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SOME SUPER EDGE-MAGIC GRAPHS

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ให้ G เป็นกราฟที่มี p จุดยอดและ q เส้น จะได้ว่า G เป็นกราฟที่มีการกำกับกลอย่างยวดยิ่ง ถ้ามีฟังก์ชันหนึ่งต่อหนึ่งและทั่วถึง f จากเซตของจุดยอดและเซตของเส้นไปยังเซต $\{1,2,\ldots,p+q\}$ ซึ่งผลรวม $f(u)+f(v)+f(uv)$ เป็นค่าคงที่ สำหรับทุกๆเส้น uv และ $f(V(G))=\{1,2,3,\ldots,p\}$ ให้ $\mu_s(G)$ แทนจำนวนจุดยอด n ที่น้อยที่สุด เมื่อเพิ่ม n จุดยอดเหล่านี้ให้กราฟ G แต่ไม่เพิ่มเส้นทำให้กราฟที่ได้มีการกำกับกลอย่างยวดยิ่งหรือในกรณีที่เป็นไปไม่ได้ $\mu_s(G)$ มีค่าเป็น $+\infty$

เราแสดงกราฟที่มีการกำกับกลอย่างยวดยิ่งบางชนิดและหาขอบเขตของ $\mu_s(G)$ สำหรับกราฟ G บางชนิด ยิ่งกว่านั้นเราเสนอการสร้างกราฟที่มีการกำกับกลอย่างยวดยิ่งจากกราฟเดิม

ภาควิชา ....คณิตศาสตร์..........	ลายมือชื่อนิสิต.......อัตถสิทธิ์........
สาขาวิชา ....คณิตศาสตร์.........	ลายมือชื่ออาจารย์ที่ปรึกษา......W. Hemakul......
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A (p,q)-graph G is super edge-magic if there exists a bijective function $f : V(G) \cup E(G) \to \{1, 2, ..., p+q\}$ such that $f(u)+f(v)+f(uv)$ is a constant for any $uv \in E(G)$ and $f(V(G)) = \{1, 2, ..., p\}$. The super edge-magic deficiency $\mu_s(G)$ of a graph G is the smallest nonnegative integer n with the property that the graph $G \cup nK_1$ is super edge-magic or $+\infty$ if there exists no such integer n.

We show some new super edge-magic graphs and investigate bounds for the super edge-magic deficiency of some graphs. Moreover, a new construction of super edge-magic graphs from the old ones is presented.

Department ...Mathematics... Student's Signature อัฐศิษฐ์......

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CHAPTER I

INTRODUCTION

1.1 Definitions

In this thesis we consider finite undirected graphs without loops and multiple edges. $V(G)$ and $E(G)$ stand for the vertex set and edge set of a graph G, respectively. We denote by (p, q)-graph G a graph with p vertices and q edges.

Definition 1.1.1. A $(p,\ q)$-graph G is *edge-magic* if there exists a bijective function $f : V(G) \cup E(G) \to \{1,\ 2,\ 3, \ldots,\ p+q\}$ such that $f(u)+f(v)+f(uv) = c(f)$ is a constant for any edge uv in G and f is called the *edge-magic labeling* of G and $c(f)$ is called the *magic constant* of f.

Definition 1.1.2. A $(p,\ q)$ graph G is *super edge-magic* if there exists an edge-magic labeling f such that $f(V(G)) = \{1,\ 2, \ldots,\ p\}$.

Definition 1.1.3. The *super edge-magic deficiency* $\mu_s(G)$ of a graph G is the smallest nonnegative integer n with the property that the graph $G \cup nK_1$ is super edge-magic or $+\infty$ if there exists no such integer n.

Definition 1.1.4. Let G be a super edge-magic graph. The *super edge-magic strength* of G, $sm(G)$ is defined as the minimum of all $c(f)$ where the minimum is taken over all super edge-magic labelings f of G. That is,

$$sm(G) = min\{c(f) : f \text{ is a super edge-magic labeling of } G\}.$$

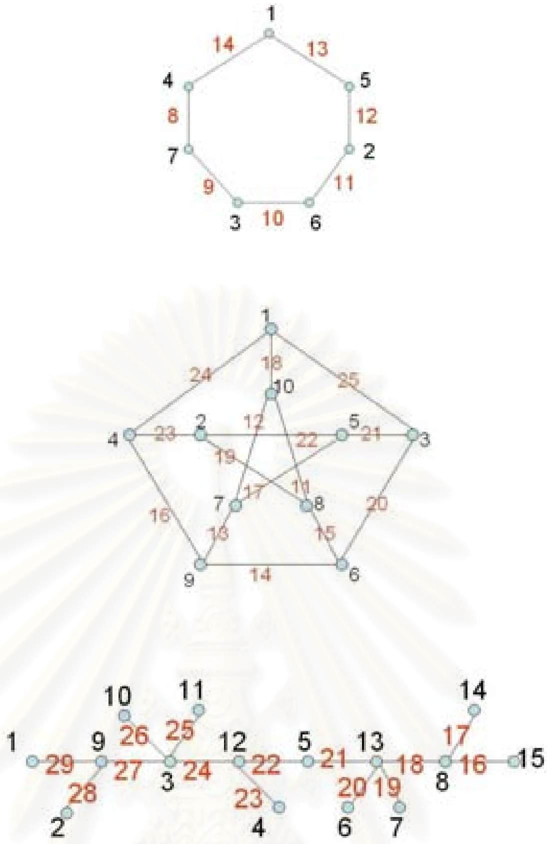


Figure 1.1: Example of super edge-magic graphs

1.2 History and Overview

The seminal paper in edge-magic labelings was published in 1970 by Kotzig and Rosa[8], who called these labelings: magic valuations; these were rediscovered by Ringel and Llado, who coined one of the now popular terms for them: edge-magic labelings. More recently, they have also been referred to as edge-magic total labelings by Wallis. In 1998, Enamoto, Llado, Nakamigawa and Ringel[2] defined a super edge-magic labeling f of a graph G. Gallian[7] surveyed some of latest developments of super edge-magic graphs as shown in the following table:

Table 1: **Summary of Super Edge-magic Labelings**

Graph	Notes
C_n	iff n is odd[Enamoto et al]
caterpillars	[Enamoto et al]
trees	?[Enamoto et al]
$K_{m,n}$	iff $m = 1$ or $n = 1$[Enamoto et al]
K_n	iff $n = 1, 2$ or 3[Enamoto et al]
nK_2	if n is odd[Kotzig and Rosa]
nG	if G is a bipartite or tripartite super edge-magic graph
	and n is odd[Figuaroa-Centeno et al]
$K_{1,m} \cup K_{1,n}$	iff m is multiple of $n + 1$[Figuaroa et al],[Lee and Kong]
$P_m \cup K_{1,n}$	if $m \geq 4$ is even[Figuaroa-Centeno et al]
$2P_n$	iff n is not 2 or 3[Figuaroa-Centeno et al]
$2P_{4n}$	for all n[Figuaroa et al]
$K_{1,m} \cup 2nK_{1,2}$	for all m and n[Figuaroa-Centeno et al]
$C_3 \cup C_n$	iff $n \geq 6$ is even[Figuaroa-Centeno et al]
$C_4 \cup C_n$	iff $n \geq 5$ is odd[Figuaroa-Centeno et al]
$C_5 \cup C_n$	iff $n \geq 5$ is even[Figuaroa-Centeno et al]
$C_m \cup C_n$	if $m \geq 6$ is even and n is odd and $n \geq \frac{m}{2} + 2$[Figuaroa-Centeno et al]
$C_4 \cup P_n$	iff $n \neq 3$[Figuaroa-Centeno et al]
$C_5 \cup P_n$	iff $n \neq 4$[Figuaroa-Centeno et al]
$C_m \cup P_n$	if $m \geq 6$ is even and $n \geq \frac{m}{2} + 2$[Figuaroa-Centeno et al]
$P_m \cup P_n$	iff $(m, n) \neq (2, 2)$ or $(3, 3)$[Figuaroa-Centeno et al]

Table 1: **Summary of Super Edge-magic Labelings**

Graph	Notes
$K_{1,1} \cup K_{1,k} \cup K_{1,n}$	$k = 1, 2$ or n[Lee and Kong]
$K_{1,2} \cup K_{1,k} \cup K_{1,n}$	$k = 2, 3$[Lee and Kong]
$K_{1,1} \cup K_{1,1} \cup K_{1,k} \cup K_{1,n}$	$k = 2, 3$[Lee and Kong]
$K_{1,k} \cup K_{1,2} \cup K_{1,2} \cup K_{1,n}$	$k = 1, 2$[Lee and Kong]
friendship graph of n triangles	iff n=3,4,5 or 7[Slamin et al]
generalized Petersen graph $P(n, 2)$	if $n \geq 3$ and n is odd[Fukuchi]
nP_3	$n \geq 4$ and n is even[Baskoro and Ngurah]
P_n^2	[Figuaroa et al]
$P_3 \cup kP_2$	for all k[Figuaroa et al]
$k(P_2 \cup P_n)$	if k is odd and $n = 3, 4$ [Figuaroa-Centeno et al]
fan F_n	iff $n \leq 6$[Figuaroa-Centeno et al]
kP_2	iff k is odd[Figuaroa-Centeno et al]
tree with α-labeling	[Figuaroa-Centeno et al]
$P_{2m+1} \times P_2$	for all m[Figuaroa-Centeno et al]
$C_{2m+1} \times P_n$	for all m, n[Figuaroa-Centeno et al]
$G \odot \overline{K}_n$	if G is super edge-magic 2-regular graph [Figuaroa-Centeno et al]
$C_m \odot \overline{K}_n$	$m \geq 3$ and $n \geq 1$
join of K_1 with any subgraph of star	[Chen]
if G is k-regular super edge-magic graph	then $k \leq 3$[Chen]
G is connected 3-regular graph on p vertices	iff $p \equiv 2 \pmod 4$[Chen]

Kotzig and Rosa[8] defined the edge-magic deficiency, $\mu(G)$, of a graph G as the smallest nonnegative integer n with the property that the graph $G \cup nK_1$ is edge-magic. In 1999, Figueroa-Centeno, Ichishima and Muntaner-Batle[5], [6] used the concept of edge-magic deficiency to define super edge-magic deficiency. They proved the following super edge-magic deficiency of graphs:

Table 2: **Summary of Super Edge-magic Deficiency**

Graph	Deficiency	Notes
nK_2	0	n is odd
	1	n is even
C_n	0	if $n \equiv 1, 3 \pmod 4$
	1	if $n \equiv 0 \pmod 4$
	$+\infty$	if $n \equiv 2 \pmod 4$
K_n	0	$n = 1, 2, 3$
	1	$n = 4$
	$+\infty$	$n \geq 5$
$K_{m,n}$	$\leq (m-1)(n-1)$	for any positive integer m, n
$K_{2,n}$	$n-1$	for any positive integer n
Forests	finite	
$K_{1,m} \cup K_{1,n}$	0	either m is multiple of $n+1$ or n is multiple of $m+1$
	1	otherwise
$P_m \cup P_n$	1	if $(m, n) = (2, 2)$ or $(3, 3)$
	0	otherwise

Table 2: **Summary of Super Edge-magic Deficiency**

Graph	Deficiency	Notes
$P_m \cup K_{1,n}$	1	$m = 2$ and n is odd or $m = 3$ and $n \equiv 1, 2 \pmod 3$
	0	otherwise
$2C_n$	1	if n is even
	$+\infty$	if n is odd
$3C_n$	0	if n is odd
	1	if $n \equiv 0 \pmod 4$
	$+\infty$	if $n \equiv 2 \pmod 4$
$4C_n$	1	for all integers $n \equiv 0 \pmod 4$

In 2000, Avadayappan, Jeyanthi and Vasuki[1] defined the super edge-magic strength and proved the super edge-magic strength of path P_n, star $K_{1,n}$, the n-bistar $B_{n,n}$ obtained from two disjoint copies of $K_{1,n}$ by joining the center vertices by an edge, odd cycle C_{2n+1}, P_n^2 and the disjoint union of odd copies of P_2.

There are five chapters in this thesis. In chapter I, we introduce definitions that will be used in and the history and overview of super edge-magic graphs and the super edge magic deficiency.

In Chapter II, super edge-magic graphs and bounds for the super edge-magic strength of some graphs are shown.

In Chapter III, we show a construction of new super edge-magic graphs from the old ones.

In Chapter IV, we investigate bounds for the super edge-magic deficiency of some graphs.

In Chapter V, we introduce the super edge-magic redundency and find bounds for the super edge-magic redundency of some graphs.

CHAPTER II

SUPER EDGE-MAGIC GRAPHS

Our purpose in this chapter is to show some new super edge-magic graphs and investigate bounds for their super edge-magic strengths. We separate this chapter into four sections. The first section contains theorems and corollary which are used in this thesis. The second section shows a super edge-magic labeling of the P-tree. The third section shows a super edge-magic labeling of the product of the caterpillar and path P_2. The last section shows a super edge-magic labeling of the product of SF-graph and path P_n.

2.1 Preliminary Tools

Theorem 2.1.1. [3] *A $(p,\ q)$-graph G is super edge-magic if and only if there exists a bijective function $f : V(G) \to \{1,\ 2,\ 3, \ldots,\ p\}$ such that the set*

$$S = \{f(u) + f(v) : uv \in E(G)\}$$

consists of q consecutive integers. In such a case, f extends to a super edge-magic labeling of G with magic constant $k = p + q + s$, where $s = min(S)$ and

$$S = \{k - (p + q),\ k - (p + q - 1),\ \ldots,\ k - (p + 1)\}$$

Corollary 2.1.2. [3] *If a $(p,\ q)$-graph G is a super edge-magic with a super edge-magic labeling f, then*

$$\sum_{v \in V(G)} f(v) deg\ v = qs + \binom{q}{2}$$

where s is defined as in theorem 2.1.1.

Theorem 2.1.3. [2] *If a $(p,\ q)$-graph is super edge-magic, then $q \le 2p-3$.*

2.2 Super edge-magic labeling of the P-tree

First, we introduce the definition of the P-tree.

Definition 2.2.1. Let r, s and t be positive integers. The *P-tree* $P(r,s,t)$ is a rooted tree with root z and $deg\ z = r$ and $deg\ c = s+1$ for every child c of z and one grandchild of z has degree $t+1$.

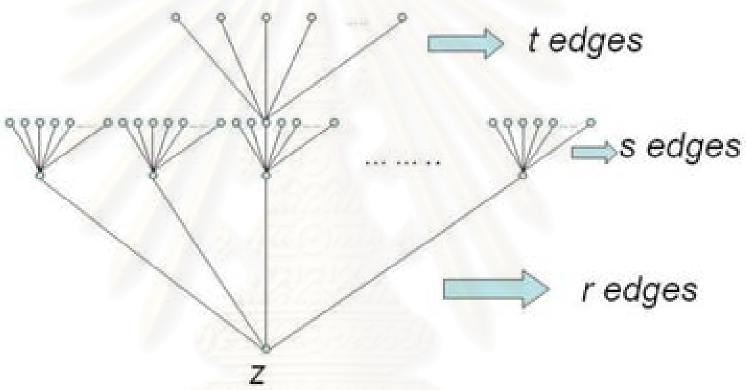


Figure 2.1: P-tree $P(r,s,t)$.

Example 2.2.2. P-tree $P(3,3,1)$ and P-tree $P(5,4,6)$ are shown below.

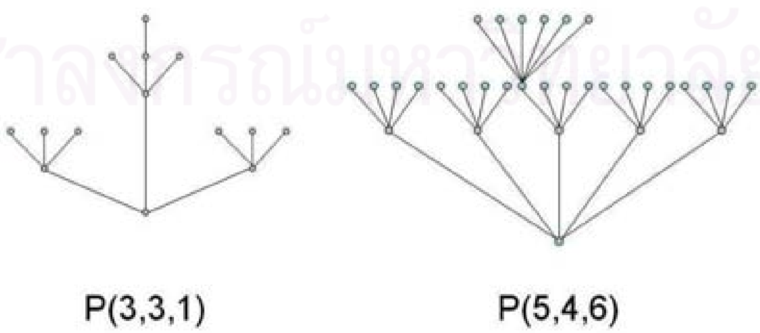


Figure 2.2: Example of P-trees.

Definition 2.2.3. Let G be a super edge-magic graph. The *super edge-magic strength* of G, $sm(G)$ is defined as the minimum of all $c(f)$ where the minimum is taken over all super edge-magic labelings f of G. That is,

$$sm(G) = min\{c(f) : f \text{ is a super edge-magic labeling of } G\}.$$

Next, we show the specific P-tree is super edge-magic.

Theorem 2.2.4. *The P-tree $P(2m+1, n, m)$ is super edge-magic with $sm(P(2m+1, n, m)) \leq 4mn + 2n + 9m + 6$ for any positive integers m, n.*

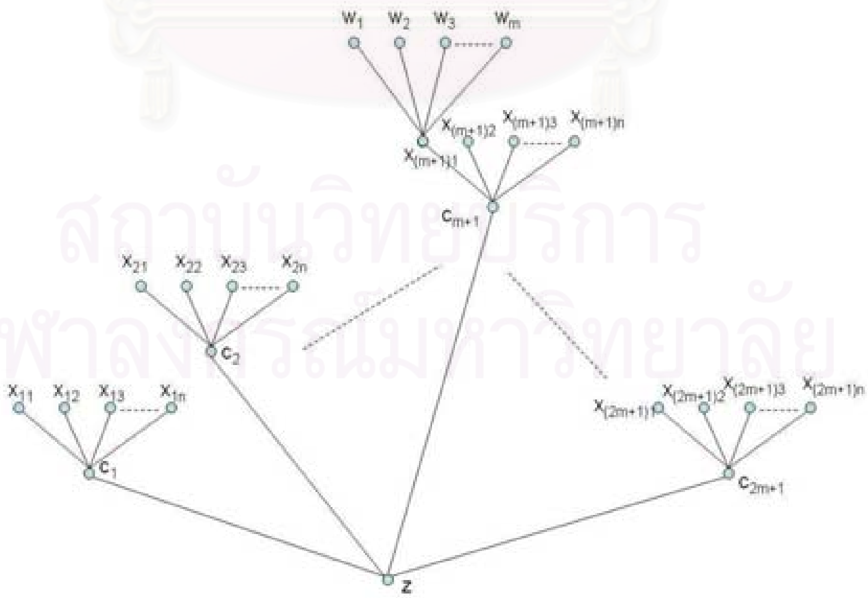
Proof. Let $G \cong P(2m+1, n, m)$ with

$V(G) = \{z\} \cup \{c_i : 1 \leq i \leq 2m+1\} \cup \{w_k : 1 \leq k \leq m\}$

$\cup \{x_{ij} : 1 \leq i \leq 2m+1, 1 \leq j \leq n\}$ and

$E(G) = \{zc_i : 1 \leq i \leq 2m+1\} \cup \{c_i x_{ij} : 1 \leq i \leq 2m+1, 1 \leq j \leq n\}$

$\cup \{x_{(m+1)1} w_k : 1 \leq k \leq m\}$.



