ปัญหาฟองสบู่เชิงระนาบที่มีเงื่อนไขบางประการเกี่ยวกับความดัน

นายบัญญัติ สร้อยแสง

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรดุษฎีบัณฑิต
สาขาวิชาคณิตศาสตร์ ภาควิชาคณิตศาสตร์
คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย
ปีการศึกษา 2551

PLANAR SOAP BUBBLE PROBLEM WITH SOME CONDITIONS ON PRESSURES

Mr. Banyat Sroysang

A Dissertation Submitted in Partial Fulfillment of the Requirements
for the Degree of Doctor of Philosophy Program in Mathematics

Department of Mathematics
Faculty of Science
Chulalongkorn University

Academic Year 2008

Thesis Title	PLANAR SOAP BUBBLE PROBLEM WITH SOME CONDITIONS ON PRESSURES
By	Mr. Banyat Sroysang
Field of Study	Mathematics
Advisor	Assistant Professor Wacharin Wichiramala, Ph.D.

Accepted by the Faculty of Science, Chulalongkorn University in Partial Fulfillment of the Requirements for the Doctoral Degree

.... Dean of the Faculty of Science

(Professor Supot Hannongbua, Dr.rer.nat.)

THESIS COMMITTEE

.... Chairman

(Associate Professor Wanida Hemakul, Ph.D.)

.... Advisor

(Assistant Professor Wacharin Wichiramala, Ph.D.)

.... Examiner

(Associate Professor Imchit Termwuttipong, Ph.D.)

.... Examiner

(Assistant Professor Nataphan Kitisin, Ph.D.)

.... External Examiner

(Assistant Professor Chaiwat Maneesawarng, Ph.D.)

บัญญัติ สร้อยแสง : ปัญหาฟองสบู่เชิงระนาบที่มีเงื่อนไขบางประการเกี่ยวกับความดัน.
(PLANAR SOAP BUBBLE PROBLEM WITH SOME CONDITIONS ON PRESSURES)
อ. ที่ปรึกษาวิทยานิพนธ์หลัก : ผ.ศ. ดร. วัชรินทร์ วิชิรมาลา, 96 หน้า.

ในวิทยานิพนธ์นี้ เราแก้ปัญหาฟองสบู่เชิงระนาบที่มีความดันเท่ากันสำหรับอาณาบริเวณสี่ ห้า และหกบริเวณ และหาสมบัติของผลเฉลยสำหรับปัญหาฟองสบู่เชิงระนาบในกรณีที่มีความดันเท่ากันหมดยกเว้นหนึ่งความดัน เราพบว่า

(1) สำหรับอาณาบริเวณสี่ ห้า และหกบริเวณ ทุกฟองสบู่ที่มีความยาวรวมสั้นสุด ซึ่งมีความดันเท่ากันและไม่มีช่องว่าง จะเป็นแบบมาตรฐาน

(2) ในกรณีอาณาบริเวณสี่บริเวณ สำหรับฟองสบู่ที่มีความยาวรวมสั้นสุด ไม่มีช่องว่าง และความดันที่น้อยที่สุดสามความดันมีค่าเท่ากัน ถ้าอาณาบริเวณที่มีความดันมากที่สุดเป็นอาณาบริเวณเชื่อมโยง จะได้ว่าฟองสบู่นี้เป็นแบบมาตรฐาน

(3) ในกรณีอาณาบริเวณสี่บริเวณ สำหรับฟองสบู่ที่มีความยาวรวมสั้นสุด ไม่มีช่องว่าง และความดันที่มากที่สุดสามความดันมีค่าเท่ากัน อาณาบริเวณที่มีความดันน้อยที่สุด ต้องมีอย่างน้อยหนึ่งชิ้นอยู่ข้างนอก

ภาควิชา..คณิตศาสตร์............... ลายมือชื่อนิสิต ....

สาขาวิชา..คณิตศาสตร์............ ลายมือชื่อ อ.ที่ปรึกษาวิทยานิพนธ์หลัก ....

ปีการศึกษา..2551......................

4873823023 : MAJOR MATHEMATICS

KEYWORDS : SOAP BUBBLE / MINIMIZING ENCLOSURE

BANYAT SROYSANG : PLANAR SOAP BUBBLE PROBLEM WITH SOME CONDITIONS ON PRESSURES. ADVISOR : ASST.PROF. WACHARIN WICHIRAMALA, Ph.D., 96 pp.

In this thesis, we solve the planar soap bubble problem with equal pressures for four, five and six regions, and find properties of solutions for the planar soap bubble problem in the case that all but one pressures are equal. We obtain the following results:

(1) for four, five and six regions, every minimizing bubble with equal pressures and without empty chambers is standard;

(2) in case of four regions, for a minimizing bubble without empty chambers and with three equal lowest pressures, if the region of the highest pressure is connected, then the bubble is standard;

(3) in case of four regions, for a minimizing bubble without empty chambers and with three equal highest pressures, the region of the lowest pressure must have at least one external component.

Department:.....Mathematics... Student's Signature:..........

Field of Study:..Mathematics... Advisor's Signature:..........

Academic Year:........2008.........

ACKNOWLEDGEMENTS

The author thanks Assistant Professor Dr. Wacharin Wichiramala, his thesis advisor, for the assistance and all suggestions in this thesis. Next, the author thanks Associate Professor Dr. Wanida Hemakul, Associate Professor Dr. Imchit Termwuttipong, Assistant Professor Dr. Nataphan Kitisin and Assistant Professor Dr. Chaiwat Maneesawarng, his thesis committee, for all suggestions. Finally, the author thanks his family for the love and the goodwill.

CONTENTS

page

LIST OF FIGURES

Figure page

Figure page

Figure page

Figure page

Figure page

Figure page

CHAPTER I

Introduction

People tend to believe that the soap bubbles enclose and separate the given volumes of air using the least surface area. The ancient Greeks believed that a circle is the best way to enclose a single given area on the plane but mathematicians could not prove this until much later in the late nineteenth century. The planar soap bubble problem seeks for the least-perimeter way to enclose and separate open regions $R_1, \ldots, R_m$ of given areas $A_1, \ldots, A_m$ on the plane.

Intuitively, humans believe that the problem have natural solutions which keep each region in a single connected component. But this is the main difficulty to show that each region of a minimizing enclosure is connected.

In 1976, Almgren [1] proved that there is a solution of the soap bubble problem for any dimension greater than two, and gave some basic regularities of such solutions. Moreover, Taylor [22] added more regularity for the case of dimension three.

In 1985, Bleicher [3] proved that solutions of the problem have a great regularity properties without giving a rigorous proof of their existence.

In 1992, Morgan [18] proved the existence of solutions to the problem with a new regularity result.

In 1993, using a new developed approach to get rid of empty chambers, Foisy, Alfaro, Brock, Hodges and Zimba [11] solved the planar double bubble problem; the solution as in Figure 1.0.1.

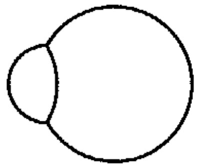


Figure 1.0.1: A standard double bubble.

In 1994, Cox, Harrison, Hutchings, Kim, Light, Mauer and Tilton [10] proved that, among enclosures of connected regions (including the exterior region), the standard triple bubble in Figure 1.0.2 is uniquely shortest. Impressively, the understanding on pressures of regions and signed curvatures of edges is the result from this paper.

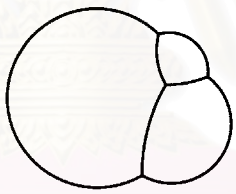


Figure 1.0.2: A standard triple bubble.

In 1996, Bleicher [4] proved that any two components in minimizing bubble may meet at most once. As a result, the sense of uniqueness of the edge between two components helps to reduce many combinatorial possibilities for candidate bubbles.

In 1998, Vaughn [25] proved in his Ph.D. dissertation that any minimizing triple bubble with equal pressures and without empty chambers is standard as in Figure 1.0.3. Moreover, he showed why the new approach is valid for the case of three areas.

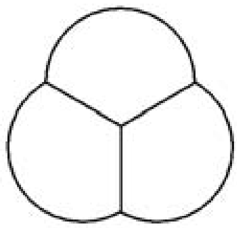


Figure 1.0.3: A standard triple bubble with equal pressures.

In 2002, the planar triple bubble conjecture was proven completely by Wichiramala [27]; the solution as in Figure 1.0.2. Impressively, he could find the bound on the number of convex components and then use it to make the bound on the total number of components. Moreover, he improved the new approach to the bubble problem that automatically eliminates the possibility of having empty chambers. In 2003, Wichiramala [26] presented a shorter proof of the planar triple bubble conjecture.

In 2005, Morgan [17] considered on planar double bubbles which can overlap with multiplicity. In the same year, Kaewkao [13, 14] found some properties of solutions for the planar quadruple bubble problem where the three highest pressures are equal. He obtained the following results: (1) each component of the lower pressure region in the solution may have at most nine sides and (2) if the lower pressure region in the solution is connected then it must be an external component.

For higher dimensions, the existence and uniqueness of standard bubble clusters of given volumes are considered by Montesinos [16] in 2001. The double bubble problem for three dimensions is solved by Hutchings, Morgan, Ritoré and Ros [12] in 2002. A year later, in 2003, Reicharht, Heilmann, Lai and Spielman [21] solved the double bubble problem in four dimensions. Lately, in 2008, Reicharht [20] proved that the least-area hypersurface enclosing and separating two given volumes in $\mathbb{R}^n$ is the standard double bubble.

Moreover, researchers focus on the double bubble problem for special spaces. For examples, the double bubble problem on a cone [15], the double bubble problem in a circle [5], the double bubble problem on a flat 2-torus [8], the double bubble problem in a three-torus [2] , the double bubble problem in the spherical space S^3 and the hyperbolic space H^3 [7, 9], the double bubble problem in the Gauss space [6], etc.

In chapter II, we list all basic results and main definitions that will be used throughout this thesis, starting with the existence and regularity of solutions to the planar soap bubble. Next, we mention the approach that helps to eliminate the possibility of having empty chamber, called the weak approach. Variations of bubbles are also described. We refer to an important tool which is the component bounds for minimizing planar bubbles. The last section focuses on geometry of planar bubbles.

In chapter III, we show the results whose proofs are similar to the proofs of the results in [26]. This consists of basic results on well-related admissible functions and a π-edge function.

In chapter IV, we discuss properties of bubbles with equal pressures and then, finally, conclude the main result that, for $m \in \{4, 5, 6\}$, every weakly minimizing m-bubble is standard (every region is connected). It follows that the work left to do is showing that every weakly minimizing m-bubble with unequal pressures is standard.

In chapter V, we find some properties of solutions for the planar quadruple bubble problem in case that the three lower pressures are equal. Our main result is that a minimizing 4-bubble with $p_1 > p_2 = p_3 = p_4$ and without empty chambers is standard (every region is connected) if R_1 is connected.

In chapter VI, we identify some important properties of solutions for the planar quadruple bubble problem in case that the three highest pressures are equal. This case, we have some results from [13, 14]. We prove our main result that a minimizing 4-bubble with $p_1 = p_2 = p_3 > p_4 \geq 0$ and without empty chambers must have at least one external component of R_4. This result extends the work by Keawkhao [13, 14] that if R_4 is connected, then it is external.

In the last chapter, we discuss some interesting conjectures related to this dissertation.

CHAPTER II

Preliminaries

In this chapter, we list all basic definitions and results, starting with the existence and regularity of solutions to the planar soap bubble, variations on bubbles, component bounds, and some geometric properties of bubbles.

2.1 Existence and regularity of planar soap bubbles

The **planar *m* soap bubble problem** is the search for the way to enclose and separate open regions $R_1, R_2, \ldots, R_m$ of m given areas in $\mathbb{R}^2$ using least perimeter.

We say that $E \subset \mathbb{R}^2$ is an **enclosure** of areas $A_1, A_2, \ldots, A_m$ in $\mathbb{R}^2$ if E is closed and bounded with finite one-dimensional Hausdorff measure and $\mathbb{R}^2 \backslash E$ contains open regions of areas $A_1, A_2, \ldots, A_m$.

For each enclosure E of areas $A_1, A_2, \ldots, A_m$, let R_i be the open region of area A_i. We call $\mathbb{R}^2 \backslash \overline{R_1 \cup \ldots \cup R_m}$ the **exterior region** R_0. Each connected component of a region is shortly called a **component**. A component is **external** if it meets R_0 and is **internal** if not. A bounded component of R_0 is specifically called an **empty chamber** as it does not contribute area to any of the m bounded regions.

We say that an enclosure E is **minimizing** if E has least Haussdorf measure among enclosures of given areas, and E is **standard** if every region is connected and if $m \in \{2, 3\}$, every two regions must meet along a single edge as in Figure 1.0.1 and 1.0.2.

We conjecture that every minimizing enclosure is standard and refer to this conjecture as the **soap bubble conjecture**.

Theorem 2.1.1. *[3, 10, 18] For positive numbers $A_1, \ldots, A_m$, there is a minimizing enclosure of areas $A_1, \ldots, A_m$. Let E be a minimizing enclosure. Then*

(1) *E is composed of finitely many circular/straight edges separating different regions and meeting only in threes at 120° angles,*

(2) *all edges in E form a connected graph, and*

(3) *there are real numbers $p_1, \ldots, p_m$, which will be called the **pressures** of the region R_i, such that every edge between R_i and R_j has curvature $|p_i - p_j|$ (bulges into the lower pressure region) where p_0 is set to be zero.*

An enclosure of m regions with properties (1), (2), and (3) is called an ***m*-bubble**. We consider an m-bubble as an embedded graph in $\mathbb{R}^2$ with labels $1, 2, \ldots, m$ and 0, and we label i to refer to components of R_i.

Existence of pressures implies the **cocycle condition**: the sum of the signed curvatures of the three edges around each vertex is zero.

Next, we follow [10] in defining variations.

Definition 2.1.2. *Let C_1 and C_2 be enclosures. We say that C_1 and C_2 has the same **combinatorial type** if there is a continuous deformation f_t on C_1 such that f_0 is the identity, f_t is injective for every $t \in [0, 1]$, and $f_1(C_1) = C_2$. For each $t \in [0, 1]$, a deformation f_t on $C_1 \subseteq \mathbb{R}^2$ we mean a function f_t on C_1 into $\mathbb{R}^2$.*

Definition 2.1.3. *A **variation** of a bubble B is a C^1 family of enclosures $\{B_t\}_{|t|<\epsilon}$ of combinatorial type B, with $B_0 = B$. Let $\ell(B)$ denote the **length** of B, which is the total length of the edges of B.*

Proposition 2.1.4. *[10] For a bubble B with pressures $p_1, \ldots, p_m$, and any variation $\{B_t\}$ of B, we have*

$$\left.\frac{d\ell(B_t)}{dt}\right|_0 = \sum_{i=1}^{m} p_i \left.\frac{dA_i(t)}{dt}\right|_0$$

where $A_i(t)$ denotes the area of the i^{th} bounded region of B_t.

We say that a bubble is **stationary** if it has no area-preserving variation that initially decreases length in first order [10], and **stable** if it is stationary and has no area-preserving variation that decreases length in second order [12].

Note that a minimizing bubble is stable and a stable bubble is stationary. According to [10], every m-bubble is stationary.

Proposition 2.1.5. *[10] A stationary bubble of areas $A_1, \ldots, A_m$ and pressures $p_1, \ldots, p_m$ has length $2\sum p_i A_i$.*

Without loss of generality, we may assume that $p_1 \geq p_2 \geq \ldots \geq p_m$. In this case, it follows from the previous proposition that $p_1 > 0$.

Definition 2.1.6. *The **sign** of curvature of a directed edge is considered positive[negative] if the edge is turning left[right].*

When considering a component C, we implicitly direct its edges counterclockwise with C thus to the left on each edge. Hence the signed curvature of an edge of a component is well-defined.

Definition 2.1.7. *The* ***turning angle*** *of a directed edge of a component is the product of its signed curvature and its length.*

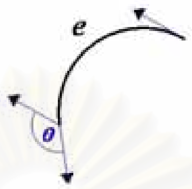


Figure 2.1.1: An edge e with absolute turning angle θ.

Lemma 2.1.8. *[26, 27] For an n-sided component of a bubble, the sum of turning angles of all edges is $\frac{6-n}{3}\pi$ if the component is bounded and $\frac{-6-n}{3}\pi$ if the component is unbounded.*

Proposition 2.1.9. *[27] For a minimizing bubble, any two regions may not share an edge of turning angle greater than π and an edge of turning angle at least π.*

2.2 The weak approach

In this section, we mention the approach that helps eliminate the possibility of having empty chamber. The idea was developed in [11] and [25] and finally completed in [26, 27] where it is called the **weak approach**. A weakly minimizing enclosure for areas $A_1, A_2, \ldots, A_m$ is a shortest enclosure for regions of areas $a_1, a_2, \ldots, a_m$ where $a_i \geq A_i$ for all i. From [26, 27], a weakly minimizing enclosure is a minimizing enclosure for areas $a_1, a_2, \ldots, a_m$. Consequently, it is also called a **weakly minimizing bubble**.

Proposition 2.2.1. *[26, 27] A weakly minimizing bubble has no empty chambers and has nonnegative pressures.*

The next theorem tells us that we need not to concern bubbles with empty chambers for $m \leq 6$.

Theorem 2.2.2. *[26, 27] For $m \leq 6$, the planar m-bubble conjecture holds if every weak minimizer is standard.*

2.3 Variations of bubbles

In this section, we list the results on variations of bubbles obtained from the work of Wichiramala [26, 27]. The main idea comes from [12] in which the study was on $\mathbb{R}^n$.

Wichiramala [26, 27] consider a planar enclosure B with smooth **interface** E_{ij} between R_i and R_j, and a continuous **variation** $V = \{B_t : B \to \mathbb{R}^2\}_{|t|<\epsilon}$ of B which is smooth on each E_{ij} up to the boundary. The **associated vector field** is $X := dB_t/dt|_0$. For each i and j, let k_{ij} be the curvature of E_{ij}, and N_{ij} the unit normal vector on E_{ij} from R_j into R_i. For each i and j, the **scalar normal component** of X from R_j into R_i is $u_{ij} := X \cdot N_{ij}$. Let k, N and u be the disjoint union functions $\dot{\bigcup\limits_{i<j}} k_{ij}$, $\dot{\bigcup\limits_{i<j}} N_{ij}$ and $\dot{\bigcup\limits_{i<j}} u_{ij}$ on $E = \dot{\bigcup\limits_{i<j}} E_{ij}$, respectively. Given a scalar or vector valued function $f = \dot{\bigcup\limits_{i<j}} f_{ij}$ defined on the interfaces of B, we define a function $Y(f)$ on the vertices of B by $Y(f)(p) = f_{ij}(p) + f_{jh}(p) + f_{hi}(p)$ if R_i, R_j, R_h meet at p (in that order counterclockwise). We say that f **agree at** p if $Y(f)(p) = 0$. Initially, the area of R_i changes at rate $\sum\limits_j \int_{E_{ji}} u_{ji}$ for all i. We say that V has **character** $(x_0, \ldots, x_m) \in \mathbb{R}^{m+1}$ if the areas of R_i initially change with

rate x_i for all i. For a real-valued function $\widetilde{u}$ on E with $\widetilde{u}_{ij} = -\widetilde{u}_{ji}$, we say $\widetilde{u}$ has **character** $(x_0, \ldots, x_m)$ if $\displaystyle\sum_j \int_{E_{ji}} \widetilde{u}_{ji} = x_i$ for all i. The character of u is equal to the character of V. If each area of B changes at a constant rate, we say that V is **steady**.

Lemma 2.3.1. *[12, 26, 27] (First variation of length for a planar enclosure). For an enclosure B with smooth interfaces E_{ij} and a variation with associated vector field X and scalar normal component u_{ij}, the initial first derivative of length is*

$$-\sum_{i<j} \int_{E_{ij}} k_{ij}u_{ij} - \sum_{vertex\ p} X(p) \cdot T(p) = -\int_B ku - \sum_{vertex\ p} X(p) \cdot T(p)$$

where $T(p)$ denote the sum of the unit tangent vectors to the edges meeting at p.

In [26, 27], a symmetric bilinear form Q for smooth scalar normal components u and v is defined by

$$Q(u, v) = \int_B (u'v' - k^2 uv) - \sum_{\text{vertex } p} Y(quv)$$

where u' is the derivative of u with respect to arc length along edges and $q_{ij} = \dfrac{k_{ih} + k_{jh}}{\sqrt{3}}$ at a vertex p where R_i, R_j, R_h meet.

For $Q(u, u)$, we shortly write $Q(u)$.

Proposition 2.3.2. *[12, 26, 27] (Second variation of length for a planar bubble). For a bubble with interfaces E_{ij} and a steady variation V, the initial second derivative of length is $Q(u)$.*

We write $Q(V)$ to mention $Q(u)$.

The next step, Wichiramala [26, 27] discussed about Q as follows. Let $u = \{u_{ij} : E_{ij} \to \mathbb{R}$ where E_{ij} is the interface between R_i and $R_j\}$ be a function such that $u_{ij} = -u_{ji}$ for all i and j. We say that u is **admissible** if u agree at every vertex ($u_{ij} + u_{jh} + u_{hi} = 0$ where R_i, R_j, R_h meet) and u_{ij} is in the Sobolev space $W^{1,2}$ for all i and j. Hence every scalar normal component is an admissible function. Let $\mathcal{F}$ be the set of admissible functions with zero characters. For piecewise C^2 functions u and v, we define an extension of Q at u and v to be

$$Q(u,v) = -\int (u'' + k^2 u)v - \sum Y((qu + u')v) - \sum [u'v]$$

where $[u'v]$ is the jump of $u'v$.

Lemma 2.3.3. *[26, 27] Let u and v be admissible functions. If at every point $u = 0$ or $v = 0$, then $Q(u,v) = 0$ and $Q(u+v) = Q(u) + Q(v)$.*

A bubble is **unstable** if it has an area-preserving smooth variation with negative second variation. An unstable bubble is clearly not minimizing.

Wichiramala [26, 27] gave conclusions that a smooth admissible function is the initial velocity of some steady smooth variation, and hence if there is an admissible function $u \in \mathcal{F}$ with $Q(u) < 0$, then the bubble is unstable.

Lemma 2.3.4. *[12, 26, 27] In a bubble B, let $S \subset B$ be closed with ∂S disjoint from the vertices, and let $K \neq 0$ be a Killing vector field such that $K \cdot N$ vanishes on ∂S. If $u = K \cdot N$ on S and $u = 0$ elsewhere, then u is admissible, $u'' + k^2 u = 0$ on ∂S, $Y((qu + u')v) = 0$ for any vertex and any admissible function v, and $Q(u) = 0$.*

According to [26, 27], from the previous lemma, if $K \neq 0$ is a rotation vector field around a point p, we say that u **rotates** S around p (see examples in Figure 2.3.1 and 2.3.2) and then we call u a **rotating function**. If p is at infinity, then K is constant and we additionally say that u **translates** S (see examples in Figure 2.3.3 and 2.3.4) and then we call u a **translating function**. Moreover, S is called the **support** of u. A **tentacle** of S is an edge of S that meets ∂S. We say that a point p is the **center** of circular arc e if e is on a circle centered at p, and e has its center at infinity if e is straight.

