2) $F \neq \mathbb{Z}_2, \dim_F V > 2$ or $\dim_F W > 1$. Let B_1 be a basis of W and B a basis of V containing B_1. Then $\mathsf{B}_1 \neq \varnothing$ and $\mathsf{B} \smallsetminus \mathsf{B}_1 \neq \varnothing$.

Case 1 : $F \neq \mathbb{Z}_2$. Let $a \in F \smallsetminus \{0,1\}, w \in \mathsf{B}_1$ and $u \in \mathsf{B} \smallsetminus \mathsf{B}_1$. Define $\alpha, \beta, \gamma \in \overline{L}_F(V,W)$ by

$$\alpha = \begin{bmatrix} u & \mathsf{B} \smallsetminus \{u\} \\ w & 0 \end{bmatrix},\ \beta = \begin{bmatrix} w & \mathsf{B} \smallsetminus \{w\} \\ aw & 0 \end{bmatrix},\ \gamma = \begin{bmatrix} u & \mathsf{B} \smallsetminus \{u\} \\ au & 0 \end{bmatrix}.$$

Then we have

$$\alpha\beta = \begin{bmatrix} u & \mathsf{B} \smallsetminus \{u\} \\ aw & 0 \end{bmatrix} = \gamma\alpha.$$

Since $a \neq 1$, we have $\alpha\beta \neq \alpha$. By Proposition 1.2(i), $\alpha\beta \in (\alpha)_q$. Suppose that $\alpha\beta \in (\alpha)_b$. By Proposition 1.2(ii), $\alpha\beta = \alpha\eta\alpha$ for some $\eta \in \overline{L}_F(V,W)$. Then

$$aw = u\alpha\beta = u\alpha\eta\alpha = (w\eta)\alpha.$$

But

$$w\eta \in W \quad \text{and} \quad W\alpha = \langle \mathsf{B}_1 \rangle \, \alpha \subseteq \langle \mathsf{B} \smallsetminus \{u\} \rangle \, \alpha = \{0\},$$

so $aw = 0$ which is contrary to $a \neq 0$. Thus $(\alpha)_q \neq (\alpha)_b$, so $\overline{L}_F(V, W)$ does not have the $\mathcal{BQ}$-property by Proposition 1.9.

Case 2 : $\dim_F W > 1$. Then $|\mathsf{B}_1| > 1$. Let $w_1, w_2 \in \mathsf{B}_1$ be such that $w_1 \neq w_2$ and $u \in \mathsf{B} \smallsetminus \mathsf{B}_1$. Define $\alpha, \beta, \gamma \in \overline{L}_F(V, W)$ by

$$\alpha = \begin{bmatrix} w_1 & u & \mathsf{B} \smallsetminus \{w_1, u\} \\ w_2 & w_1 & 0 \end{bmatrix}, \; \beta = \begin{bmatrix} w_1 & \mathsf{B} \smallsetminus \{w_1\} \\ w_1 & 0 \end{bmatrix}, \; \gamma = \begin{bmatrix} u & \mathsf{B} \smallsetminus \{u\} \\ u & 0 \end{bmatrix}.$$

Then we have

$$\alpha\beta = \begin{bmatrix} u & \mathsf{B} \smallsetminus \{u\} \\ w_1 & 0 \end{bmatrix} = \gamma\alpha \neq \alpha,$$

so $\alpha\beta \in (\alpha)_q$. If $\alpha\beta \in (\alpha)_b$, then $\alpha\beta = \alpha\eta\alpha$ for some $\eta \in \overline{L}_F(V, W)$. Thus

$$w_1 = u\alpha\beta = u\alpha\eta\alpha = (w_1\eta)\alpha.$$

Since $w_1\eta \in W = \langle \mathsf{B}_1 \rangle$, we have

$$w_1\eta = aw_1 + v \quad \text{for some} \quad a \in F \quad \text{and} \quad v \in \langle \mathsf{B}_1 \smallsetminus \{w_1\} \rangle \, .$$

But $\mathsf{B}_1 \smallsetminus \{w_1\} \subseteq \mathsf{B} \smallsetminus \{w_1, u\}$, so $v\alpha = 0$. Consequently, $w_1 = (aw_1 + v)\alpha = aw_2$ which is contrary to the independence of w_1 and w_2. By Proposition 1.9, $\overline{L}_F(V, W)$ does not have the $\mathcal{BQ}$-property.

Case 3 : $\dim_F V > 2$ and $\dim_F W = 1$. Then $|\mathsf{B}_1| = 1$ and $|\mathsf{B} \smallsetminus \mathsf{B}_1| > 1$. Let $\mathsf{B}_1 = \{w\}$ and $u_1, u_2 \in \mathsf{B} \smallsetminus \mathsf{B}_1$ be such that $u_1 \neq u_2$. Let $\alpha, \beta, \gamma \in \overline{L}_F(V, W)$ be defined by

$$\alpha = \begin{bmatrix} u_1 & u_2 & \mathsf{B} \smallsetminus \{u_1, u_2\} \\ w & u_1 & 0 \end{bmatrix}, \; \beta = \begin{bmatrix} w & \mathsf{B} \smallsetminus \{w\} \\ w & 0 \end{bmatrix}, \; \gamma = \begin{bmatrix} u_1 & \mathsf{B} \smallsetminus \{u_1\} \\ u_1 & 0 \end{bmatrix}.$$

Then we have

$$\alpha\beta = \begin{bmatrix} u_1 & \mathsf{B} \smallsetminus \{u_1\} \\ w & 0 \end{bmatrix} = \gamma\alpha \neq \alpha,$$

so $\alpha\beta \in (\alpha)_q$. Suppose that $\alpha\beta \in (\alpha)_b$. It follows that $\alpha\beta = \alpha\eta\alpha$ for some $\eta \in \overline{L}_F(V, W)$. Thus

$$w = u_1\alpha\beta = u_1\alpha\eta\alpha = (w\eta)\alpha.$$

But

$$w\eta \in W = \langle w \rangle \quad \text{and} \quad w\alpha = 0,$$

so $w = (w\eta)\alpha = 0$, a contradiction. Hence $(\alpha)_q \neq (\alpha)_b$, so $\overline{L}_F(V, W)$ does not have the $\mathcal{BQ}$-property, as before.