Note that, $|V(G)| = 2mn + n + 3m + 2$.

Define a vertex labeling $f : V(G) \to \{1,\ 2,\ 3, \ldots, 2mn + n + 3m + 2\}$ by:

$$f(u) = \begin{cases} i + j(2m+1), & \text{if } u = x_{ij}; \\ 2m + 2 - \frac{i+1}{2}, & \text{if } u = c_i, i \text{ is odd}; \\ 2m + 2 - \frac{2m+i+2}{2}, & \text{if } u = c_i, i \text{ is even}; \\ 2mn + n + 3m + 2, & \text{if } u = z; \\ 2mn + n + 3m + 2 - k, & \text{if } u = w_k. \end{cases}$$

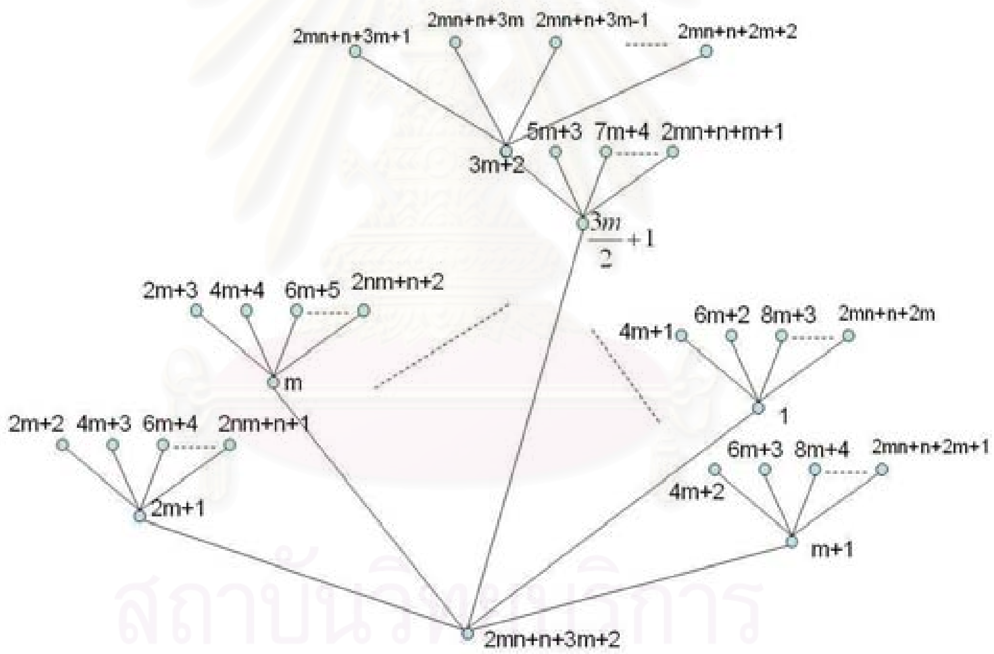


Figure 2.3: A vertex labeling of P-tree $P(2m+1, n, m)$.

In order to show that f extends to a super edge-magic labeling of P-tree $P(2m+1, n, m)$, it suffices to verify by Theorem 2.1.1:

a) $f(V(G)) = \{1,\ 2,\ 3, \ldots, 2mn + n + 3m + 2\}$

b) $S = \{f(x) + f(y) : xy \in E(G)\}$ consists of $2mn + 3m + n + 1$ consecutive integers.

To show that $f(V(G)) = \{1,\ 2,\ 3, \ldots, 2mn+n+3m+2\}$, we consider the labels of vertices as follows:

Vertices $c_2, c_4, c_6..., c_{2m}$ are labeled by numbers $m, m-1, m-2, ..., 1$, respectively and $c_1, c_3, c_5..., c_{2m+1}$ are labeled by numbers $2m+1, 2m, 2m-1, ..., m+1$, respectively and $x_{11}, x_{21}, ..., x_{(2m+1)1}, x_{12}, x_{22}, ..., x_{(2m+1)2}, ..., x_{1n}, x_{2n}, ..., x_{(2m+1)n}$ are labeled by numbers $2m+2, 2m+3, ..., 4m+2, 4m+3, 4m+4, ..., 6m+3, ..., 2mn+n+1, 2mn+n+2, ..., 2mn+n+2m+1$, respectively and $z, w_1, w_2, ..., w_m$ are labeled by number $2mn+n+3m+2, 2mn+n+3m+1, 2mn+n+3m, ..., 2mn+n+2m+2$.

Hence $f(V(G)) = \{1,\ 2,\ 3, \ldots, 2mn+n+3m+2\}$.

To show that S consists of $2mn+3m+n+1$ consecutive integers, we consider $f(x)+f(y)$ for all edges xy in G.

For edge $c_i x_{ij}$,

when i is odd, $f(c_i)+f(x_{ij}) = (2m+2-\frac{i+1}{2}) + (i+j(2m+1))$

$$= j(2m+1)+2m+\tfrac{i-1}{2}+2,$$

when i is even, $f(c_i)+f(x_{ij}) = (2m+2-\frac{2m+i+2}{2}) + (i+j(2m+1))$

$$= j(2m+1)+m+\tfrac{i}{2}+1.$$

For edge zc_i,

when i is odd, $f(z)+f(c_i) = (2mn+n+3m+2)+(2m+2-\frac{i+1}{2})$

$$= 2mn+n+5m+4-\tfrac{i+1}{2},$$

when i is even, $f(z)+f(c_i) = (2mn+n+3m+2)+2m+2-\frac{2m+i+2}{2}$

$$= 2mn+n+5m+4-\tfrac{2m+i+2}{2}.$$

For edge $x_{(m+1)1}w_k$,

$f(x_{(m+1)1})+f(w_k) = ((m+1)+(2m+1))+(2mn+n+3m+2-k)$

$$= 2mn+n+6m+4-k.$$

We note that

$$
\begin{aligned}
S =&\{f(x)+f(y) : xy \in E(G)\}\\
=&\bigcup_{j=1}^{n}\{f(c_i)+f(x_{ij}) : i \text{ is odd}\} \cup \bigcup_{j=1}^{n}\{f(c_i)+f(x_{ij}) : i \text{ is even}\} \cup \{f(z)+f(c_i) : i \text{ is odd}\}\\
&\cup \{f(z)+f(c_i) : i \text{ is even}\} \cup \{f(w_k)+f(x_{m+1,1})\}
\end{aligned}
$$

and

$$
\begin{aligned}
\bigcup_{j=1}^{n}\{f(c_i)+f(x_{ij}) : i \text{ is odd}\} =&\{4m+3, 4m+4, .., 5m+3\} \cup \{6m+4, 6m+5, ...,\\
&7m+4\} \cup \cdots \cup \{2mn+n+2m+2,\\
&2mn+n+2m+3, ..., 2mn+n+3m+2\},\\
\bigcup_{j=1}^{n}\{f(c_i)+f(x_{ij}) : i \text{ is even}\} =&\{3m+3, 3m+4, ..., 4m+2\} \cup \{5m+4, 5m+5, ...,\\
&6m+3\} \cup \cdots \cup \{2mn+n+m+2,\\
&2mn+n+m+3, ..., 2mn+n+2m+1\},\\
\{f(z)+f(c_i) : i \text{ is odd}\} =&\{2mn+n+4m+3, 2mn+n+4m+4, ...,\\
&2mn+n+5m+3\},\\
\{f(z)+f(c_i) : i \text{ is even}\} =&\{2mn+n+3m+3, 2mn+n+3m+4, ...,\\
&2mn+n+4m+2\},\\
\{f(w_k)+f(x_{(m+1)1})\} =&\{2mn+n+5m+4, 2mn+n+5m+5, ...,\\
&2mn+n+6m+3\}.
\end{aligned}
$$

Then $S = \{3m+3, 3m+4, \ldots, 2mn+n+6m+3\}$ is a set of $2mn+n+3m+1$ consecutive integers. Therefore, f extends to a super edge-magic labeling of G with magic constant $(2mn+n+3m+2)+(2mn+n+3m+1)+(3m+3) = 4mn+2n+9m+6$. Hence $sm(G) \leq 4mn+2n+9m+6$. $\square$

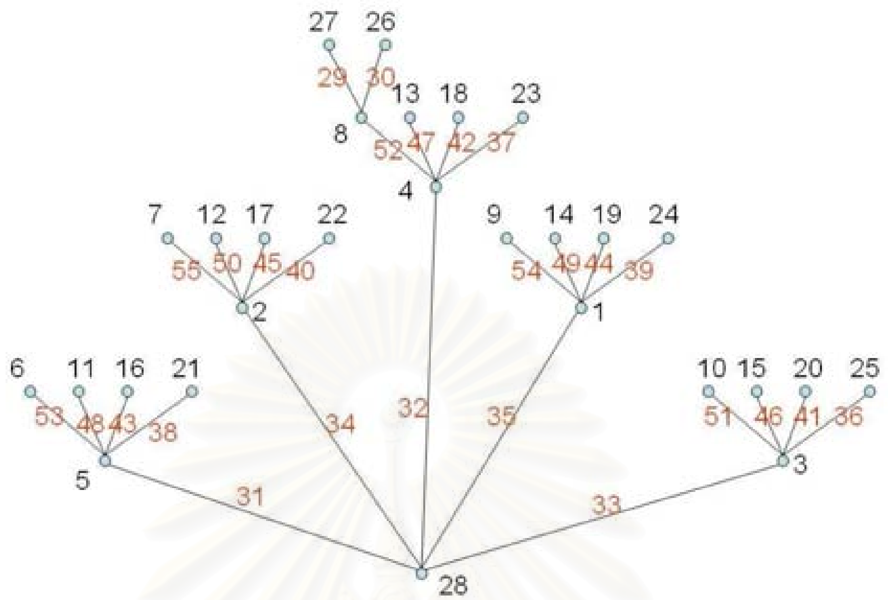


Figure 2.4: A super edge-magic labeling of the P-tree $P(5,4,2)$ with magic constant 64

2.3 Super edge-magic labeling of the product of caterpillar and path P_2

In this section, we show the super edge-magic labeling of the product of caterpillar and path P_2.

Definition 2.3.1. A *caterpillar graph* $CP_{n_1,n_2,\ldots,n_t}$ is a graph which the vertex-set is $\{c_i : 1 \leq i \leq t\} \cup \{x_{ij} : 1 \leq i \leq t, 1 \leq j \leq n_i\}$ and the edge-set is $\{c_{i+1}c_i : 1 \leq i \leq t-1\} \cup \{c_i x_{ij} : 1 \leq i \leq t, 1 \leq j \leq n_i\}$.

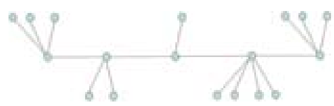


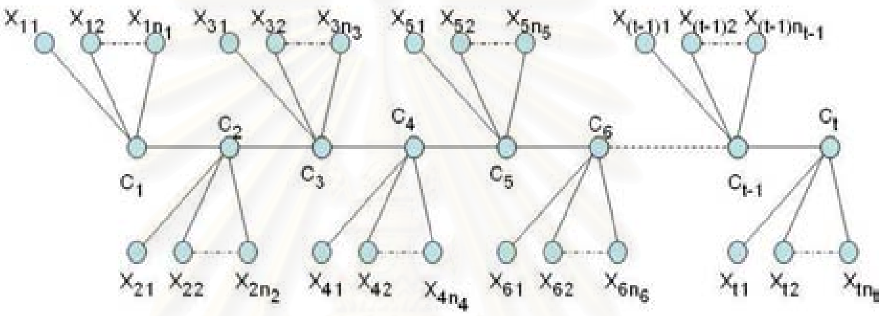
Figure 2.5: $CP_{3,2,1,4,3}$

Theorem 2.3.2. *Let $CP_{n_1,n_2,\ldots,n_t}$ be a caterpillar with t is odd.*
If $\displaystyle\sum_{\substack{k=1\\ k\text{ is odd}}}^{t} n_k = \sum_{\substack{k=2\\ k\text{ is even}}}^{t-1} n_k$, *then the graph $CP_{n_1,n_2,\ldots,n_t} \times P_2$ is super edge-magic.*

Proof. Let $G \cong CP_{n_1,n_2,\ldots,n_t}$ with

$V(G) = \{c_i : 1 \le i \le t\} \cup \{x_{ij} : 1 \le i \le t, 1 \le j \le n_i\}$ and

$E(G) = \{c_{i+1}c_i : 1 \le i \le t-1\} \cup \{c_i x_{ij} : 1 \le i \le t, 1 \le j \le n_i\}$.



Let p be the number of vertices of G. Then $p = t + \displaystyle\sum_{k=1}^{t} n_k$.

First, define a vertex labeling $f : V(G) \to \{1, 2, ..., t + \displaystyle\sum_{k=1}^{t} n_k\}$ by

$$
f(w) = \begin{cases}
1, & \text{if } w = c_1;\\
\frac{i+1}{2} + \displaystyle\sum_{\substack{k=2\\ k\text{ is even}}}^{i-1} n_k, & \text{if } w = c_i, i \text{ is odd and } i \ge 3;\\
\frac{i+t+1}{2} + \displaystyle\sum_{\substack{k=2\\ k\text{ is even}}}^{t-1} n_k + \sum_{\substack{k=1\\ k\text{ is odd}}}^{i-1} n_k, & \text{if } w = c_i, i \text{ is even};\\
j+1, & \text{if } w = x_{2j}, 1 \le j \le n_2;\\
\frac{i}{2} + j + \displaystyle\sum_{\substack{k=2\\ k\text{ is even}}}^{i-2} n_k, & \text{if } w = x_{ij}, i \text{ is even}, 1 \le j \le n_i, i \ge 4;\\
\frac{t+1}{2} + j + \displaystyle\sum_{\substack{k=2\\ k\text{ is even}}}^{t-1} n_k, & \text{if } w = x_{1j}, 1 \le j \le n_1;\\
\frac{i+t}{2} + j + \displaystyle\sum_{\substack{k=2\\ k\text{ is even}}}^{t-1} n_k + \sum_{\substack{k=1\\ k\text{ is odd}}}^{i-2} n_k, & \text{if } w = x_{ij}, i \text{ is odd}, 1 \le j \le n_i, i \ge 3;
\end{cases}
$$

Next, define a vertex labeling $g : V(G) \to \{1, 2, ..., t + \sum_{k=1}^{t} n_k\}$ by

$$g(w) = \begin{cases} \frac{t+1}{2} + \sum\limits_{\substack{k=1 \\ k \text{ is odd}}}^{t} n_k, & \text{if } w = c_1; \\ \frac{i+t}{2} + \sum\limits_{\substack{k=1 \\ k \text{ is odd}}}^{t} n_k + \sum\limits_{\substack{k=2 \\ k \text{ is even}}}^{i-1} n_k, & \text{if } w = c_i, i \text{ is odd and } i \geq 3; \\ \frac{i}{2} + \sum\limits_{\substack{k=1 \\ k \text{ is odd}}}^{i-1} n_k, & \text{if } w = c_i, i \text{ is even}; \\ j, & \text{if } w = x_{1j}, 1 \leq j \leq n_1; \\ \frac{t+1}{2} + j + \sum\limits_{\substack{k=1 \\ k \text{ is odd}}}^{t} n_k, & \text{if } w = x_{2j}, 1 \leq j \leq n_2; \\ \frac{i+t-1}{2} + j + \sum\limits_{\substack{k=1 \\ k \text{ is odd}}}^{t} n_k + \sum\limits_{\substack{k=2 \\ k \text{ is even}}}^{i-2} n_k, & \text{if } w = x_{ij}, i \text{ is even}, 1 \leq j \leq n_i, i \geq 4; \\ \frac{i-1}{2} + j + \sum\limits_{\substack{k=1 \\ k \text{ is odd}}}^{i-2} n_k, & \text{if } w = x_{ij}, i \text{ is odd}, 1 \leq j \leq n_i, i \geq 3. \end{cases}$$

For instance, Figures 2.6 and 2.7 show vertex labelings f and g of $CP_{2,2,1,2,1}$.

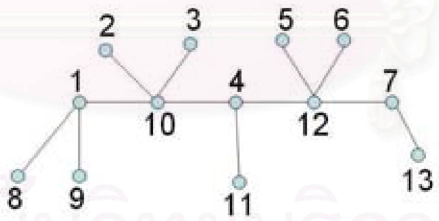


Figure 2.6: A vertex labeling f of $CP_{2,2,1,2,1}$.

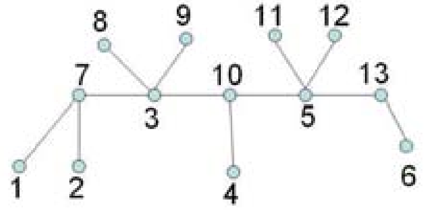


Figure 2.7: A vertex labeling g of $CP_{2,2,1,2,1}$.

In order to show that f and g extend to super edge-magic labelings of G, it suffices to verify by Theorem 2.1.1:

a) $f(V(G)) = g(V(G)) = \{1, 2, 3, ...p\}$

b) $S_f = \{f(x) + f(y) : xy \in E(G)\}$ and $S_g = \{g(x) + g(y) : xy \in E(G)\}$ consist of $p - 1$ consecutive integers.