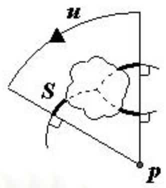


Figure 2.3.1: The function u rotates S (the bold part) around p and vanishes at ∂S. (Figure 6.1 in [26])

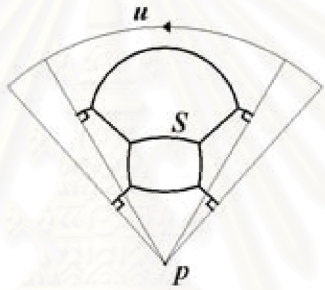


Figure 2.3.2: The function u rotates S (the bold part) around p and vanishes at ∂S.

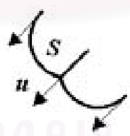


Figure 2.3.3: The function u translates S (the bold part) and vanishes at ∂S away from vertices. (Figure 6.3 in [26])

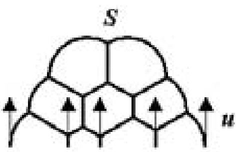


Figure 2.3.4: The function u translates S (the bold part) and vanishes at ∂S away from vertices.

Remark 2.3.5. *In fact, it is sufficient to replace S by $\overline{\{u \neq 0\}}$.*

Lemma 2.3.6. *[26, 27] Let B be a stable bubble and u_i be rotating function of S_i around p_i. Suppose that the S_i's are disjoint and $u = \sum \alpha_i u_i \in \mathcal{F}$ for some $\alpha_i \neq 0$. Then each p_i is the center of every tentacle of S_i.*

We use definitions in [26, 27] as follows. A component is **convex** if its edges have nonnegative curvatures. For a bubble B with a component C, let V be a steady variation on B that has inward normal component identically 1 on the edges of C and vanishes elsewhere. Hence $Q(V) < 0$ if C is nonhexagonal convex, and $Q(V) = 0$ if C is hexagonal. We call the scalar normal component of this variation a **uniform function**.

An edge is **long** if it has absolute turning angle greater than π (see an example in Figure 2.3.5). The **long-edge variation** on a long edge e is the variation V that moves only e using circular edges and fixing both endpoints. Hence a steady variation V has second variation $Q(V) < 0$. We call the scalar normal component of a long-edge variation a **long-edge function**. In fact, if e has absolute turning angle exactly π (see an example in Figure 2.3.5), we have $Q(V) = 0$ which we will again mention in the next chapter.

The idea of using long edges to help getting linear independence was introduced by Morgan [19].

Figure 2.3.5: Two edges with absolute turning angle greater than π and exactly π, respectively.

Any two edges are **disjoint** if they are distinct. An edge and a component are **disjoint** if the edge is not an edge of the component. Any two components are

disjoint if they have no common edge. For a bubble without 2-sided components, any long edge is disjoint from a convex component.

We say that admissible functions u_i are **well-related** if each u_i is a uniform function on a nonhexagonal convex component, a long-edge function, a rotating (or translating) function and among these functions, we have the following:

(1) The tentacles of the support of each rotating around p must not all have center at p. For each translating function, its tentacles are not parallel.

(2) The supports of these functions must satisfy:

- The uniform functions and long-edge functions are on disjoint convex components and long edges.
- The rotating functions have disjoint supports.
- As the support of a long-edge function, a long edge must be either entirely in or disjoint from the support of any rotating (or translating) function.
- As the support of a uniform function, a convex component must be either entirely in or disjoint from the support of any rotating (or translating) function.

Lemma 2.3.7. *[26, 27] Let u_i be well-related admissible functions and $\alpha_i \in \mathbb{R}$. Then* $Q(\sum \alpha_i u_i) = \sum \alpha_i^2 Q(u_i)$.

2.4 Bounds on the number of convex components

In this section, we recall properties of structures of each component in an m-bubble.

Proposition 2.4.1. *[4] For a minimizing bubble, any two components may meet at most once, along a single edge.*

Corollary 2.4.2. *[11] For $m \geq 3$, a minimizing m-bubble has no 2-sided component.*

We say that two components of a region are **identical** if there is an isometry from one to the other, preserving labels of surrounding components.

Proposition 2.4.3. *[26, 27] A stable bubble may not have two identical nonhexagonal convex components.*

The next theorem is very important as we name it the **component bound**.

Theorem 2.4.4. *[26, 27] A stable m-bubble B has at most m disjoint nonhexagonal convex components.*

Proposition 2.4.5. *[27] If a stable m-bubble B has a convex component C, then B has at most $m-1$ disjoint nonhexagonal convex components away from C.*

2.5 Geometry of planar soap bubble

In this section, we recall some geometric properties of planar soap bubbles obtained from [26, 27].

Lemma 2.5.1. *[26, 27] Let e, f, g be consecutive edges of a component with internal angles $120°$. Suppose e and g have the same (signed) curvature and they have different centers p and q. Let l be the line that perpendicularly bisects the segment $\overline{pq}$. Then l perpendicularly bisects f.*

We say that two circular arcs are **parallel** if they have the same center, and **cocircle** if they are on the same circle or the same straight line.

Lemma 2.5.2. *[26, 27] Let e and f be nonparallel arcs and k a given curvature. Unless e and f are straight lines with intersecting angle* $60°$ *and* $k = 0$ *(see Figure* 2.5.1*), there are at most two possible arcs with curvature k that cut e and f with same orientated angles* $120°$*. In particular, if there are two possibilities, one can be reflected through the line l perpendicular to e and f to be cocircular with the other (see Figure* 2.5.2*).*

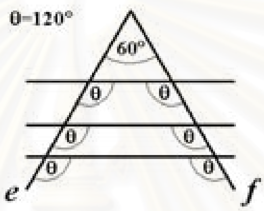


Figure 2.5.1: If e and f are straight lines with intersecting angle $60°$ and $k = 0$, then there are infinitely many possible arcs with curvature k that cut e and f with same orientated angles $120°$

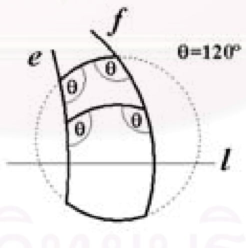


Figure 2.5.2: If there are two possibilities, then one can be reflected through the line l perpendicular to e and f to be cocircular with the other.

Lemma 2.5.3. *[25] A* 3*-sided component is uniquely determined by its three vertex positions and orientation. It is also uniquely determined, up to rigid motion, by its three curvatures.*

An edge e is said to be **incident** to a vertex v if v is an endpoint of e. An **incident** edge of a component C is an incident edge at a vertex of C that is not an edge on the boundary of C.

The next lemma tells us the hidden geometry in a bubble. In particular, we can reduce a 3-sided component in a bubble to form another bubble. This process is called a **reduction** of a 3-sided component, see Figure 2.5.3. Conversely, we can add a 3-sided component at a vertex in a bubble to form another bubble. This process is called a **decoration** of a 3-sided component.

Lemma 2.5.4. *[26, 27] For a 3-sided component, if its three incident edges are prolonged into the component, they will meet at a point satisfying the cocycle condition and thus forming a bubble (with different areas).*

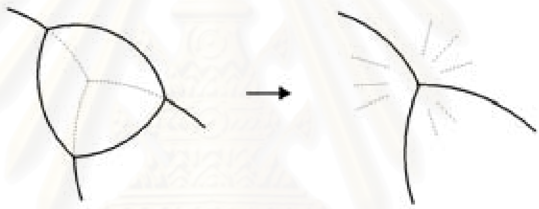


Figure 2.5.3: The reduction of a 3-sided component.

Proposition 2.5.5. *[26, 27] A 4-sided component with two parallel (but not co-circular) edges must look like the component in Figure* 2.5.4 *where it is composed of two opposite, parallel edges with opposite signs of curvatures (or both straight), and two opposite, equal curvature edges as caps for both ends.*

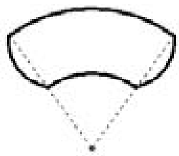


Figure 2.5.4: A 4-sided component with two parallel edges and two edges as caps for both ends.

CHAPTER III

Basic results

In this chapter, we list all basic results needed in the following chapters.

3.1 Basic results on well-related admissible functions

The next lemma is extended from Lemma 6.2 in [26] which states that a triple bubble with four well-related admissible functions is not minimizing. Its proof is also valid for m-bubbles, where $m > 3$.

Lemma 3.1.1. *For $m \geq 3$, an m-bubble with $m+1$ well-related admissible functions is not minimizing.*

Proof. This proof is similar to the proof of Lemma 6.2 in [26]. Suppose to get a contradiction that a stable m-bubble B has $m+1$ distinct well-related admissible functions u_i. There is a nontrivial linear combination $u = \sum \alpha_i u_i \in \mathcal{F}$. By Lemma 2.3.7, $Q(u) = \sum \alpha_i^2 Q(u_i)$. If some uniform function or long-edge function u_i has nonzero α_i, then $Q(u) < 0$ and thus B is unstable. Hence we may assume that only a rotating function u_i may have nonzero α_i. By Lemma 2.3.6, the tentacles of each function u_i have their center at p_i, contradicting the definition of well-related. □

3.2 Basic results on a π-edge function

In this section, we investigate the properties of an m-bubble with edges of absolute turning angle π.

A **π-edge** is an edge of absolute turning angle π. The **π-edge variation** on a π-edge e is the variation V that moves only e using circular edges and fixing both end points. Hence a steady variation V has second variation $Q(V) = 0$ [26, 27]. We call the scalar normal component of a π-edge variation a **π-edge function**.

Let $\{u_i\}$ be a collection of well-related admissible functions. We say that a π-edge function w, on a π-edge e, is **well-related to** $\{u_i\}$ if we have the following:

(1) For each uniform function in $\{u_i\}$, each edge in its nonhexagonal convex component and e are disjoint and have no common endpoints.

(2) For each long-edge function in $\{u_i\}$, its long-edge and e are disjoint.

(3) Each rotating (or translating) function in $\{u_i\}$ and the π-edge variation of w have disjoint supports.

In this case, we may say that $\{w\} \cup \{u_i\}$ is **well-related**, or say that w and u_i's are **well-related**.

Lemma 3.2.1. *Let $\{u_i\}$ be a collection of well-related admissible functions and w a π-edge function such that w is well-related to $\{u_i\}$. For each $\{\beta\} \cup \{\alpha_i\} \subseteq \mathbb{R}$, we have $Q(\beta w + \sum \alpha_i u_i) = \beta^2 Q(w) + \sum \alpha_i^2 Q(u_i)$.*

Proof. This proof is similar to the proof of Lemma 6.1 in [26]. By definition, $Q(\beta w+\sum \alpha_i u_i) = \beta^2 Q(w)+\sum \alpha_i^2 Q(u_i)+2\sum \alpha_i \beta Q(u_i, w)+2\sum_{i<j} \alpha_i \alpha_j Q(u_i, u_j)$. By Lemma 2.3.3, it suffices to show that $Q(v, u) = 0$ for a rotating function v with support S and a uniform function u on a component C where either C is entirely in S or C is disjoint from S. Thus ∂S is disjoint from C. Hence $[v'u] = 0$

on ∂S, and then

$$Q(v,u) = -\int (v'' + k^2 v)u - \sum Y((qv + v')u).$$

By Lemma 2.3.4, $Q(v,u) = 0$. □

Lemma 3.2.2. *Let B be an m-bubble with m well-related admissible functions u_i. If there is a π-edge function w in B such that w is well-related to $\{u_i\}$, then B is not minimizing.*

Proof. This proof is similar to the proof of Lemma 6.2 in [26]. Assume that there is a π-edge function w in B such that w is well-related to $\{u_i\}$. Suppose to get a contradiction that B is minimizing. There is a nontrivial linear combination $u = \beta w + \sum \alpha_i u_i \in \mathcal{F}$. By Lemma 3.2.1, $Q(u) = \beta^2 Q(w) + \sum \alpha_i^2 Q(u_i)$. If $\beta \neq 0$, then $\{\alpha_i\} \neq \{0\}$. If some uniform function or long-edge function u_i has nonzero α_i, then $Q(u) < 0$ and hence B is unstable. Hence we may assume that only a rotating function u_i can have nonzero α_i. By Lemma 2.3.6, the tentacles of each function u_i have their center at p_i, contradicting the definition of well-related.

□

Corollary 3.2.3. *Let B be a minimizing m-bubble. If B has a π-edge e, then B has at most m-1 disjoint nonhexagonal convex components each of which is disjoint from e and has no common endpoints to e's.*

Proof. This follows from Lemma 3.2.2. □

CHAPTER IV

Bubbles with equal pressure regions

In this chapter, we discuss properties of bubbles with equal pressures and then, finally, conclude the main result that, for $m = 4, 5, 6$, every weakly minimizing m-bubble is standard [23, 24].

As areas have no connection to these results, to reduce difficulty in calculation, we scale bubbles so that the external edges have curvature 1. For convenience, we denote the number of n-sided components by N_n.

4.1 Properties of bubbles with equal pressures

In his Ph.D. dissertation [25], Vaughn lists basic properties of bubbles with equal pressures. Now, we start with those properties and new results needed for the main theorem.

Lemma 4.1.1. *[25] In a minimizing m-bubble with equal pressures, any component of* $R_1, R_2, \ldots, R_m$ *is convex and hence has at most* 6 *sides. In addition, one that shares an edge with the exterior region has at most* 5 *sides.*

Lemma 4.1.2. *[25] In a minimizing bubble with equal pressures, there is a unique shape for a* 3*-sided component, and if there are no empty chambers, then there is a one-parametered family of possible* 4*-sided components, and a two-parametered family of possible* 5*-sided components.*(*See Figures* 4.1.1, 4.1.2 *and* 4.1.3)

Note that an internal component has 6 straight sides.

Figure 4.1.1: The unique shape for a 3-sided component. (Figure 4.1 in [25])

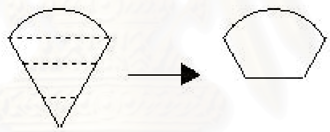


Figure 4.1.2: A choice determines a 4-sided component. (Figure 4.2 in [25])

Figure 4.1.3: Two choices determine a 5-sided component. (Figure 4.3 in [25])

Corollary 4.1.3. *For a minimizing bubble with equal pressures and without empty chambers, if a* 3*-sided component is between two* 5*-sided components, then the* 5*-sided components are isometric.*

Proof. This follows from Lemma 4.1.2. □

Corollary 4.1.4. *For a minimizing bubble with equal pressures and without empty chambers, any two consecutive* 4*-sided components are isometric.*

Proof. This follows from Lemma 4.1.2. □

We define an **external edge** to be an edge on the boundary of R_0.

The next proposition tells that a bubble with equal pressures and without empty chambers must be almost convex when we consider its external edges.

Proposition 4.1.5. *Let B be a minimizing m-bubble with equal pressures and without empty chambers and e an external edge in B. Then e intersects the boundary of the convex hull of B.*

Proof. This follows from Lemmas 4.1.1 and 4.1.2. □

Proposition 4.1.6. *Let C be a component of a bubble with equal pressures and without empty chambers such that C is adjacent to three consecutive 6-sided components. Then C is 6-sided.*

Proof. Using Figure 4.1.4 and Lemma 4.1.1, it is obvious that C has six sides since every angle is 120°. □

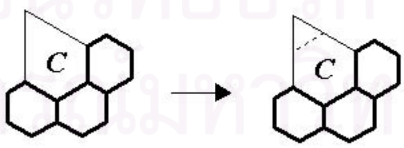


Figure 4.1.4: A choice determines a component which is adjacent to three consecutive 6-sided components.

Remark 4.1.7. *By the previous proposition, we may conclude that, for a bubble with equal pressures, all internal components form a convex networks of hexagons.*

Lemma 4.1.8. *A region of a minimizing m-bubble with equal pressures and without empty chambers has at most one 3-sided component.*

Proof. Suppose R_i has two 3-sided components. According to Figure 4.1.5, a 3-sided component of R_i has area less than π and its external edge has length π.

Figure 4.1.5: A bubble with two 3-sided components of R_i can be improved to a shorter enclosure.

The new enclosure of the same areas on the right has $d < \frac{\pi}{2}$ since the width of the rectangle is 2. By Proposition 4.1.5, the new edge can not meet the other part of the bubble. Hence we create a shorter enclosure, a contradiction.

□

Lemma 4.1.9. *A region of a minimizing m-bubble with equal pressures and without empty chambers may not have both a 3-sided component and a 4-sided component.*

Proof. Suppose R_i has a 3-sided component and a 4-sided component. According to Figure 4.1.6, a 3-sided component of R_i has area less than $\frac{\sqrt{3}}{2}\pi$ and its external edge has length π.

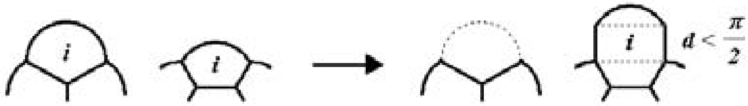


Figure 4.1.6: A bubble with a 3-sided component and a 4-sided component of R_i can be improved to a shorter enclosure.

The new enclosure of the same areas on the right has $d < \frac{\pi}{2}$ since the width of the rectangle is $\sqrt{3}$. By Proposition 4.1.5, the new edge can not meet the other part of the bubble. Hence we create a shorter enclosure, a contradiction.

□

Lemma 4.1.10. *A region of a minimizing m-bubble with equal pressures and without empty chambers has at most one 4-sided component.*

Proof. Suppose R_i has two 4-sided components. According to Figure 4.1.7, a 4-sided component of R_i has area less than $\frac{\pi}{3} + \frac{\sqrt{3}}{2}$ and its external edge has length $\frac{2\pi}{3}$.

Figure 4.1.7: A bubble with two 4-sided components of R_i can be improved to a shorter enclosure.

The new enclosure of the same length on the right has the same perimeter but the upper half of the new component has area $\pi - 2\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right) = \frac{\pi}{3} + \frac{\sqrt{3}}{2}$. Hence, by tiny adjustment, we can make a shorter enclosure of the same areas. By Proposition 4.1.5, the new edge can not meet the other part of the bubble. Hence we create a shorter enclosure, a contradiction.

□

Theorem 4.1.11. *A minimizing m-bubble with equal pressures and without empty chambers has $N_3 + N_4 \leq m$. Moreover, their labels are different.*

Proof. This follows from Lemmas 4.1.8, 4.1.9 and 4.1.10. □

Lemma 4.1.12. *A minimizing m-bubble with equal pressures and without empty chambers has* $2N_3 + N_4 = 6$.

Proof. This follows from Lemma 2.1.8. □

Lemma 4.1.13. *For* $m \geq 4$*, if a minimizing m-bubble with equal pressures and without empty chambers has* $N_3 = 2$*,* $N_4 = 2$ *and* $N_6 = 0$*, then* $N_5 \leq m - 4$.

Proof. Suppose that $N_3 = 2$, $N_4 = 2$, $N_6 = 0$ and $N_5 > m - 4$. Hence the bubble has configuration straight with 2 layers as examples in Figure 4.1.8. By Theorem 4.1.11, all 3-sided and 4-sided components have different labels.

Figure 4.1.8: Some bubbles with $N_3 = 2$, $N_4 = 2$ and $N_6 = 0$.

If $N_5 = 1$ and $m = 4$, it is not possible to label the 5-sided component as it is adjacent to the other four components of different labels, a contradiction.

Now suppose that $N_5 > 1$. For each 3-sided component, since $N_6 = 0$, the two adjacent components can not be both 5-sided, and since $N_3 = 2$, they can not be both 4-sided. Thus each 3-sided component is adjacent to a 4-sided component and a 5-sided component. Together with the assumption, all 5-side components are isometric. Hence we can exchange their labels so that there are two consecutive components with the same label. Hence we create a nonminimizing bubble with the same perimeter, a contradiction.

□

Lemma 4.1.14. *A region of a minimizing bubble with equal pressures and without empty chambers may not have both a* 3*-sided component and a* 5*-sided component.*

Proof. Suppose R_i has a 3-sided component and a 5-sided component. The lower part of a 3-sided component has area $\frac{1}{\sqrt{3}}$ and the external edge of a 5-side component has length $\frac{\pi}{3}$. According to Figure 4.1.9, the new enclosure on the right of the same areas has $d < \frac{\pi}{6}$ since the width of the rectangle is 2.

Figure 4.1.9: A bubble with a 3-sided component and a 5-sided component of R_i can be improved to a shorter enclosure.

By Proposition 4.1.5, we have a shorter enclosure, a contradiction. □

Lemma 4.1.15. *A region of a minimizing m-bubble with equal pressures and without empty chambers may not have both a* 5*-sided component and a* 4*-sided component that the length of its bottom edge is greater or equal to* $\frac{\sqrt{3}}{2}$.

Proof. Suppose R_i has a 5-sided component and a 4-sided component that the length of its bottom edge is greater or equal to $\frac{\sqrt{3}}{2}$. The extended edge of these two components have total length π and the area of the 4-sided component is at most $\frac{\pi}{3} + \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{4}\left(\frac{\sqrt{3}}{2}\right)^2$. According to Figure 4.1.10, the new enclosure on the right has the same perimeter and the upper part of the new enclosure has area greater than $\frac{\pi}{2} + 0.02 > \frac{\pi}{3} + \frac{5}{16}\sqrt{3}$.

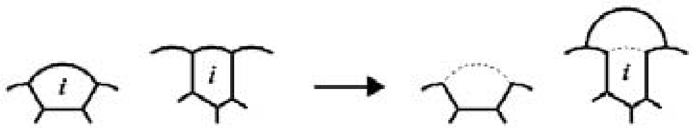


Figure 4.1.10: A bubble with a short 4-sided component and a 5-sided component of R_i can be improved to a shorter enclosure.

Hence, by smaller adjustment and Proposition 4.1.5, we can create a shorter enclosure of the original areas, a contradiction. □

Lemma 4.1.16. *A minimizing m-bubble with equal pressures and without empty chambers may not have two pairs of* 4*-sided and* 5*-sided components of* R_j *and* R_j *as in the left of Figure* 4.1.11.

Proof. Suppose the contrary. By Lemma 4.1.15, the two edges between four components of R_i and R_j has total length greater than $\sqrt{3}$. According to Figure 4.1.11, we create an enclosure of the same areas by removing the two edges and adding a new edge to balance areas of R_i and R_j. This new edge has length less than $\sqrt{3}$. Hence the new enclosure is shorter, a contradiction. □

Figure 4.1.11: A bubble with two pairs of components can be improved to a shorter enclosure.