For the converse, if (i) or (ii) holds, then $\overline{L}_F(V, W) = L_F(V)$ which has the $\mathcal{BQ}$-property by Proposition 1.5. If (iii) holds, then $\overline{L}_F(V, W)$ has the $\mathcal{BQ}$-property by Proposition 1.9 and Lemma 5.6. □

The following corollaries follow directly from Proposition 1.5, Corollary 3.4, Theorem 5.7 and the proof of Lemma 5.6.

Corollary 5.8. *If $F \neq \mathbb{Z}_2$, $\dim_F V \neq 2$ or $\dim_F W \neq 1$, then the following statements are equivalent.*
(i) *$\overline{L}_F(V, W)$ is a $\mathcal{BQ}$-semigroup.*
(ii) *$W = V$ or $W = \{0\}$.*
(iii) *$\overline{L}_F(V, W)$ is a regular semigroup.*

Corollary 5.9. *The semigroup $\overline{L}_F(V, W)$ is a nonregular $\mathcal{BQ}$-semigroup if and only if $F = \mathbb{Z}_2, \dim_F V = 2$ and $\dim_F W = 1$. Hence if $F = \mathbb{Z}_2$ and $\dim_F V = 2$, there are exactly 3 semigroups $\overline{L}_F(V, W)$ which are nonregular $\mathcal{BQ}$-semigroups, and each of such $\overline{L}_F(V, W)$ contains 8 elements.*

Remark 5.10. (i) By Corollary 3.2, Corollary 3.6, Theorem 5.3 and Theorem 5.4, we have that if $\{0\} \neq W \subsetneq V$, then $L_F(V, W)$ and $K_F(V, W)$ are $\mathcal{BQ}$-semigroups which are not regular.

(ii) By Corollary 3.13, Corollary 3.17 and Corollary 5.5, we have that if $k < n$, then $C_n(F, k)$ and $R_n(F, k)$ are $\mathcal{BQ}$-semigroups which are not regular.

(iii) We also have that $L_F(V,W)$ is an ideal of $\overline{L}_F(V,W)$ (see Remark 4.10). Consequently, if $\{0\} \neq W \subsetneq V$, then $\overline{L}_F(V,W)$ is neither left nor right 0-simple. Hence if $F = \mathbb{Z}_2, \dim_F V = 2$ and $\dim_F W = 1$, then $\overline{L}_F(V,W)$ is a $\mathcal{BQ}$-semigroup which is neither regular nor left [right] 0-simple.

CHAPTER VI

THE $\mathcal{BQ}$-PROPERTY OF RINGS OF LINEAR TRANSFORMATIONS

We consider the rings $(L_F(V,W),+,\circ)$, $(\overline{L}_F(V,W),+,\circ)$ and $(K_F(V,W),+,\circ)$ in this chapter. We characterize when they have the $\mathcal{BQ}$-property.

It is shown that for a ring $(R,+,\cdot)$, if $(R,\cdot)$ is a $\mathcal{BQ}$-semigroup, then $(R,+,\cdot)$ is a $\mathcal{BQ}$-ring. However, the converse is not true in general. It is shown by the ring $(\overline{L}_F(V,W),+,\circ)$ for some V,W and F.

Throughout this chapter, V is a vector space over a field F and W is a subspace of V.

Since for nonempty subsets A,B of a ring $(R,+,\cdot)$, we have that

$$\text{in the semigroup } (R,\cdot),\ AB=\{ab \mid a\in A \text{ and } b\in B\},$$
$$\text{in the ring } (R,+,\cdot),\ AB=\left\{\sum_{i=1}^{k} a_ib_i \;\middle|\; a_i\in A, b_i\in B \text{ and } k\in\mathbb{N}\right\},$$

the following lemma is immediately obtained.

Lemma 6.1. *Let $(R,+,\cdot)$ be a ring and $A\subseteq R$. Then :*
(i) *If A is a bi-ideal [quasi-ideal] of the ring $(R,+,\cdot)$, then A is a bi-ideal [quasi-ideal] of the semigroup $(R,\cdot)$.*
(ii) *If A is a bi-ideal [quasi-ideal] of the semigroup $(R,\cdot)$ and A is a subring of the ring $(R,+,\cdot)$, then A is a bi-ideal [quasi-ideal] of the ring $(R,+,\cdot)$.*

Note that this fact is also true for left ideals, right ideals and ideals.

The following result is obtained directly from Lemma 6.1.

Lemma 6.2. *Let $(R,+,\cdot)$ be a ring. If $(R,\cdot)$ is a $\mathcal{BQ}$-semigroup, then $(R,+,\cdot)$ is a $\mathcal{BQ}$-ring.*

Theorem 6.3. *The ring $(L_F(V,W),+,\circ)$ always has the $\mathcal{BQ}$-property.*

Proof. This follows directly from Theorem 5.3 and Lemma 6.2. □

Theorem 6.4. *The ring $(K_F(V,W),+,\circ)$ always has the $\mathcal{BQ}$-property.*

Proof. It follows from Theorem 5.4 and Lemma 6.2. □

From Theorem 5.7, we have that the semigroup $(\overline{L}_F(V,W),\circ)$ has the $\mathcal{BQ}$-property if and only if (i) $W=V$, (ii) $W=\{0\}$ or (iii) $F=\mathbb{Z}_2$, $\dim_F W=1$ and $\dim_F V=2$. By Lemma 6.2, if one of (i), (ii) and (iii) hold, then the ring $(\overline{L}_F(V,W),+,\circ)$ has the $\mathcal{BQ}$-property. Our main result of this chapter is to show that the ring $(\overline{L}_F(V,W),+,\circ)$ has the $\mathcal{BQ}$-property if and only if one of the following statements holds.

(i) $W=V$.

(ii) $W=\{0\}$.