Note that, $\frac{t+1}{2} + \sum\limits_{\substack{k=2 \\ k \text{ is even}}}^{t-1} n_k = \dfrac{t+1}{2} + \sum\limits_{\substack{k=1 \\ k \text{ is odd}}}^{t} n_k = \dfrac{p+1}{2}$.

To show that $f(V(G) = \{1, 2, 3, ..., p\}$, we consider the labels of vertices as follows:

Vertices $c_1, x_{21}, x_{22}, ..., x_{2n_1}, c_3, x_{41}, ..., x_{4n_3}, c_5, ..., c_t$ are labeled by numbers $1, 2, 3, ...,$ $n_1+1, n_1+2, n_1+3, ..., n_1+n_3+3, n_1+n_3+4, ..., \sum\limits_{\substack{k=2 \\ k \text{ is even}}}^{t-1} n_k + \dfrac{t+1}{2} + 1 = \dfrac{p+1}{2}$, respectively and $x_{11}, x_{12}, ..., x_{1n_2}, c_2, x_{31}, ..., x_{3n_3}, c_4, ..., x_{(t-1)n_{t-1}}$ are labeled by numbers $\sum\limits_{\substack{k=2 \\ k \text{ is even}}}^{t-1} n_k + \dfrac{t+1}{2} + 2 = \dfrac{p+1}{2}+1, \dfrac{p+1}{2}+2, \dfrac{p+1}{2}+3, ..., \dfrac{p+1}{2}+n_2+1, \dfrac{p+1}{2}+$ $n_2+2, \dfrac{p+1}{2}+n_2+3, ..., \dfrac{p+1}{2}+n_2+n_4+3, \dfrac{p+1}{2}+n_2+n_4+4, ..., p$, respectively.

Hence $f(V(G)) = \{1, 2, 3, ..., p\}$.

To show that S_f consists of $p-1$ consecutive integers, we consider $f(x)+f(y)$ for all edges xy in G.

For edge $c_i x_{ij}$,

when $i = 1$, $f(c_1) + f(x_{1j}) = 1 + (\frac{t+1}{2} + j + \sum\limits_{\substack{k=2 \\ k \text{ is even}}}^{t-1} n_k) = \dfrac{p+1}{2} + j + 1$,

when $i = 3, 4, ..., t$,

$$\begin{aligned} f(c_i) + f(x_{ij}) =& (\frac{i+1}{2} + \sum_{\substack{k=2 \\ k \text{ is even}}}^{i-1} n_k) + (\frac{i+t}{2} + j + \sum_{\substack{k=2 \\ k \text{ is even}}}^{t-1} n_k + \sum_{\substack{k=1 \\ k \text{ is odd}}}^{i-2} n_k) \\ =& \frac{p+1}{2} + i + \sum_{k=1}^{i-1} n_k + j, \end{aligned}$$

when $i = 2$,

$$f(c_2) + f(x_{2j}) =(\frac{t+3}{2} + \sum_{\substack{k=2 \\ k \text{ is even}}}^{t-1} n_k + n_1) + (j+1)$$
$$=\frac{p+1}{2} + n_1 + 2 + j,$$

when $i = 4, 6, ..., t-1$,

$$f(c_i) + f(x_{ij}) =(\frac{i+t+1}{2} + \sum_{\substack{k=2 \\ k \text{ is even}}}^{t-1} n_k + \sum_{\substack{k=1 \\ k \text{ is odd}}}^{i-1} n_k) + (\frac{i}{2} + j + \sum_{\substack{k=2 \\ k \text{ is even}}}^{i-2} n_k)$$
$$=\frac{p+1}{2} + i + \sum_{k=1}^{i-1} n_k + j.$$

Note that, $f(c_i) + f(x_{i(j+1)}) = f(c_i) + f(x_{ij}) + 1$.

For edge c_ic_{i+1},

when $i = 1$, $f(c_1) + f(c_2) = 1 + \frac{t+3}{2} + \sum_{\substack{k=2 \\ k \text{ is even}}}^{t-1} n_k + n_1 = \frac{p+1}{2} + n_1 + 2$,

when $i = 3, 5, ..., t-2$,

$$f(c_i) + f(c_{i+1}) =(\frac{i+1}{2} + \sum_{\substack{k=2 \\ k \text{ is even}}}^{i-1} n_k) + (\frac{i+t+2}{2} + \sum_{\substack{k=2 \\ k \text{ is even}}}^{t-1} n_k + \sum_{\substack{k=1 \\ k \text{ is odd}}}^{i} n_k)$$
$$=\frac{p+1}{2} + i + 1 + \sum_{k=1}^{i} n_k,$$

when $i = 2, 4, ..., t-1$,

$$f(c_i) + f(c_{i+1}) =(\frac{i+t+1}{2} + \sum_{\substack{k=2 \\ k \text{ is even}}}^{t-1} n_k + \sum_{\substack{l=1 \\ l \text{ is odd}}}^{i-1} n_l) + (\frac{i+2}{2} + \sum_{\substack{k=2 \\ k \text{ is even}}}^{i} n_k).$$
$$=\frac{p+1}{2} + i + 1 + \sum_{k=1}^{i} n_k.$$

Note that, $f(c_i) + f(c_{i+1}) = f(c_i) + f(x_{in_i}) + 1$

and $f(c_{(i+1)}) + f(x_{(i+1)1}) = f(c_i) + f(c_{i+1}) + 1$.

Hence $S_f = \{\frac{p+1}{2} + 2,\ \frac{p+1}{2} + 3, ...,\ \frac{p+1}{2} + p\}$ is a set of $p-1$ consecutive integers.

From Theorem 2.1.1, f extends to a super edge-magic labeling of G.

Similarly, we can show that $g(V(G)) = \{1, 2, 3, ..., p\}$ and $S_g = \{g(x) + g(y) : xy \in E(G)\} = \{\frac{p+1}{2} + 1,\ \frac{p+1}{2} + 2, ...,\ \frac{p+1}{2} + p - 1\}$ is a set of $p - 1$ consecutive integers. From Theorem 2.1.1, g extends to a super edge-magic labeling of G.

We will construct a super edge-magic labeling of $CP_{n_1,n_2,...,n_t} \times P_2$ as follows. Let $V(P_2) = \{1, 2\}$ and $E(P_2) = \{12\}$ and $H = G \times P_2$. Then
$V(H) = \{(c_i, k) : 1 \leq i \leq t, k = 1, 2\} \cup \{(x_{ij}, k) : 1 \leq i \leq t, 1 \leq j \leq n_i, k = 1, 2\}$.
Define a vertex labeling $h : V(H) \to \{1, 2, ..., 2p\}$ by

$$h(w) = \begin{cases} f(c_i), & \text{if } w = (c_i, 1); \\ f(x_{ij}), & \text{if } w = (x_{ij}, 1); \\ p + g(c_i), & \text{if } w = (c_i, 2); \\ p + g(x_{ij}), & \text{if } w = (x_{ij}, 2). \end{cases}$$

For instance, Figure 2.8 shows the vertex labeling h of $CP_{2,2,1,2,1} \times P_2$ constructed from f and g in Figure 2.6 and Figure 2.7.

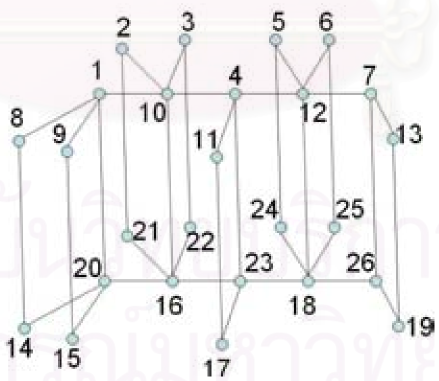


Figure 2.8: A vertex labeling of $CP_{2,2,1,2,1} \times P_2$.

In order to show that h extends to a super edge-magic labeling of H, it suffices to verify by Theorem 2.1.1:

a) $h(V(H)) = \{1, 2, 3, ..., 2p\}$

b) $S = \{h(x) + h(y) : xy \in E(H)\}$ consists of $3p - 2$ consecutive integers.

We note that

$$h(V(H)) = \{h(u,1) : (u,1) \in V(H))\} \cup \{h(u,2) : (u,2) \in V(H))\}$$

and

$$\begin{aligned}\{h(u,1) : (u,1) \in V(H))\} =& \{f(u) : u \in V(G)\} \\ =& \{1, 2, ..., p\} \\ \{h(u,2) : (u,2) \in V(H))\} =& \{p + g(u) : u \in V(G)\} \\ =& \{p+1, p+2, .., 2p\}.\end{aligned}$$

Then $h(V(H)) = \{1, 2, 3, ..., 2p\}$.

To show that S consists of $3p-2$ consecutive integers, we consider $h(u,1) + h(u,2)$ for all edges $(u,1)(u,2)$, where $u \in V(G)$.

For edge $(c_1,1)(c_1,2)$,

$$\begin{aligned}h(c_1,1) + h(c_1,2) &= f(c_1) + p + g(c_1) \\ &= 1 + p + (\frac{t+1}{2} + \sum_{\substack{k=1 \\ k \text{ is odd}}}^{t} n_k) \\ &= \frac{p+1}{2} + p + 1.\end{aligned}$$

For edge $(c_i,1)(c_i,2)$ when $i = 3, 5, ..., t$,

$$\begin{aligned}h(c_i,1) + h(c_i,2) &= f(c_i) + p + g(c_i) \\ &= (\frac{i+1}{2} + \sum_{\substack{k=2 \\ k \text{ is even}}}^{i-1} n_k) + p + (\frac{i+t}{2} + \sum_{\substack{k=1 \\ k \text{ is odd}}}^{t} n_k + \sum_{\substack{k=2 \\ k \text{ is even}}}^{i-1} n_k) \\ &= \frac{p+1}{2} + p + i + 2\sum_{\substack{k=2 \\ k \text{ is even}}}^{i-1} n_k.\end{aligned}$$

For edge $(x_{2j},1)(x_{2j},2)$,

$$
\begin{aligned}
h(x_{2j},1)+h(x_{2j},2) &= f(x_{2j})+p+g(x_{2j}) \\
&= (j+1)+p+\Big(\frac{t+1}{2}+j+\sum_{\substack{k=1\\ k \text{ is odd}}}^{t} n_k\Big) \\
&= \frac{p+1}{2}+p+2j+1.
\end{aligned}
$$

For edge $(x_{ij},1)(x_{ij},2)$ when $i=4,6,...,t-1$,

$$
\begin{aligned}
h(x_{ij},1)+h(x_{ij},2) &= f(x_{ij})+p+g(x_{ij}) \\
&= \Big(\frac{i}{2}+j+\sum_{\substack{k=2\\ k \text{ is even}}}^{i-2} n_k\Big)+p+\Big(\frac{i+t-1}{2}+j+\sum_{\substack{k=1\\ k \text{ is odd}}}^{t} n_k+\sum_{\substack{k=2\\ k \text{ is even}}}^{i-2} n_k\Big) \\
&= \frac{p+1}{2}+p+i-1+2\sum_{\substack{k=2\\ k \text{ is even}}}^{i-2} n_k+2j.
\end{aligned}
$$

Note that, for any i is odd,

$h(c_i,1)+h(c_i,2)=h(x_{(i+1)1},1)+h(x_{(i+1)1},2)-2$,

$h(x_{(i-1)n_{i-1}},1)+h(x_{(i-1)n_{i-1}},2)=h(c_i,1)+h(c_i,2)-2$,

$h(x_{(i+1)j},1)+h(x_{(i+1)j},2)=h(x_{(i+1)(j+1)},1)+h(x_{(i+1)(j+1)},2)-2.$

For edge $(c_i,1)(c_i,2)$ when $i=2,4,...,t-1$,

$$
\begin{aligned}
h(c_i,1)+h(c_i,2) &= f(c_i)+p+g(c_i) \\
&= \Big(\frac{i+t+1}{2}+\sum_{\substack{k=2\\ k \text{ is even}}}^{t-1} n_k+\sum_{\substack{k=1\\ k \text{ is odd}}}^{i-1} n_k\Big)+p+\Big(\frac{i}{2}+\sum_{\substack{k=1\\ k \text{ is odd}}}^{i-1} n_k\Big) \\
&= \frac{p+1}{2}+p+i+2\sum_{\substack{k=1\\ k \text{ is odd}}}^{i-1} n_k.
\end{aligned}
$$

For edge $(x_{1j},1)(x_{1j},2)$,

$$
\begin{aligned}
h(x_{1j},1)+h(x_{1j},2) &= f(x_{1j})+p+g(x_{1j}) \\
&= \Big(\frac{t+1}{2}+j+\sum_{\substack{k=2\\ k \text{ is even}}}^{t-1} n_k\Big)+p+j \\
&= \frac{p+1}{2}+p+2j.
\end{aligned}
$$

For edge $(x_{ij},1)(x_{ij},2)$ when $i=3,5,...,t$,

$$\begin{aligned} h(x_{ij},1)+h(x_{ij},2) &= f(x_{ij})+p+g(x_{ij}) \\ &= (\frac{i+t}{2}+j+\sum_{\substack{k=2 \\ k \text{ is even}}}^{t-1} n_k+\sum_{\substack{k=1 \\ k \text{ is odd}}}^{i-2} n_k)+p+(\frac{i-1}{2}+j+\sum_{\substack{k=1 \\ k \text{ is odd}}}^{i-2} n_k) \\ &= \frac{p+1}{2}+p+i+2\sum_{\substack{k=1 \\ k \text{ is odd}}}^{i-2} n_k+2j-1. \end{aligned}$$

Note that, for any i is even,

$h(c_i,1)+h(c_i,2)=h(x_{(i+1)1},1)+h(x_{(i+1)1},2)-2,$

$h(x_{(i-1)n_{i-1}},1)+h(x_{(i-1)n_{i-1}},2)=h(c_i,1)+h(c_i,2)-2,$

$h(x_{(i+1)j},1)+h(x_{(i+1)j},2)=h(x_{(i+1)(j+1)},1)+h(x_{(i+1)(j+1)},2)-2.$

Thus

$$\{h(c_i,1)+h(c_i,2):i=1,3,..,t\}\cup\bigcup_{\substack{i=2 \\ i \text{ is even}}}^{t-1}\{h(x_{ij},1)+h(x_{ij},2):j=1,2,...,n_i\}$$

$$=\{\frac{p+1}{2}+p+1,\frac{p+1}{2}+p+3,\frac{p+1}{2}+p+5,...,\frac{p+1}{2}+2p\}$$

and

$$\{h(c_i,1)+h(c_i,2):i=2,4,..,t-1\}\cup\bigcup_{\substack{i=1 \\ i \text{ is odd}}}^{t}\{h(x_{ij},1)+h(x_{ij},2):j=1,2,...,n_i\}$$

$$=\{\frac{p+1}{2}+p+2,\frac{p+1}{2}+p+4,\frac{p+1}{2}+p+6,...,\frac{p+1}{2}+2p-1\}.$$

Hence $\{h(u,1)+h(u,2):(u,1)(u,2)\in V(H)\}=\{\frac{p+1}{2}+p+1,\frac{p+1}{2}+p+2,\frac{p+1}{2}+p+3,...,\frac{p+1}{2}+2p\}$

We note that

$$\begin{aligned} S=&\{h(u)+h(v):uv\in E(H)\} \\ =&\{h(u,1)+h(v,1):(u,1)(v,1)\in V(H)\}\cup\{h(u,2)+h(v,2):(u,2)(v,2)\in V(H)\}\cup \\ &\{h(u,1)+h(u,2):(u,1)(u,2)\in V(H)\} \end{aligned}$$

and

$$
\begin{aligned}
\{h(u,1)+h(v,1):(u,1)(v,1)\in V(H)\} =& \{f(u)+f(v):uv\in E(CP_{n_1,n_2,\ldots,n_t})\}\\
=& \{\frac{p+1}{2}+2,\ \frac{p+1}{2}+3,\ldots,\ \frac{p+1}{2}+p\}\\
\{h(u,2)+h(v,2):(u,2)(v,2)\in V(H)\} =& \{2p+g(u)+g(v):uv\in E(CP_{n_1,n_2,\ldots,n_t})\}\\
=& \{\frac{p+1}{2}+2p+1,\ \frac{p+1}{2}+2p+2,\ldots,\\
& \frac{p+1}{2}+3p-1\}\\
\{h(u,1)+h(u,2):(u,1)(u,2)\in V(H)\} =& \{\frac{p+1}{2}+p+1,\frac{p+1}{2}+p+2,\ldots,\\
& \frac{p+1}{2}+2p\}.
\end{aligned}
$$

Then $S=\{\frac{p+1}{2}+2,\ \frac{p+1}{2}+3,\ldots,\ \frac{p+1}{2}+3p-1\}$ is a set of $3p-2$ consecutive integers. From Theorem 2.1.1, h extends to a super edge-magic labeling of H. $\square$

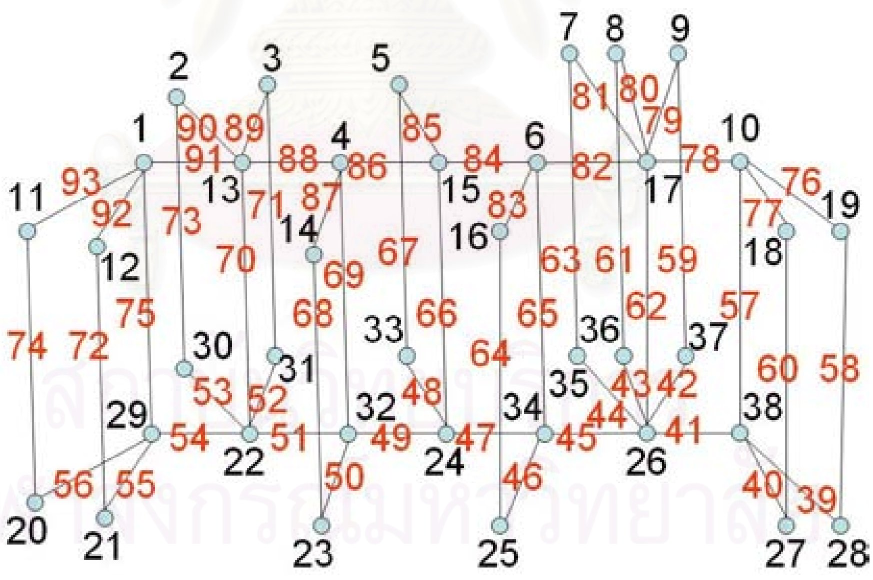


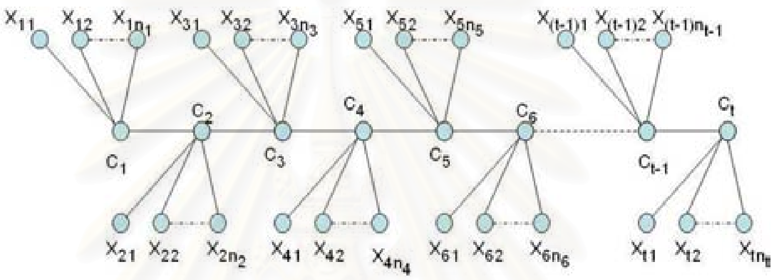
Figure 2.9: A super edge-magic labeling of $CP_{2,2,1,1,1,3,2}\times P_2$ with magic constant 105.