Lemma 4.1.17. *A minimizing m-bubble with equal pressures and without empty chambers may not have three consecutive components (a* 4*-sided component of* R_i*, a* 5*-sided component of* R_j*, a* 5*-sided component of* R_i*, respectively) as in the left of Figure* 4.1.12.

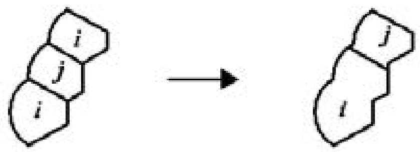


Figure 4.1.12: A bubble with a consecutive sequence of components can be improved to a shorter enclosure.

Proof. By Lemma 4.1.15, the bottom edge of the 4-sided component in the sequence has length less than $\frac{\sqrt{3}}{2}$. It follows that we can create a shorter enclosure of the original areas as in Figure 4.1.12, a contradiction. $\square$

4.2 Bubbles for four or five areas with equal pressure regions

In this section, we prove the main result that, for $m = 4$ or 5, every weakly minimizing m-bubble with equal pressures is standard. However we prove a stronger result for every minimizing m-bubble with equal pressures and without empty chambers.

Lemma 4.2.1. *For $m = 4$ or 5, a minimizing m-bubble with equal pressures and without empty chambers may not have $N_3 = 2$, $N_4 = 2$ and $N_6 = 1$.*

Proof. Suppose that $N_3 = 2$, $N_4 = 2$ and $N_6 = 1$. All possible configurations are in Figure 4.2.1.

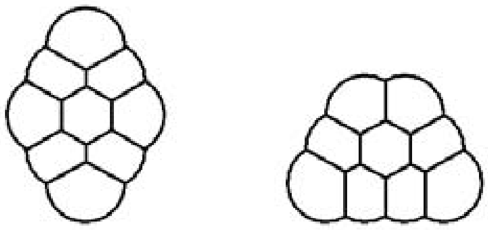


Figure 4.2.1: Some bubbles with only one hexagonal component.

By Theorem 4.1.11, all 3-sided and 4-sided components have different labels. Since all 5-side components in Figure 4.2.1 are isometric, we can exchange their labels so that there are two consecutive components with the same label. Hence we create a nonminimizing bubble with the same perimeter, a contradiction.

□

Lemma 4.2.2. *For $m = 4$ or 5, a minimizing m-bubble with equal pressures and without empty chambers may not have $N_3 = 3$ and $N_4 = 0$.*

Proof. Suppose $N_3 = 3$ and $N_4 = 0$. By Theorem 2.4.4, there are at most m disjoint nonhexagonal convex components. All possible configurations are in Figure 4.2.2. By Lemma 4.1.8, all 3-sided components have different labels. Since all 5-side components in Figure 4.2.2 are isometric, we can exchange their labels so that there are two consecutive components with the same label. Hence we create a nonminimizing bubble with the same perimeter, a contradiction. □

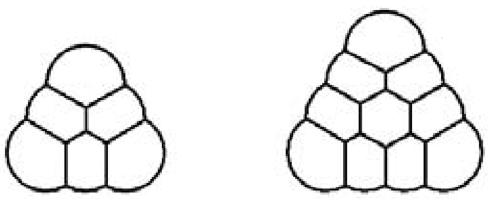


Figure 4.2.2: Some bubbles with $N_3 = 3$ and $N_4 = 0$.

Proposition 4.2.3. *Every minimizing* 4*-bubble with equal pressures and without empty chambers is standard.*

Proof. By Theorem 4.1.11, we have $N_3 + N_4 \leq 4$. By Lemmas 4.1.12 and 4.2.2, we have $N_3 = 2$ and $N_4 = 2$. We will divide into cases according to N_6.

Case $N_6 = 0$. By Lemma 4.1.13, we have $N_5 = 0$. The only one possibility is in Figure 4.2.3.

Figure 4.2.3: The standard 4-bubble.

Case $N_6 = 1$. By Lemma 4.2.1, this case is impossible.

Case $N_6 > 1$. By Theorem 2.4.4, there are at most 4 disjoint nonhexagonal convex components. The only one possibility is in Figure 4.2.4.

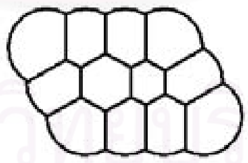


Figure 4.2.4: A 4-bubble with two hexagonal components.

By Theorem 4.1.11 and Lemma 4.1.14, without loss of generality, we have only one choice of labeling in Figure 4.2.5, a contradiction to Lemma 4.1.16.

Therefore being standard is the only one possibility. □

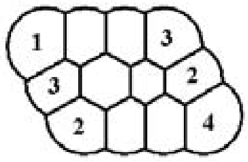


Figure 4.2.5: A labeled 4-bubble with two hexagonal components.

Proposition 4.2.4. *Every minimizing* 5*-bubble with equal pressures and without empty chambers is standard.*

Proof. By Theorem 4.1.11, we have $N_3 + N_4 \leq 5$. By Lemmas 4.1.12 and 4.2.2, we have two cases: (1) $N_3 = 2$ and $N_4 = 2$; (2) $N_3 = 1$ and $N_4 = 4$.

Case 1: $N_3 = 2$ and $N_4 = 2$. We will divide into subcases according to N_6.

Subcase 1A: $N_6 = 0$. By Lemma 4.1.13, we have $N_5 \leq 1$. The only one possibility is in Figure 4.2.6.

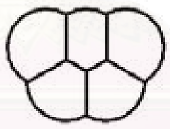


Figure 4.2.6: The standard 5-bubble.

Subcase 1B: $N_6 = 1$. By Lemma 4.2.1, this case is impossible.

Subcase 1C: $N_6 > 1$. By Theorem 2.4.4, there are at most 5 disjoint nonhexagonal convex components. All possible configurations are in Figure 4.2.7. Now, all 3-sided and 4-sided components have different labels. Consider possibility (a), (b) and (c). By Corollary 4.1.3 and Lemma 4.1.14, without loss of generality, we may label them as Figure 4.2.8. By Lemma 4.1.16, configurations (a), (b) and (c) are not possible. By Corollary 4.1.3, 4.1.4 and Lemma 4.1.14, configuration (d) must have $\{\alpha, \beta, \gamma\} = \{1, 2\}$, a contradiction.

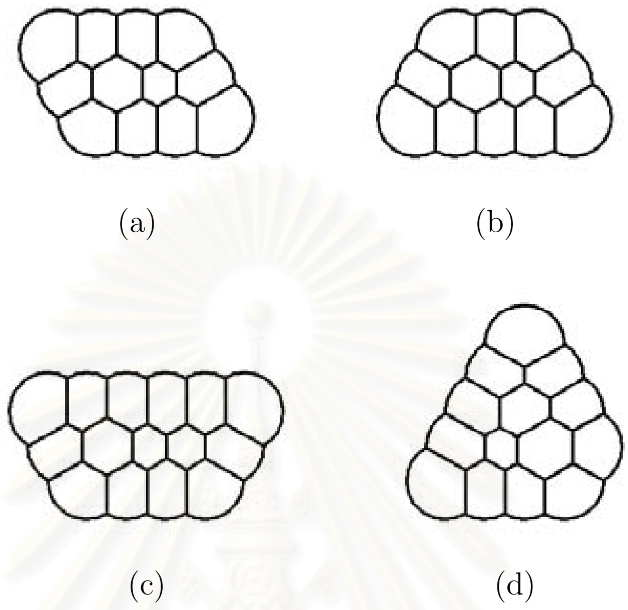


Figure 4.2.7: Some 5-bubbles with two or three hexagonal components.

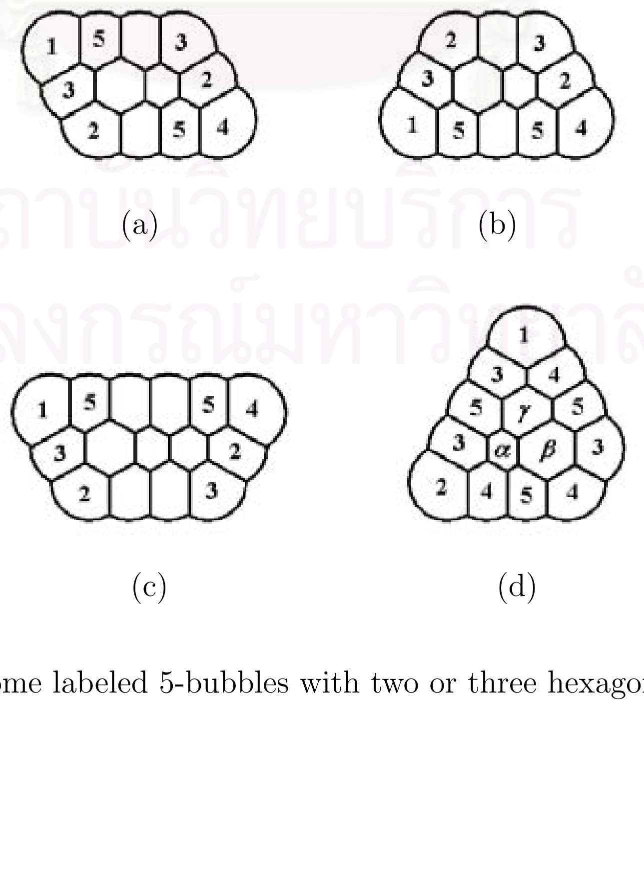


Figure 4.2.8: Some labeled 5-bubbles with two or three hexagonal components.

Case 2: $N_3 = 1$ and $N_4 = 4$. We will divide into subcases according to N_6.

Subcase 2A: $N_6 \leq 2$. All possible configurations are in Figure 4.2.9. Now, all 3-sided and 4-sided components have different labels. Without loss of generality, we may label them as Figure 4.2.10. By Corollary 4.1.4 and Lemma 4.1.14, it is not possible to assign any label to i.

Figure 4.2.9: Some 5-bubbles with one or two hexagonal components.

Figure 4.2.10: Some labeled 5-bubbles with one or two hexagonal components.

Subcase 2B: $N_6 > 2$. By Theorem 2.4.4, there are at most 5 disjoint non-hexagonal convex components. All possible configurations are in Figure 4.2.11.

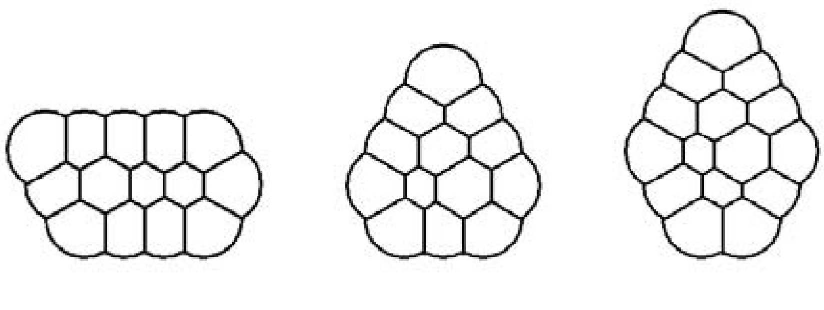


Figure 4.2.11: Some 5-bubbles with one or two hexagonal components.

Since all 3-sided and 4-sided components have different labels, without loss of generality, we may label them as Figure 4.2.12.

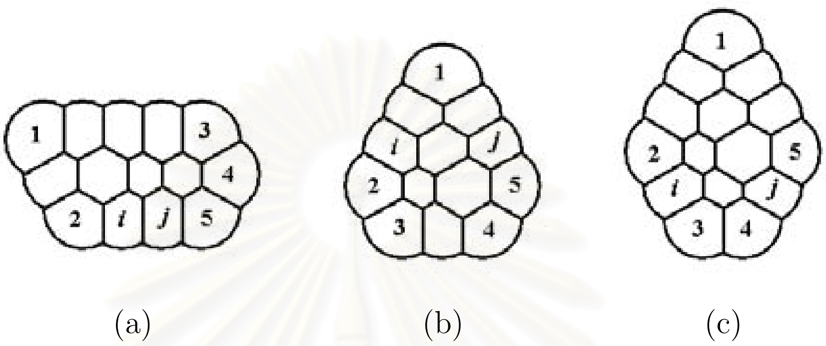
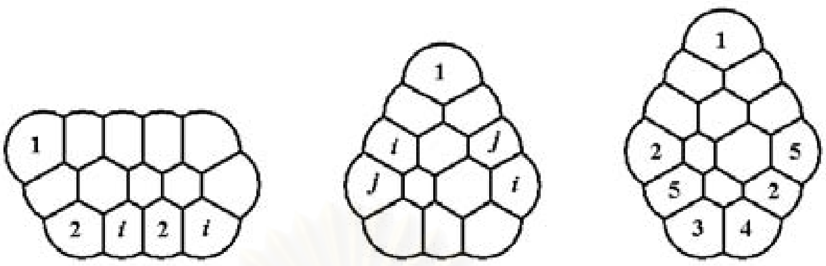


Figure 4.2.12: Some labeled 5-bubbles with one or two hexagonal components.

We will use Lemma 4.1.14 to determine labels on 5-sided components and then use Corollary 4.1.4 to show that consecutive 4-sided components are isometric, so that they can exchange their labels. In order to avoid creating another enclosure of the same areas but having some edge between the same region, labeling must be as follows.

For Figure 4.2.12 (a), we must have $j \neq 1, 3, 4, 5$. Hence $j = 2$ and we may exchange labels to be in Figure 4.2.13 (a). For Figure 4.2.12 (b), we must have $i \neq 1, 2, 3$ and $j \neq 1, 4, 5$. Hence we may exchange labels to be in Figure 4.2.13 (b). For Figure 4.2.12 (c), we must have $i \neq 1, 2, 3, 4$ and $j \neq 1, 3, 4, 5$. Hence $i = 5$, $j = 2$ and we may exchange labels to be in Figure 4.2.13 (c). In every possibility in Figure 4.2.13, there is a contradiction to Lemma 4.1.16.

□



(a) (b) (c)

Figure 4.2.13: Some labeled 5-bubbles with one or two hexagonal components.

Theorem 4.2.5. *For $m = 4$ or 5, a minimizing m-bubble with equal pressures and without empty chambers is standard.*

Proof. This follows from Propositions 4.2.3 and 4.2.4. □

Corollary 4.2.6. *For $m = 4$ or 5, a weakly minimizing m-bubble with equal pressures is standard.*

For $m = 4$ or 5, to solve the planar m-bubble problem completely, the work left to do is showing that every weakly minimizing m-bubble with unequal pressures is standard.

4.3 Bubbles for six areas with equal pressure regions

In this section, we prove the main result that every weakly minimizing 6-bubble with equal pressures is standard. However we prove a stronger result for every minimizing 6-bubble with equal pressures and without empty chambers.

Lemma 4.3.1. *A minimizing 6-bubble with equal pressures and without empty chambers may not have $N_6 = 1$.*

Proof. Suppose that $N_6 = 1$. All possible configurations are in Figure 4.3.1. By Lemma 4.1.10, all 4-sided components have different labels. By Corollary 4.1.4, we can exchange their labels. Since all 5-sided components in Figure 4.3.1 are isometric, we can exchange their labels. Without loss of generality, we may label all 4-sided and 5-sided components as Figure 4.3.2. Hence it is not possible to assign any label to i, a contradiction. □

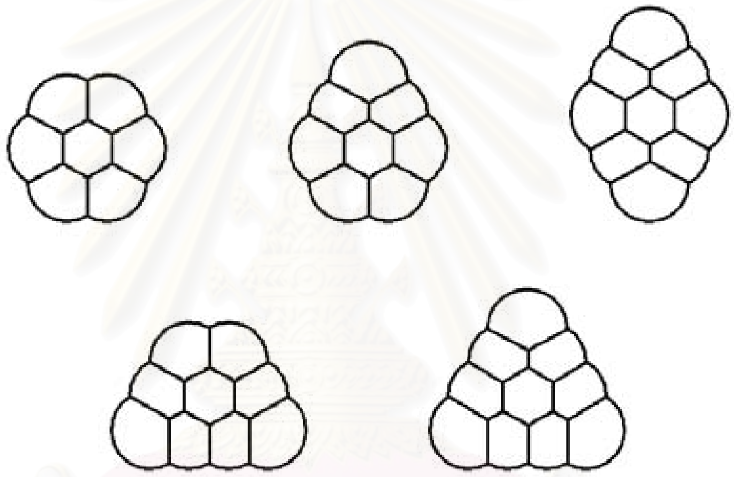


Figure 4.3.1: Some 6-bubbles with only one hexagonal component.

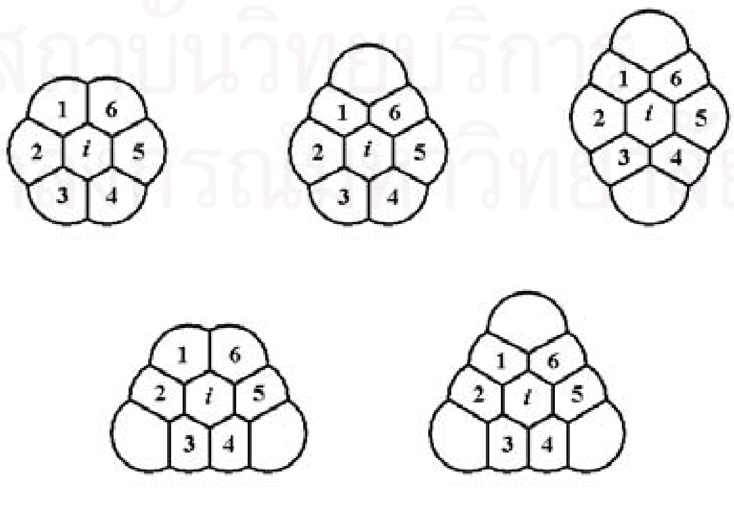


Figure 4.3.2: Some labeled 6-bubbles with one hexagonal component.

Lemma 4.3.2. *A minimizing* 6*-bubble with equal pressures and without empty chambers may not have* $N_6 = 2$*.*

Proof. Suppose that $N_6 = 2$. All possible configurations are in Figure 4.3.3. By Theorem 4.1.11, all 3-sided 4-sided components have different labels. Without loss of generality, we may label all 3-sided and 4-sided components as Figure 4.3.4.

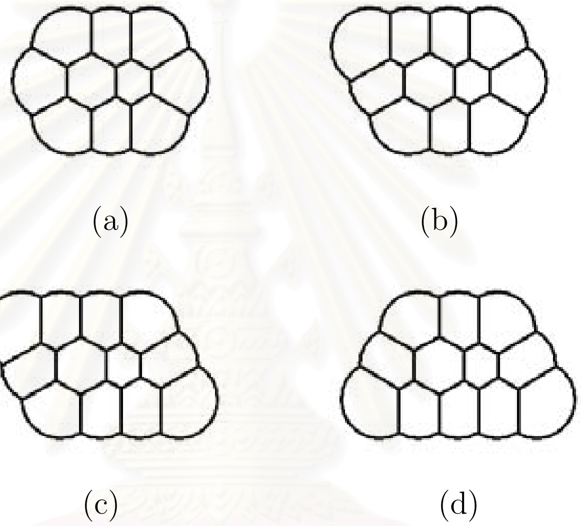


Figure 4.3.3: Some 6-bubbles with two hexagonal components.

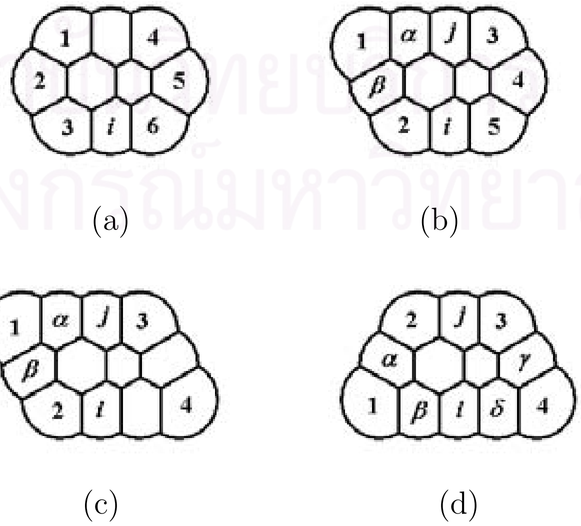


Figure 4.3.4: Some labeled 6-bubbles with two hexagonal components.

Consider possibility (a). By Corollary 4.1.4, we can exchange their labels. Hence it is not possible to assign any label to i.

Consider possibility (b). By Corollary 4.1.4 and Lemma 4.1.14, we have $i = 6$. Since the 5-sided components with labels i and j are isometric, we can exchange their labels. Together with Lemma 4.1.14, we have $j = 6$. By Corollary 4.1.3, we can assume that $\alpha \in \{3, 4, 5\}$. By Corollary 4.1.4, we may exchange labels to be Figure 4.3.5, a contradiction to Lemma 4.1.17.

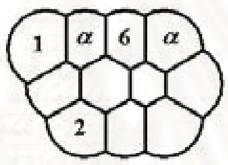


Figure 4.3.5: A labeled 6-bubble of possibility (b) in Figure 4.3.3.

Consider possibility (c). Since the 5-sided components with labels i and j are isometric, we can exchange their labels. Together with Lemma 4.1.14, we have $j \in \{5, 6\}$. By Corollary 4.1.3 and Lemma 4.1.14, we can assume that $\{\alpha, \beta, j\} = \{3, 5, 6\}$. We may exchange labels to be Figure 4.3.6, a contradiction to Lemma 4.1.17.

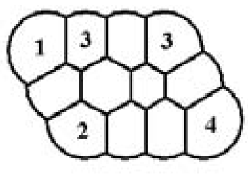


Figure 4.3.6: A labeled 6-bubble of possibility (c) in Figure 4.3.3.

Consider possibility (d). Since the 5-sided components with labels i and j are isometric, we can exchange their labels. Together with Lemma 4.1.14, we have $i \in \{5, 6\}$. By Corollary 4.1.3 and Lemma 4.1.14, we can assume that $\{\alpha, \beta, i\} = \{3, 5, 6\}$ and $\{\gamma, \delta, i\} = \{2, 5, 6\}$. We may exchange labels to be Figure 4.3.7, a contradiction to Lemma 4.1.16.

□

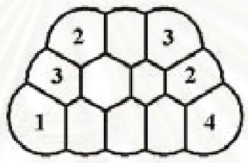


Figure 4.3.7: A labeled 6-bubble of possibility (d) in Figure 4.3.3.

Lemma 4.3.3. *A minimizing* 6*-bubble with equal pressures and without empty chambers may not have* $N_6 = 3$.

Proof. Suppose that $N_6 = 3$. All possible configurations are in Figure 4.3.8.