(iii) $F=\mathbb{Z}_p$ for some prime p and $\dim_F W=1$.

(iv) $F=\mathbb{Z}_p$ for some prime p and $\dim_F(V/W)=1$.

Hence we deduce that the converse of Lemma 6.2 need not be generally true.

Lemma 6.5. *If B is a bi-ideal of a semigroup [ring] A, then $(BA\cap AB)\cap Reg\,(A)\subseteq B$.*

Proof. Let $x\in(BA\cap AB)\cap\mathrm{Reg}\,(A)$. Then $x=xyx$ for some $y\in A$. This implies that

$$x=xyx\in BAyAB\subseteq BAB\subseteq B.$$

□

Lemma 6.6. *If $\{0\}\neq W\subsetneq V$ and the ring $(\overline{L}_F(V,W),+,\circ)$ has the $\mathcal{BQ}$-property, then $F=\mathbb{Z}_p$ for some prime p.*

Proof. Let B_1 be a basis of W and B a basis of V containing B_1. By assumption, $\mathsf{B}_1 \neq \varnothing$ and $\mathsf{B} \smallsetminus \mathsf{B}_1 \neq \varnothing$. Let $w \in \mathsf{B}_1$ and $u \in \mathsf{B} \smallsetminus \mathsf{B}_1$.

Assume that $F \neq \mathbb{Z}_p$ for any prime p. This implies that $\mathbb{Z}1_F \subsetneq F$. Let $a \in F \smallsetminus \mathbb{Z}1_F$. Define $\alpha, \beta, \gamma \in L_F(V, W)$ by

$$\alpha = \begin{bmatrix} u & \mathsf{B} \smallsetminus \{u\} \\ w & 0 \end{bmatrix},\ \beta = \begin{bmatrix} w & \mathsf{B} \smallsetminus \{w\} \\ aw & 0 \end{bmatrix},\ \gamma = \begin{bmatrix} u & \mathsf{B} \smallsetminus \{u\} \\ au & 0 \end{bmatrix}.$$

Then

$$\alpha\beta = \begin{bmatrix} u & \mathsf{B} \smallsetminus \{u\} \\ aw & 0 \end{bmatrix} = \gamma\alpha \in \alpha\overline{L}_F(V, W) \cap \overline{L}_F(V, W)\alpha \subseteq (\alpha)_q.$$

Suppose that $\alpha\beta \in (\alpha)_b$. Since $(\alpha)_b = \mathbb{Z}\alpha + \alpha\overline{L}_F(V, W)\alpha$, we have $\alpha\beta = n\alpha + \alpha\lambda\alpha$ for some $n \in \mathbb{Z}$ and $\lambda \in \overline{L}_F(V, W)$. Consequently,

$$\begin{aligned} aw = u(\alpha\beta) = {} & u(n\alpha + \alpha\lambda\alpha) \\ = {} & nw + (w\lambda)\alpha \\ = {} & nw + 0 \qquad \text{since } \ w\lambda \in W \ \text{ and } \ W\alpha = \{0\} \\ = {} & nw. \end{aligned}$$

But $w \neq 0$, so $a = n1_F \in \mathbb{Z}1_F$ which is a contradiction. Hence $\alpha\beta \notin (\alpha)_b$. This proves that $(\overline{L}_F(V, W), +, \circ)$ does not have the $\mathcal{BQ}$-property.

Therefore the lemma is proved. □

Lemma 6.7. *Assume that* $\dim_F W = 1$ *and* $W = \langle w \rangle$ *and* $\alpha \in K_F(V, W) \smallsetminus$ *Reg* $(\overline{L}_F(V, W))$. *Then the following statements hold.*

(i) $w \in$ *ker* $\alpha \cap$ *ran* α.

(ii) *Let* B_1 *be a basis of ker* α *containing* w, B_2 *a basis of ran* α *containing* w *and for each* $v \in \mathsf{B}_2$, *let* $v' \in v\alpha^{-1}$. *If* $\alpha_1, \alpha_2 \in L_F(V)$ *are defined on the basis* $\mathsf{B}_1 \cup \{v' \mid v \in \mathsf{B}_2\}$ *of* V *by*

$$\alpha_1 = \begin{bmatrix} \mathsf{B}_1 & w' & v' \\ 0 & w & 0 \end{bmatrix}_{v \in \mathsf{B}_2 \smallsetminus \{w\}} \quad \textit{and} \quad \alpha_2 = \begin{bmatrix} \mathsf{B}_1 & w' & v' \\ 0 & 0 & v \end{bmatrix}_{v \in \mathsf{B}_2 \smallsetminus \{w\}},$$

then

$$\alpha = \begin{bmatrix} \mathsf{B}_1 & w' & v' \\ 0 & w & v \end{bmatrix}_{v \in \mathsf{B}_2 \smallsetminus \{w\}} = \alpha_1 + \alpha_2, \tag{1}$$

$$\alpha_1 \in L_F(V,W) \cap K_F(V,W) \quad \text{and} \quad \alpha_2 \in K_F(V,W) \cap (\alpha \overline{L}_F(V,W)\alpha). \tag{2}$$

Proof. First, we note that $W = Fw$.

(i) Since $\alpha \in K_F(V,W)$, $w \in \ker\alpha$. Since $\alpha \notin \mathrm{Reg}\,(\overline{L}_F(V,W))$, by Theorem 3.3, $\mathrm{ran}\,\alpha \cap W \neq W\alpha = \{0\}$. But $\mathrm{ran}\,\alpha \cap W$ is a subspace of W and $\dim_F W = 1$, so $\mathrm{ran}\,\alpha \cap W = W = Fw$. Thus $w \in \mathrm{ran}\,\alpha$.