Theorem 2.3.3. *Let $CP_{n_1,n_2,\ldots,n_t}$ be a caterpillar with t is odd.*
If $\displaystyle\sum_{\substack{k=1\\ k \text{ is odd}}}^{t} n_k = \sum_{\substack{k=2\\ k \text{ is even}}}^{t-1} n_k + 2$, *then the graph $CP_{n_1,n_2,\ldots,n_t} \times P_2$ is super edge-magic.*

Proof. Let $G \cong CP_{n_1,n_2,\ldots,n_t}$ with

$V(G) = \{c_i : 1 \leq i \leq t\} \cup \{x_{ij} : 1 \leq i \leq t, 1 \leq j \leq n_i\}$ and

$E(G) = \{c_{i+1}c_i : 1 \leq i \leq t-1\} \cup \{c_i x_{ij} : 1 \leq i \leq t, 1 \leq j \leq n_i\}$.



Let p be the number of vertices of G. Then $p = t + \displaystyle\sum_{k=1}^{t} n_k$.

First, define a vertex labeling $f : V(G) \rightarrow \{1, 2, \ldots, t + \displaystyle\sum_{k=1}^{t} n_k\}$ by

$$f(w) = \begin{cases} \frac{t+1}{2} + \displaystyle\sum_{\substack{k=1\\ k \text{ is odd}}}^{t} n_k, & \text{if } w = c_1; \\ \frac{i+t}{2} + \displaystyle\sum_{\substack{k=1\\ k \text{ is odd}}}^{t} n_k + \sum_{\substack{k=2\\ k \text{ is even}}}^{i-2} n_k, & \text{if } w = c_i, i \text{ is odd and } i \geq 3; \\ \frac{i}{2} + \displaystyle\sum_{\substack{k=1\\ k \text{ is odd}}}^{i-1} n_k, & \text{if } w = c_i, i \text{ is even}; \\ j, & \text{if } w = x_{1j}, 1 \leq j \leq n_1; \\ \frac{i+t-1}{2} + j + \displaystyle\sum_{\substack{k=1\\ k \text{ is odd}}}^{t} n_k, & \text{if } w = x_{2j}, 1 \leq j \leq n_2; \\ \frac{i+t-1}{2} + j + \displaystyle\sum_{\substack{k=1\\ k \text{ is odd}}}^{t} n_k + \sum_{\substack{k=2\\ k \text{ is even}}}^{i-2} n_k, & \text{if } w = x_{ij}, i \text{ is even}, 1 \leq j \leq n_i, i \geq 4; \\ \frac{i-1}{2} + j + \displaystyle\sum_{\substack{k=1\\ k \text{ is odd}}}^{i-2} n_k, & \text{if } w = x_{ij}, i \text{ is odd}, 1 \leq j \leq n_i, i \geq 3. \end{cases}$$

Next, define a vertex labeling $g : V(G) \to \{1, 2, ..., t + \sum_{k=1}^{t} n_k\}$ by

$$g(w) = \begin{cases} 1, & \text{if } w = c_1; \\ \frac{i+1}{2} + \sum_{\substack{k=2 \\ k \text{ is even}}}^{i-1} n_k, & \text{if } w = c_i, i \text{ is odd and } i \geq 3; \\ \frac{i+t+1}{2} + \sum_{\substack{k=2 \\ k \text{ is even}}}^{t-1} n_k + \sum_{\substack{k=1 \\ k \text{ is odd}}}^{i-1} n_k, & \text{if } w = c_i, i \text{ is even}; \\ j+1, & \text{if } w = x_{2j}, 1 \leq j \leq n_2; \\ \frac{i}{2} + j + \sum_{\substack{k=2 \\ k \text{ is even}}}^{i-2} n_k, & \text{if } w = x_{ij}, i \text{ is even}, 1 \leq j \leq n_i, i \geq 4; \\ \frac{t+1}{2} + j + \sum_{\substack{k=2 \\ k \text{ is even}}}^{t-1} n_k, & \text{if } w = x_{1j}, 1 \leq j \leq n_1; \\ \frac{i+t}{2} + j + \sum_{\substack{k=2 \\ k \text{ is even}}}^{t-1} n_k + \sum_{\substack{k=1 \\ k \text{ is odd}}}^{i-2} n_k, & \text{if } w = x_{ij}, i \text{ is odd}, 1 \leq j \leq n_i, i \geq 3. \end{cases}$$

For instance, Figures 2.10 and 2.11 show vertex labelings f and g of $CP_{3,2,2,2,2,1}$.

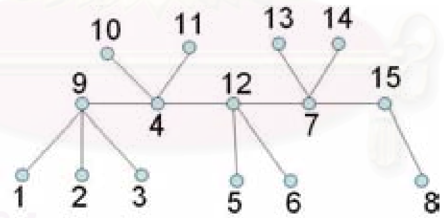


Figure 2.10: A vertex labeling f of $CP_{3,2,2,2,2,1}$.

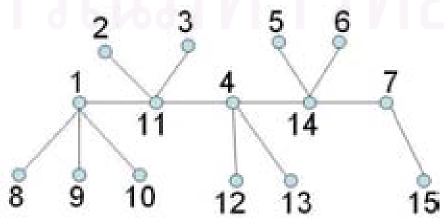


Figure 2.11: A vertex labeling g of $CP_{3,2,2,2,2,1}$.

Similarly to Theorem 2.3.2, we can show that $f(V(G)) = g(V(G)) = \{1, 2, 3, ..., p\}$

and

$$\{f(x)+f(y): xy \in E(G)\} = \{\frac{p+1}{2}+2,\ \frac{p+1}{2}+3, ...,\ \frac{p+1}{2}+p\}$$
$$\{g(x)+g(y): xy \in E(G)\} = \{\frac{p+1}{2}+1,\ \frac{p+1}{2}+2, ...,\ \frac{p+1}{2}+p-1\}$$

are sets of $p-1$ consecutive integers. From Theorem 2.1.1, f and g extend to super edge-magic labelings of G.

Let $V(P_2)=\{1,2\}$ and $E(P_2)=\{12\}$ and $H=G\times P_2$. Thus

$V(H)=\{(c_i,k): 1\le i\le t, k=1,2\}\cup\{(x_{ij},k): 1\le i\le t, 1\le j\le n_i, k=1,2\}$.

Define a vertex labeling $h: V(H)\to\{1,2,...,2p\}$ by

$$h(w)=\begin{cases} f(c_i), & \text{if } w=(c_i,1);\\ f(x_{ij}), & \text{if } w=(x_{ij},1);\\ p+g(c_i), & \text{if } w=(c_i,2);\\ p+g(x_{ij}), & \text{if } w=(x_{ij},2).\end{cases}$$

For instance, Figure 2.12 shows the vertex labeling h of $CP_{3,2,2,2,1}\times P_2$ constructed from f and g in Figure 2.10 and Figure 2.11.

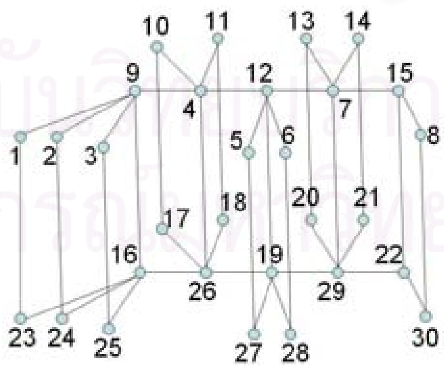


Figure 2.12: A vertex labeling of $CP_{3,2,2,2,1}\times P_2$.

Similar to Theorem 2.3.1, we can show that $h(V(H)) = \{1, 2, 3, ..., 2p\}$ and $\{h(x) + h(y) : xy \in E(H)\} = \{\frac{p+1}{2} + 2,\ \frac{p+1}{2} + 3, ...,\ \frac{p+1}{2} + 3p - 1\}$ is a set of $3p-2$ consecutive integers. From Theorem 2.1.1, h extends to a super edge-magic labeling of H. $\square$

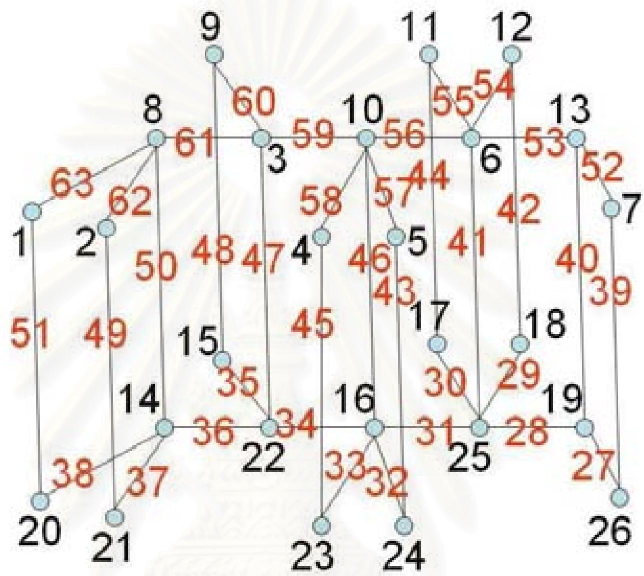


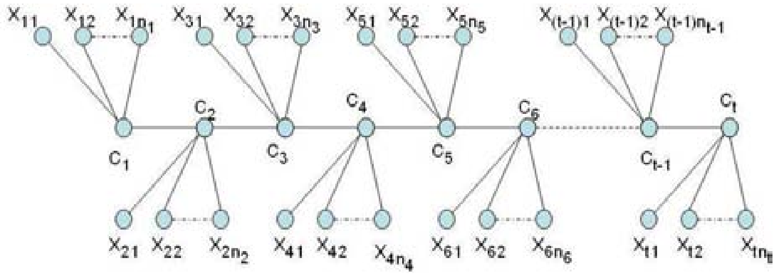
Figure 2.13: A super edge-magic labeling of $CP_{2,1,2,2,1} \times P_2$ with magic constant 72.

Theorem 2.3.4. *Let $CP_{n_1,n_2,...,n_t}$ be a caterpillar with t is even.*
If $\displaystyle\sum_{\substack{k=1 \\ k \text{ is odd}}}^{t-1} n_k = \sum_{\substack{k=2 \\ k \text{ is even}}}^{t} n_k + 1$, *then the graph $CP_{n_1,n_2,...,n_t} \times P_2$ is super edge-magic.*

Proof. Let $G \cong CP_{n_1,n_2,...,n_t}$ with

$V(G) = \{c_i : 1 \le i \le t\} \cup \{x_{ij} : 1 \le i \le t, 1 \le j \le n_i\}$ and

$E(G) = \{c_{i+1}c_i : 1 \le i \le t-1\} \cup \{c_i x_{ij} : 1 \le i \le t, 1 \le j \le n_i\}$.



Let p be the number of vertices of G. Then $p = t + \sum\limits_{k=1}^{t} n_k$.

First, define a vertex labeling $f : V(G) \rightarrow \{1, 2, ..., t + \sum\limits_{k=1}^{t} n_k\}$ by

$$f(w) = \begin{cases} \frac{t+1}{2} + \sum\limits_{\substack{k=1 \\ k \text{ is odd}}}^{t-1} n_k, & \text{if } w = c_1; \\ \frac{i+t}{2} + \sum\limits_{\substack{k=1 \\ k \text{ is odd}}}^{t-1} n_k + \sum\limits_{\substack{k=2 \\ k \text{ is even}}}^{i-2} n_k, & \text{if } w = c_i, i \text{ is odd and } i \geq 3; \\ \frac{i}{2} + \sum\limits_{\substack{k=1 \\ k \text{ is odd}}}^{i-1} n_k, & \text{if } w = c_i, i \text{ is even}; \\ j, & \text{if } w = x_{1j}, 1 \leq j \leq n_1; \\ \frac{i+t-1}{2} + j + \sum\limits_{\substack{k=1 \\ k \text{ is odd}}}^{t-1} n_k, & \text{if } w = x_{2j}, 1 \leq j \leq n_2; \\ \frac{i+t-1}{2} + j + \sum\limits_{\substack{k=1 \\ k \text{ is odd}}}^{t-1} n_k + \sum\limits_{\substack{k=2 \\ k \text{ is even}}}^{i-2} n_k, & \text{if } w = x_{ij}, i \text{ is even}, 1 \leq j \leq n_i, i \geq 4; \\ \frac{i-1}{2} + j + \sum\limits_{\substack{k=1 \\ k \text{ is odd}}}^{i-2} n_k, & \text{if } w = x_{ij}, i \text{ is odd}, 1 \leq j \leq n_i, i \geq 3. \end{cases}$$

Next, define a vertex labeling $g : V(G) \to \{1, 2, ..., t + \sum_{k=1}^{t} n_k\}$ by

$$g(w) = \begin{cases} 1, & \text{if } w = c_1; \\ \frac{i+1}{2} + \sum\limits_{\substack{k=2 \\ k \text{ is even}}}^{i-1} n_k, & \text{if } w = c_i, i \text{ is odd and } i \geq 3; \\ \frac{i+t+1}{2} + \sum\limits_{\substack{k=2 \\ k \text{ is even}}}^{t} n_k + \sum\limits_{\substack{k=1 \\ k \text{ is odd}}}^{i-1} n_k, & \text{if } w = c_i, i \text{ is even}; \\ j+1, & \text{if } w = x_{2j}, 1 \leq j \leq n_2; \\ \frac{i}{2} + j + \sum\limits_{\substack{k=2 \\ k \text{ is even}}}^{i-2} n_k, & \text{if } w = x_{ij}, i \text{ is even}, 1 \leq j \leq n_i, i \geq 4; \\ \frac{t+1}{2} + j + \sum\limits_{\substack{k=2 \\ k \text{ is even}}}^{t} n_k, & \text{if } w = x_{1j}, 1 \leq j \leq n_1; \\ \frac{i+t}{2} + j + \sum\limits_{\substack{k=2 \\ k \text{ is even}}}^{t} n_k + \sum\limits_{\substack{k=1 \\ k \text{ is odd}}}^{i-2} n_k, & \text{if } w = x_{ij}, i \text{ is odd}, 1 \leq j \leq n_i, i \geq 3. \end{cases}$$

For instance, Figures 2.14 and 2.15 show vertex labelings f and g of $CP_{2,2,1,0,2,2}$.

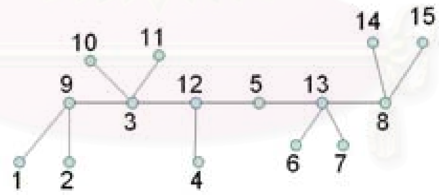


Figure 2.14: A vertex labeling f of $CP_{2,2,1,0,2,2}$.

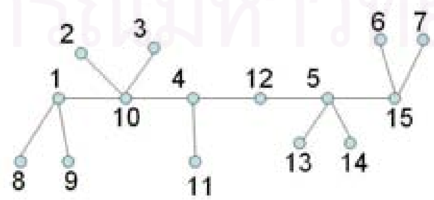


Figure 2.15: A vertex labeling g of $CP_{2,2,1,0,2,2}$.

Similar to Theorem 2.3.2, we can show that $f(V(G)) = g(V(G)) = \{1, 2, 3, ..., p\}$

and

$$\{f(x)+f(y):xy\in E(G)\} =\{\frac{p+1}{2}+2,\ \frac{p+1}{2}+3,...,\ \frac{p+1}{2}+p\}$$
$$\{g(x)+g(y):xy\in E(G)\} =\{\frac{p+1}{2}+1,\ \frac{p+1}{2}+2,...,\ \frac{p+1}{2}+p-1\}$$

are sets of $p-1$ consecutive integers. From Theorem 2.1.1, f and g extend to super edge-magic labelings of G.

Let $V(P_2)=\{1,2\}$ and $E(P_2)=\{12\}$ and $H=G\times P_2$. Thus

$V(H)=\{(c_i,k):1\leq i\leq t, k=1,2\}\cup\{(x_{ij},k):1\leq i\leq t, 1\leq j\leq n_i, k=1,2\}$.

Define a vertex labeling $h:V(H)\to\{1,2,...,2p\}$ by

$$h(w)=\begin{cases} f(c_i), & \text{if } w=(c_i,1);\\ f(x_{ij}), & \text{if } w=(x_{ij},1);\\ p+g(c_i), & \text{if } w=(c_i,2);\\ p+g(x_{ij}), & \text{if } w=(x_{ij},2).\end{cases}$$

For instance, Figure 2.16 shows the vertex labeling h of $CP_{2,2,1,0,2,2}\times P_2$ constructed from f and g in Figure 2.14 and Figure 2.15.

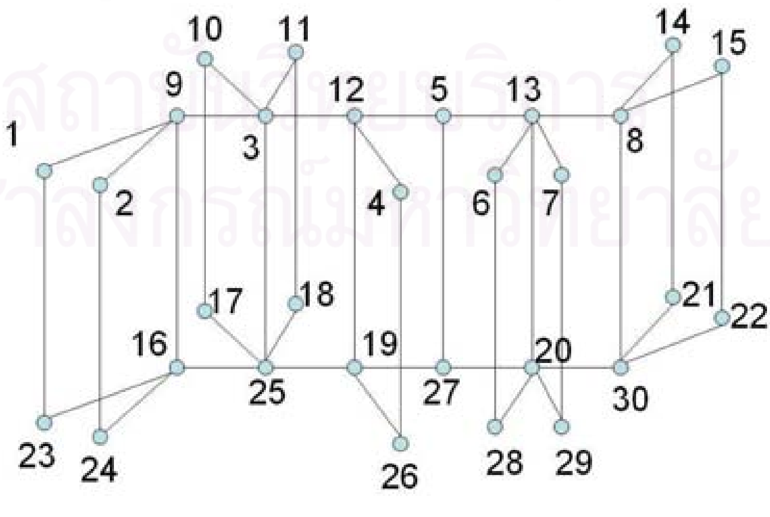


Figure 2.16: A vertex labeling h of $CP_{2,2,1,0,2,2}\times P_2$.