Consider possibilities (a) and (e). By Lemma 4.1.10, all 4-sided components have different labels. Without loss of generality, we may label all 4-sided components as Figures 4.3.9 (a) and 4.3.10 (a). By Corollary 4.1.4, we may exchange labels to be Figures 4.3.9 (b) and 4.3.10 (b) which contradict to Lemma 4.1.16.

Consider possibility (f). By Theorem 4.1.11, all 3-sided 4-sided components have different labels. Without loss of generality, we may label all 4-sided components as Figure 4.3.11. By Corollaries 4.1.4 and 4.1.3, we may exchange labels to be Figure 4.3.12 (a) or (b), a contradiction to Lemma 4.1.17.

Possibilities (b), (c), (d), (g), and (h) are not minimizing by Lemma 3.1.1 using admissible functions shown in Figure 4.3.13. □

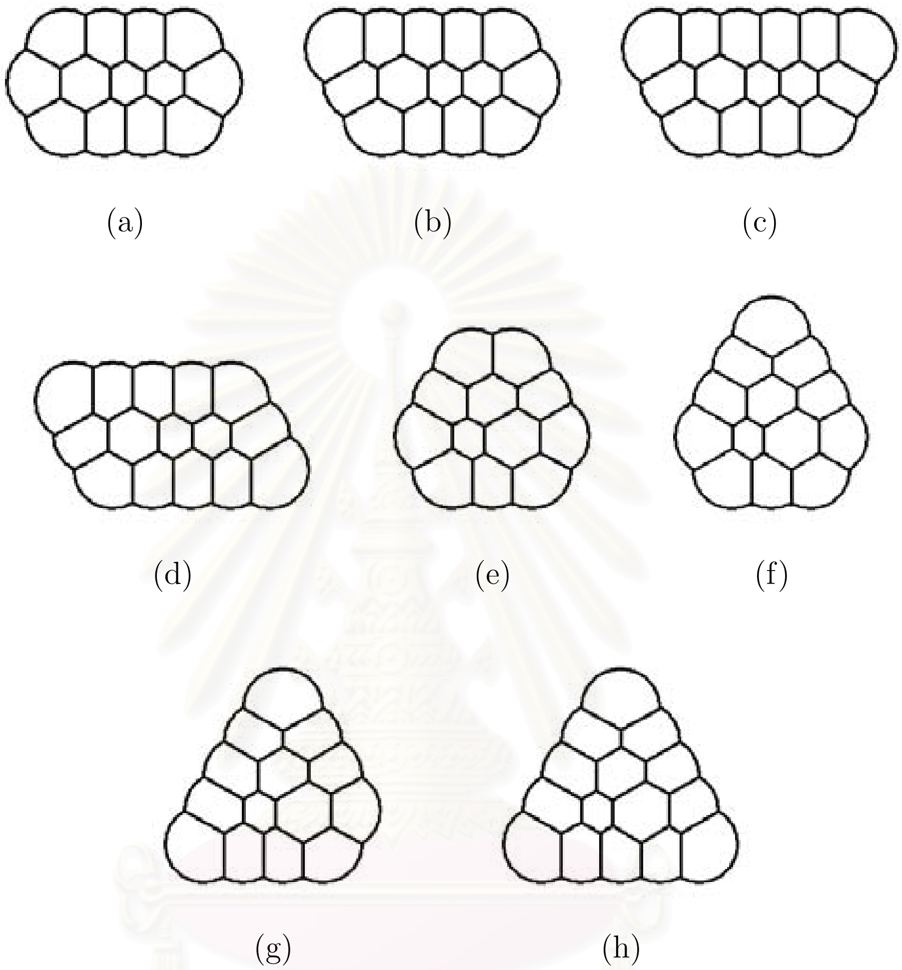


Figure 4.3.8: Some 6-bubbles with three hexagonal components.

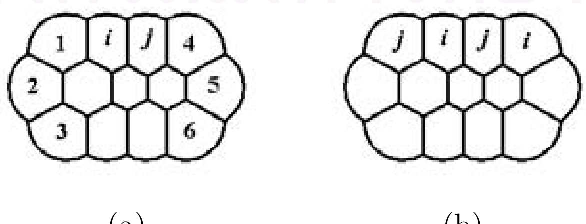
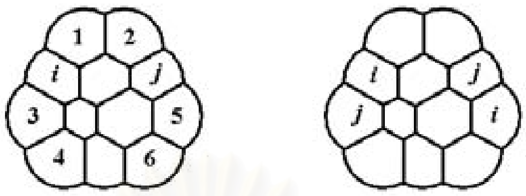


Figure 4.3.9: Some labeled 6-bubbles with three hexagonal components.



(a) (b)

Figure 4.3.10: Some labeled 6-bubbles with three hexagonal components.

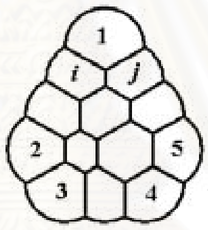
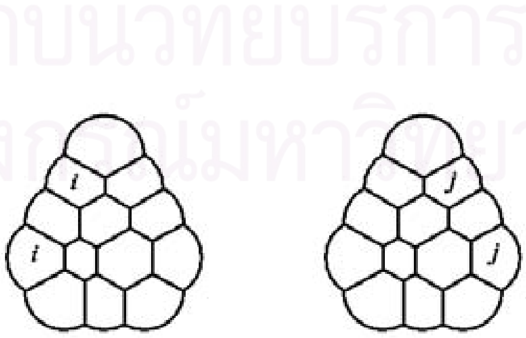


Figure 4.3.11: A labeled 6-bubble with three hexagonal components.



(a) (b)

Figure 4.3.12: Some labeled 6-bubbles with three hexagonal components.

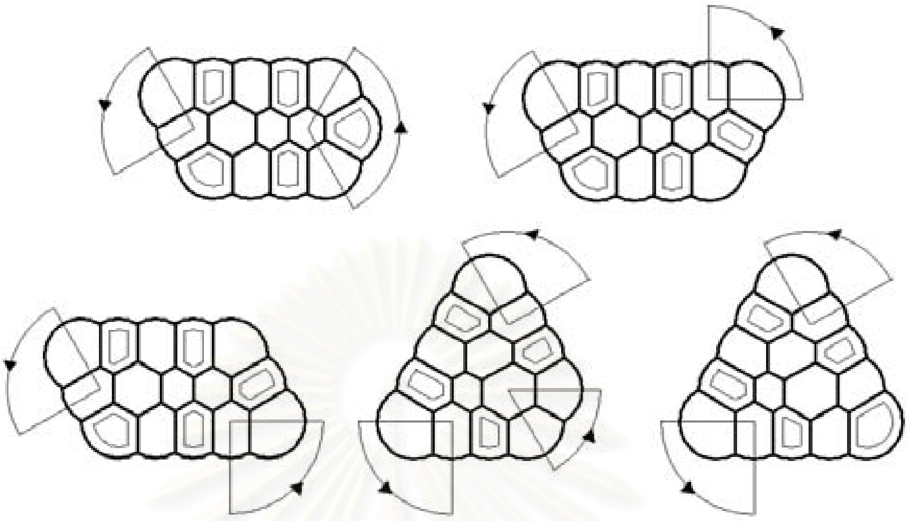


Figure 4.3.13: Some 6-bubbles with seven admissible functions.

Lemma 4.3.4. *A minimizing* 6*-bubble with equal pressures and without empty chambers may not have* $N_6 = 4$*.*

Proof. Suppose that $N_6 = 4$. By Theorem 2.4.4, there are at most 6 disjoint nonhexagonal convex components. All possible configurations are in Figure 4.3.14 which are not minimizing by Lemma 3.1.1 using admissible functions shown in Figure 4.3.15. □

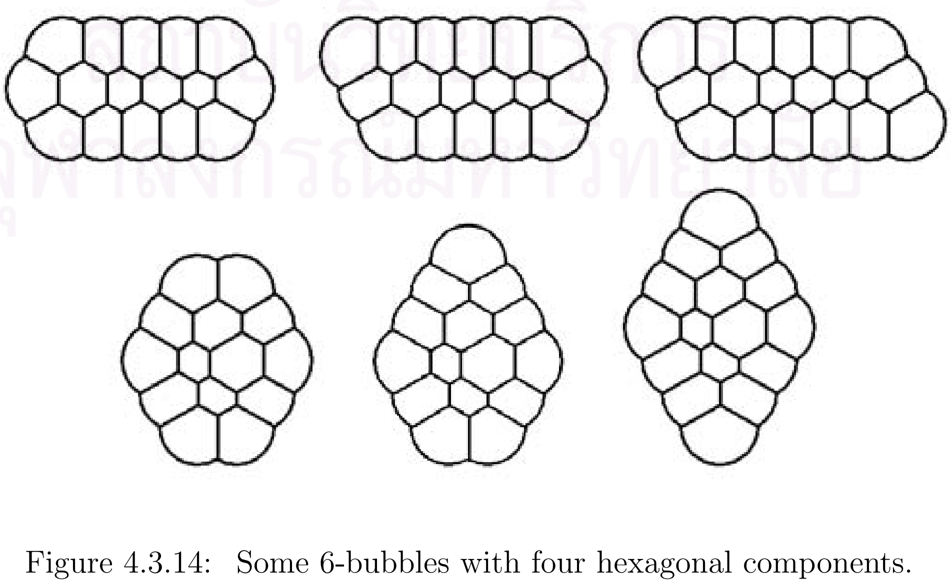


Figure 4.3.14: Some 6-bubbles with four hexagonal components.

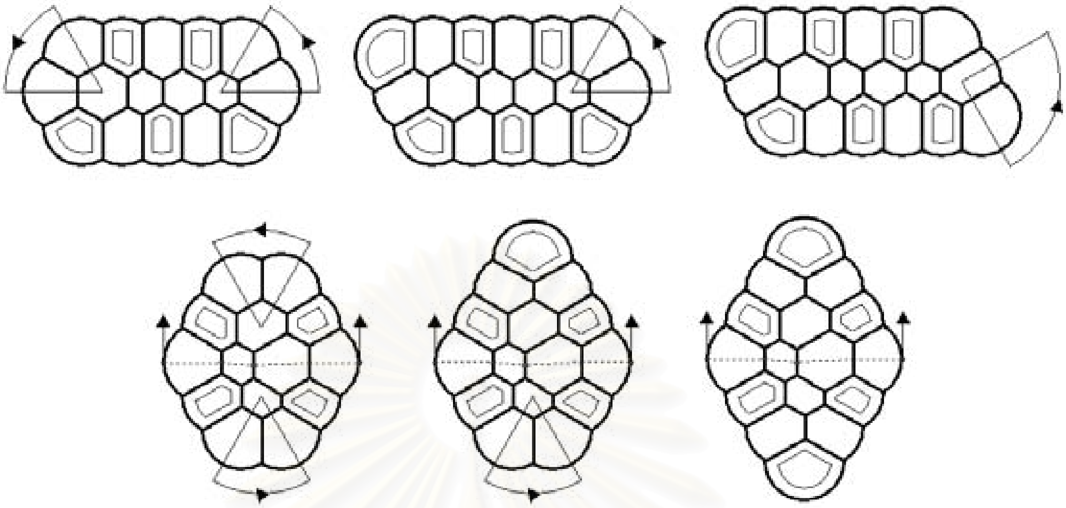
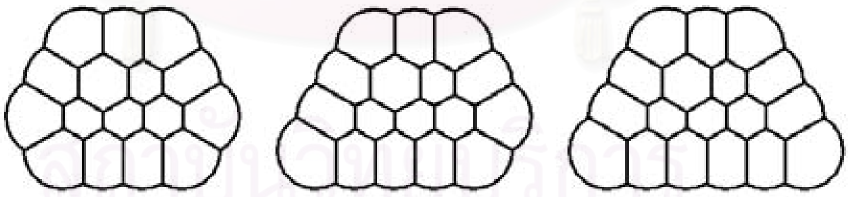


Figure 4.3.15: Some 6-bubbles with seven admissible functions.

Lemma 4.3.5. *A minimizing* 6*-bubble with equal pressures and without empty chambers may not have* $N_6 = 5$.

Proof. Suppose that $N_6 = 5$. By Theorem 2.4.4, there are at most 6 disjoint nonhexagonal convex components. All possible configurations are in Figure 4.3.16.



(a) (b) (c)

Figure 4.3.16: Some 6-bubbles with five hexagonal components.

Consider possibility (a). By Lemma 4.1.10, all 4-sided components have different labels. Without loss of generality, we may label all 4-sided components as Figure 4.3.17. By Corollary 4.1.4, we can assume that $i \in \{4, 5, 6\}$ and $j \in \{1, 2, 3\}$. Now, we may exchange labels to be Figure 4.3.18 (a) or (b) or (c) or (d), a contradiction to Lemma 4.1.16.

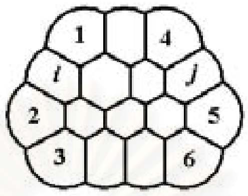


Figure 4.3.17: A labeled 6-bubble with five hexagonal components.

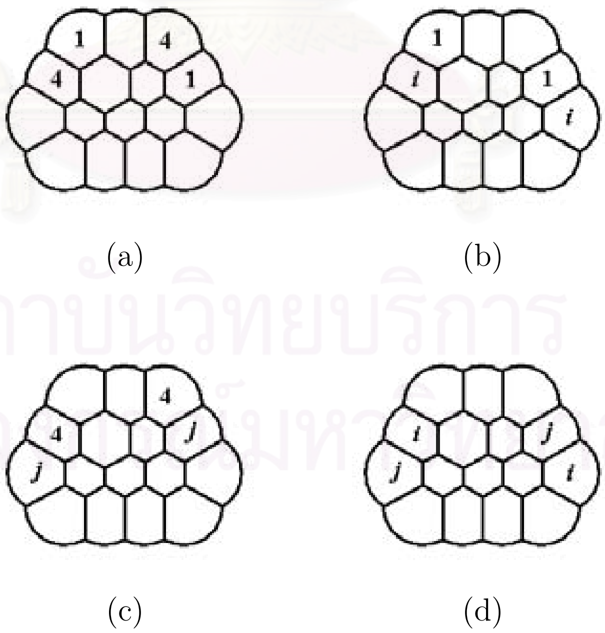


Figure 4.3.18: Some labeled 6-bubbles with five hexagonal components.

Possibilities (b) and (c) are not minimizing by Lemma 3.1.1 using admissible functions shown in Figure 4.3.19. □

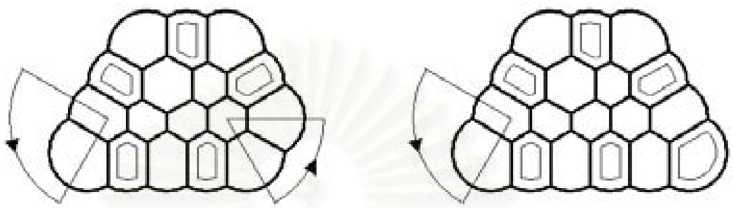


Figure 4.3.19: Some 6-bubbles with seven admissible functions.

Lemma 4.3.6. *A minimizing* 6*-bubble with equal pressures and without empty chambers may not have* $N_6 = 6$*.*

Proof. Suppose that $N_6 = 6$. By Theorem 2.4.4, there are at most 6 disjoint nonhexagonal convex components. All possible configurations are in Figure 4.3.20.

Consider possibility (a). By Lemma 4.1.10, all 4-sided components have different labels. Without loss of generality, we may label all 4-sided components as Figure 4.3.21. By Corollary 4.1.4, we can assume that $i \in \{4, 5, 6\}$ and $j \in \{1, 2, 3\}$. Now, we may exchange labels to be Figure 4.3.22 (a) or (b) or (c) or (d), a contradiction to Lemma 4.1.16.

Consider possibility (d). By Lemma 4.1.10, all 4-sided components have different labels. Without loss of generality, we may label all 4-sided components as Figure 4.3.23. By Corollary 4.1.4, we can assume that $i \in \{3, 4, 5, 6\}$ and $j \in \{1, 2, 5, 6\}$. If $i \in \{3, 4\}$ and $j \in \{1, 2\}$, then we may exchange labels to be Figure 4.3.24 (a), a contradiction to Lemma 4.1.16. If $\{i, j\} \cap \{5, 6\} \neq \phi$, then we may exchange labels to be Figure 4.3.24 (b) or (c), a contradiction to Lemma 4.1.17.

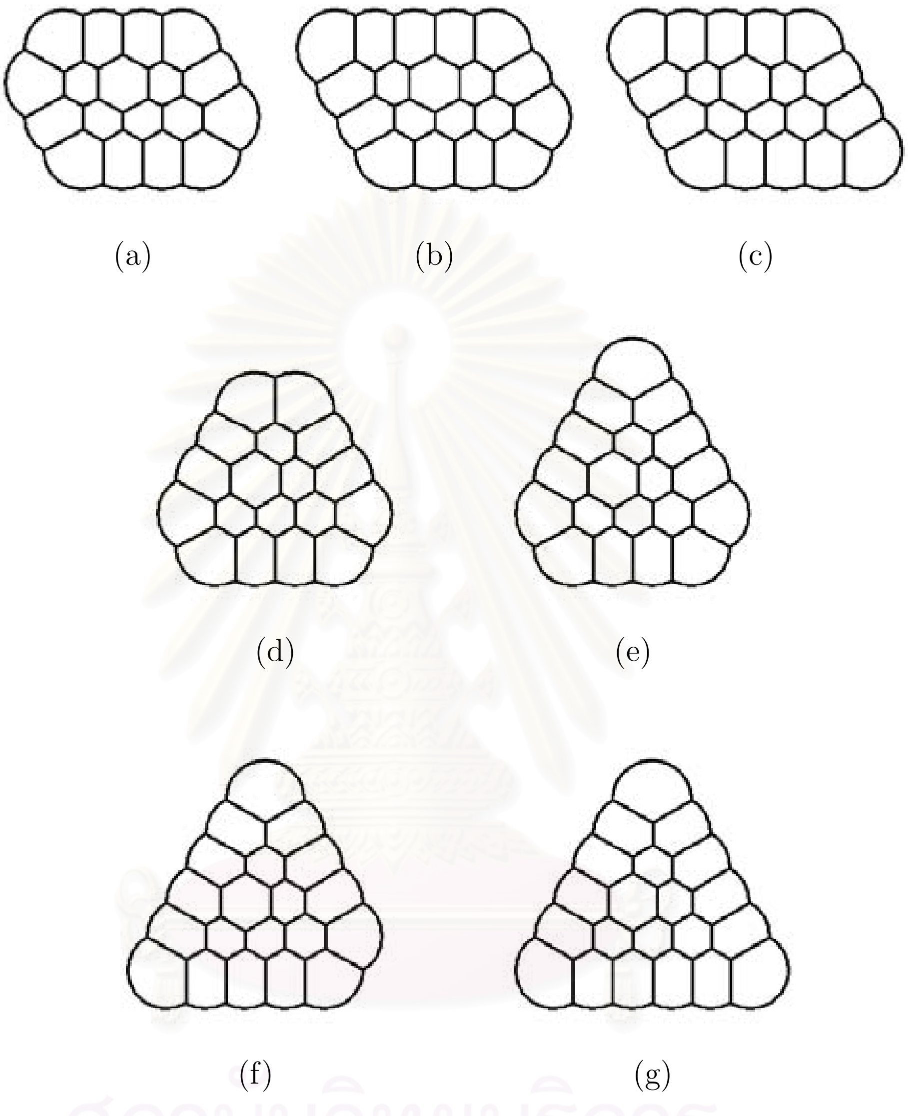


Figure 4.3.20: Some 6-bubbles with six hexagonal components.

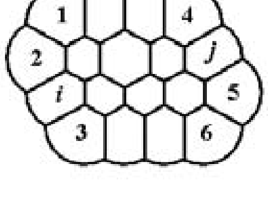


Figure 4.3.21: A labeled 6-bubble with six hexagonal components.

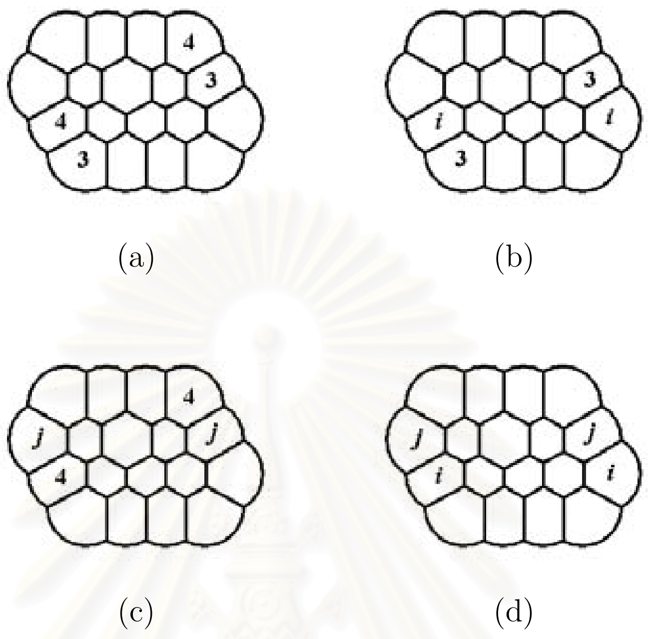


Figure 4.3.22: Some labeled 6-bubbles with six hexagonal components.

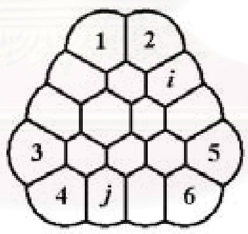


Figure 4.3.23: A labeled 6-bubble with six hexagonal components.

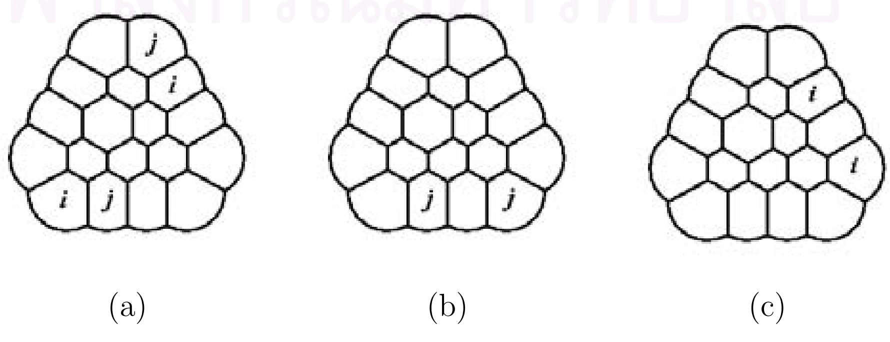


Figure 4.3.24: Some labeled 6-bubbles with six hexagonal components.