(ii) Clearly, (1) holds, $\alpha_1 \in L_F(V,W) \cap K_F(V,W)$ and $\alpha_2 \in K_F(V,W)$. To prove (2), it remains to show that $\alpha_2 \in \alpha\overline{L}_F(V,W)\alpha$. Let B_3 be a basis of V containing B_2. Define $\beta \in \overline{L}_F(V,W)$ by

$$\beta = \begin{bmatrix} w & v & \mathsf{B}_3 \smallsetminus \mathsf{B}_2 \\ 0 & v' & 0 \end{bmatrix}_{v \in \mathsf{B}_2 \smallsetminus \{w\}}.$$

Then

$$\mathsf{B}_1\alpha\beta\alpha = \{0\} = \mathsf{B}_1\alpha_2, \quad w'\alpha\beta\alpha = w\beta\alpha = \{0\} = w'\alpha_2,$$
$$\text{for every } v \in \mathsf{B}_2 \smallsetminus \{w\},\ v'\alpha\beta\alpha = v\beta\alpha = v'\alpha = v = v'\alpha_2,$$

so we deduce that $\alpha_2 = \alpha\beta\alpha \in \alpha\overline{L}_F(V,W)\alpha$. □

Lemma 6.8. *Assume that* $F = \mathbb{Z}_p$ *and* $\dim_F W = 1$. *If* B *is a bi-ideal of* $(\overline{L}_F(V,W),+,\circ)$ *and* $B \subseteq K_F(V,W)$, *then*

$$BL_F(V,W) \subseteq B + BK_F(V,W).$$

Proof. Let $w \in W \smallsetminus \{0\}$. Then $W = \mathbb{Z}_p w$. Since $W\alpha\beta \subseteq W\beta = \{0\}$ for all $\alpha \in \overline{L}_F(V,W)$ and $\beta \in K_F(V,W)$, we have that $K_F(V,W)$ is a left ideal of $(\overline{L}_F(V,W),+,\circ)$. Hence by Lemma 5.2, $K_F(V,W)$ is an ideal of $(\overline{L}_F(V,W),+,\circ)$. Since $BK_F(V,W) \subseteq B + BK_F(V,W)$, it remains to show that $B(L_F(V,W) \smallsetminus$

$K_F(V,W)) \subseteq B + BK_F(V,W)$. Let $\alpha \in B$ and $\beta \in L_F(V,W) \smallsetminus K_F(V,W)$. Then $w\beta \in W \smallsetminus \{0\}$, so $w\beta = kw$ for some $k \in \mathbb{Z}_p \smallsetminus \{0\}$.

Case 1 : $\alpha \in$ Reg $(\overline{L}_F(V,W))$. Then $\alpha \in \alpha\overline{L}_F(V,W)\alpha$ and thus

$$\begin{aligned} \alpha\beta &\in B\overline{L}_F(V,W)B(L_F(V,W) \smallsetminus K_F(V,W)) \\ &\subseteq B\overline{L}_F(V,W)K_F(V,W)(L_F(V,W) \smallsetminus K_F(V,W)) \quad \text{since } B \subseteq K_F(V,W) \\ &\subseteq BK_F(V,W) \qquad \text{since } K_F(V,W) \text{ is an ideal of } \overline{L}_F(V,W) \\ &\qquad\qquad \text{and } L_F(V,W) \subseteq \overline{L}_F(V,W) \\ &\subseteq B + BK_F(V,W). \end{aligned}$$

Case 2 : $\alpha \notin$ Reg $(\overline{L}_F(V,W))$. Since $\alpha \in B \subseteq K_F(V,W)$, we have $\alpha \in K_F(V,W) \smallsetminus$Reg $(\overline{L}_F(V,W))$. Define $\mathsf{B}_1, \mathsf{B}_2, \mathsf{B}_3$, α_1, α_2 and β as in the assumption and the proof of Lemma 6.7(ii). Then

$$\alpha = \alpha_1 + \alpha_2, \quad \alpha_1 \in L_F(V,W) \cap K_F(V,W)$$

and

$$\alpha_2 \in K_F(V,W) \cap (\alpha\overline{L}_F(V,W)\alpha).$$

Then we deduce that $\alpha_2 \in B\overline{L}_F(V,W)B \subseteq B$, so $\alpha_1 = \alpha - \alpha_2 \in B$. Thus $k\alpha_1 \in B$. Let $\beta' \in L_F(V)$ be defined by

$$\beta' = \begin{bmatrix} w & v \\ 0 & v\beta \end{bmatrix}_{v \in \mathsf{B}_3 \smallsetminus \{w\}}.$$

Then $\beta' \in K_F(V,W)$. But since

$$\begin{aligned} &\mathsf{B}_1(k\alpha_1 + \alpha_2\beta') = \{0\} = \mathsf{B}_1(\alpha\beta), \\ &w'(k\alpha_1 + \alpha_2\beta') = kw = w\beta = w'\alpha\beta \\ &\text{and for all } v \in \mathsf{B}_2 \smallsetminus \{w\},\ v'(k\alpha_1 + \alpha_2\beta') = v'\alpha_2\beta' = v\beta' = v\beta = v'\alpha\beta, \end{aligned}$$

it follows that $\alpha\beta = k\alpha_1 + \alpha_2\beta' \in B + BK_F(V,W)$.

Therefore the lemma is proved. □

Lemma 6.9. *If* $F = \mathbb{Z}_p$ *and* $\dim_F W = 1$*, then the ring* $(\overline{L}_F(V,W), +, \circ)$ *has the* $\mathcal{BQ}$*-property.*

Proof. Let $w \in W \smallsetminus \{0\}$. Then $W = \mathbb{Z}_p w$. Let B be a bi-ideal of $(\overline{L}_F(V,W), +, \circ)$. Then $B\overline{L}_F(V,W)B \subseteq B$. To show that B is a quasi-ideal of $(\overline{L}_F(V,W), +, \circ)$, let $\alpha \in B\overline{L}_F(V,W) \cap \overline{L}_F(V,W)B$. If $\alpha \in \text{Reg}\,(\overline{L}_F(V,W))$, then by Lemma 6.5, $\alpha \in B$.