Similar to Theorem 2.3.1, we can show that $h(V(H)) = \{1, 2, 3, ..., 2p\}$ and $\{h(x) + h(y) : xy \in E(H)\} = \{\frac{p+1}{2} + 2, \frac{p+1}{2} + 3, ..., \frac{p+1}{2} + 3p - 1\}$ is a set of $3p-2$ consecutive integers. From Theorem 2.1.1, h extends to a super edge-magic labeling of H. $\square$

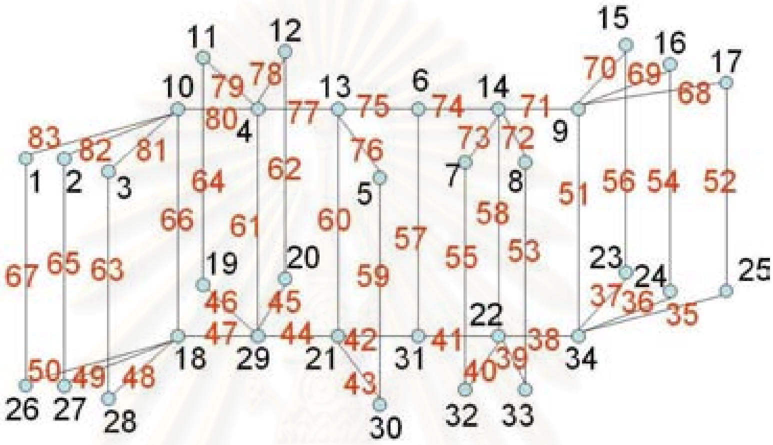


Figure 2.17: A super edge-magic labeling of $CP_{3,2,1,0,2,3} \times P_2$ with magic constant 94.

2.4 Super edge-magic labeling of the product of SF-graph and path P_n

In this section, we show the super edge-magic labeling of the product of specific SF-graph and path P_n.

Definition 2.4.1. A *SF-graph* $SF_{n_1,n_2,...,n_t}$ is a graph which the vertex-set is $\{c_i : 1 \leq i \leq t\} \cup \{x_{ij} : 1 \leq i \leq t, 1 \leq j \leq n_i\}$ and the edge-set is

$\{c_{i+1}c_i : 1 \leq i \leq t-1\} \cup \{c_1c_t\} \cup \{c_ix_{ij} : 1 \leq i \leq t, 1 \leq j \leq n_i\}$.

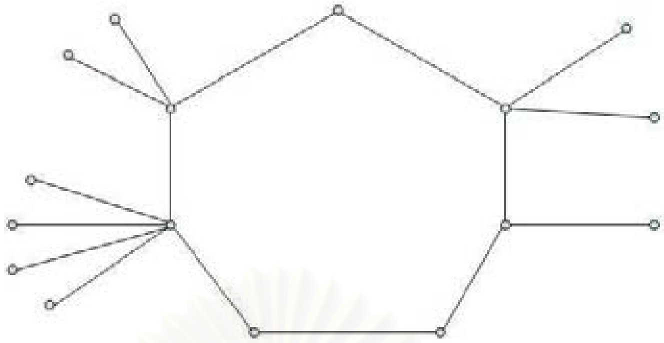


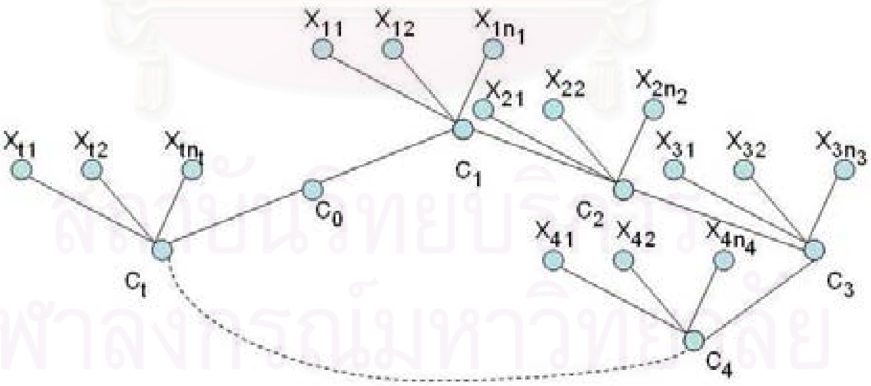
Figure 2.18: SF-graph $SF_{4,2,0,2,1,0,0}$

Theorem 2.4.2. *Let G be the $SF_{0,n_1,n_2,\ldots,n_t}$ and t is even. If* $\displaystyle\sum_{\substack{k=1\\ k \text{ is odd}}}^{t-1} n_k = \sum_{\substack{k=2\\ k \text{ is even}}}^{t} n_k,$ *then the graph $G \times P_n$ is super edge-magic for all $n \in \mathbb{N}$.*

Proof. Let $G \cong SF_{0,n_1,n_2,\ldots,n_t}$ with

$V(G) = \{c_i : 0 \leq i \leq t\} \cup \{x_{ij} : 1 \leq i \leq t, 1 \leq j \leq n_i\}$ and

$E(G) = \{c_i c_{i+1} : 0 \leq i \leq t-1\} \cup \{c_0 c_t\} \cup \{c_i x_{ij} : 1 \leq i \leq t, 1 \leq j \leq n_i\}$.



Let p be the number of vertices of G. Then $p = t + 1 + \sum_{k=1}^{t} n_k$.

First, define a vertex labeling $f : V(G) \to \{1, 2, ..., t + 1 + \sum_{i=1}^{t} n_i\}$ by

$$f(w) = \begin{cases} 1, & \text{if } w = c_0; \\ j + 1, & \text{if } w = x_{1j}, 1 \leq j \leq n_1; \\ \frac{i+1}{2} + j + \sum_{\substack{k=1 \\ k \text{ is odd}}}^{i-2} n_k, & \text{if } w = x_{ij}, i \text{ is odd}, 1 \leq j \leq n_i, i \geq 3; \\ \frac{i}{2} + 1 + \sum_{\substack{k=1 \\ k \text{ is odd}}}^{i-1} n_k, & \text{if } w = c_i, i \text{ is even and } i \geq 2; \\ \frac{t}{2} + 2 + \sum_{\substack{k=1 \\ k \text{ is odd}}}^{t-1} n_k, & \text{if } w = c_1; \\ \frac{t}{2} + 2 + j + \sum_{\substack{k=1 \\ k \text{ is odd}}}^{t-1} n_k & \text{if } w = x_{2j}, 1 \leq j \leq n_2; \\ \frac{i+t+1}{2} + 1 + \sum_{\substack{k=1 \\ k \text{ is odd}}}^{t-1} n_k + \sum_{\substack{k=2 \\ k \text{ is even}}}^{i-1} n_k, & \text{if } w = c_i, i \text{ is odd}, i \geq 3; \\ \frac{i+t}{2} + j + 1 + \sum_{\substack{k=1 \\ k \text{ is odd}}}^{t-1} n_k + \sum_{\substack{k=2 \\ k \text{ is even}}}^{i-2} n_k, & \text{if } w = x_{ij}, i \text{ is even}, 1 \leq j \leq n_i, i \geq 4. \end{cases}$$

For instance, Figure 2.19 shows vertex labeling f of $SF_{0,1,2,3,2}$.

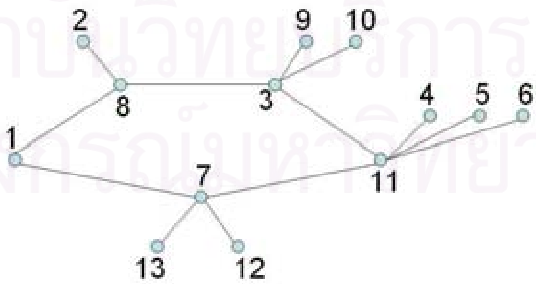


Figure 2.19: A vertex labeling f of $SF_{0,1,2,3,2}$.

Next, define a vertex labeling $g : V(G) \to \{1, 2, ..., t + 1 + \sum_{i=1}^{t} n_i\}$ by

$$g(w) = \begin{cases} 1, & \text{if } w = c_1; \\ j + 1, & \text{if } w = x_{2j}, 1 \leq j \leq n_2; \\ \frac{i}{2} + j + \sum\limits_{\substack{k=2 \\ k \text{ is even}}}^{i-2} n_k, & \text{if } w = x_{ij}, i \text{ is even}, 1 \leq j \leq n_i, i \geq 4; \\ \frac{i+1}{2} + \sum\limits_{\substack{k=2 \\ k \text{ is even}}}^{i-1} n_k, & \text{if } w = c_i, i \text{ is odd and } i \geq 3; \\ \frac{t}{2} + 1 + \sum\limits_{\substack{k=2 \\ k \text{ is even}}}^{t} n_k, & \text{if } w = c_0; \\ \frac{t}{2} + 1 + j + \sum\limits_{\substack{k=2 \\ k \text{ is even}}}^{t} n_k & \text{if } w = x_{1j}, 1 \leq j \leq n_1; \\ \frac{i+t}{2} + 1 + \sum\limits_{\substack{k=2 \\ k \text{ is even}}}^{t} n_k + \sum\limits_{\substack{k=1 \\ k \text{ is odd}}}^{i-1} n_k, & \text{if } w = c_i, i \text{ is even}, i \geq 2; \\ \frac{i+t-1}{2} + j + 1 + \sum\limits_{\substack{k=2 \\ k \text{ is even}}}^{t} n_k + \sum\limits_{\substack{k=1 \\ k \text{ is odd}}}^{i-2} n_k, & \text{if } w = x_{ij}, i \text{ is odd}, 1 \leq j \leq n_i, i \geq 3. \end{cases}$$

For instance, Figure 2.20 shows vertex labeling g of $SF_{0,1,2,3,2}$.

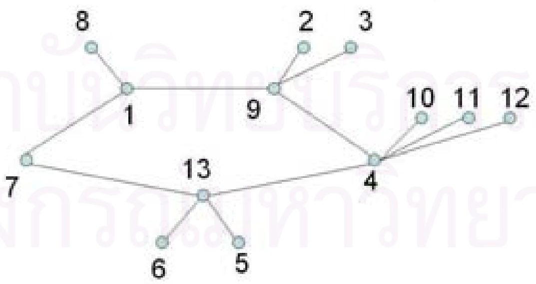


Figure 2.20: A vertex labeling g of $SF_{0,1,2,3,2}$.

In order to show that f and g extend to super edge-magic labelings of G, it suffices to verify by Theorem 2.1.1:

a) $f(V(G)) = g(V(G)) = \{1, 2, 3, ..., p\}$

b) $S_f = \{f(x) + f(y) : xy \in E(G)\}$ and $S_g = \{g(x) + g(y) : xy \in E(G)\}$ consist of p consecutive integers.

Note that, $\sum\limits_{\substack{k=1 \\ k \text{ is odd}}}^{t-1} n_k + \frac{t}{2} + 1 = \sum\limits_{\substack{k=2 \\ k \text{ is even}}}^{t} n_k + \frac{t}{2} + 1 = \frac{p+1}{2}$.

To show that $f(V(G)) = \{1, 2, ..., p\}$, we consider the labels of vertices as follows:

Vertices $c_0, x_{11}, x_{12}, ..., x_{1n_1}, c_2, x_{31}, ..., x_{3n_3}, c_4, ..., c_t$ are labeled by the numbers $1, 2, 3, ..., n_1+1, n_1+2, n_1+3, ..., n_1+n_3+3, n_1+n_3+4, ..., \sum\limits_{\substack{k=1 \\ k \text{ is odd}}}^{t-1} n_k + \frac{t}{2} + 1 = \frac{p+1}{2}$, respectively, and $c_1, x_{21}, x_{22}, ..., x_{2n_2}, c_3, x_{41}, ..., x_{4n_4}, c_5, ..., x_{tn_t}$ are labeled by the numbers $\sum\limits_{\substack{k=1 \\ k \text{ is odd}}}^{t-1} n_k + \frac{t}{2} + 2 = \frac{p+1}{2} + 1, \frac{p+1}{2} + 2, \frac{p+1}{2} + 3, ..., \frac{p+1}{2} + n_2 + 1, \frac{p+1}{2} + n_2 + 2, \frac{p+1}{2} + n_2 + 3, ..., \frac{p+1}{2} + n_2 + n_4 + 3, \frac{p+1}{2} + n_2 + n_4 + 4, ..., p$, respectively. Hence $f(V(G)) = \{1, 2, ..., p\}$.

To show that S_f consists of p consecutive integers, we have

$$
\begin{aligned}
f(c_0) + f(c_t) &= 1 + (\frac{t}{2} + 1 + \sum_{\substack{k=1 \\ k \text{ is odd}}}^{t-1} n_k) \\
&= \frac{p+1}{2} + 1 \\
f(c_0) + f(c_1) &= 1 + (\frac{t}{2} + 2 + \sum_{\substack{k=1 \\ k \text{ is odd}}}^{t-1} n_k) \\
&= \frac{p+1}{2} + 2.
\end{aligned}
$$

Similar to Theorem 2.3.2, we can verify that

$$f(c_i) + f(x_{ij}) = f(c_i) + f(x_{i(j+1)}) - 1$$

$$f(c_i) + f(x_{in_i}) = f(c_i) + f(c_{i+1}) - 1$$

$$f(c_i) + f(c_{i+1}) = f(c_i) + f(x_{(i+1)j}) - 1.$$

Then $S_f = \{\frac{p+1}{2}+1, \frac{p+1}{2}+2, \ldots, \frac{p+1}{2}+p\}$ is a set of p consecutive integers. From Theorem 2.1.1, f extends to a super edge-magic labeling of G.

Similarly, we can show that $g(V(G)) = \{1, 2, 3, ..., p\}$ and $S_g = \{g(x) + g(y) : xy \in E(G)\} = \{\frac{p+1}{2}+1,\ \frac{p+1}{2}+2, ...,\ \frac{p+1}{2}+p\}$ is a set of p consecutive integers. From Theorem 2.1.1, g extends to a super edge-magic labeling of G.

Note that, $S_f = S_g$.

We will construct a super edge-magic labeling of $SF_{0,n_1,n_2,\ldots,n_t} \times P_n$ as follows.

Let $V(P_n) = \{1, 2, ..., n\}$ and $E(P_n) = \{12, 23, 34, ..., (n-1)n\}$ and $H \cong G \times P_n$. Then

$V(H) = \{(c_i, k) : 1 \le i \le t, 1 \le k \le n\} \cup \{(x_{ij}, k) : 1 \le i \le t, 1 \le j \le n_i,$
$1 \le k \le n\}$.

Define a vertex labeling $h : V(H) \to \{1, 2, \ldots, np\}$ by

$$h(w) = \begin{cases} (k-1)p + f(c_i), & \text{if } w = (c_i, k), k \text{ is odd;} \\ (k-1)p + f(x_{ij}), & \text{if } w = (x_{ij}, k), k \text{ is odd;} \\ (k-1)p + g(c_i), & \text{if } w = (c_i, k), k \text{ is even;} \\ (k-1)p + g(x_{ij}), & \text{if } w = (x_{ij}, k), k \text{ is even.} \end{cases}$$

For instance, Figure 2.21 shows the vertex labeling h of $SF_{0,1,2,3,2} \times P_3$ constructed from f and g in Figure 2.19 and Figure 2.20.

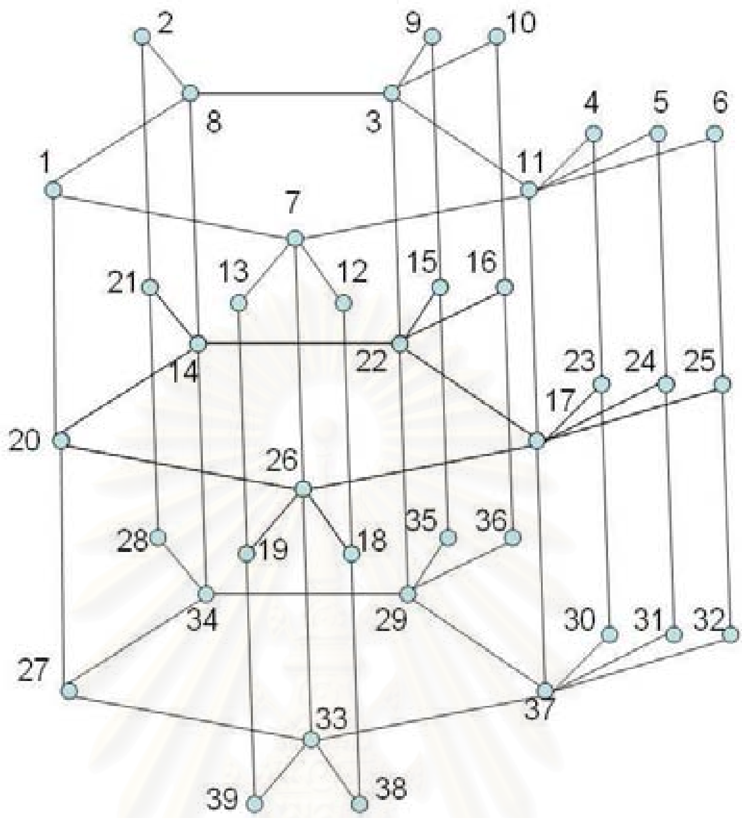


Figure 2.21: A vertex labeling of $SF_{0,1,2,3,2} \times P_3$.

In order to show that h extends to a super edge-magic labeling of H, it suffices to verify by Theorem 2.1.1:

a) $h(V(H)) = \{1, 2, 3, ..., np\}$

b) $S = \{h(x) + h(y) : xy \in E(H)\}$ consists of $2np - p$ consecutive integers.