Possibilities (b), (c), (e), and (g) are not minimizing by Lemma 3.1.1 using admissible functions shown in Figure 4.3.25. □

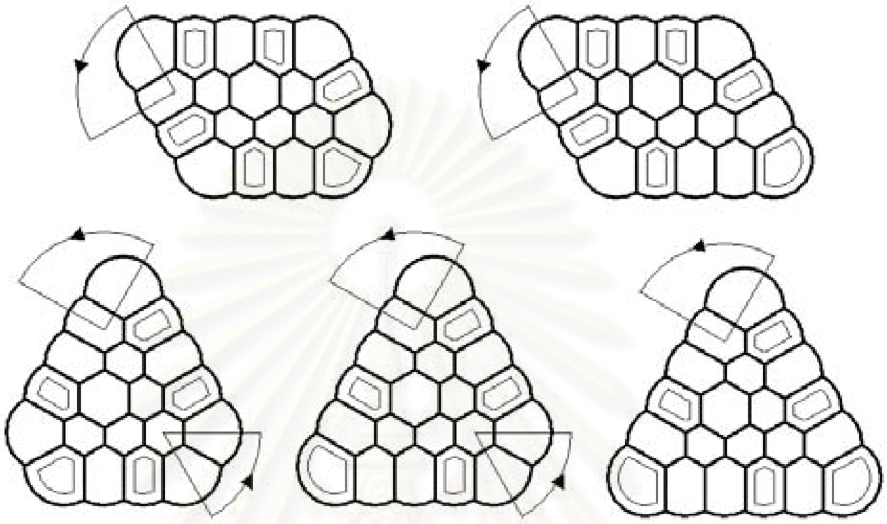


Figure 4.3.25: Some 6-bubbles with seven admissible functions.

Lemma 4.3.7. *A minimizing* 6*-bubble with equal pressures and without empty chambers may not have* $N_6 > 6$*.*

Proof. Suppose that $N_6 > 6$. By Theorem 2.4.4, there are at most 6 disjoint nonhexagonal convex components. All possible configurations are in Figure 4.3.26 which are not minimizing by Lemma 3.1.1 using admissible functions shown in Figure 4.3.27. □

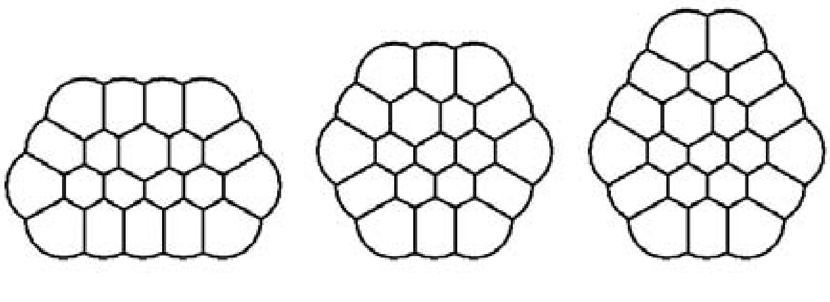


Figure 4.3.26: Some labeled 6-bubbles with seven or eight hexagonal components.

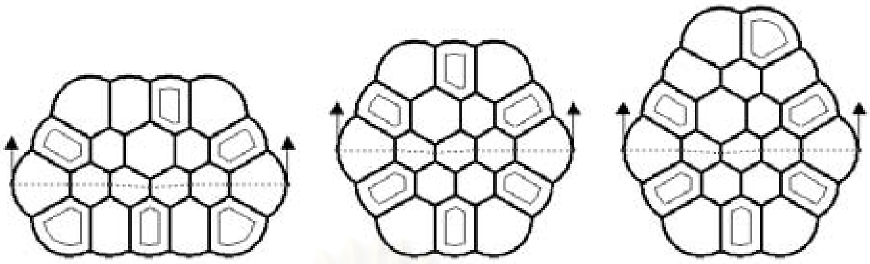


Figure 4.3.27: Some 6-bubbles with seven admissible functions.

Proposition 4.3.8. *Every minimizing* 6*-bubble with equal pressures and without empty chambers is standard.*

Proof. By Lemmas 4.3.1, 4.3.2, 4.3.3, 4.3.4, 4.3.5, 4.3.6 and 4.3.7, every minimizing 6-bubble with equal pressures and without empty chambers must have $N_6 = 0$. By Lemma 4.1.12, we have $2N_3 + N_4 = 6$. Now, we have four cases: (1) $N_3 = 3$ and $N_4 = 0$; (2) $N_3 = 2$ and $N_4 = 2$; (3) $N_3 = 1$ and $N_4 = 4$; (4) $N_3 = 0$ and $N_4 = 6$.

Case 1: $N_3 = 3$ and $N_4 = 0$. The only one possibility is in Figure 4.3.28 (a).

Case 2: $N_3 = 2$ and $N_4 = 2$. By Lemma 4.1.13, we have $N_5 \leq 2$. The only one possibility is in Figure 4.3.28 (b).

Case 3: $N_3 = 1$ and $N_4 = 4$. This case is impossible.

Case 4: $N_3 = 0$ and $N_4 = 6$. This case is impossible.

Therefore being standard is the only one possibility. □

(a) (b)

Figure 4.3.28: Standard 6-bubbles.

Corollary 4.3.9. *A weakly minimizing* 6*-bubble with equal pressures is standard.*

To solve the planar 6-bubble problem completely, the work left to do is showing that every weakly minimizing 6-bubble with unequal pressures is standard.

CHAPTER V

Quadruple bubbles with $p_1 > p_2 = p_3 = p_4$

In this chapter, we discuss properties of 4-bubbles with $p_1 > p_2 = p_3 = p_4$. We conclude the main result that if R_1 is connected, then a minimizing 4-bubble with $p_1 > p_2 = p_3 = p_4$ and without empty chambers is standard. In particular, it must be one of the standard 4-bubbles shown in Figure 5.0.1.

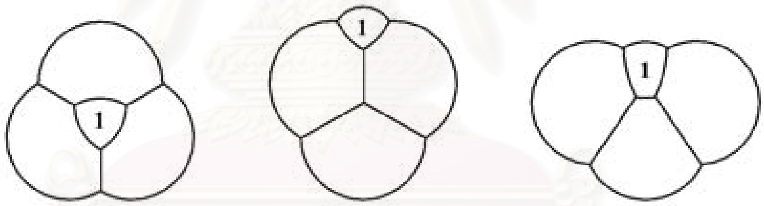


Figure 5.0.1: The standard 4-bubbles with $p_1 > p_2 = p_3 = p_4$.

5.1 Properties of m-bubbles with $p_1 > p_2 = p_3 = \ldots = p_m$

In this section, we discuss basic properties of m-bubbles with $p_1 > p_2 = p_3 = \ldots = p_m$ for $m \geq 4$.

Proposition 5.1.1. *For $m \in \{4, 5, 6\}$, a stable m-bubble with $p_1 > p_2 = p_3 = \ldots = p_m$ must have $p_m > 0$.*

Proof. Suppose that $p_m \leq 0$. By Proposition 2.1.5, we have $p_1 > 0$. Let C be the unbounded component of the bubble and n the number of sides on the boundary of C. By Lemma 2.1.8, the sum of turning angles of all edges of C is $\frac{-6-n}{3}\pi$. Let n_1 be the number of edges between C and R_1. By Proposition 2.4.1 and Theorem 2.4.4, we have $n_1 \leq m$. Now, $p_1 > 0 \geq p_2 = p_3 = \ldots = p_m$. For each edge e on the boundary of C, the turning angle of e is in $(-\pi, 0)$ if e is between C and R_1, and the turning angle of e is nonnegative otherwise. Thus $\frac{-6-n}{3}\pi > (n - n_1)(0) + n_1(-\pi)$ and then $n < 3n_1 - 6$. Note that $n \geq 2n_1$. Hence $6 < n_1 \leq m$, a contradiction. □

Lemma 5.1.2. *Consider a minimizing m-bubble with $p_1 > p_2 = p_3 = \ldots = p_m$ and without empty chambers for $m \geq 4$. A 4-sided component of a type in Figure* 5.1.1 *is symmetric as shown. Moreover, the shape of the first type is uniquely determined by its curvatures and the length of the side edges.*

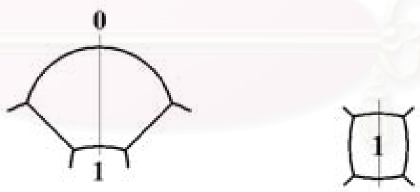


Figure 5.1.1: Two types of 4-sided components that are symmetric.

Proof. This proof is similar to the proof of Lemma 5.18 in [26, 27]. Let C be a 4-sided component of a type in Figure 5.1.1. Since $p_2 = p_3 = \ldots = p_m$, it follows that the side edges have same curvature.

Case 1: C is of the first type. Suppose that the side edges of C are cocircular. Since the two side edges of C are straight, the bottom edge would have turning angle $\frac{2\pi}{3}$ as in Figure 5.1.2. Since $p_1 > p_2 = p_3 = \ldots = p_m$ and Lemma 2.1.8, the convex component C_0 of R_1 under C is a 2-sided component as in Figure 5.1.2.

This contradicts to Corollary 2.4.2. Hence C is not cocircular. Therefore, by Lemma 2.5.1, C is also symmetric vertically.

Figure 5.1.2: To make a circular 4-sided component of the first type, the bottom edge would have turning angle $\frac{2\pi}{3}$. In case $p_1 > p_2 = p_3 = \ldots = p_m$, the component is a 2-sided component.

Next, we will show that the shape of C is uniquely determined by curvatures and a side length. By symmetry, the two side edges of C are straight and their lengths are equal. By Proposition 2.5.5 and that the signs of the curvatures of its edges, the top and bottom edges can not be parallel. Given four curvature of edges and a side length, by Lemma 2.5.2, the shape of C is unique.

Case 2: C is of the second type. Suppose that the side edges of C are cocircular, say on a circle O. Consider the construction of C by clipping O. Since $p_1 > p_2 = p_3 = \ldots = p_m$, to make the bottom edge, we have to clip out the perimeter of O for an angle $\frac{4\pi}{3}$ (see Figure 5.1.3). Similarly, to make the top edge, we have to clip out the perimeter of O for an angle $\frac{4\pi}{3}$ (see Figure 5.1.3). Thus we have no perimeter of O left to make C, a contradiction. Hence the side edges of C are not cocircular. Therefore, by Lemma 2.5.1, C is symmetric vertically.

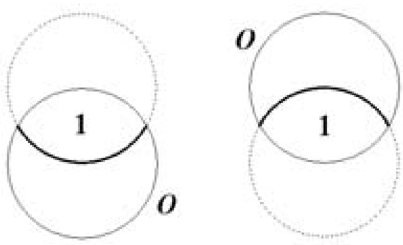


Figure 5.1.3: To make a circular 4-sided component of the second type, we have to clip out the perimeter of the circle.

This proof is completed. □

Corollary 5.1.3. *Consider a minimizing m-bubble with* $p_1 > p_2 = p_3 = \ldots = p_m$ *and without empty chambers for* $m \geq 4$. *The 4-sided internal component of* R_1 *in Figure* 5.1.4 *is bisymmetric as shown.*

Figure 5.1.4: A bisymmetric 4-sided component.

Proof. This follows from Lemma 5.1.2. □

Proposition 5.1.4. *Consider a minimizing m-bubble B with* $p_1 > p_2 = p_3 = \ldots = p_m$ *where* $m \in \{4, 5, 6\}$ *and a component C of* R_i *in B where* $i \neq 0, 1$. *Let n be the number of sides of C, and* n_1 *the number of edges between C and* R_1. *If C is internal and all components of* R_1 *adjacent to C are internal 3-sided, then* $n = n_1 + 6$. *Otherwise, we have* $\max\{2n_1, 3\} \leq n \leq n_1 + 5$.

Proof. By Proposition 5.1.1, we have $p_m > 0$. For each edge e on the boundary of C, the turning angle of e is

1. $-\frac{\pi}{3}$ if e is between C and an internal 3-sided component of R_1,
2. in $(-\frac{\pi}{3}, 0]$ if e is between C and a component of R_1 which is not internal 3-sided,
3. positive if e is between C and R_0, and
4. zero otherwise.

By Lemma 2.1.8, the sum of turning angles of all edges of C is $\frac{6-n}{3}\pi$.

If C is internal and all components of R_1 which are adjacent to C are internal 3-sided, then $\frac{6-n}{3}\pi = n_1(-\frac{\pi}{3})$ and then $n = n_1 + 6$.

If C is external or C is internal and there is a component of R_1 adjacent to C but it is not internal 3-sided. Thus $\frac{6-n}{3}\pi > n_1(-\frac{\pi}{3})$ and then $n < n_1 + 6$. Hence $n \leq n_1 + 5$. Note that $n \geq 2n_1$. By Corollary 2.4.2, we have $n \geq \max\{2n_1, 3\}$.

This proof is completed. $\square$

5.2 Properties of 4-bubbles with $p_1 > p_2 = p_3 = p_4$

In this section, we discuss properties of 4-bubbles with $p_1 > p_2 = p_3 = p_4$ and without empty chambers. Note that $p_4 > 0$ according to Proposition 5.1.1. Moreover, by Lemma 2.1.8, every external component of R_2, R_3 or R_4 not adjacent to R_1 is of a type in Figure 5.2.1.

(a) (b) (c)

Figure 5.2.1: An external 3-sided component, an external 4-sided component and an external 5-sided component.

Proposition 5.2.1. *A minimizing 4-bubble with $p_1 > p_2 = p_3 = p_4$ and without empty chambers has at most one internal 3-sided component of R_1.*

Proof. Suppose that a minimizing 4-bubble with $p_1 > p_2 = p_3 = p_4$ and without empty chambers has at least two internal 3-sided components of R_1. By Lemma 2.5.3, there are two identical nonhexagonal convex components, contradicts to Proposition 2.4.3. $\square$

Lemma 5.2.2. *A minimizing* 4*-bubble with* $p_1 > p_2 = p_3 = p_4$ *and without empty chambers has at most one external 3-sided component of the regions* R_2, R_3 *or* R_4 *not adjacent to* R_1 *(see Figure* 5.2.1 *(a).)*

Proof. Suppose that a minimizing 4-bubble with $p_1 > p_2 = p_3 = p_4$ and without empty chambers has at least two external 3-sided components as in Figure 5.2.1 (a). By Lemma 2.5.3, all 3-sided components as in Figure 5.2.1 (a) must have a unique shape. Since $p_1 > p_2 = p_3 = p_4$, it follows that the 3-sided component as in Figure 5.2.1 (a) is not a component of R_1 and is not adjacent to R_1. If two 3-sided components as in Figure 5.2.1 (a) has different labels, then we can exchange the labels of them and then create a shorter enclosure of the original areas by removing redundant edges, which is a contradiction. Thus all 3-sided components as in Figure 5.2.1 (a) must have same labels. Hence there are two identical nonhexagonal convex components which contradicts to Proposition 2.4.3. □

Lemma 5.2.3. *A minimizing* 4*-bubble with* $p_1 > p_2 = p_3 = p_4$ *and without empty chambers may not have three consecutive 4-sided components of the first type in Figure* 5.1.1*.*

Proof. Suppose that a minimizing 4-bubble with $p_1 > p_2 = p_3 = p_4$ and without empty chambers has three consecutive 4-sided components of the first type in Figure 5.1.1. By Lemma 5.1.2, we can exchange all labels of them until there is an edge between the same region and then create a shorter enclosure of the original areas by deleting the edge, this is a contradiction. □

Lemma 5.2.4. *A minimizing* 4*-bubble with* $p_1 > p_2 = p_3 = p_4$ *and without empty chambers may not have a local configuration in Figure* 5.2.2*.*

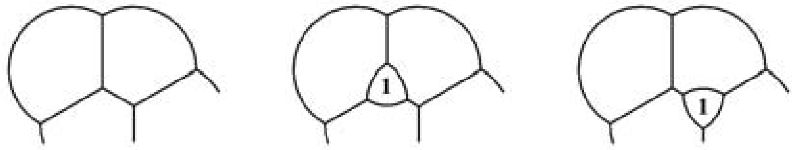


Figure 5.2.2: Three local configurations on 4-bubbles with $p_1 > p_2 = p_3 = p_4$.

Proof. Suppose that a minimizing 4-bubble with $p_1 > p_2 = p_3 = p_4$ and without empty chambers has a local configuration in Figure 5.2.2. We label the local configurations as in Figure 5.2.3.

Figure 5.2.3: Labeling of the local configurations in Figure 5.2.2.

By decoration of a 3-sided component, we may extend the idea in [25] for the first two local configurations to create a shorter enclosure of the original areas as in Figure 5.2.4.

Figure 5.2.4: A way to improve the first two local configurations in Figure 5.2.3.

It is clear that the third local configuration has the same length and areas as the second local configuration. Hence the third local configuration is not minimizing either. □

Lemma 5.2.5. *A minimizing* 4*-bubble with* $p_1 > p_2 = p_3 = p_4$ *and without empty chambers may not have a local configuration in Figure* 5.2.5*.*

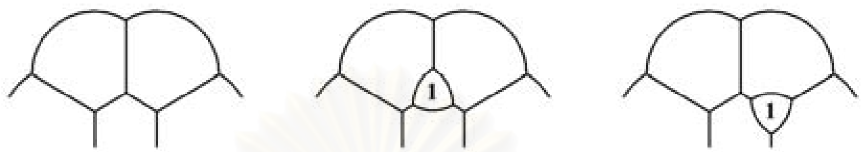


Figure 5.2.5: Three local configurations on 4-bubbles with $p_1 > p_2 = p_3 = p_4$.

Proof. Suppose that a minimizing 4-bubble with $p_1 > p_2 = p_3 = p_4$ and without empty chambers has a local configuration in Figure 5.2.5. We label the local configurations as in Figure 5.2.6.

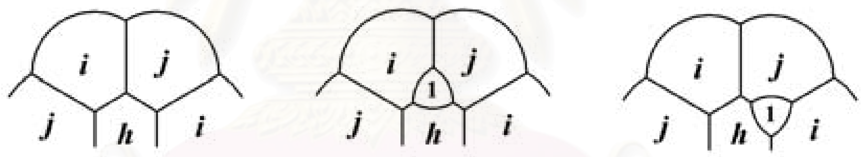


Figure 5.2.6: Labeling of the local configurations in Figure 5.2.5.

Note that the two consecutive 4-sided components of the first local configuration in Figure 5.2.6 are isometric. By decoration of a 3-sided component, we may extend the idea in [25] for the first two local configurations to create a shorter enclosure of the original areas as in Figure 5.2.7.

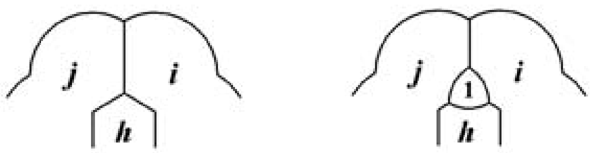


Figure 5.2.7: A way to improve the first two local configurations in Figure 5.2.6.

It is clear that the third local configuration has the same length and areas as the second local configuration. Hence the third local configuration is not minimizing either. □

Lemma 5.2.6. *A minimizing* 4*-bubble with* $p_1 > p_2 = p_3 = p_4$ *and without empty chambers may not have a local configuration in Figure* 5.2.8*.*

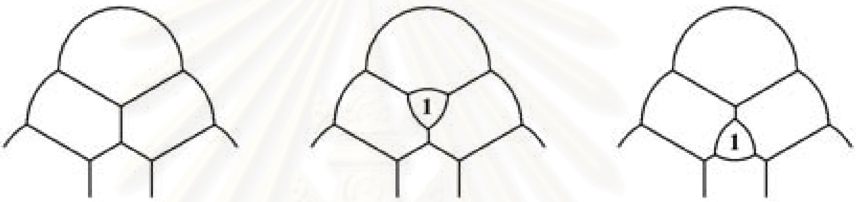


Figure 5.2.8: Three local configurations on 4-bubbles with $p_1 > p_2 = p_3 = p_4$.

Proof. Suppose that a minimizing 4-bubble with $p_1 > p_2 = p_3 = p_4$ and without empty chambers has a local configuration in Figure 5.2.8. We label the local configurations as in Figure 5.2.9.

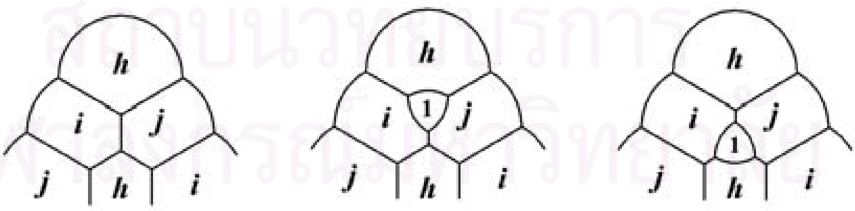


Figure 5.2.9: Labeling of the local configurations in Figure 5.2.8.

By decoration of a 3-sided component, we may extend the idea in [25] for the local configurations to create a shorter enclosure of the original areas as in Figure 5.2.10. This leads to a contradiction. □

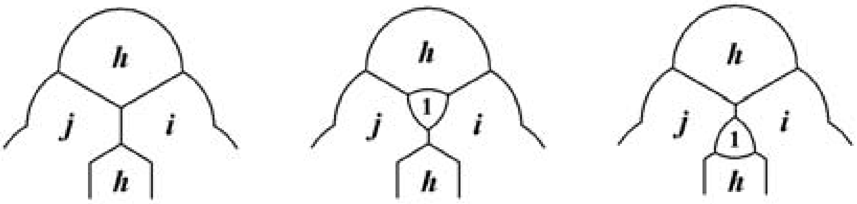


Figure 5.2.10: A way to improve the local configurations in Figure 5.2.9.

Lemma 5.2.7. *A minimizing* 4*-bubble with* $p_1 > p_2 = p_3 = p_4$ *and without empty chambers may not have a local configuration in Figure* 5.2.11*.*

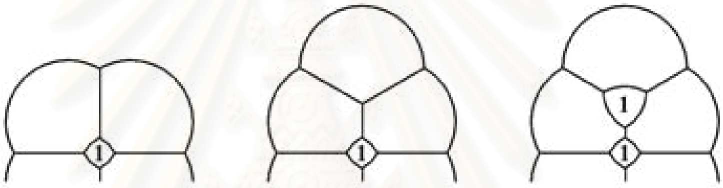


Figure 5.2.11: Three local configurations on 4-bubbles with $p_1 > p_2 = p_3 = p_4$.

Proof. Suppose that a minimizing 4-bubble with $p_1 > p_2 = p_3 = p_4$ and without empty chambers has a local configuration in Figure 5.2.11. Then we can label Figure 5.2.11 as in Figure 5.2.12.