Next, assume that $\alpha \notin \text{Reg}\,(\overline{L}_F(V,W))$. Then $\text{ran}\,\alpha \cap W \neq W\alpha$. Since $W\alpha \subseteq W$ and $\dim_F W = 1$, it follows that $W\alpha = W$ or $W\alpha = \{0\}$. If $W\alpha = W$, then

$$W\alpha = W\alpha \cap W \subseteq \text{ran}\,\alpha \cap W = \text{ran}\,\alpha \cap W\alpha = W\alpha,$$

so we have $\text{ran}\,\alpha \cap W = W\alpha$, a contradiction. Thus $W\alpha = \{0\}$. Hence $\alpha \in K_F(V,W) \smallsetminus \text{Reg}\,(\overline{L}_F(V,W))$. Let $\mathsf{B}_1, \mathsf{B}_2$, α_1 and α_2 be as in the assumption of Lemma 6.7(ii). Then by Lemma 6.7(ii),

$$\alpha = \alpha_1 + \alpha_2, \quad \alpha_1 \in L_F(V,W) \cap K_F(V,W)$$

and

$$\alpha_2 \in K_F(V,W) \cap (\alpha\overline{L}_F(V,W)\alpha).$$

Since $\alpha \in B\overline{L}_F(V,W) \cap \overline{L}_F(V,W)B$ and $1_V \in \overline{L}_F(V,W)$, it follows that

$$\begin{aligned} \alpha_2 \in \alpha\overline{L}_F(V,W)\alpha \;&\subseteq\; B\overline{L}_F(V,W)\overline{L}_F(V,W)\overline{L}_F(V,W)B \\ &\subseteq B \;\subseteq\; B\overline{L}_F(V,W) \cap \overline{L}_F(V,W)B. \end{aligned} \tag{1}$$

Hence we have $\alpha_1 = \alpha - \alpha_2 \in B\overline{L}_F(V,W) \cap \overline{L}_F(V,W)B$. We claim that $\alpha_1 \in B$.

Case 1 : There is a $\beta \in B$ such that $w\beta \neq 0$. Since $W\beta \subseteq W = \mathbb{Z}_p w$, we have $w\beta = kw$ for some $k \in \mathbb{Z}_p \smallsetminus \{0\}$. We have that

$$\alpha_1 = \begin{bmatrix} \mathsf{B}_1 & w' & v' \\ 0 & w & 0 \end{bmatrix}_{v \in \mathsf{B}_2 \smallsetminus \{w\}}.$$

Then

$$\alpha_1\beta = \begin{bmatrix} \mathsf{B}_1 & w' & v' \\ 0 & kw & 0 \end{bmatrix}_{v \in \mathsf{B}_2 \smallsetminus \{w\}}.$$

But $\alpha_1 \in B\overline{L}_F(V,W)$, thus $\alpha_1\beta \in B\overline{L}_F(V,W)B \subseteq B$. Since $k\mathbb{Z}_p = \mathbb{Z}_p$, it follows that

$$\alpha_1 \in \mathbb{Z}_p \begin{bmatrix} \mathsf{B}_1 & w' & v' \\ 0 & kw & 0 \end{bmatrix}_{v \in \mathsf{B}_2 \smallsetminus \{w\}} \subseteq B.$$

Case 2 : $w\beta = 0$ for all $\beta \in B$. Then $B \subseteq K_F(V,W)$ and hence B is a bi-ideal of the ring $(K_F(V,W),+,\circ)$. By Theorem 6.4, $(K_F(V,W),+,\circ)$ is a $\mathcal{BQ}$-ring. It follows that B is a quasi-ideal of the ring $(K_F(V,W),+,\circ)$ and thus $BK_F(V,W) \cap K_F(V,W)B \subseteq B$. Since $L_F(V)$ is regular, $\alpha_1 \in \alpha_1 L_F(V)\alpha_1$. But $\alpha_1 \in L_F(V,W) \cap K_F(V,W)$ and $L_F(V,W)$ and $K_F(V,W)$ are a left ideal and a right ideal of $L_F(V)$, respectively and $\alpha_1 \in B\overline{L}_F(V,W) \cap \overline{L}_F(V,W)B$, so we have

$$\alpha_1 \in \alpha_1 L_F(V)\alpha_1 \in B\overline{L}_F(V,W)L_F(V)L_F(V,W) \subseteq BL_F(V,W), \tag{2}$$

$$\alpha_1 \in \alpha_1 L_F(V)\alpha_1 \in K_F(V,W)L_F(V)\overline{L}_F(V,W)B \subseteq K_F(V,W)B. \tag{3}$$

Since $B \subseteq K_F(V,W)$, by Lemma 6.8, $BL_F(V,W) \subseteq B + BK_F(V,W)$. From (2), we have $\alpha_1 = \gamma + \lambda$ for some $\gamma \in B$ and $\lambda \in BK_F(V,W)$. Thus

$$\begin{aligned} \lambda &= \alpha_1 - \gamma \\ &\in K_F(V,W)B + B && \text{by (3)} \\ &\subseteq \overline{L}_F(V,W)B && \text{since } 1_V \in \overline{L}_F(V,W). \end{aligned}$$

Therefore we have $\lambda \in BK_F(V,W) \cap \overline{L}_F(V,W)B$. Since $L_F(V)$ is regular, $\lambda \in \lambda L_F(V)\lambda$. Thus

$$\begin{aligned} \lambda \in\ & \lambda L_F(V)\lambda \\ \subseteq\ & BK_F(V,W)L_F(V)\overline{L}_F(V,W)B \\ \subseteq\ & K_F(V,W)B && \text{since } K_F(V,W) \text{ is a right ideal of } (L_F(V),+,\circ). \end{aligned}$$

Consequently, $\lambda \in BK_F(V,W) \cap K_F(V,W)B \subseteq B$ since $K_F(V,W) \subseteq \overline{L}_F(V,W)$. Thus $\alpha_1 = \gamma + \lambda \in B + B \subseteq B$.

Hence $\alpha = \alpha_1 + \alpha_2 \in B + B \subseteq B$ by (1). This shows that $B\overline{L}_F(V,W) \cap \overline{L}_F(V,W)B \subseteq B$, as required.