To show $h(V(H)) = \{1, 2, 3, ..., np\}$, we have

$$h(V(H)) = \bigcup_{k=1}^{n} \{h(u,k) : (u,k) \in V(H))\}$$

$$=\{1, 2, ..., p\} \cup \{p+1, p+2, ..., 2p\} \cup \{(n-1)p+1, (n-1)p+2, ..., np\}$$

$$=\{1, 2, ..., np\}.$$

To show that S consists of $2np - p$ consecutive integers, we consider $h(u, k) + h(u, k+1)$ for all edges $(u, k)(u, k+1)$, where $u \in V(G)$ and $k = 1, 2, ..., n-1$.

For edge $(c_0, k)(c_0, k+1)$,

$$\begin{aligned} h(c_0,k)+h(c_0,k+1) &= (k-1)p + kp + f(c_0) + g(c_0) \\ &= (2k-1)p + 1 + (\frac{t}{2} + 1 + \sum_{\substack{k=2 \\ k \text{ is even}}}^{t} n_k) \\ &= (2k-1)p + \frac{p+1}{2} + 1. \end{aligned}$$

For edge $(c_i, k)(c_i, k+1)$ when $i = 2, 4, ..., t$,

$$\begin{aligned} h(c_i,k)+h(c_i,k) &= (2k-1)p + f(c_i) + g(c_i) \\ &= (2k-1)p + (\frac{i}{2} + 1 + \sum_{\substack{k=1 \\ k \text{ is odd}}}^{i-1} n_k) + (\frac{i+t}{2} + 1 \sum_{\substack{k=2 \\ k \text{ is even}}}^{t} n_k + \sum_{\substack{k=1 \\ k \text{ is odd}}}^{i-1} n_k) \\ &= (2k-1)p + \frac{p+1}{2} + i + 1 + 2\sum_{\substack{k=1 \\ k \text{ is odd}}}^{i-1} n_k. \end{aligned}$$

For edge $(x_{1j}, k)(x_{1j}, k+1)$,

$$\begin{aligned} h(x_{1j},k)+h(x_{1j},k) &= (2k-1)p + f(x_{1j}) + g(x_{1j}) \\ &= (2k-1)p + (j+1) + (\frac{t}{2} + 1 + j + \sum_{\substack{k=2 \\ k \text{ is even}}}^{t} n_k) \\ &= (2k-1)p + \frac{p+1}{2} + 2j + 1. \end{aligned}$$

For edge $(x_{ij}, 1)(x_{ij}, 2)$ when $i = 3, 5, ..., t-1$,

$$\begin{aligned} h(x_{ij},1)+h(x_{ij},2) &= (2k-1)p + f(x_{ij}) + g(x_{ij}) \\ &= (2k-1)p + (\frac{i+1}{2} + j + \sum_{\substack{k=1 \\ k \text{ is odd}}}^{i-2} n_k) \\ &\quad + (\frac{i+t-1}{2} + j + \sum_{\substack{k=2 \\ k \text{ is even}}}^{t} n_k + \sum_{\substack{k=1 \\ k \text{ is odd}}}^{i-2} n_k) \\ &= (2k-1)p + \frac{p+1}{2} + i + 2\sum_{\substack{k=1 \\ k \text{ is odd}}}^{i-2} n_k + 2j. \end{aligned}$$

Note that, for any i is even,

$h(c_i, k) + h(c_i, k+1) = h(x_{(i+1)1}, k) + h(x_{(i+1)1}, k+1) - 2,$

$h(x_{(i-1)n_{i-1}}, k) + h(x_{(i-1)n_{i-1}}, k+1) = h(c_i, k) + h(c_i, k+1) - 2,$

$h(x_{(i+1)j}, k) + h(x_{(i+1)j}, k+1) = h(x_{(i+1)(j+1)}, k) + h(x_{(i+1)(j+1)}, k+1) - 2.$

For edge $(c_1, k)(c_1, k+1)$,

$$\begin{aligned} h(c_1, k) + h(c_1, k+1) &= (2k-1)p + f(c_1) + g(c_1) \\ &= (2k-1)p + (\frac{t}{2} + 2 + \sum_{\substack{k=1 \\ k \text{ is odd}}}^{t-1} n_k) + 1 \\ &= (2k-1)p + \frac{p+1}{2} + 2. \end{aligned}$$

For edge $(c_i, k)(c_i, k+1)$ when $i = 3, 5, ..., t-1$,

$$\begin{aligned} h(c_i, k) + h(c_i, k+1) &= (2k-1)p + f(c_i) + g(c_i) \\ &= (2k-1)p + (\frac{i+t+1}{2} + 1 + \sum_{\substack{k=1 \\ k \text{ is odd}}}^{t-1} n_k + \sum_{\substack{k=2 \\ k \text{ is even}}}^{i-1} n_k) \\ &+ (\frac{i+1}{2} + \sum_{\substack{k=2 \\ k \text{ is even}}}^{i-1} n_k) \\ &= (2k-1)p + \frac{p+1}{2} + i + 1 + 2 \sum_{\substack{k=2 \\ k \text{ is even}}}^{i-1} n_k. \end{aligned}$$

For edge $(x_{2j}, k)(x_{2j}, k+1)$,

$$\begin{aligned} h(x_{1j}, k) + h(x_{1j}, k+1) &= (2k-1)p + f(x_{2j}) + g(x_{2j}) \\ &= (2k-1)p + (\frac{t}{2} + 2 + j + \sum_{\substack{k=1 \\ k \text{ is odd}}}^{t-1} n_k) + (j+1) \\ &= (2k-1)p + \frac{p+1}{2} + 2j + 2. \end{aligned}$$

For edge $(x_{ij},k)(x_{ij},k+1)$ when $i=4,6,...,t$,

$$\begin{aligned}
h(x_{ij},k)+h(x_{ij},k+1) &= (2k-1)p+f(x_{ij})+g(x_{ij}) \\
&= (2k-1)p+\Big(\frac{i+t}{2}+j+1+\sum_{\substack{k=1\\ k \text{ is odd}}}^{t-1} n_k+\sum_{\substack{k=2\\ k \text{ is even}}}^{i-2} n_k\Big) \\
&+\Big(\frac{i}{2}+j+\sum_{\substack{k=2\\ k \text{ is even}}}^{i-2} n_k\Big) \\
&= (2k-1)p+\frac{p+1}{2}+i+2\sum_{\substack{k=2\\ k \text{ is even}}}^{i-2} n_k+2j.
\end{aligned}$$

Note that, for any i is odd,

$h(c_i,k)+h(c_i,k+1)=h(x_{(i+1)1},k)+h(x_{(i+1)1},k+1)-2,$

$h(x_{(i-1)n_{i-1}},k)+h(x_{(i-1)n_{i-1}},k+1)=h(c_i,k)+h(c_i,k+1)-2,$

$h(x_{(i+1)j},k)+h(x_{(i+1)j},k+1)=h(x_{(i+1)(j+1)},k)+h(x_{(i+1)(j+1)},k+1)-2.$

Thus

$$\{h(c_i,k)+h(c_i,k+1): i \text{ is even}\}\cup\bigcup_{\substack{i=1\\ i \text{ is odd}}}^{t-1}\{h(x_{ij},k)+h(x_{ij},k+1): j=1,2,...,n_i\}$$

$$=\{\frac{p+1}{2}+(2k-1)p+1,\frac{p+1}{2}+(2k-1)p+3,...,\frac{p+1}{2}+2kp\}$$

and

$$\{h(c_i,k)+h(c_i,k+1): i \text{ is odd}\}\cup\bigcup_{\substack{i=2\\ i \text{ is even}}}^{t}\{h(x_{ij},k)+h(x_{ij},k+1): j=1,2,...,n_i\}$$

$$=\{\frac{p+1}{2}+(2k-1)p+2,\frac{p+1}{2}+(2k-1)p+4,...,\frac{p+1}{2}+2kp-1\}.$$

Hence $\{h(u,k)+h(u,k+1):(u,k)(u,k+1)\in V(H)\}=\{\frac{p+1}{2}+(2k-1)p+1,\frac{p+1}{2}+(2k-1)p+2,...,\frac{p+1}{2}+2kp\}$

We note that

$$
\begin{aligned}
S =& \{h(u)+h(v) : uv \in E(H)\} \\
=& \bigcup_{k=1}^{n} \{h(u,k)+h(v,k) : (u,k)(v,k) \in V(H)\} \cup \\
& \bigcup_{k=1}^{n-1} \{h(u,k)+h(u,k+1) : (u,k)(v,k+1) \in V(H)\}
\end{aligned}
$$

and

$$
\begin{aligned}
& \bigcup_{k=1}^{n} \{h(u,k)+h(v,k) : (u,k)(v,k) \in V(H)\} \\
=& \bigcup_{k=1}^{n} \{\frac{p+1}{2}+(2k-2)p+1, \frac{p+1}{2}+(2k-2)p+2, ..., \frac{p+1}{2}+(2k-1)p\} \\
=& \{\frac{p+1}{2}+1, \frac{p+1}{2}+2, ..., \frac{p+1}{2}+p\} \cup \\
& \{\frac{p+1}{2}+2p+1, \frac{p+1}{2}+2p+2, ..., \frac{p+1}{2}+3p\} \cup \cdots \cup \\
& \{\frac{p+1}{2}+(2n-2)p+1, \frac{p+1}{2}+(2n-2)p+2, ..., \frac{p+1}{2}+(2n-1)p\}
\end{aligned}
$$

and

$$
\begin{aligned}
& \bigcup_{k=1}^{n-1} \{h(u,k)+h(u,k+1) : (u,k)(v,k+1) \in V(H)\} \\
=& \bigcup_{k=1}^{n-1} \{\frac{p+1}{2}+(2k-1)p+1, \frac{p+1}{2}+(2k-1)p+2, ..., \frac{p+1}{2}+2kp\} \\
=& \{\frac{p+1}{2}+p+1, \frac{p+1}{2}+p+2, ..., \frac{p+1}{2}+2p\} \cup \\
& \{\frac{p+1}{2}+3p+1, \frac{p+1}{2}+3p+2, ..., \frac{p+1}{2}+4p\} \cup \cdots \cup \\
& \{\frac{p+1}{2}+(2n-3)p+1, \frac{p+1}{2}+(2n-3)p+2, ..., \frac{p+1}{2}+(2n-2)p\}.
\end{aligned}
$$

Then $S = \{\frac{p+1}{2}+1,\ \frac{p+1}{2}+2, ...,\ \frac{p+1}{2}+(2n-1)p\}$ is a set of $2np-p$ consecutive integers. From Theorem 2.1.1, h extends to a super edge-magic labeling of H. $\square$

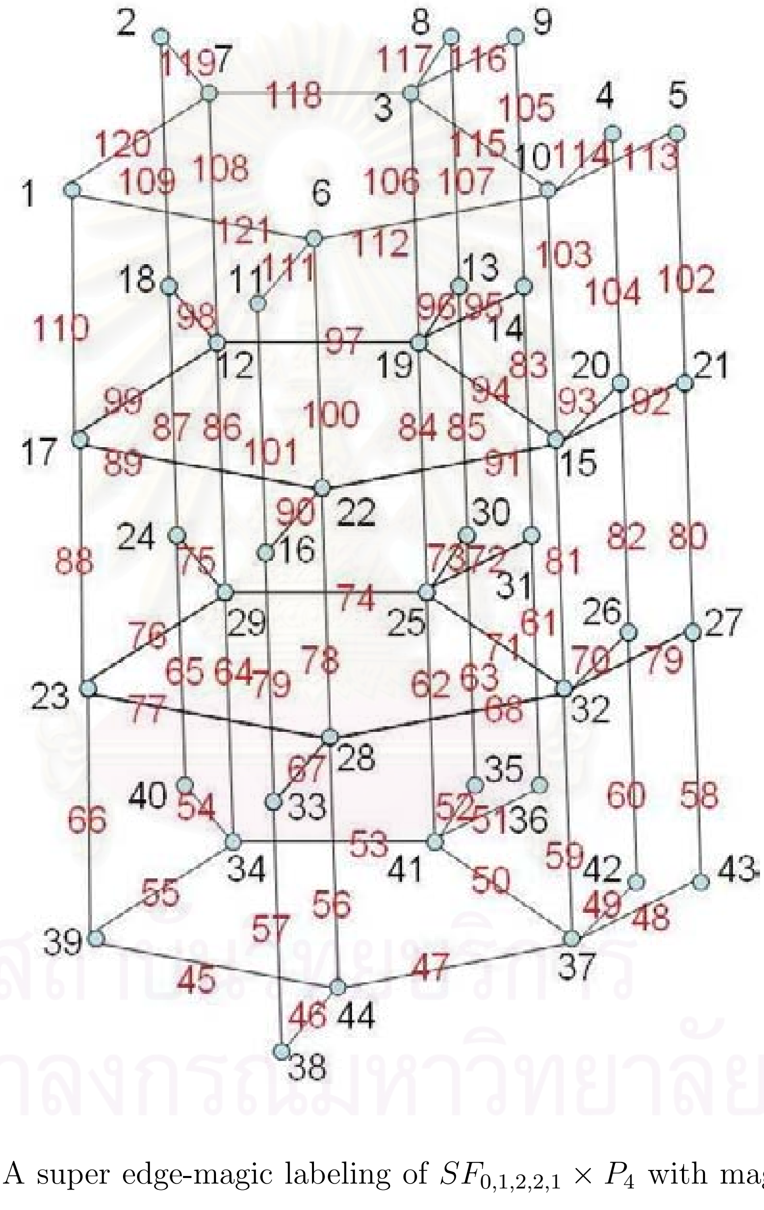


Figure 2.22: A super edge-magic labeling of $SF_{0,1,2,2,1} \times P_4$ with magic constant 128.

CHAPTER III

CREATING NEW SUPER EDGE-MAGIC GRAPHS FROM OLD ONES

Some algorithms to construct new super edge-magic graphs from the old ones done by Sudarsana, Baskoro, Ismaimuza and Assiyatun are given in Theorem 3.1, 3.3 and 3.5. Examples of these algorithms are shown in Example 3.2, 3.4 and 3.6. Then we give a generalization of these algorithms in Theorem 3.7.

Theorem 3.1. [9] *Let a $(p,\ q)$-graph G be super edge-magic with magic constant k and $k \geq 2p+2$. If n is odd and $n = 6p+5-2k$ then the new graph, formed from G and path P_n by joining all vertices of P_n to a vertex x_0 of G labeled by $k-2p-1$, is super edge-magic with magic constant $k+3n-1$.*

Example 3.2. Let G be a graph in figure 3.1(left) which is super edge-magic with magic constant 16. Let x_0 be the vertex labeled by 3, the new graph, formed from G and path P_9 by joining all vertices of P_9 to vertex x_0 of G, is super edge-magic with magic constant 42 as shown in figure 3.1(right).

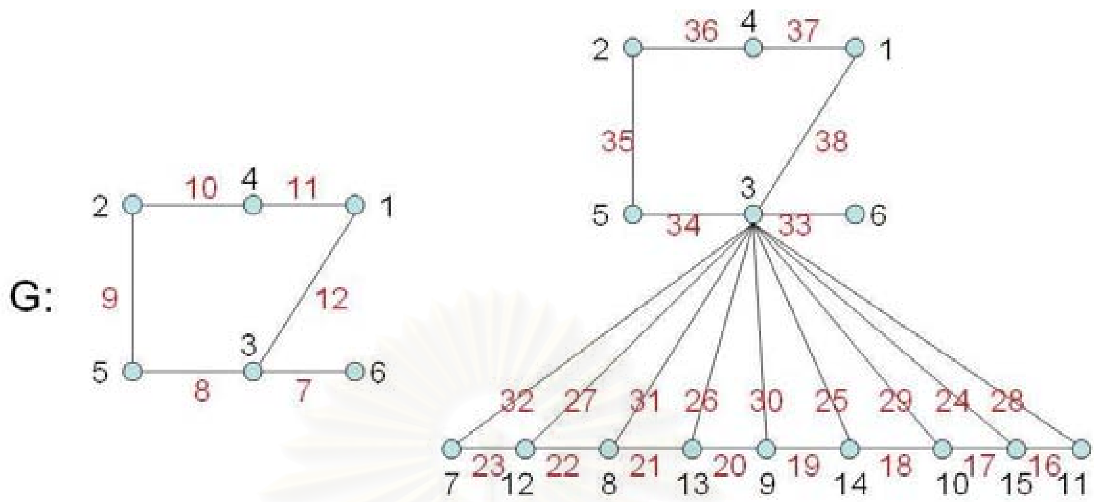


Figure 3.1: The new graph, formed from a super edge-magic graph G with magic constant 16 and path P_9, is super edge-magic with magic constant 42.

Theorem 3.3. [9] *Let a $(p,\ q)$-graph G be super edge-magic with magic constant k and $k \geq 2p+2$. If n is even and $n = 6p+4-2k$ then the new graph, formed from G and path P_n by joining all vertices of P_n to a vertex x_0 of G labeled by $k-2p-1$, is super edge-magic with magic constant $k+3n-1$.*

Example 3.4. Let G be a graph in figure 3.2(left) which is super edge-magic with magic constant 16. Let x_0 be the vertex labeled by 3, the new graph, formed from G and path P_8 by joining all vertices of P_8 to vertex x_0 of G, is super edge-magic with magic constant 39 as shown in figure 3.2(right).