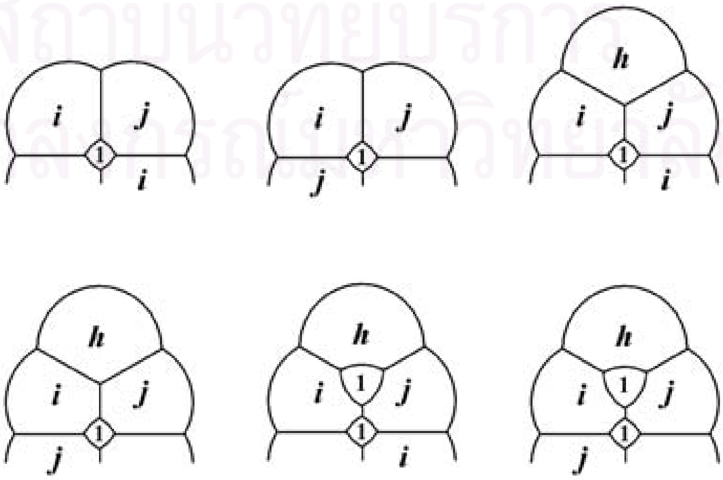


Figure 5.2.12: Labeling of the local configurations in Figure 5.2.11.

By Lemma 5.1.2, the two consecutive 4-sided components of the first local configuration in Figure 5.2.11 are isometric. By decoration of a 3-sided component, we can create a shorter enclosure of the original areas as in Figure 5.2.13. So the supposition is not possible. □

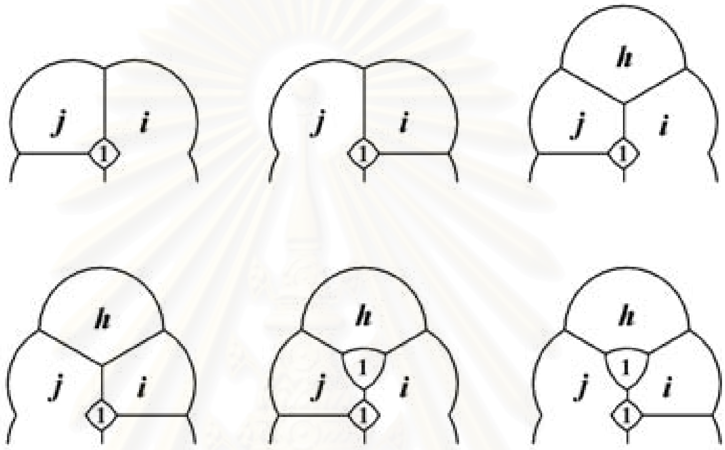


Figure 5.2.13: A way to improve the local configurations in Figure 5.2.12.

Lemma 5.2.8. *For the measurement in Figure* 5.2.14, *we have that the length* d *is increasing on the length* L.

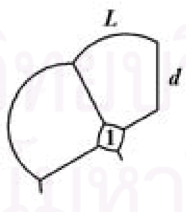


Figure 5.2.14: A local configuration on 4-bubbles with $p_1 > p_2 = p_3 = p_4$.

Proof. This is by exhaustive, directed, elementary calculation as follows.

We scale bubbles with $p_1 > p_2 = p_3 = p_4$ so that the external edges of R_2, R_3 and R_4 have curvature one. Let r be the radius of edges between the region R_1 and the region R_i where $i \neq 0, 1$. Note that $\frac{\pi}{3} < L < \frac{2\pi}{3}$. We denote $t = \frac{L}{2} - \frac{\pi}{6}$, so $0 <$

$t < \frac{\pi}{6}$. Then $d = \frac{\sqrt{3}}{2} + \frac{\cot t}{2} - \left(\cos 2t + \frac{\sin 2t}{\sqrt{3}}\right)\left(\frac{\sqrt{3}}{2} + \frac{\cot t}{2} - \frac{2r\tan t}{1+\sqrt{3}\tan t} - \frac{2-r(1-\sqrt{3}\tan t)}{\sqrt{3}-\tan t}\right)$ for all $0 < t < \frac{\pi}{6}$. Simplifying, we get $d = \frac{6\cos 2t - 3(1+r)\cos 4t + 2\sqrt{3}(1+(1+r)\cos 2t)\sin 2t}{3(\sqrt{3}\cos t - \sin t)(\cos t + \sqrt{3}\sin t)}$ for all $0 < t < \frac{\pi}{6}$. Thus $d' = \frac{2(1+r)(9\cos 2t + \sqrt{3}(3\sin 2t + 2\sin 6t))}{3(\sqrt{3}\cos t - \sin t)^2(\cos t + \sqrt{3}\sin t)^2} > 0$ for all $0 < t < \frac{\pi}{6}$. Hence d is increasing on the length L. □

Remark 5.2.9. *Simplification of d may be obtained using Mathematics 5.1.*

Lemma 5.2.10. *A minimizing* 4*-bubble with* $p_1 > p_2 = p_3 = p_4$ *and without empty chambers may not have a local configuration in Figure* 5.2.15.

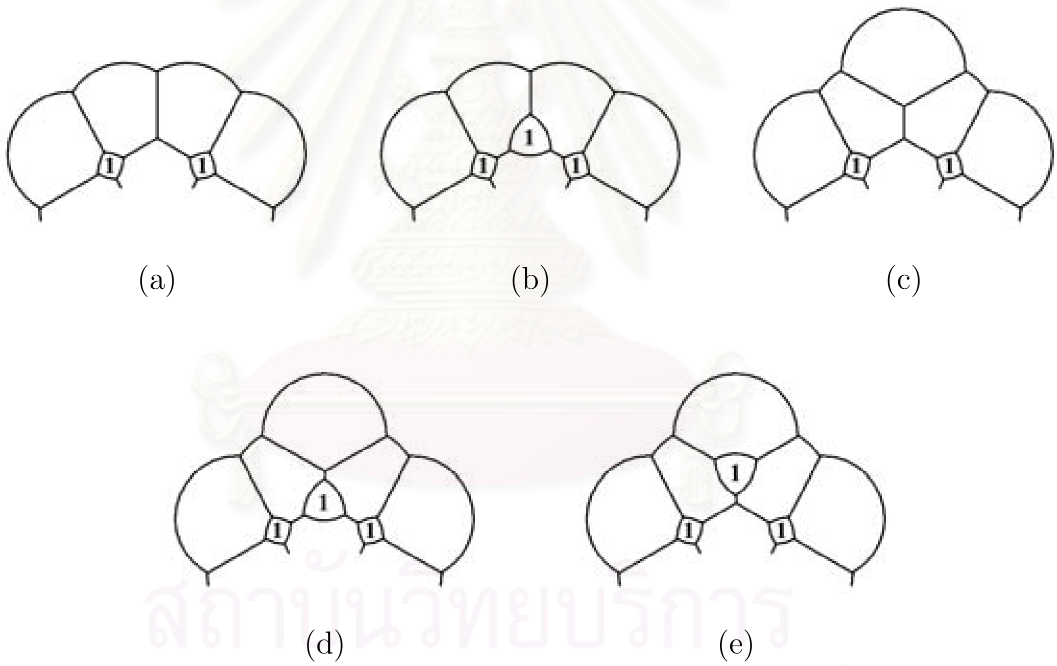


Figure 5.2.15: Five local configurations on 4-bubbles with $p_1 > p_2 = p_3 = p_4$.

Proof. Suppose that a minimizing 4-bubble with $p_1 > p_2 = p_3 = p_4$ and without empty chambers has a local configurations in Figure 5.2.15. By Lemma 5.2.8, and by decoration of 3-sided components, it follows that every local configuration in Figure 5.2.15 is symmetric as in Figure 5.2.16.

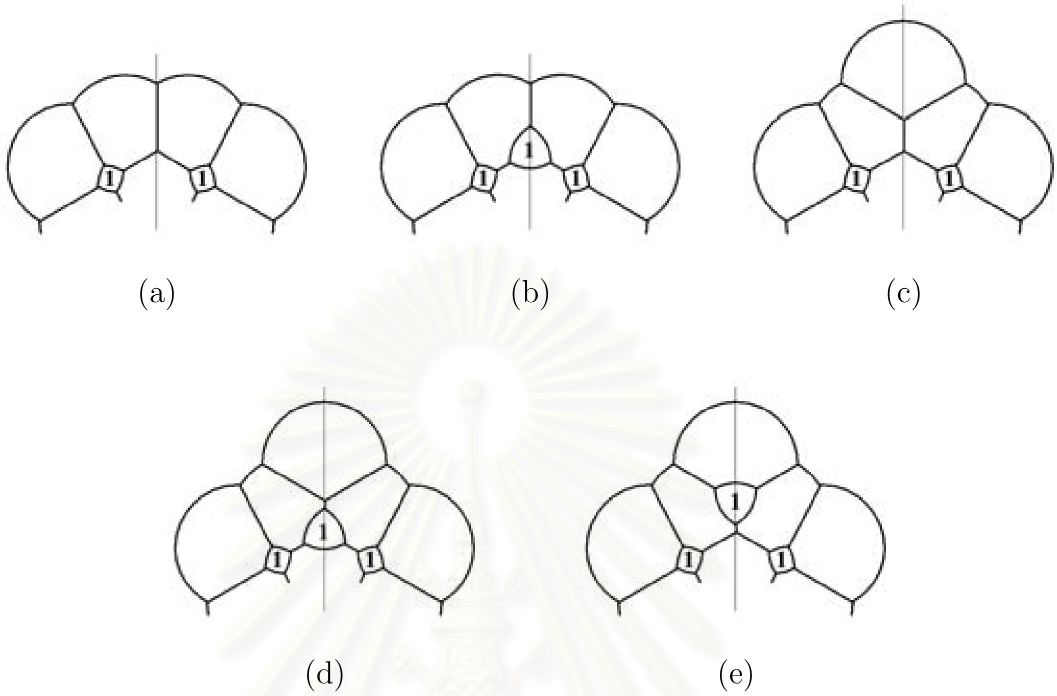


Figure 5.2.16: The symmetric line of each local configuration in Figure 5.2.15.

Consider the local configuration (a) in Figure 5.2.15. All possibilities are in Figure 5.2.17. For the first two possibilities in Figure 5.2.17, we can create a shorter enclosure of the original areas as in Figure 5.2.18, which is a contradiction. Thus the local configuration must be the third possibility in Figure 5.2.17. By Lemma 5.1.2, we have a rotating function which preserves all areas as in Figure 5.2.19. By Lemma 2.3.6, the four tentacles have the same center, which is a contradiction.

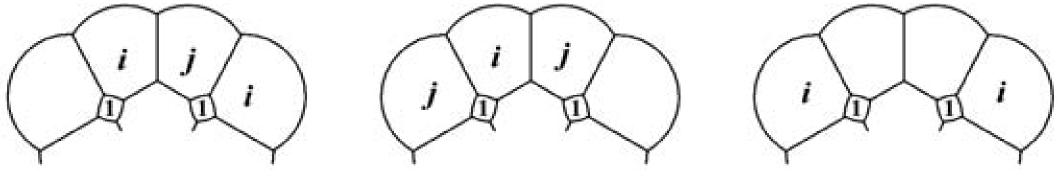


Figure 5.2.17: All possibilities from the local configuration (a) in Figure 5.2.15.

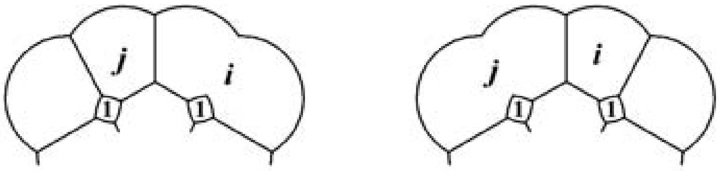


Figure 5.2.18: A way to improve the first two possibilities in Figure 5.2.17.

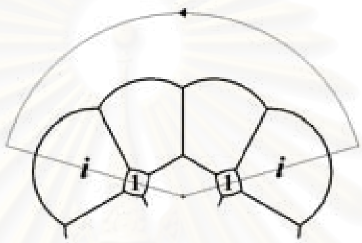


Figure 5.2.19: A rotating function on the third possibility in Figure 5.2.17.

Consider the local configuration (b) in Figure 5.2.15. All possibilities are in Figure 5.2.20. For the first two possibilities in Figure 5.2.20, we can create a shorter enclosure of the original areas as in Figure 5.2.21, which is a contradiction. Thus the local configuration must be the third possibility in Figure 5.2.20. By Lemma 5.1.2, we have a rotating function which preserves all areas as in Figure 5.2.22. By Lemma 2.3.6, the four tentacles have the same center, which is a contradiction.

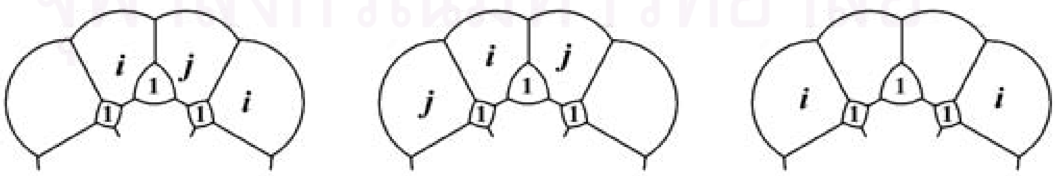


Figure 5.2.20: All possibilities from the local configuration (b) in Figure 5.2.15.

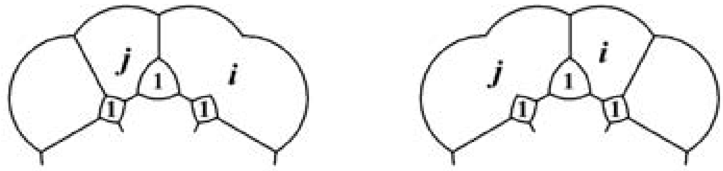


Figure 5.2.21: A way to improve the first two possibilities in Figure 5.2.20.

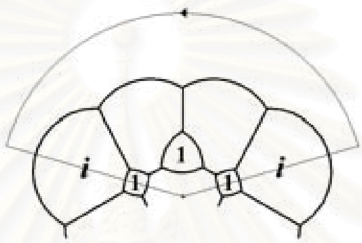


Figure 5.2.22: A rotating function on the third possibility in Figure 5.2.20.

Consider the local configuration (c) in Figure 5.2.15. All possibilities are in Figure 5.2.23. For the first two possibilities in Figure 5.2.23, we can create a shorter enclosure of the original areas as in Figure 5.2.24, which is a contradiction. Thus the local configuration must be the third possibility in Figure 5.2.23. By Lemma 5.1.2, we have a rotating function which preserves all areas as in Figure 5.2.25. By Lemma 2.3.6, the four tentacles have the same center, which is a contradiction.

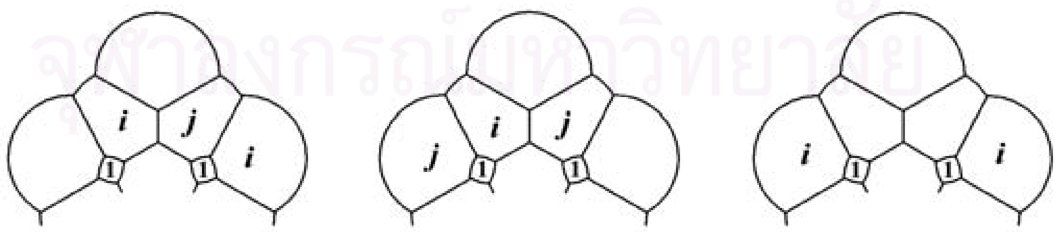


Figure 5.2.23: All possibilities from the local configuration (c) in Figure 5.2.15.

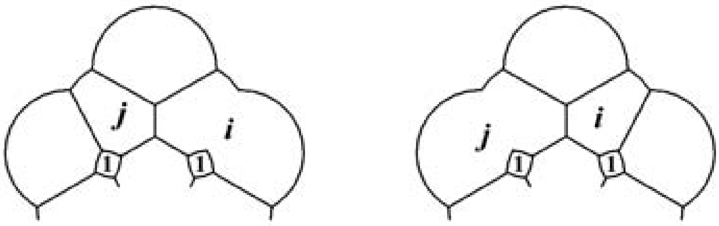


Figure 5.2.24: A way to improve the first two possibilities in Figure 5.2.23.

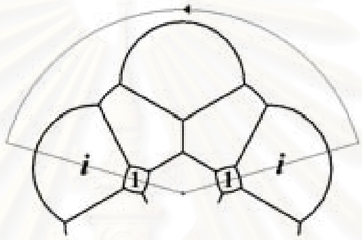


Figure 5.2.25: A rotating function on the third possibility in Figure 5.2.23.

Consider the local configuration (d) in Figure 5.2.15. All possibilities are in Figure 5.2.26. For the first two possibilities in Figure 5.2.26, we can create a shorter enclosure of the original areas as in Figure 5.2.27, which is a contradiction. Thus the local configuration must be the third possibility in Figure 5.2.26. By Lemma 5.1.2, we have a rotating function which preserves all areas as in Figure 5.2.28. By Lemma 2.3.6, the four tentacles have the same center, which is a contradiction.

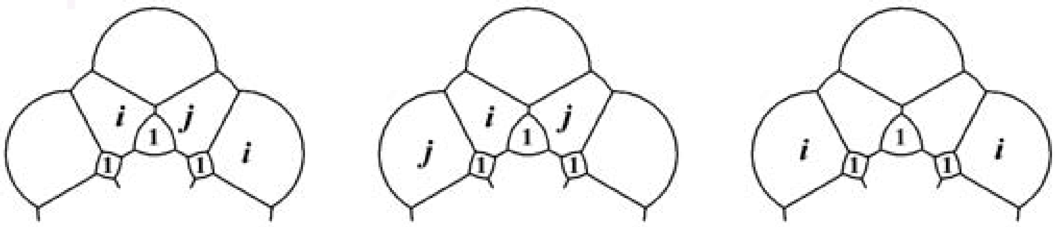


Figure 5.2.26: All possibilities from the local configuration (d) in Figure 5.2.15.

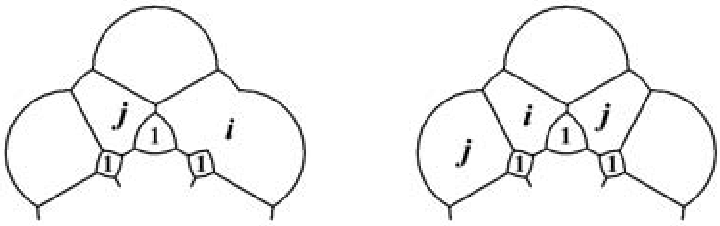


Figure 5.2.27: A way to improve the first two possibilities in Figure 5.2.26.

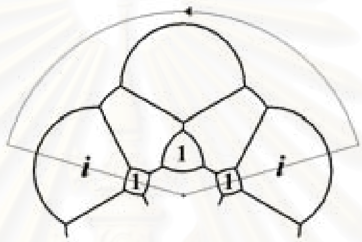


Figure 5.2.28: A rotating function on the third possibility in Figure 5.2.26.

Consider the local configuration (e) in Figure 5.2.15. All possibilities are in Figure 5.2.29. For the first two possibilities in Figure 5.2.29, we can create a shorter enclosure of the original areas as in Figure 5.2.30, which is a contradiction. Thus the local configuration must be the third possibility in Figure 5.2.29. By Lemma 5.1.2, we have a rotating function which preserves all areas as in Figure 5.2.31. By Lemma 2.3.6, the four tentacles have the same center, which is a contradiction.

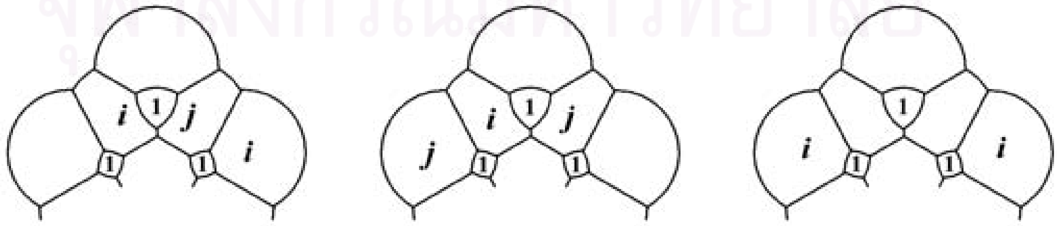


Figure 5.2.29: All possibilities from the local configuration (e) in Figure 5.2.15.

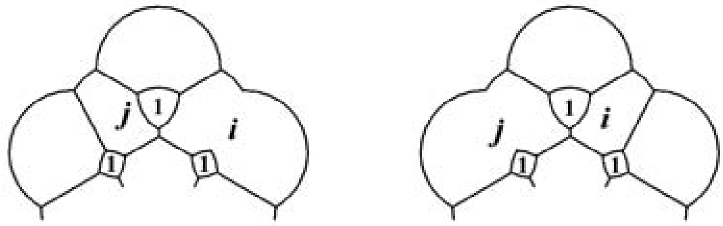


Figure 5.2.30: A way to improve the first two possibilities in Figure 5.2.29.

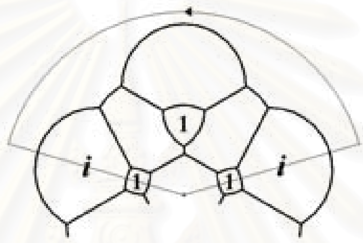


Figure 5.2.31: A rotating function on the third possibility in Figure 5.2.29.

This proof is completed. □

Theorem 5.2.11. *A minimizing* 4*-bubble with* $p_1 > p_2 = p_3 = p_4$ *and without empty chambers may not have a local configuration in Figure* 5.2.32*.*

Proof. This follows Lemmas 5.2.4, 5.2.5, 5.2.6, 5.2.7 and 5.2.10. □

Theorem 5.2.12. *Suppose that* B *is a minimizing* 4*-bubble with* $p_1 > p_2 = p_3 = p_4$ *and without empty chambers. If* B *has four disjoint nonhexagonal convex components, then* B *may not additionally have a local configuration (above the dotted curve) in Figure* 5.2.33*.*

Proof. Assume that B has four disjoint nonhexagonal convex components. Suppose that B additionally has a local configuration as in the Figure 5.2.33.