Therefore the proof is completed. $\square$

Lemma 6.10. *Assume that $V = W + \langle u \rangle$ where $u \in V \smallsetminus W$ and $\alpha \in \overline{L}_F(V,W) \smallsetminus$ Reg $(\overline{L}_F(V,W))$. Then the following statements hold.*

(i) $u\alpha \in W \smallsetminus W\alpha$.

(ii) *$ker\ \alpha \subseteq W$.*

(iii) *Let B_1 be a basis of $ker\ \alpha$, B_2 a basis of $W\alpha$ and for each $w \in \mathsf{B}_2$, let $w' \in w\alpha^{-1} \cap W$, then $\mathsf{B}_1 \cup \{w' \mid w \in \mathsf{B}_2\}$ is a basis of W, $\mathsf{B}_2 \cup \{u\alpha\}$ is a basis of ran α and $\mathsf{B}_1 \cup \{u\} \cup \{w' \mid w \in \mathsf{B}_2\}$ is a basis of V.*

(iv) *If $\alpha_1, \alpha_2 \in L_F(V)$ are defined on the basis $\mathsf{B}_1 \cup \{u\} \cup \{w' \mid w \in \mathsf{B}_2\}$ of V by*

$$\alpha_1 = \begin{bmatrix} \mathsf{B}_1 & u & w' \\ 0 & 0 & w \end{bmatrix}_{w \in \mathsf{B}_2} \quad \text{and} \quad \alpha_2 = \begin{bmatrix} \mathsf{B}_1 & u & w' \\ 0 & u\alpha & 0 \end{bmatrix}_{w \in \mathsf{B}_2},$$

then

$$\alpha = \begin{bmatrix} \mathsf{B}_1 & u & w' \\ 0 & u\alpha & w \end{bmatrix}_{w \in \mathsf{B}_2} = \alpha_1 + \alpha_2,$$

$\alpha_1 \in \alpha\overline{L}_F(V,W)\alpha$ and $\alpha_2 \in L_F(V,W) \cap K_F(V,W)$.

Proof. First, we note that by assumption, $V = W \,\dot{\cup}\, (W + (F \smallsetminus \{0\})u)$.

(i) Since $\alpha \notin$ Reg $(\overline{L}_F(V,W))$, by Theorem 3.3, ran $\alpha \cap W \neq W\alpha$. Since $V = W + \langle u \rangle$, it follows that ran $\alpha = V\alpha = W\alpha \cup (W + (F \smallsetminus \{0\})u)\,\alpha$. Hence

$$\begin{aligned} W\alpha \neq \text{ran } \alpha \cap W &= \big(W\alpha \cup (W + (F \smallsetminus \{0\})u)\,\alpha\big) \cap W \\ &= \big(W\alpha \cup (W\alpha + (F \smallsetminus \{0\})u\alpha)\,\big) \cap W \\ &= W\alpha \cup \big(\,(W\alpha + (F \smallsetminus \{0\})u\alpha) \cap W\big) \quad \text{since } W\alpha \subseteq W \end{aligned}$$

which implies that $w\alpha + a(u\alpha) \in W\smallsetminus W\alpha$ for some $w \in W$ and $a \in F\smallsetminus\{0\}$. Consequently, $a(u\alpha) \in W\smallsetminus W\alpha$ and thus $u\alpha \in W\smallsetminus W\alpha$.

(ii) If $w \in W$ and $a \in F\smallsetminus\{0\}$, then by (i),

$$(w + au)\alpha \;=\; w\alpha + a(u\alpha) \;\in\; W\smallsetminus W\alpha.$$

But $V = W \,\dot\cup\, (W + (F\smallsetminus\{0\})u)$, so we have $\ker\alpha \subseteq W$.

(iii) is clearly seen from (i) and (ii). Note that $\ker\alpha = \ker(\alpha_{|W})$.

(iv) It is clear that

$$\alpha = \begin{bmatrix} \mathsf{B}_1 & u & w' \\ 0 & u\alpha & w \end{bmatrix}_{w\in\mathsf{B}_2} = \alpha_1 + \alpha_2.$$

Since $W = \langle \mathsf{B}_1 \cup \{w' \mid w \in \mathsf{B}_2\}\rangle$, by the definition of α_2, we have $\alpha_2 \in L_F(V,W) \cap K_F(V,W)$.

We note that $B_2 \cup \{u\alpha\} \subseteq W\alpha \cap W \subseteq W$. Next, to show that $\alpha_1 \in \alpha\overline{L}_F(V,W)\alpha$, let B_3 be a basis of W containing $\mathsf{B}_2 \cup \{u\alpha\}$. This implies that $\mathsf{B}_3 \cup \{u\}$ is a basis of V. Define $\beta \in L_F(V)$ by

$$\beta = \begin{bmatrix} w & (\mathsf{B}_3\smallsetminus\mathsf{B}_2) \cup \{u\alpha\} \\ w' & 0 \end{bmatrix}_{w\in\mathsf{B}_2}.$$

Then $\operatorname{ran}\beta \subseteq W$, so $\beta \in \overline{L}_F(V,W)$. Since

$$\begin{aligned} \mathsf{B}_1\alpha\beta\alpha &= \{0\} = \mathsf{B}_1\alpha_1, \\ u\alpha\beta\alpha &= (u\alpha)\beta\alpha = 0\alpha = 0 = u\alpha_1, \\ w'\alpha\beta\alpha &= w\beta\alpha = w'\alpha = w = w'\alpha_1 \text{ for all } w \in \mathsf{B}_2, \end{aligned}$$

we have $\alpha_1 = \alpha\beta\alpha \in \alpha\overline{L}_F(V,W)\alpha$, as desired. □

Lemma 6.11. *If* $F = \mathbb{Z}_p$ *and* $\dim_F(V/W) = 1$, *then the ring* $(\overline{L}_F(V,W), +, \circ)$ *has the* $\mathcal{BQ}$*-property.*

Proof. Let B be a bi-ideal of $(\overline{L}_F(V,W),+,\circ)$. Then $B\overline{L}_F(V,W)B \subseteq B$. To show that B is a quasi-ideal of $(\overline{L}_F(V,W),+,\circ)$, let $\alpha \in B\overline{L}_F(V,W) \cap \overline{L}_F(V,W)B$. If $\alpha \in \text{Reg}\,(\overline{L}_F(V,W))$, then by Lemma 6.5, $\alpha \in B$.