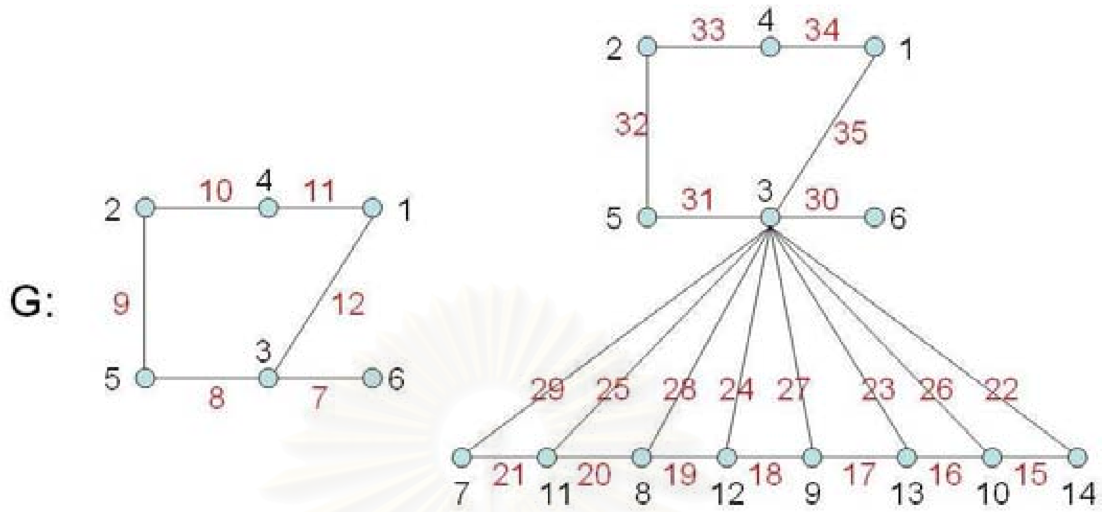


Figure 3.2: The new graph, formed from a super edge-magic graph G with magic constant 16 and path P_8, is super edge-magic with magic constant 39.

Theorem 3.5. [9] *Let a $(p,\ q)$-graph G be super edge-magic with magic constant k and $k \geq 2p+2$. If $n = 3p+2-k$ then the new graph, formed from G and star $K_{1,n}$ by joining all vertices of $K_{1,n}$ to a vertex x_0 of G labeled by $k-2p-1$, is super edge-magic with magic constant $k = k+3n+2$.*

Example 3.6. Let G be a graph in figure 3.3(left) which is super edge-magic with magic constant 16. Let x_0 be the vertex labeled by 3, the new graph, formed from G and a star $K_{1,4}$ by joining all vertices of $K_{1,4}$ to vertex x_0 of G, is super edge-magic with the magic constant 30 as shown in figure 3.3(right).

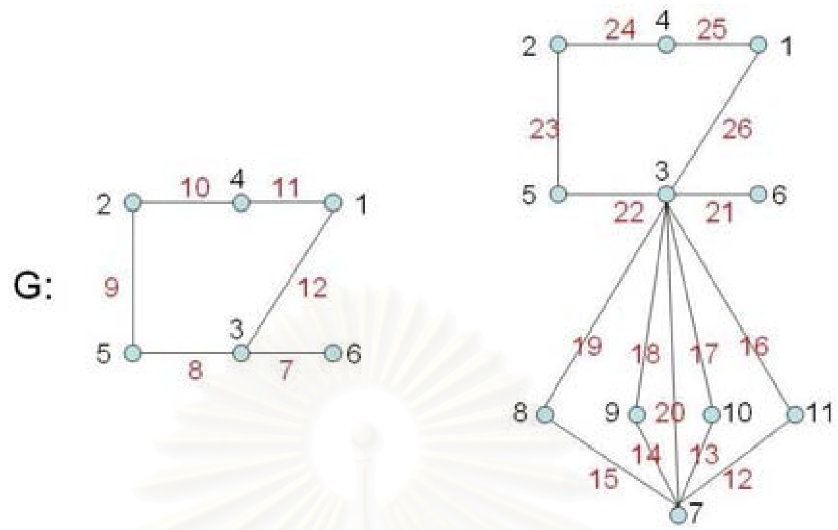


Figure 3.3: The new graph, formed from a super edge-magic graph G with magic constant 16 and a star $K_{1,4}$, is super edge-magic with magic constant 30.

We present a generalization of the above algorithms to construct the super edge-magic graph from the old ones.

Theorem 3.7. *Let G_1 and G_2 be super edge-magic $(p_1,\ q_1)$-graph and $(p_2,\ q_2)$-graph with magic constants k_1 and k_2, respectively. If $k_1 \geq 2p_1+2$ and $k_1 - 3p_1 = k_2 - 2p_2 - q_2$, then the new graph, formed from G_1 and G_2 by joining all vertices of G_2 to a vertex x_0 of G_1 labeled by $k_1 - 2p_1 - 1$, is super edge-magic with magic constant $k_1 + 2p_2 + q_2$.*

Proof. Since G_1 and G_2 are super edge-magic, By Theorem 2.1.1, there exist super edge-magic labelings λ_1 on G_1 and λ_2 on G_2 such that

$\{\lambda_1(u)+\lambda_1(v) : uv \in E(G_1)\} = \{k_1-(p_1+q_1),\, k_1-(p_1+q_1-1),\ldots,\, k_1-(p_1+1)\}$,

$\{\lambda_2(u)+\lambda_2(v) : uv \in E(G_2)\} = \{k_2-(p_2+q_2),\, k_2-(p_2+q_2-1),\ldots,\, k_2-(p_2+1)\}$,

respectively.

Let x_0 be the vertex of G_1 labeled by $k_1 - 2p_1 - 1$ and G be the new graph , formed from G_1 and G_2 by joining all vertices of G_2 to vertex x_0.

Define a vertex labeling $\lambda : V(G) \rightarrow \{1, 2, \ldots, p_1 + p_2\}$ by

$$\lambda(u) = \begin{cases} \lambda_1(u), & \text{if } u \in V(G_1); \\ p_1 + \lambda_2(u), & \text{if } u \in V(G_2). \end{cases}$$

Since $\{\lambda(u) : u \in V(G)\} = \{\lambda(u) : u \in V(G_1)\} \cup \{\lambda(u) : u \in V(G_2)\}$

and $\{\lambda(u) : u \in V(G_1)\} = \{\lambda_1(u) : u \in V(G_1)\} = \{1, 2, ..., p_1\}$

and $\{\lambda(u) : u \in V(G_2)\} = \{p_1 + \lambda_2(u) : u \in V(G_2)\} = \{p_1 + 1, p_1 + 2, ..., p_1 + p_2\}$,

$\{\lambda(u) : u \in V(G)\} = \{1, 2, ..., p_1 + p_2\}$.

Consider

$$\begin{aligned} \{\lambda(u) + \lambda(v) : uv \in E(G)\} =& \{\lambda(u) + \lambda(v) : uv \in E(G_1)\} \cup \{\lambda(x_0) + \lambda(v) : v \in V(G_2)\} \\ & \cup \{\lambda(u) + \lambda(v) : uv \in E(G_2)\} \\ =& \{\lambda_1(u) + \lambda_1(v) : uv \in E(G_1)\} \cup \{\lambda_1(x_0) + \lambda_2(v) : v \in V(G_2)\} \\ & \cup \{2p_1 + \lambda_2(u) + \lambda_2(v) : uv \in E(G_2)\}. \end{aligned}$$

Note that, for all $v \in V(G_2)$,

$\lambda_1(x_0) + \lambda_2(v) = (k_1 - 2p_1 - 1) + (p_1 + \lambda_2(v)) = k_1 - p_1 + \lambda_2(v) - 1$.

Since $1 \leq \lambda_2(v) \leq p_2$ for all $v \in V(G_2)$,

$\{\lambda_1(x_0) + \lambda_2(v) : v \in V(G_2)\} = \{k_1 - p_1, k_1 - p_1 + 1, \ldots, k_1 - p_1 + p_2 - 1\}$.

Since $k_1 - 3p_1 = k_2 - 2p_2 - q_2$, we have $2p_1 + k_2 - (p_2 + q_2) = k_1 - p_1 + p_2$,

$$\begin{aligned} & \{2p_1 + \lambda_2(u) + \lambda_2(v) : uv \in E(G_2)\} \\ =& \{2p_1 + k_2 - (p_2 + q_2), 2p_1 + k_2 - (p_2 + q_2 - 1), \ldots, 2p_1 + k_2 - (p_2 + 1)\} \\ =& \{k_1 - p_1 + p_2, k_1 - p_1 + p_2 + 1, \ldots, k_1 - p_1 + p_2 + q_2 - 1\}. \end{aligned}$$

Hence $\{\lambda(u)+\lambda(v) : uv \in E(G)\} = \{k_1-(p_1+q_1), k_1-(p_1+q_1-1), \ldots, k_1-(p_1+1)\}$ $\cup \{k_1 - p_1, k_1 - p_1 + 1, \ldots, k_1 - p_1 + p_2 - 1\} \cup \{k_1 - p_1 + p_2, k_1 - p_1 + p_2 + 1, \ldots, k_1 - p_1 + p_2 + q_2 - 1\}$ which is the set of $q_1 + q_2 + p_2$ consecutive integers. Then G is super edge-magic with magic constant $(p_1+p_2)+(q_1+q_2+p_2)+(k_1-(p_1+q_1)) = k_1+2p_2+q_2$. $\square$

Example 3.8. Let G_1 and G_2 be graphs in figure 3.4(left) which are super edge-magic with magic constant 16 and 33, respectively. Let x_0 be the vertex labeled by 3 in G_1, the new graph, formed from G_1 and G_2 by joining all vertices of G_2 to vertex x_0 of G_1, is super edge-magic with magic constant 51 as shown in figure 3.4(right).

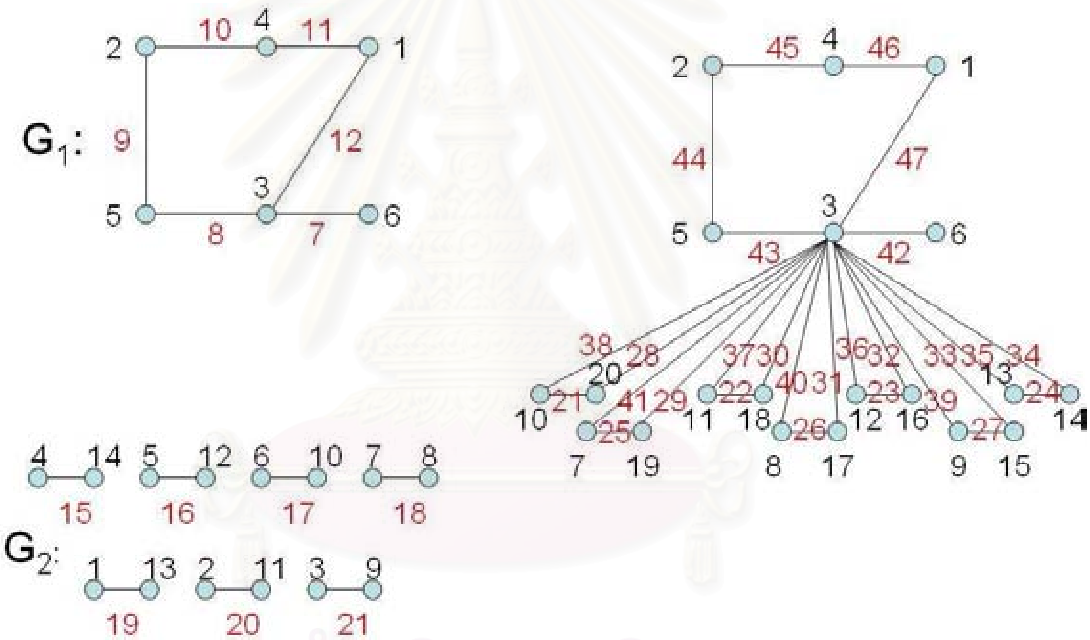


Figure 3.4: The new graph, formed from a super edge-magic graph G_1 with magic constant 16 and G_2 with magic constant 33, is super edge-magic with magic constant 51.

Corollary 3.9. *Let a $(p,\ q)$-graph G be a super edge-magic with magic constant k and $k \geq 2p+2$. If n is odd and $n = 6p+3-2k$ then the new graph, formed from G and cycle C_n by joining all vertices of C_n to a vertex x_0 of G labeled by $k-2p-1$, is super edge-magic with the magic constant $k+3n$.*

Proof. It is known that [2] every odd cycle C_n is super edge-magic with magic constant $\frac{5n+3}{2}$. Let p', q', k' be number of vertices, number of edges and magic constant of C_n, respectively. Thus $k' = \frac{5n+3}{2} = \frac{5(6p+3-2k)+3}{2} = 15p+9-5k$. Then $k'-2p'-q' = (15p+9-5k)-2(6p+3-2k)-(6p+3-2k) = k-3p$. By Theorem 3.7, the new graph, formed from G and cycle C_n by joining all vertices of C_n to a vertex x_0, is super edge-magic with magic constant $k+2p'+q' = k+3n$. □

Example 3.10. Let G be a graph in figure 3.5(left) is super edge-magic with magic constant 16. Let x_0 be the vertex labeled by 3, the new graph, formed from G and a cycle C_7 by joining all vertices of C_7 to vertex x_0 of G, is super edge-magic with the magic constant 37 as shown in figure 3.5(right).

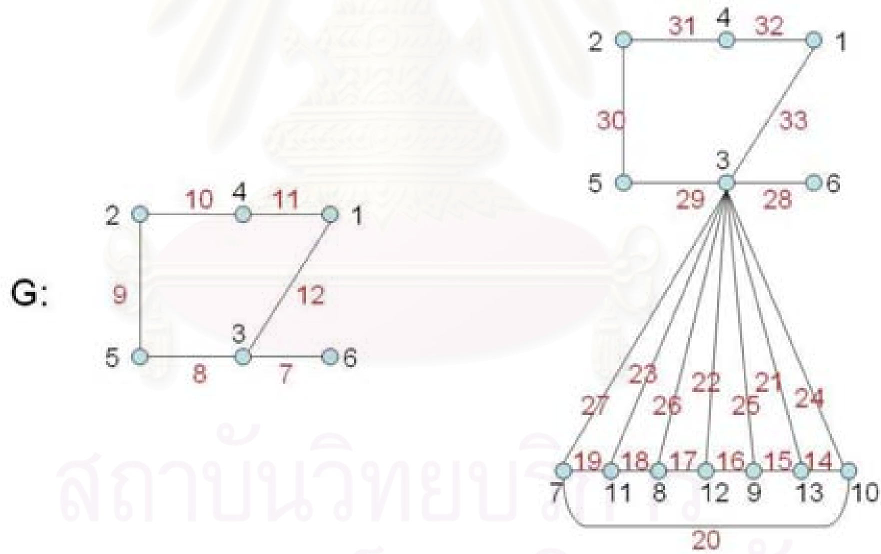


Figure 3.5: The new graph, formed from a super edge-magic graph G with magic constant 16 and a cycle C_7, is super edge-magic with magic constant 37.

CHAPTER IV

SUPER EDGE-MAGIC DEFICIENCY OF SOME GRAPHS

Our purpose in this chapter is to investigate bounds for the super edge-magic deficiency of some graphs.

Definition 4.1. The *super edge-magic deficiency* $\mu_s(G)$ of a graph G is the smallest nonnegative integer n with the property that the graph $G \cup nK_1$ is super edge-magic or $+\infty$ if there exists no such integer n.

Example 4.2. Since cycle C_4 is not super edge-magic and $C_4 \cup K_1$ is super edge-magic, then $\mu_s(G) = 1$.

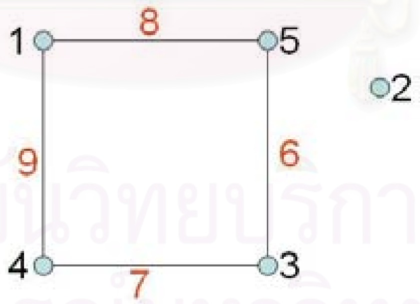


Figure 4.1: $C_4 \cup K_1$ is super edge-magic with magic constant 14.

Figuaroa-Centeno, Ichishima and Muntaner-Batle showed the following theorem.

Theorem 4.3. [5] *If G is a graph with even degree and q edges, where $\frac{q}{2}$ is odd, then $\mu_s(G) = +\infty$.*

We investigate a lower bound for the super edge-magic deficiency of the join of cycle C_n and m isolated vertices.

Theorem 4.4. *For all integers $m \geq 1$ and $n \geq 3$,*

$$\mu_s(mK_1 \vee C_n) \geq \frac{(m-1)(n-2)+1}{2}.$$

Proof. Let G be the join of m copies of K_1 and n-cycle C_n with

$|V(G)| = m + n$ and $|E(G)| = n + mn$.

Thus

$$\begin{aligned}|E(G)| &= mn + n = m(n-2+2) + n = m(n-2) + 2m + n \\ &\geq (n-2) + 2m + n > 2m + 2n - 3 = 2(m+n) - 3 = 2|V(G)| - 3.\end{aligned}$$

By Theorem 2.1.3, G is not super edge-magic.

Let k be a positive integer such that $G \cup kK_1$ is super edge-magic.

By Theorem 2.1.3, $|E(G \cup kK_1)| \leq 2|V(G \cup kK_1)| - 3$.

Thus $mn + n \leq 2(m+n+k) - 3$, then $k \geq \dfrac{(m-1)(n-2)+1}{2}$.

Hence $\mu_s(G) \geq \dfrac{(m-1)(n-2)+1}{2}$. □

We investigate an upper bound for the super edge-magic deficiency of the join of odd cycle C_n and m isolated vertices.

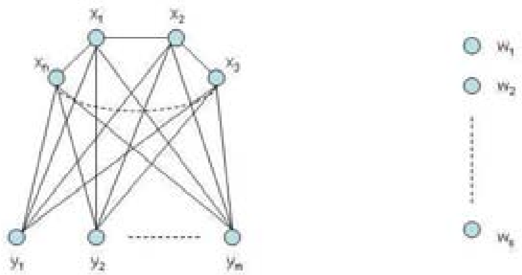
Theorem 4.5. *For all positive integers m, n and n is odd,*

$$\mu_s(mK_1 \vee C_n) \leq \frac{(2m-1)(n-1)}{2}.$$

Proof. Let $s = \frac{(2m-1)(n-1)}{2}$ and $G \cong (mK_1 \vee C_n) \cup sK_1$ be the graph with

$V(G) = \{x_i : 1 \leq i \leq n\} \cup \{y_j : 1 \leq j \leq m\} \cup \{w_k : 1 \leq k \leq s\}$ and

$E(G) = \{x_i x_{i+1} : 1 \leq i \leq n-1\} \cup \{x_n x_1\} \cup \{y_j x_i : 1 \leq j \leq m, 1 \leq i \leq n\}$.