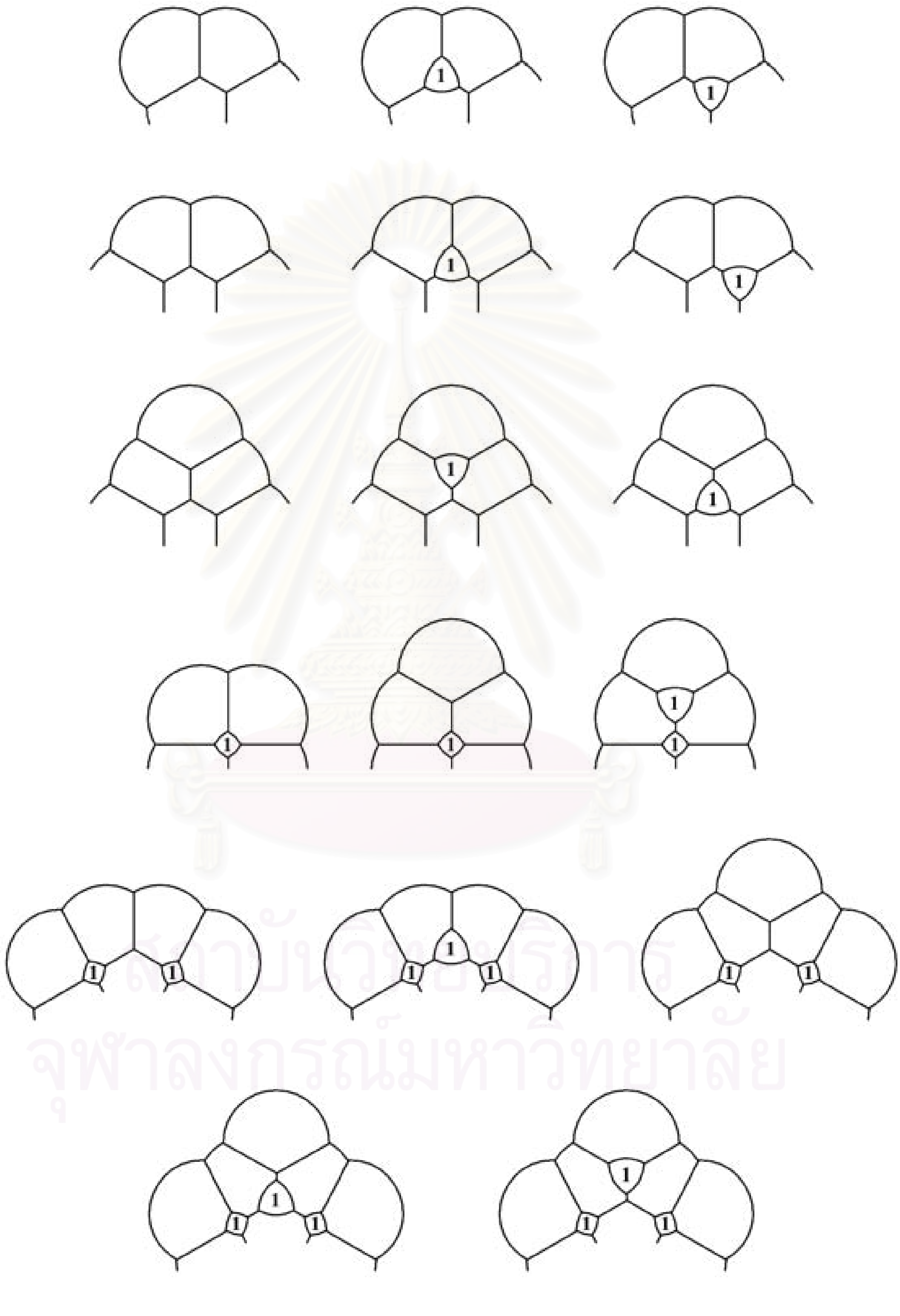


Figure 5.2.32: Seventeen local configurations on 4-bubbles with $p_1 > p_2 = p_3 = p_4$.

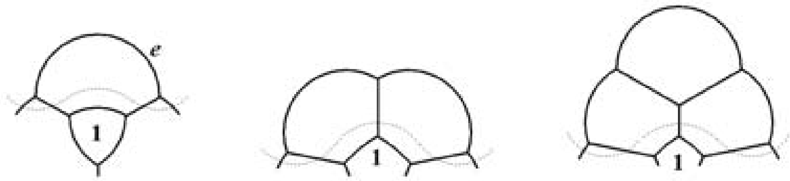


Figure 5.2.33: Three local configurations on 4-bubbles with $p_1 > p_2 = p_3 = p_4$.

For the first local configuration, it is not minimizing by Corollary 3.2.3 since the turning angle of e is π, a contradiction.

For each the other local configuration, we can find a translating function as in Figure 5.2.34.

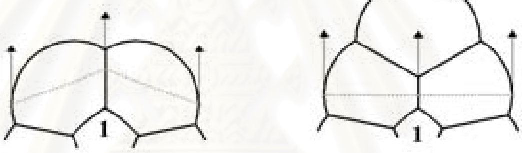


Figure 5.2.34: A translating function for each of last two local configurations in Figure 5.2.33.

Thus B has five admissible functions which contradicts to Lemma 3.1.1.

This proof is completed. $\square$

5.3 Main results on 4-bubbles with $p_1 > p_2 = p_3 = p_4$

In this section, we prove main result that a minimizing 4-bubble with $p_1 > p_2 = p_3 = p_4$ and without empty chambers is standard if R_1 is connected. Recall that $p_4 > 0$ and every external component of R_2, R_3, or R_4 not adjacent to R_1 is of a type in Figure 5.2.1.

Theorem 5.3.1. *[13, 14] A minimizing bubble may not have a local configuration with straight internal edges in Figure* 5.3.1.

Figure 5.3.1: Two local configurations that are not in minimizing bubbles.

Corollary 5.3.2. *A minimizing* 4*-bubble with* $p_1 > p_2 = p_3 = p_4$ *may not have a local configuration in Figure* 5.3.2.

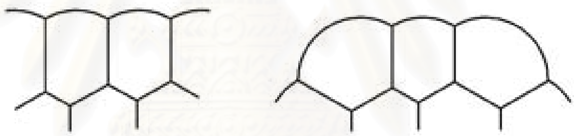


Figure 5.3.2: Two local configurations that are not in every minimizing 4-bubbles with $p_1 > p_2 = p_3 = p_4$.

Proof. This follows from Theorem 5.3.1. □

Lemma 5.3.3. *Suppose that* B *is a minimizing* 4*-bubble with* $p_1 > p_2 = p_3 = p_4$ *and without empty chambers. If* R_1 *is connected then every internal component of* R_2, R_3 *or* R_4 *is of a type in Figure* 5.3.3.

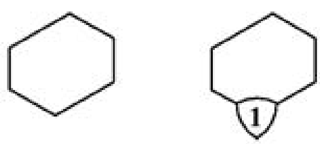


Figure 5.3.3: Two types of internal components of R_2, R_3 and R_4.

Proof. Assume that R_1 is connected. Suppose that B has an internal component C of R_2, R_3 or R_4 is not of a type in Figure 5.3.3. Let n be the number of edges of C and n_1 the number of edges of the component of R_1. Then $n_1 = 1$. If C is not adjacent to R_1, then C must be hexagonal because $p_2 = p_3 = p_4$. Thus C is adjacent to R_1. Moreover, by Lemma 2.1.8, we have $n \geq 7$. Since C is not of the second type in Figure 5.3.3, we have $n \neq 7$. By Proposition 5.1.4, the component of R_1 is not internal 3-sided and then $3 \leq n \leq 6$, a contradiction. □

Lemma 5.3.4. *Suppose that B is a minimizing 4-bubble with $p_1 > p_2 = p_3 = p_4$ and without empty chambers. If R_1 is connected then every external 4-sided component of R_2, R_3 or R_4 is adjacent to R_1.*

Proof. Assume that R_1 is connected. Suppose that B has an external 4-sided component C of R_2, R_3 or R_4 not adjacent to R_1 as in Figure 5.3.4.

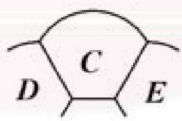


Figure 5.3.4: An external 4-sided component not adjacent to R_1.

By Lemma 5.3.3, every internal component of R_2, R_3 or R_4 is of a type in Figure 5.3.3.

Since R_1 is connected, without loss of generality, the component D in Figure 5.3.4 is not adjacent to R_1. By Lemma 5.2.4 and 5.2.5, the component D is 5-sided as in Figure 5.3.5 (a). We have two cases for Figure 5.3.5 (a) since R_1 is connected.

Case 1. The component F in Figure 5.3.5 (a) is not adjacent to R_1. By Corollary 5.3.2, the component F is 3-sided as in Figure 5.3.5 (d). By Lemma

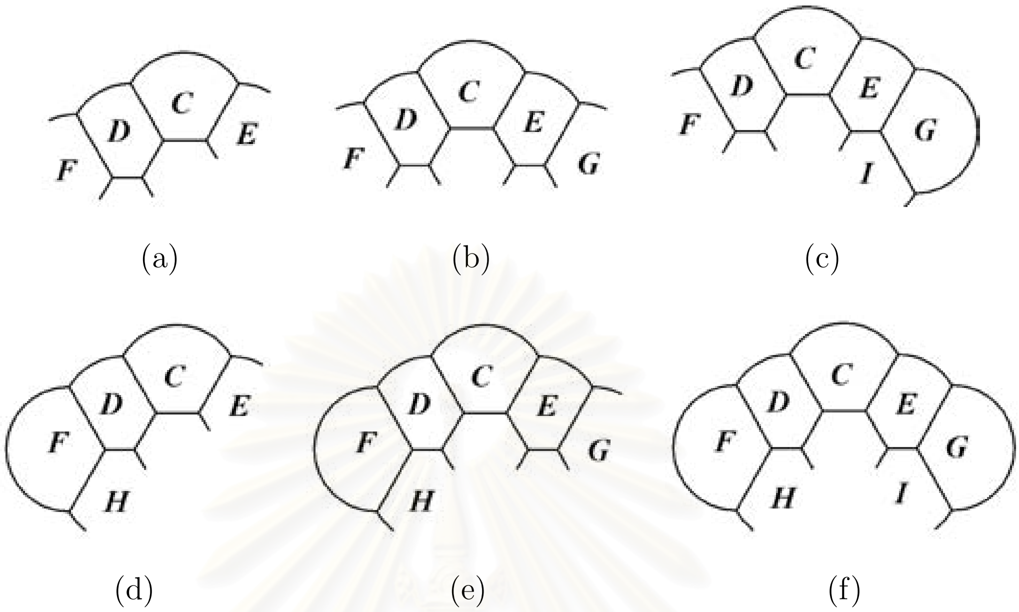


Figure 5.3.5: Local configurations with an external 4-sided component C not adjacent to R_1.

5.2.4 and 5.2.6, the component H in Figure 5.3.5 (d) is adjacent to R_1. Since R_1 is connected, it follows that the component E in Figure 5.3.5 (d) is not adjacent to R_1. By Lemma 5.2.4 and 5.2.5, the component E is 5-sided as in Figure 5.3.5 (e). Since R_1 is connected, it follows that the component G in Figure 5.3.5 (e) is not adjacent to R_1. By Corollary 5.3.2, the component G is 3-sided as in Figure 5.3.5 (f). This is a contradiction to Lemma 5.2.2.

Case 2. The component E in Figure 5.3.5 (a) is not adjacent to R_1. By Lemma 5.2.4 and 5.2.5, E is 5-sided as in Figure 5.3.5 (b). We have two subcases for Figure 5.3.5 (b) since R_1 is connected.

Subcase 2A. The component F in Figure 5.3.5 (b) is not adjacent to R_1. By Corollary 5.3.2, the component F is 3-sided as in Figure 5.3.5 (e). By Lemma 5.2.4 and 5.2.6, the component H in Figure 5.3.5 (e) is adjacent to R_1. Since R_1 is connected, it follows that the component G in Figure 5.3.5 (e) is not adjacent to R_1. By Corollary 5.3.2, the component G is 3-sided as in Figure 5.3.5 (f). This is a contradiction to Lemma 5.2.2.

Subcase 2B. The component G in Figure 5.3.5 (b) is not adjacent to R_1. By Corollary 5.3.2, the component G is 3-sided as in Figure 5.3.5 (c). By Lemma 5.2.4 and 5.2.6, the component I in Figure 5.3.5 (c) is adjacent to R_1. Since R_1 is connected, it follows that the component F in Figure 5.3.5 (c) is not adjacent to R_1. By Corollary 5.3.2, the component F is 3-sided as in Figure 5.3.5 (f). This is a contradiction to Lemma 5.2.2.

This proof is completed. □

Lemma 5.3.5. *Suppose that B is a minimizing 4-bubble with $p_1 > p_2 = p_3 = p_4$ and without empty chambers. If R_1 is connected, then every external 5-sided component of R_2, R_3 or R_4 is adjacent to R_1.*

Proof. Assume that R_1 is connected. Suppose that B has an external 5-sided component C of R_2, R_3 or R_4 not adjacent to R_1 as in Figure 5.3.6.

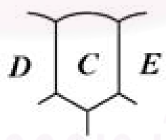


Figure 5.3.6: An external 5-sided component which is not adjacent to R_1.

By Lemma 5.3.3, every internal component of R_2, R_3 or R_4 is of a type in Figure 5.3.3.

Since R_1 is connected, without loss of generality, the component D in Figure 5.3.6 is not adjacent to R_1. By Corollary 5.3.2 and Lemma 5.3.4, the component D is 3-sided as in Figure 5.3.7.

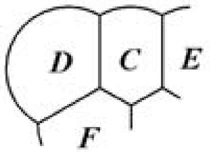


Figure 5.3.7: A local configuration with an external 5-sided component C not adjacent to R_1.

By Lemma 5.2.4 and 5.2.6, the component F in Figure 5.3.7 is adjacent to R_1. Since R_1 is connected, it follows that the component E in Figure 5.3.7 is not adjacent to R_1. By Corollary 5.3.2 and Lemma 5.3.4, the component E is 3-sided as in Figure 5.3.8.

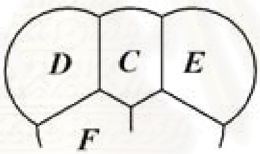


Figure 5.3.8: A local configuration with an external 5-sided component C not adjacent to R_1.

This is a contradiction to Lemma 5.2.2. □

Theorem 5.3.6. *Suppose that B is a minimizing 4-bubble with $p_1 > p_2 = p_3 = p_4$ and without empty chambers. If R_1 is connected, then B is standard.*

Proof. Assume that R_1 is connected. Suppose that B is not standard. By Proposition 5.1.4, the number of sides of each external component of R_2, R_3 or R_4 not adjacent to R_1 is $3, 4$ or 5.

By Lemma 5.3.4, every external 4-sided component of R_2, R_3 or R_4 is adjacent to R_1.

By Lemma 5.3.5, every external 5-sided component of R_2, R_3 or R_4 is adjacent to R_1.

By Lemma 5.2.2, B has at most one external 3-sided component of the regions R_2, R_3 or R_4 not adjacent to R_1.

By Lemma 5.3.3, every internal component of R_2, R_3 or R_4 is of a type in Figure 5.3.3. Since R_1 is connected, it follows that if there is an internal component of R_2, R_3 or R_4 then there is an external 4-sided component or external 5-sided component of R_2, R_3 or R_4 which is not adjacent to R_1. Thus B has no any internal component of R_2, R_3 or R_4.

Hence B has at most one component of R_2, R_3 or R_4 which is not adjacent to R_1. Moreover, it is external and 3-sided. Note that the number of sides of R_1 is $3, 4$ or 5. All possible configurations are in Figure 5.3.9.

For possibility (a), we label it as in Figure 5.3.10. We create a new enclosure of the same areas and the same length as in Figure 5.3.11. Since the new enclosure is not minimizing, it follows that the bubble in Figure 5.3.10 is not minimizing, a contradiction.

By Lemma 5.2.7, possibilities (b) and (c) are not minimizing, a contradiction.

For possibility (d), by proposition 2.1.9, we have the way to label it as in Figure 5.3.12. We create a new enclosure of the same areas and the same length as in Figure 5.3.13. Since the new enclosure is not minimizing, it follows that Figure 5.3.12 is not minimizing, a contradiction.

By Lemma 5.2.3, possibilities (e) and (f) are not minimizing, a contradiction.

For possibility (g), by proposition 2.1.9, we have only two ways to label it as in Figure 5.3.14. By Lemma 5.1.2, the two consecutive 4-sided component in Figure 5.3.14 have the same area. For each label in Figure 5.3.14, we create a shorter enclosure of the same areas as in Figure 5.3.15, a contradiction.

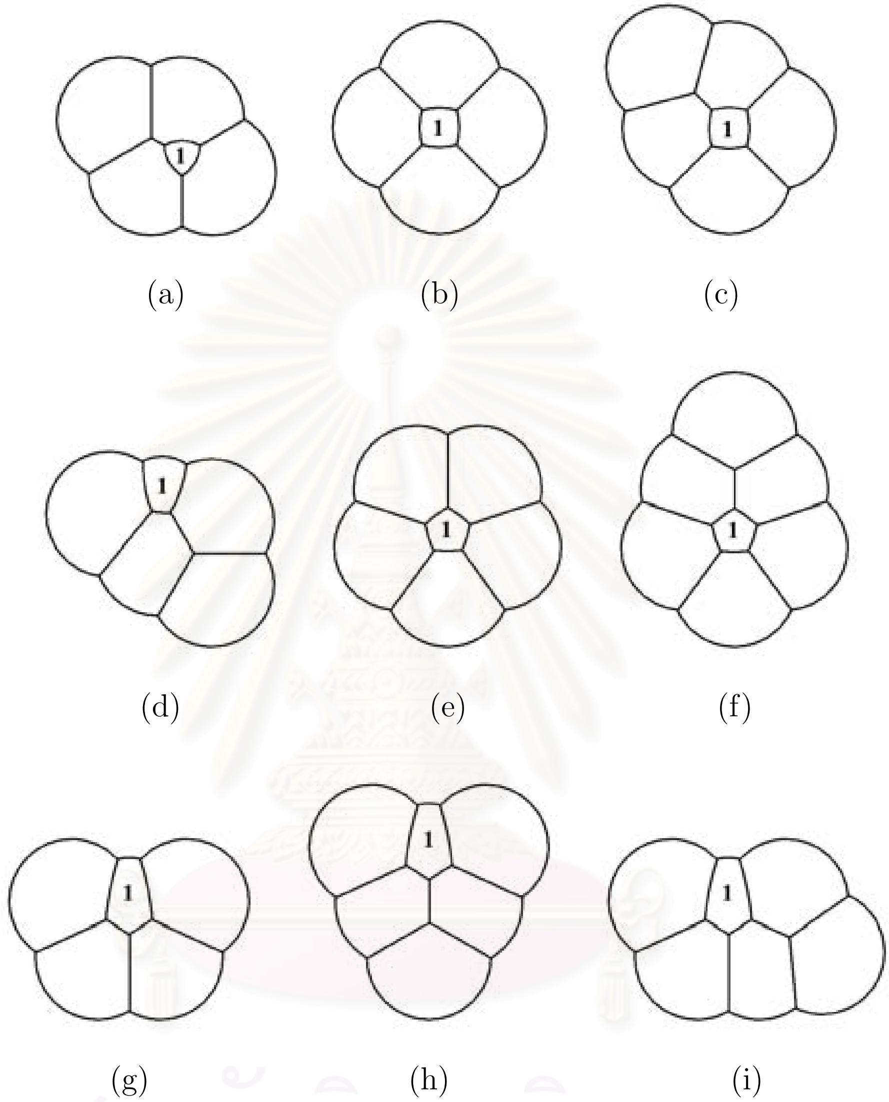


Figure 5.3.9: Nine configurations on 4-bubbles with $p_1 > p_2 = p_3 = p_4$.

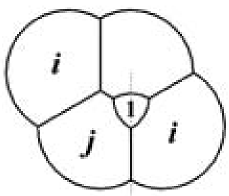


Figure 5.3.10: A label for possibility (a) in Figure 5.3.9.

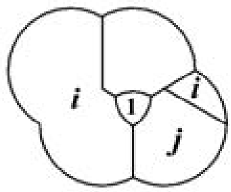


Figure 5.3.11: A new enclosure of the same areas and the same length as Figure 5.3.10.

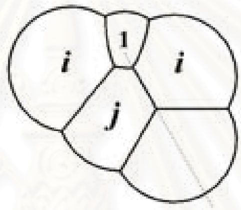


Figure 5.3.12: A label for possibility (d) in Figure 5.3.9.

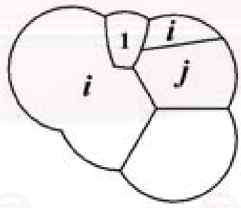


Figure 5.3.13: A new enclosure of the same areas and the same length as Figure 5.3.12.

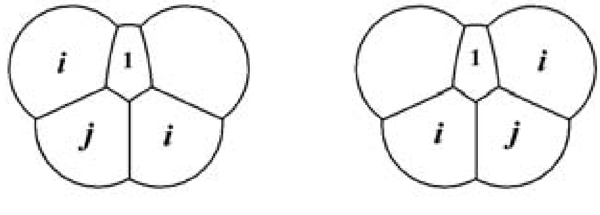


Figure 5.3.14: Labels for possibility (g) in Figure 5.3.9.

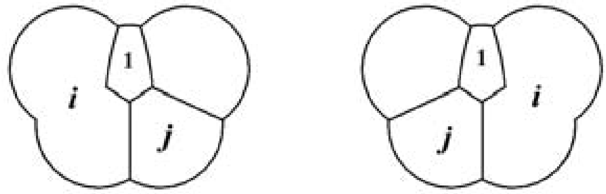


Figure 5.3.15: Shorter enclosures of the same areas as Figure 5.3.14.

For possibility (h), by proposition 2.1.9, we have only two ways to label it as in Figure 5.3.16. By the decoration of a 3-sided component, the two consecutive 5-sided component of R_2, R_3 or R_4 in Figure 5.3.16 have the same area. For each label in Figure 5.3.16, we create a shorter enclosure of the same areas as in Figure 5.3.17, a contradiction.

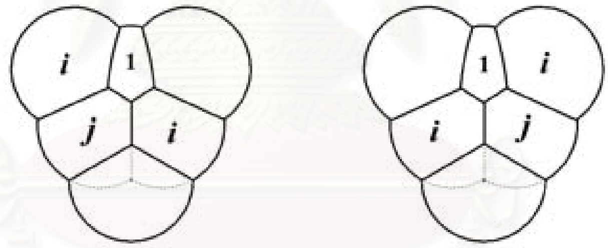


Figure 5.3.16: Labels for possibility (h) in Figure 5.3.9.

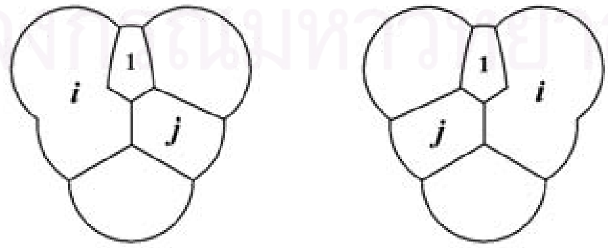


Figure 5.3.17: Shorter enclosures of the same areas as Figure 5.3.16.

Note that the 5-sided component of R_1 is symmetric about the vertical line.

For possibility (i), by proposition 2.1.9, we have only two ways to label it as in Figure 5.3.18. For each label in Figure 5.3.18, we create a new enclosure of the same areas and the same length as in Figure 5.3.19. Since the new enclosures is not minimizing, it follows that each label in Figure 5.3.18 is not minimizing, a contradiction.