Next, assume that $\alpha \notin \text{Reg}\,(\overline{L}_F(V,W))$. Since $\dim_F(V/W) = 1$, we have $V = W + \langle u \rangle$ for some $u \in V \smallsetminus W$. By Lemma 6.10(i), $u\alpha \in W \smallsetminus W\alpha$. Define $\mathsf{B}_1, \mathsf{B}_2$, α_1 and α_2 be as in the assumption of Lemma 6.10(iii) and (iv). Then

$$\alpha = \alpha_1 + \alpha_2, \quad \alpha_1 \in \alpha\overline{L}_F(V,W)\alpha \ \text{ and } \ \alpha_2 \in L_F(V,W) \cap K_F(V,W).$$

Thus

$$\begin{aligned} \alpha_1 \in \alpha\overline{L}_F(V,W)\alpha \ &\subseteq \ B\overline{L}_F(V,W)\overline{L}_F(V,W)\overline{L}_F(V,W)B \\ &\subseteq B \ \subseteq \ B\overline{L}_F(V,W) \cap \overline{L}_F(V,W)B \end{aligned}$$

which implies that

$$\alpha_2 = \alpha - \alpha_1 \in B\overline{L}_F(V,W) \cap \overline{L}_F(V,W)B. \tag{1}$$

Since $\alpha_1 \in B$, to show that $\alpha \in B$, it suffices to show that $\alpha_2 \in B$. Since $\alpha_2 \in \overline{L}_F(V,W)B$ by (1), we have that

$$\alpha_2 = \sum_{k=1}^{n} \gamma_k\beta_k \quad \text{for some } \ \gamma_k \in \overline{L}_F(V,W) \ \text{ and } \ \beta_k \in B.$$

Without loss of generality, assume that $u\gamma_1, \ldots, u\gamma_m \in V \smallsetminus W$ and $u\gamma_{m+1}, \ldots, u\gamma_n \in W$. Then for $i \in \{1, \ldots, m\}$,

$$v\gamma_i = w_i + l_i u \ \text{ for some } w_i \in W \text{ and } l_i \in \mathbb{Z}_p \smallsetminus \{0\}. \tag{2}$$

Since $(B,+)$ is an abelian group, we have

$$\sum_{i=1}^{m} l_i\beta_i \ \in \ B. \tag{3}$$

Let B_4 be a basis of V containing $u\alpha$. For each $i \in \{1, \ldots, m\}$, let

$$\lambda_i = \begin{bmatrix} u\alpha & \mathsf{B}_4 \smallsetminus \{u\alpha\} \\ w_i & 0 \end{bmatrix}$$

and for each $j \in \{m+1, \ldots, n\}$, let

$$\mu_j = \begin{bmatrix} u\alpha & \mathsf{B}_4 \setminus \{u\alpha\} \\ u\gamma_j & 0 \end{bmatrix}.$$

Then $\lambda_i, \mu_j \in \overline{L}_F(V,W)$ for all $i \in \{1, \ldots, m\}$ and $j \in \{m+1, \ldots, n\}$. From (1), we have

$$\alpha_2\lambda_i\beta_i,\ \alpha_2\mu_j\beta_j \in\ B\overline{L}_F(V,W)\overline{L}_F(V,W)B\ \subseteq\ B \tag{4}$$

for all $i \in \{1, \ldots, m\}$ and $j \in \{m+1, \ldots, n\}$. By (3) and (4),

$$\theta\ =\ \sum_{i=1}^{m} l_i\beta_i + \sum_{i=1}^{m} \alpha_2\lambda_i\beta_i + \sum_{j=m+1}^{n} \alpha_2\mu_j\beta_j\ \in\ B. \tag{5}$$

We also have that

$$\begin{aligned} u\theta &= \sum_{i=1}^{m} l_i(u\beta_i) + \sum_{i=1}^{m}(u\alpha_2)\lambda_i\beta_i + \sum_{j=m+1}^{n}(u\alpha_2)\mu_j\beta_j \\ &= \sum_{i=1}^{m} l_i(u\beta_i) + \sum_{i=1}^{m}(u\alpha)\lambda_i\beta_i + \sum_{j=m+1}^{n}(u\alpha)\mu_j\beta_j \quad \text{since}\ \ u\alpha_2 = u\alpha \\ &= \sum_{i=1}^{m} l_i(u\beta_i) + \sum_{i=1}^{m} w_i\beta_i + \sum_{j=m+1}^{n}(u\gamma_j)\beta_j \\ &= \sum_{i=1}^{m}(l_iu + w_i)\beta_i + \sum_{j=m+1}^{n}(u\gamma_j)\beta_j \\ &= \sum_{i=1}^{m}(u\gamma_i)\beta_i + \sum_{j=m+1}^{n}(u\gamma_j)\beta_j \qquad \text{from (2)} \\ &= u(\sum_{k=1}^{n}\gamma_k\beta_k) = u\alpha_2 = u\alpha \qquad \text{since}\ \ \alpha_2 = \sum_{k=1}^{n}\gamma_k\beta_k. \end{aligned} \tag{6}$$

Case 1 : $\theta \in \text{Reg}\ (\overline{L}_F(V,W))$. Then ran $\theta \cap W = W\theta$. Since $u\theta = u\alpha \in$ ran $\theta \cap W$ by (6) and Lemma 6.10(i), there is an element $z \in W$ such that $z\theta = u\alpha$. Define $\eta \in \overline{L}_F(V,W)$ on the basis B_4 of V by

$$\eta = \begin{bmatrix} u\alpha & \mathsf{B}_4 \setminus \{u\alpha\} \\ z & 0 \end{bmatrix}.$$

Since

$$\begin{aligned}
\alpha_2\eta\theta &= \begin{bmatrix} \mathsf{B}_1 & u & w' \\ 0 & u\alpha & 0 \end{bmatrix}_{w\in\mathsf{B}_2} \begin{bmatrix} u\alpha & \mathsf{B}_4\setminus\{u\alpha\} \\ z & 0 \end{bmatrix}\theta \\
&= \begin{bmatrix} \mathsf{B}_1 & u & w' \\ 0 & z & 0 \end{bmatrix}_{w\in\mathsf{B}_2} \theta \\
&= \begin{bmatrix} \mathsf{B}_1 & u & w' \\ 0 & z\theta & 0 \end{bmatrix}_{w\in\mathsf{B}_2} = \begin{bmatrix} \mathsf{B}_1 & u & w' \\ 0 & u\alpha & 0 \end{bmatrix}_{w\in\mathsf{B}_2} = \alpha_2,
\end{aligned}$$

it follows that $\alpha_2 = \alpha_2\eta\theta \in B\overline{L}_F(V,W)\overline{L}_F(V,W)B \subseteq B$ by (1) and (5).

Case 2 : $\theta \notin \text{Reg}\ (\overline{L}_F(V,W))$. By Lemma 6.10(iv), there are $\theta_1 \in \theta\overline{L}_F(V,W)\theta, \theta_2 \in L_F(V,W) \cap K_F(V,W)$ with $u\theta_2 = u\theta$ such that $\theta = \theta_1 + \theta_2$. Since $\theta \in B$, we have $\theta_1 \in B$ which implies that $\theta_2 = \theta - \theta_1 \in B$. But

$$\begin{aligned}
(\mathsf{B}_1 \cup \{w' \mid w \in \mathsf{B}_2\})\,\theta_2 &\subseteq W\theta_2 \\
&= \{0\} && \text{since } \theta_2 \in K_F(V,W) \\
&= (\mathsf{B}_1 \cup \{w' \mid w \in \mathsf{B}_2\})\,\alpha_2 && \text{by the definition of } \alpha_2
\end{aligned}$$

and $u\theta_2 = u\theta = u\alpha_2$ by (6), so we deduce that $\alpha_2 = \theta_2 \in B$.

Hence the lemma is proved. □

Theorem 6.12. *The ring* $(\overline{L}_F(V,W), +, \circ)$ *has the* $\mathcal{BQ}$*-property if and only if one of the following statements holds.*

(i) $W = V$.

(ii) $W = \{0\}$.

(iii) $F = \mathbb{Z}_p$ *for some prime* p *and* $\dim_F W = 1$.

(iv) $F = \mathbb{Z}_p$ *for some prime* p *and* $\dim_F (V/W) = 1$.

Proof. Assume that (i), (ii), (iii) and (iv) are false. Then $\{0\} \neq W \subsetneq V$ and (1) $F \neq \mathbb{Z}_p$ for all prime p or (2) $\dim_F W > 1$ and $\dim_F(V/W) > 1$. Let B_1 be a basis of W and B a basis of V containing B_1. Then $\mathsf{B}_1 \neq \varnothing$ and $\mathsf{B}\setminus\mathsf{B}_1 \neq \varnothing$.

Case 1 : $F \neq \mathbb{Z}_p$ for all prime p. By Lemma 6.6, the ring $(\overline{L}_F(V,W),+,\circ)$ does not have the $\mathcal{BQ}$-property.

Case 2 : $\dim_F W > 1$ and $\dim_F(V/W) > 1$. Then $|\mathsf{B}_1| > 1$ and $|\mathsf{B}\smallsetminus\mathsf{B}_1| > 1$. Let $w_1, w_2 \in \mathsf{B}_1$ and $u_1, u_2 \in \mathsf{B}\smallsetminus\mathsf{B}_1$ be such that $w_1 \neq w_2$ and $u_1 \neq u_2$. Let $\alpha, \beta, \gamma \in \overline{L}_F(V,W)$ be defined by

$$\alpha = \begin{bmatrix} u_1 & u_2 & v \\ w_1 & w_2 & 0 \end{bmatrix}_{v\in\mathsf{B}\smallsetminus\{u_1,u_2\}}, \beta = \begin{bmatrix} w_2 & v \\ w_1 & 0 \end{bmatrix}_{v\in\mathsf{B}\smallsetminus\{w_2\}}, \gamma = \begin{bmatrix} u_2 & v \\ u_1 & 0 \end{bmatrix}_{v\in\mathsf{B}\smallsetminus\{u_2\}}.$$

Then we have

$$\alpha\beta = \begin{bmatrix} u_2 & v \\ w_1 & 0 \end{bmatrix}_{v\in\mathsf{B}\smallsetminus\{u_2\}} = \gamma\alpha \neq \alpha,$$

so $\alpha\beta \in \alpha\overline{L}_F(V,W) \cap \overline{L}_F(V,W)\alpha \subseteq (\alpha)_q$ by Proposition 1.3. Suppose that $\alpha\beta \in (\alpha)_b$. By Proposition 1.4, $\alpha\beta = a\alpha + \alpha\eta\alpha$ for some $\eta \in \overline{L}_F(V,W)$ and $a \in F$. Thus

$$w_1 = u_2\alpha\beta = u_2(a\alpha + \alpha\eta\alpha) = a(u_2\alpha) + (u_2\alpha)\eta\alpha = aw_2 + (w_2\eta)\alpha.$$

But $w_2\eta \in W$ and $W\alpha = \{0\}$, so $(w_2\eta)\alpha = 0$. Hence $w_1 = aw_2$ which is contrary to the independence of w_1 and w_2. Hence $(\alpha)_q \neq (\alpha)_b$, so the ring $(\overline{L}_F(V,W),+,\circ)$ does not have the $\mathcal{BQ}$-property.

For the converse, if (i) or (ii) holds, then $\overline{L}_F(V,W) = L_F(V)$ which has the $\mathcal{BQ}$-property. If (iii) or (iv) holds, then the ring $(\overline{L}_F(V,W),+,\circ)$ has the $\mathcal{BQ}$-property by Lemma 6.9 and Lemma 6.11, respectively.

Hence the theorem is proved. □

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