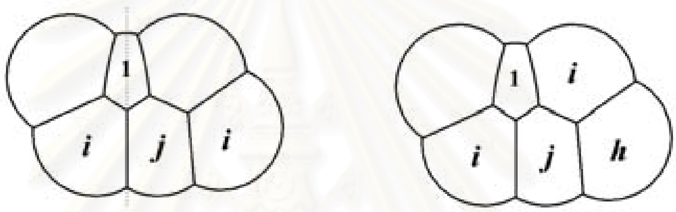


Figure 5.3.18: Labels for possibility (i) in Figure 5.3.9.

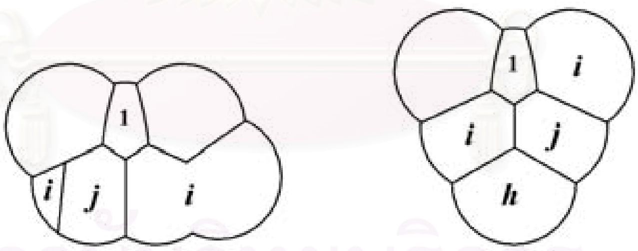


Figure 5.3.19: New enclosures of the same areas and the same length as Figure 5.3.18.

This proof is completed. □

CHAPTER VI

Quadruple bubbles with $p_1 = p_2 = p_3 > p_4$

In this chapter, we focus on minimizing 4-bubbles with $p_1 = p_2 = p_3 > p_4$. Keawkhao [13, 14] showed that in a minimizing 4-bubble with $p_1 = p_2 = p_3 > p_4 \geq 0$ and without empty chambers, if R_4 is connected then it is external. Our goal is to show that a minimizing 4-bubble with $p_1 = p_2 = p_3 > p_4 \geq 0$ and without empty chambers must have at least one external component of R_4. This extends the work by Keawkhao.

6.1 Properties of 4-bubbles with $p_1 = p_2 = p_3 > p_4$

In this section, we discuss properties of 4-bubbles with $p_1 = p_2 = p_3 > p_4$ and without empty chambers.

Theorem 6.1.1. *[13, 14] In a minimizing 4-bubble with $p_1 = p_2 = p_3 > p_4 \geq 0$ and without empty chambers, if R_4 is connected then it is external.*

Theorem 6.1.2. *[13, 14] In a minimizing 4-bubble with $p_1 = p_2 = p_3 > p_4 \geq 0$ and without empty chambers, every component of R_4 has at most nine sides.*

In [13, 14], Keawkhao names the components (a), (b) and (c) in Figure 6.1.1 that a π**-cell**, a $\frac{2\pi}{3}$**-cell** and a $\frac{\pi}{3}$**-cell**, respectively. Note that the turning angles of the circular edges of a π-cell, a $\frac{2\pi}{3}$-cell and a $\frac{\pi}{3}$-cell are π, $\frac{2\pi}{3}$ and $\frac{\pi}{3}$, respectively.

(a) (b) (c)

Figure 6.1.1: A π-cell, a $\frac{2\pi}{3}$-cell and a $\frac{\pi}{3}$-cell.

Proposition 6.1.3. *[13, 14] In a minimizing* 4*-bubble with* $p_1 = p_2 = p_3 > p_4 \geq 0$ *and without empty chambers, if there is internal component* C *of* R_4*, then* C *is not adjacent to an internal* π*-cell.*

Lemma 6.1.4. *Suppose that* B *is a minimizing* 4*-bubble with* $p_1 = p_2 = p_3 > p_4 \geq 0$ *and without empty chambers. If* B *has an internal component* C *of* R_4*, then* C *is not adjacent to an internal* $\frac{2\pi}{3}$*-cell.*

Proof. Assume that B has an internal component C of R_4.

Suppose that C is adjacent to an internal $\frac{2\pi}{3}$-cell D as in Figure 6.1.2.

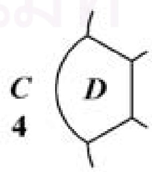


Figure 6.1.2: An internal component C of R_4 adjacent to an internal $\frac{2\pi}{3}$-cell.

Let n be the number of sides of C. By Theorem 6.1.2, we have $n \leq 9$. Let $t_1, t_2, \ldots, t_n$ be the turning angles of all edges of C where t_n is of the edge of D.

Since C is an internal component of R_4 and $p_1 = p_2 = p_3 > p_4$, it follows that $t_n = -\frac{2\pi}{3}$ and $t_i < 0$ for all i. By Lemma 2.1.8, the sum of all these turning angles is $\frac{6-n}{3}\pi = -\frac{2\pi}{3} + \sum_{i=1}^{n-1} t_i < -\frac{2\pi}{3}$. Thus $n = 9$. We scale the bubble B so that these edges have curvature 1. Then for each i, $|t_i|$ is the length of the edge with turning angle t_i. Note that the length of the edge between C and D is less than the sum of the length of these edges that is not adjacent to D. Thus $\frac{2\pi}{3} = |t_n| < \sum_{i=1}^{n-1} |t_i| = \left|\frac{6-n}{3}\pi\right| - \left|-\frac{2\pi}{3}\right| = \frac{\pi}{3}$, a contradiction. □

Lemma 6.1.5. *Suppose that B is a minimizing 4-bubble with $p_1 = p_2 = p_3 > p_4 \geq 0$ and without empty chambers. If B has an internal component C of R_4, then C must not be adjacent to an internal $\frac{\pi}{3}$-cell.*

Proof. Assume that B has an internal component C of R_4.

Suppose that C is adjacent to an internal $\frac{\pi}{3}$-cell D as in Figure 6.1.3.

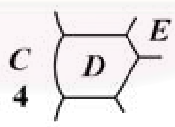


Figure 6.1.3: An internal component C of R_4 such that C is adjacent to an internal $\frac{\pi}{3}$-cell D.

Let n be the number of sides of C. By Theorem 6.1.2, we have $n \leq 9$. Let $t_1, t_2, \ldots, t_n$ be the turning angle of all edges of C where t_n is of the edge of D. Since C is an internal component of R_4 and $p_1 = p_2 = p_3 > p_4$, it follows that $t_n = -\frac{\pi}{3}$ and $t_i < 0$ for all i. By Lemma 2.1.8, the sum of these turning angles is $\frac{6-n}{3}\pi = -\frac{\pi}{3} + \sum_{i=1}^{n-1} t_i < -\frac{\pi}{3}$. Thus $n \in \{8, 9\}$.

Case 1: $n = 8$. We scale the bubble B so that the edges of C have curvature 1. Then for each i, $|t_i|$ is the length of the edge with turning angle t_i. Note that the length of the edge between C and D is less than the sum of the length of these edges that is not adjacent to D. Thus $\frac{\pi}{3} = |t_n| < \sum_{i=1}^{n-1} |t_i| = \left|\frac{6-n}{3}\pi\right| - \left|-\frac{\pi}{3}\right| = \frac{\pi}{3}$, a contradiction.

Case 2: $n = 9$. Since the component E in Figure 6.1.3 is a component of R_1 or R_2 or R_3 and $p_1 = p_2 = p_3 > p_4$, it follows that E is convex. Thus we have four disjoint nonhexagonal convex components away from E as in Figure 6.1.4, a contradiction to Proposition 2.4.5.

Figure 6.1.4: Four disjoint nonhexagonal convex components away from the convex component E where the component C is 9-sided.

This proof is completed. □

Theorem 6.1.6. *Suppose that B is a minimizing 4-bubble with $p_1 = p_2 = p_3 > p_4 \geq 0$ and without empty chambers. If B has an internal component C of R_4, then every component adjacent to C must be of a type in Figure* 6.1.5.

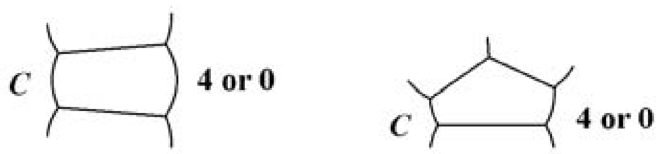


Figure 6.1.5: Two possible types for a component adjacent to the internal component C of R_4.

Proof. Assume that B has an internal component C of R_4. Since $p_1 = p_2 = p_3 > p_4 \geq 0$ and every component adjacent to C must be nonhexagonal convex, it follows that each of these components is a π-cell or a $\frac{2\pi}{3}$-cell or a $\frac{\pi}{3}$-cell or a type in Figure 6.1.5.

By Proposition 6.1.3, C is not adjacent to an internal π-cell.

By Lemma 6.1.4, C is not adjacent to an internal $\frac{2\pi}{3}$-cell.

By Lemma 6.1.5, C is not adjacent to an internal $\frac{\pi}{3}$-cell.

Hence every component adjacent to C must be of a type in Figure 6.1.5. □

6.2 Main results on 4-bubbles with $p_1 = p_2 = p_3 > p_4$

In this section, we prove the main result that a minimizing 4-bubble with $p_1 = p_2 = p_3 > p_4 \geq 0$ and without empty chambers must have at least one external component of R_4. This extends the result by Keawkhao [13, 14] that if R_4 is connected then it is external.

Lemma 6.2.1. *Consider a minimizing m-bubble with $p_1 = p_2 = p_3 > p_4 \geq 0$ and without empty chambers. A 4-sided component in Figure* 6.2.1 *is symmetric as shown.*

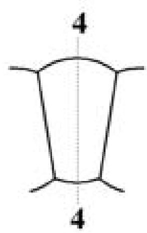


Figure 6.2.1: A type of 4-sided component that is symmetric.

Proof. This proof is similar to the proof of Lemma 5.18 in [26]. Let C be a 4-sided component in Figure 6.2.1. Since $p_1 = p_2 = p_3 > p_4 \geq 0$, it follows that the side edges have the same curvature.

Suppose that the side edges of C are cocircular. Since the two side edges of C are straight, the top edge or the bottom edge would have turning angle $\frac{4\pi}{3}$. Thus the sum of turning angles of all edges of C is greater than $\frac{4\pi}{3}$, contradicting Lemma 2.1.8. Therefore, by Lemma 2.5.1, C is symmetric vertically. □

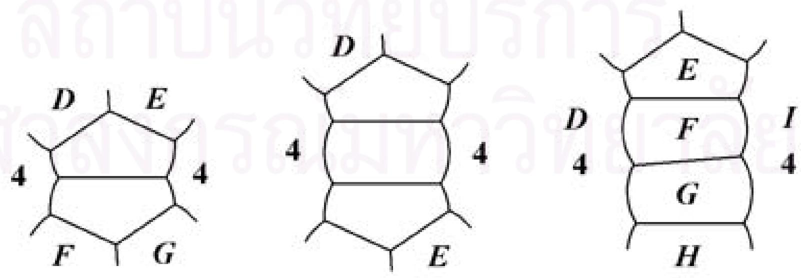
Theorem 6.2.2. *A minimizing* 4*-bubble with* $p_1 = p_2 = p_3 > p_4 \geq 0$ *and without empty chambers must have at least one external component of* R_4

Proof. Assume that B is a minimizing 4-bubble with $p_1 = p_2 = p_3 > p_4 \geq 0$ and without empty chambers.

By Theorem 6.1.1, if R_4 is connected then it is external.

Hence we may assume that R_4 is not connected and its components are internal.

By Theorem 2.1.1 and 6.1.6, B must have a local configuration in Figure 6.2.2.



(a) (b) (c)

Figure 6.2.2: Local configurations between two components of R_4.

Case 1. B has the local configuration (a) in Figure 6.2.2.

By Theorem 6.1.6, the components D, E, F and G are 5-sided as in Figure 6.2.3.

Figure 6.2.3: A local configuration that extend from the local configuration (a) in Figure 6.2.2.

Since the number of sides of each internal components of R_4 is greater than or equal to 7, we have five disjoint nonhexagonal convex components as in Figure 6.2.4, a contradiction to Theorem 2.4.4.

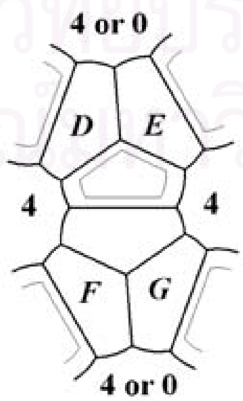


Figure 6.2.4: Five disjoint nonhexagonal convex components in Figure 6.2.3.

Case 2. B has the local configuration (b) in Figure 6.2.2.

By Theorem 6.1.6, the components D and E are 5-sided as in Figure 6.2.5.

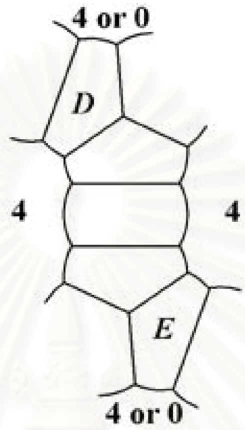


Figure 6.2.5: A local configuration that extend from the local configuration (b) in Figure 6.2.2.

Since the number of sides of each internal components of R_4 is greater than or equal to 7, we have five disjoint nonhexagonal convex components as in Figure 6.2.6, a contradiction to Theorem 2.4.4.

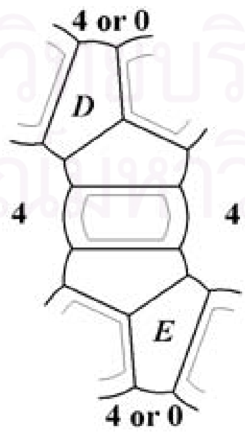


Figure 6.2.6: Five disjoint nonhexagonal convex components in Figure 6.2.5.

Case 3. B has the local configuration (c) in Figure 6.2.2.

By Theorem 6.1.6, the component H is 4-sided or 5-sided as in Figure 6.1.5. Let t_1, t_2, t_3, t_4 be the turning angle of edges between D and E, F, G, H, respectively. Let t'_1, t'_2, t'_3, t'_4 be the turning angle of edges between I and E, F, G, H, respectively. Let T and T' be the sums of turning angles of all edges of D and I, respectively. Note that $t_i < 0$ and $t'_i < 0$ for all i. By Lemma 2.1.8, we have $t_1 + t'_1 = -\frac{\pi}{3} \geq t_4 + t'_4$ and $t_2 + t'_2 = -\frac{2\pi}{3} = t_3 + t'_3$. Hence, using Lemma 2.1.8 again, $\min\{T, T'\} < \min\{t_1 + t_2 + t_3 + t_4, t'_1 + t'_2 + t'_3 + t'_4\} \leq \frac{1}{2}\left(-\frac{\pi}{3} - \frac{2\pi}{3} - \frac{2\pi}{3} - \frac{\pi}{3}\right) = -\pi$. Thus, by Lemma 2.1.8, D or I have at least ten sides, a contradiction to Theorem 6.1.2.

This proof is completed. $\square$

CHAPTER VII

Future study

In this chapter, we discuss some interesting work to be studied in the future.

7.1 Quadruple bubbles with $p_1 > p_2 = p_3 = p_4$

In this section, we discuss interesting conjectures on 4-bubbles with $p_1 > p_2 = p_3 = p_4$.

In chapter V, we have the main result that a minimizing 4-bubble with $p_1 > p_2 = p_3 = p_4$ and without empty chambers is standard if R_1 is connected. In case that R_1 is not connected, each external edge may not intersect the boundary of the convex hull of the bubble. Thus it would be very hard to rely on just the idea in chapter IV. However, it is possible to overcome the case that every component of R_1 is internal. Thus we believe that it is not too difficult to prove the following.

Conjecture 7.1.1. *Suppose that B is a minimizing 4-bubble with $p_1 > p_2 = p_3 = p_4$ and without empty chambers. If every component of R_1 in B is internal then B is standard.*

However, for some bubbles, we still have to find five admissible functions on each of them. A few interesting sets of five admissible functions are shown in Figure 7.1.1.

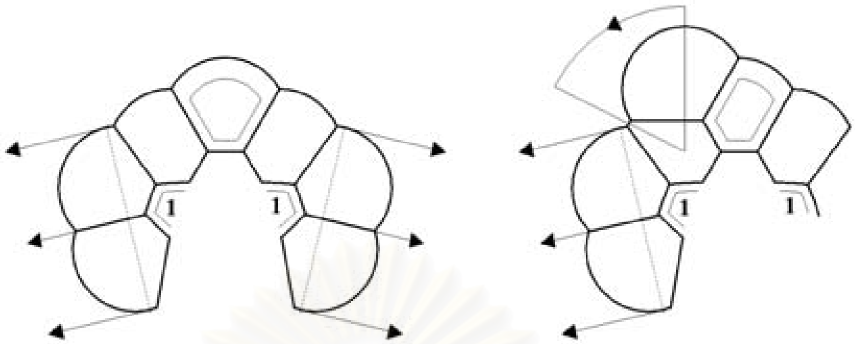


Figure 7.1.1: Five admissible functions on each local configuration in case $p_1 > p_2 = p_3 = p_4$.

7.2 Quadruple bubbles with $p_1 = p_2 = p_3 > p_4$

In this section, we discuss interesting conjectures on 4-bubbles with $p_1 = p_2 = p_3 > p_4$.

In chapter VI, we have the main result that a minimizing 4-bubble with $p_1 = p_2 = p_3 > p_4 \geq 0$ and without empty chambers must have at least one external component of R_4. In the proof, we consider the case that there are two internal components. Thus we believe that it is not too difficult to prove the following.

Conjecture 7.2.1. *A minimizing* 4*-bubble with* $p_1 = p_2 = p_3 > p_4 \geq 0$ *and without empty chambers must have at most one internal component of* R_4*.*

Moreover, we may extend the idea in chapter IV to construct some tools for eliminating the case that R_4 has at least two external components. Thus we believe that it is possible to prove that

Conjecture 7.2.2. *A minimizing* 4*-bubble with* $p_1 = p_2 = p_3 > p_4 \geq 0$ *and without empty chambers must have at most one external component of* R_4*.*

REFERENCES

[1] Almgren, F. J. Jr. Existence and regularity almost everywhere of solutions to elliptic variation problems with constraints, *Mem. AMS* **165** (1976).

[2] Álvarez, M. C., Corneli, J., Walsh, G., Beheshti, S. Double bubbles in the three-torus. *Exp. Math.* **12** (2003): 79−89.

[3] Bleicher, M. N. Isoperimetric division into a finite number of cells in the plane. *Stud. Sci. Math. Hungar.* **22** (1987): 123−137.

[4] Bleicher, M. N. Isoperimetric networks in the Euclidean plane. *Stud. Sci. Math. Hungar.* **31** (1996): 455−478.

[5] Cañete, A., and Ritoré, M. Least-perimeter partitions of the disk into three regions of given areas. *Indiana Univ. Math. J.* **53** (2004): 883−904.

[6] Corneli, J., Corwin, I., Xu, Y., Hurder, S., Sesum, V., Adams, E., Davis, D., Lee, M. Pettit, R., and Hoffman, N. Double bubbles in Gauss space and spheres. *Houston J. Math.* **34** (2008): 181−204.

[7] Corneli, J., Hoffman, N., Holt, P., Lee, G., Leger, N., Moseley, S., and Schoenfeld, E. Double bubbles in S^3 and H^3. *J Geom. Anal.* **17** (2007): 189−212.

[8] Corneli, J., Holt, P., Lee, G., Leger, N., Schoenfeld, E., and Steinhurst, B. The double bubble problem on the flat two-torus. *Trans. Amer. Math. Soc.* **356** (2004): 3769−3820.

[9] Cotton, A., and Freeman, D. The double bubble problem in spherical and hyperbolic space. *Intern. J. Math. Sci.* **32** (2002): 641−699.

[10] Cox, C., Harrison, L., Hutchings, M., Kim, S., Light, J., Mauer, A., and Tilton, M. The shortest enclosure of three connected areas in $\mathbb{R}^2$. *Real Anal. Exchange* **20** (1994/95): 313−335.

[11] Foisy, J., Alfaro, M., Brock, J., Hodges, N., and Zimba, J. The standard double soap bubble in $\mathbb{R}^2$ uniquely minimizes perimeter. *Pacific J. Math.* **159** (1993): 47−59.

[12] Hutchings, M., Morgan, F., Ritoré, M., and Ros, A. Proof of the double bubble conjecture. *Ann. Math.* **155** (2002): 459−489.

[13] Kaewkhao, An. *The shortest enclosures for four regions of given areas.* Master's thesis, Department of Mathematics, Faculty of Science, Chulalongkorn University, 2005.

[14] Kaewkhao, An., and Wichiramala, W. Shortest enclosures for four regions of given areas. *Thai J. Math.* special issue (2007): 127−146.

[15] Lopez, R., and Baker, T. B. The double bubble problem on cone. *New York J. Math.* **12** (2006): 157−167.

[16] Montesinos Amilibia, A. Existence and uniqueness of standard bubble clusters of given volumes in $\mathbb{R}^n$. *Asian J. Math.* **5** (2001): 25−31.

[17] Morgan, F. Clusters with Multiplicities in $\mathbb{R}^2$. *Pacific J. Math.* **221** (2005): 123−146.

[18] Morgan, F. Soap bubbles in $\mathbb{R}^2$ and on surfaces. *Pacific J. Math.* **165** (1994): 347−361.

[19] Morgan, F., and Wichiramala, W. The standard double is the unique stable double bubbles in $\mathbb{R}^2$. *Proc. AMS* **130** (2002): 2745−2751.

[20] Reicharht, B. W. Proof of the double bubble conjecture in $\mathbb{R}^n$. *J. Geom. Anal.* **18** (2008): 172−191.

[21] Reicharht, B. W., Heilmann, C., Lai, Y. Y., and Spielman, A. Proof of the double bubble conjecture in $\mathbb{R}^4$ and certain higher dimensional cases. *Pacific J. Math.* **208** (2003): 347−366.

[22] Taylor, J. The structure of singularities in soap-bubble-like and soap-film-like minimal surfaces. *Ann. Math.* **103** (1976).

[23] Sroysang, B., and Wichiramala, W. The planar soap bubble problem with equal pressure regions. *Thai J. Math.* special issue (2007): 57−68.

[24] Sroysang, B., and Wichiramala, W. The planar soap bubble problem with six equal pressure regions, to appear.

[25] Vaughn, R. *Planar soap bubbles.* Ph.D. thesis, Office of Graduate Studies, University of California, Davis, 1998.

[26] Wichiramala, W. Proof of the planar triple bubble conjecture. *Reine Angew. Math.* **567** (2004): 1−49.

[27] Wichiramala, W. *The planar triple bubble problem.* Ph.D. thesis, Department of Mathematics, University of Illinois at Urbana-Champaign, 2002.

VITA

Mr. Banyat Sroysang was born on September 22, 1981 in Surin, Thailand. He graduated with a Bachelor of Education Degree in Mathematics Education (second class honor) from Kasetsart University in 2003. He graduated with a Master of Science Degree in Mathematics from Chulalongkorn University in 2005.