Define a vertex labeling $f : V(G) \to \{1, 2, \ldots, m+n+s\}$ by

$$f(u) = \begin{cases} \dfrac{i+1}{2}, & \text{if } u = x_i, i \text{ is odd;} \\ \dfrac{n+1+i}{2}, & \text{if } u = x_i, i \text{ is even;} \\ \dfrac{3n+1}{2} + (j-1)n, & \text{if } u = y_j. \end{cases}$$

and

$$\begin{aligned} \{f(w_k) : k = 1, 2, ..., s\} =& \{n+1, n+2, ..., \frac{3n-1}{2}\} \cup \{\frac{3n+3}{2}, \frac{3n+5}{2}, \ldots, \frac{5n-1}{2}\} \cup \\ & \{\frac{5n+3}{2}, \frac{5n+5}{2}, \ldots, \frac{7n-1}{2}\} \cup \{\frac{7n+3}{2}, \frac{7n+5}{2}, \ldots, \frac{9n-1}{2}\} \\ & \cup \cdots \cup \{\frac{2mn-n+3}{2}, \frac{2mn-n+5}{2}, \ldots, \frac{2mn+n-1}{2}\} \\ =& \{n+1, n+2, ..., \frac{3n-1}{2}\} \cup \bigcup_{a=2}^{m} (\bigcup_{b=2}^{n} \{\frac{(2a-1)n+(2b-1)}{2}\}). \end{aligned}$$

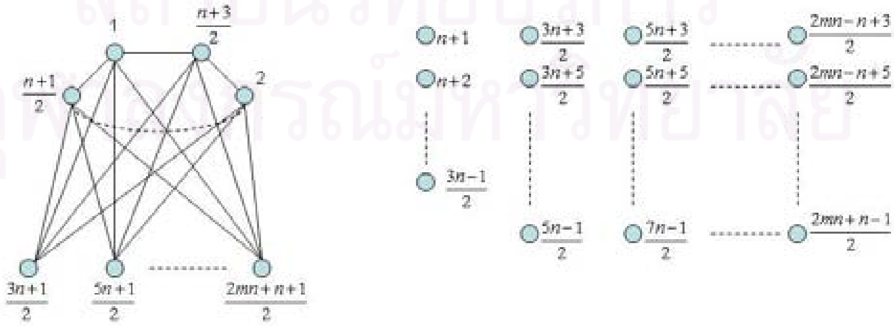


Figure 4.2: A vertex labeling of $(mK_1 \vee C_n) \cup sK_1$.

In order to show that f extends to a super edge-magic labeling of G, it suffices to verify by Theorem 2.1.1:

a) $f(V(G)) = \{1,\ 2,\ 3, \ldots, m+n+s\}$

b) $S = \{f(x)+f(y) : xy \in E(G)\}$ consists of $mn+n$ consecutive integers.

To show that $f(V(G)) = \{1,\ 2,\ 3, \ldots, m+n+s\}$, we consider the labels of vertices as follows:

Vertices $x_1, x_3, x_5 ..., x_n$ are labeled by numbers $1, 2, 3, ..., \frac{n+1}{2}$, respectively and $x_2, x_4, x_6 ..., x_{n-1}$ are labeled by numbers $\frac{n+3}{2}, \frac{n+5}{2}, \frac{n+7}{2}, ..., n$, respectively and $y_1, y_2, y_3, ..., y_m$ are labeled by numbers $\frac{3n+1}{2}, \frac{5n+1}{2}, \frac{7n+1}{2}, ..., \frac{2mn+n+1}{2}$, respectively and $w_1, w_2, ..., w_s$ are labeled by remaining numbers. Hence $f(V(G)) = \{1,\ 2,\ 3, \ldots, m+n+s\}$.

To show that S consists of $mn+n$ consecutive integers, we consider $f(x)+f(y)$ for all edges xy in G.

For edge $x_n x_1$, $f(x_n)+f(x_1) = \dfrac{n+1}{2} + 1 = \dfrac{n+3}{2}$.

For edge $x_i x_{i+1}$:$i = 1, 3, 5, \ldots, n-2$,

$f(x_i)+f(x_{i+1}) = \dfrac{i+1}{2} + \dfrac{n+i+2}{2} = \dfrac{n+3+2i}{2}$.

For edge $x_i x_{i+1}$:$i = 2, 4, 6, \ldots, n-1$,

$f(x_i)+f(x_{i+1}) = \dfrac{n+i+1}{2} + \dfrac{i+2}{2} = \dfrac{n+3+2i}{2}$.

For edge $y_j x_i$:$i = 1, 3, 5, \ldots, n,\quad j = 1, 2, ..., m$,

$f(y_j)+f(x_i) = \dfrac{3n+1}{2} + (j-1)n + \dfrac{i+1}{2} = \dfrac{(2j+1)n+i+2}{2}$.

For edge $y_j x_i$:$i = 2, 4, 6, \ldots, n-1,\quad j = 1, 2, ..., m$,

$f(y_j)+f(x_i) = \dfrac{3n+1}{2} + (j-1)n + \dfrac{n+1+i}{2} = \dfrac{(2j+2)n+i+2}{2}$.

We note that

$$
\begin{aligned}
S =& \{f(x)+f(y) : xy \in E(G)\} \\
=& \{f(x_n)+f(x_1)\} \cup \{f(x_i)+f(x_{i+1}) : i = 1, 2, ..., n-1\} \cup \\
& \bigcup_{j=1}^{m} \{f(y_j)+f(x_i) : i = 1, 3, ..., n\} \cup \bigcup_{j=1}^{m} \{f(y_j)+f(x_i) : i = 2, 4, ..., n-1\}
\end{aligned}
$$

and

$$\begin{aligned}
\{f(x_n)+f(x_1)\} =&\{\frac{n+3}{2}\}\\
\{f(x_i)+f(x_{i+1}):i=1,2,...,n-1\} =&\{\frac{n+5}{2},\frac{n+7}{2},\frac{n+9}{2},...,\frac{3n+1}{2}\}\\
\bigcup_{j=1}^{m}\{f(y_j)+f(x_i):i=1,3,...,n\} =&\{\frac{3n+3}{2},\frac{3n+5}{2},...,\frac{4n+2}{2}\}\cup\\
&\{\frac{5n+3}{2},\frac{5n+5}{2},...,\frac{6n+2}{2}\}\cup\cdots\cup\\
&\{\frac{2mn+n+3}{2},\frac{2mn+n+5}{2},...,\frac{2mn+2n+2}{2}\}
\end{aligned}$$

$$\begin{aligned}
\bigcup_{j=1}^{m}\{f(y_j)+f(x_i):i=2,4,...,n-1\} =&\{\frac{4n+4}{2},\frac{4n+6}{2},...,\frac{5n+1}{2}\}\cup\\
&\{\frac{6n+4}{2},\frac{6n+6}{2},...,\frac{7n+1}{2}\}\cup\cdots\cup\\
&\{\frac{2mn+2n+4}{2},\frac{2mn+2n+6}{2},...,\frac{2mn+3n+1}{2}\}.
\end{aligned}$$

Then $S=\{\frac{n+3}{2},\frac{n+5}{2},\frac{n+7}{2},\ldots,\frac{2mn+3n+1}{2}\}$ is a set of $mn+n$ consecutive integers. By Theorem 2.1.1, f extends to a super edge-magic labeling of G. Therefore $\mu_s(mK_1\vee C_n)\leq\frac{(2m-1)(n-1)}{2}$ when n is odd. $\square$

Example 4.6. $6\leq\mu_s(3K_1\vee C_7)\leq 15$.

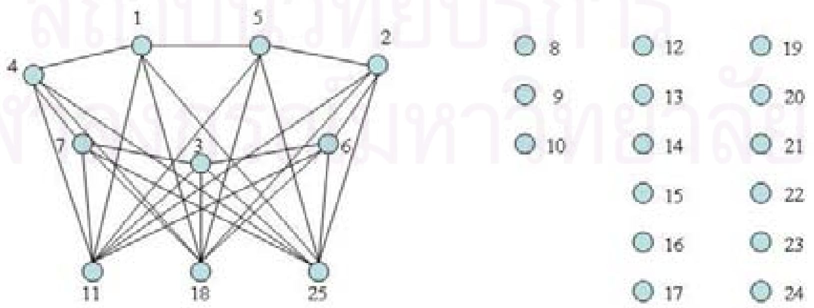


Figure 4.3: A vertex labeling of $(3K_1\vee C_7)\cup 15K_1$.

We investigate the super edge-magic deficiency of the join of specific even cycle C_n and m isolated vertices.

Theorem 4.7. *For all positive integers m, n and $m, n \equiv 2 \pmod 4$,*

$$\mu_s(mK_1 \vee C_n) = +\infty.$$

Proof. Let $m = 4s + 2$ and $n = 4t + 2$ for some positive integers s, t.

Then

$$\begin{aligned} |E(mK_1 \vee C_n)| &= mn + n \\ &= (4s+2)(4t+2) + (4t+2) \\ &= 4(4st + 2s + 3t) + 6. \end{aligned}$$

Since $mK_1 \vee C_n$ is graph with even graph degree and $\dfrac{|E(mK_1 \vee C_n)|}{2} = 2(4st + 2s + 3t) + 3$ is odd, by Theorem 4.3, $\mu_s(mK_1 \vee C_n) = +\infty$. □

We investigate a lower bound for the super edge-magic deficiency of the join of path P_n and m isolated vertices.

Theorem 4.8. *For all integers $m \geq 2$ and $n \geq 3$,*

$$\mu_s(mK_1 \vee P_n) \geq \frac{(m-1)(n-2)}{2}.$$

Proof. Let G be the join of m copies of K_1 and path P_n with

$|V(G)| = m + n$ and $|E(G)| = mn + n - 1$.

Thus

$$\begin{aligned} |E(G)| &= mn + n - 1 = m(n-2+2) + n - 1 = m(n-2) + 2m + n - 1 \\ &> (n-2) + 2m + n - 1 = 2m + 2n - 3 = 2(m+n) - 3 = 2|V(G)| - 3. \end{aligned}$$

By Theorem 2.1.3, G is not super edge-magic.

Let k be a positive integer such that $G \cup kK_1$ is super edge-magic.

By Theorem 2.1.3, $|E(G \cup kK_1)| \leq 2|V(G \cup kK_1)| - 3$.

Thus $mn + n - 1 \leq 2(m + n + k) - 3$, then $k \geq \dfrac{(m-1)(n-2)}{2}$.

Hence $\mu_s(G) \geq \dfrac{(m-1)(n-2)}{2}$. $\square$

We investigate an upper bound for the super edge-magic deficiency of the join of path P_n and m isolated vertices.

Theorem 4.9. *For all positive integers m, n*

$$\mu_s(mK_1 \vee P_n) \leq \begin{cases} \dfrac{(2m-1)(n-1)}{2}, & \text{if } n \text{ is odd;} \\ \dfrac{(2m-1)(n-1)-1}{2}, & \text{if } n \text{ is even.} \end{cases}$$

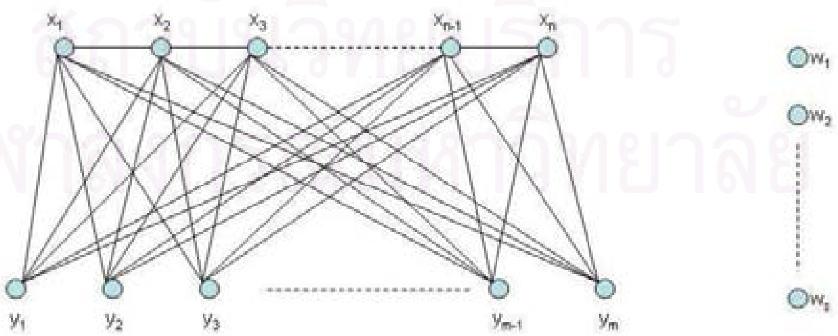
Proof. Let

$$s = \begin{cases} \dfrac{(2m-1)(n-1)}{2}, & \text{if } n \text{ is odd;} \\ \dfrac{(2m-1)(n-1)-1}{2}, & \text{if } n \text{ is even.} \end{cases}$$

and $G \cong (mK_1 \vee P_n) \cup sK_1$ be the graph with

$V(G) = \{x_i : 1 \leq i \leq n\} \cup \{y_j : 1 \leq j \leq m\} \cup \{w_k : 1 \leq k \leq s\}$ and

$E(G) = \{x_i x_{i+1} : 1 \leq i \leq n-1\} \cup \{x_1 x_n : 1\} \cup \{y_j x_i : 1 \leq j \leq m, 1 \leq i \leq n\}$.



Case 1. n is odd.

Define a vertex labeling $f : V(G) \to \{1, 2, \ldots m+n+s\}$ by

$$f(u) = \begin{cases} \dfrac{i+1}{2}, & \text{if } u = x_i, i \text{ is odd;} \\ \dfrac{n+1+i}{2}, & \text{if } u = x_i, i \text{ is even;} \\ \dfrac{3n+1}{2} + (j-1)n, & \text{if } u = y_j. \end{cases}$$

and

$$\begin{aligned} \{f(w_k) : k = 1, 2, ..., s\} =& \{n+1, n+2, ..., \frac{3n-1}{2}\} \cup \{\frac{3n+3}{2}, \frac{3n+5}{2}, \ldots, \frac{5n-1}{2}\} \cup \\ & \{\frac{5n+3}{2}, \frac{5n+5}{2}, \ldots, \frac{7n-1}{2}\} \cup \{\frac{7n+3}{2}, \frac{7n+5}{2}, \ldots, \frac{9n-1}{2}\} \\ & \cup \cdots \cup \{\frac{2mn-n+3}{2}, \frac{2mn-n+5}{2}, \ldots, \frac{2mn+n-1}{2}\} \\ =& \{n+1, n+2, ..., \frac{3n-1}{2}\} \cup \bigcup_{a=2}^{m} (\bigcup_{b=2}^{n} \{\frac{(2a-1)n+(2b-1)}{2}\}). \end{aligned}$$

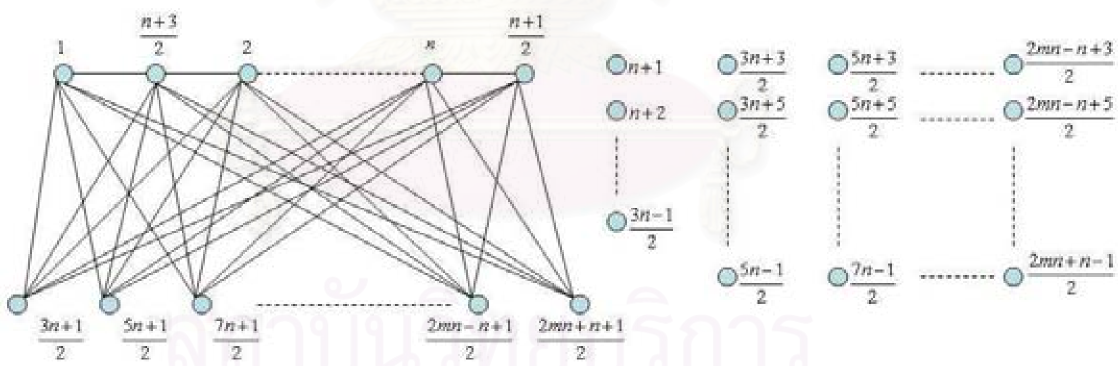


Figure 4.4: A vertex labeling of $(mK_1 \vee P_n) \cup sK_1$ when n is odd.

In order to show that f extends to a super edge-magic labeling of G, it suffices to verify by Theorem 2.1.1:

a) $f(V(G)) = \{1,\ 2,\ 3, \ldots, m+n+s\}$

b) $S = \{f(x) + f(y) : xy \in E(G)\}$ consists of $mn+n-1$ consecutive integers.

It can be verified that $f(V(G)) = \{1,\ 2,\ 3, \ldots, m+n+s\}$.

To show that S consists of $mn+n-1$ consecutive integers, we consider $f(x)+f(y)$ for all edges xy in G.

For edge x_ix_{i+1} :$i=1,3,5,\dots,n-2$,

$f(x_i)+f(x_{i+1})=\dfrac{i+1}{2}+\dfrac{n+i+2}{2}=\dfrac{n+3+2i}{2}$.

For edge x_ix_{i+1} :$i=2,4,6,\dots,n-1$,

$f(x_i)+f(x_{i+1})=\dfrac{n+i+1}{2}+\dfrac{i+2}{2}=\dfrac{n+3+2i}{2}$.

For edge y_jx_i :$i=1,3,5,\dots,n$, $\quad j=1,2,...,m$,

$f(y_j)+f(x_i)=\dfrac{3n+1}{2}+(j-1)n+\dfrac{i+1}{2}=\dfrac{(2j+1)n+i+2}{2}$.

For edge y_jx_i :$i=2,4,6,\dots,n-1$, $\quad j=1,2,...,m$,

$f(y_j)+f(x_i)=\dfrac{3n+1}{2}+(j-1)n+\dfrac{n+1+i}{2}=\dfrac{(2j+2)n+i+2}{2}$.

We note that

$$\begin{aligned}S=&\{f(x)+f(y):xy\in G\}\\ =&\{f(x_i)+f(x_{i+1}):i=1,2,...,n-1\}\cup\bigcup_{j=1}^{m}\{f(y_j)+f(x_i):i=1,3,...,n\}\cup\\ &\bigcup_{j=1}^{m}\{f(y_j)+f(x_i):i=2,4,...,n-1\}\end{aligned}$$

and

$$\begin{aligned}\{f(x_i)+f(x_{i+1}):i=1,2,...,n-1\}=&\{\frac{n+5}{2},\frac{n+7}{2},...,\frac{3n+1}{2}\}\\ \bigcup_{j=1}^{m}\{f(y_j)+f(x_i):i=1,3,...,n\}=&\{\frac{3n+3}{2},\frac{3n+5}{2},...,\frac{4n+2}{2}\}\cup\\ &\{\frac{5n+3}{2},\frac{5n+5}{2},...,\frac{6n+2}{2}\}\cup\cdots\cup\\ &\{\frac{2mn+n+3}{2},\frac{2mn+n+5}{2},...,\frac{2mn+2n+2}{2}\}\\ \bigcup_{j=1}^{m}\{f(y_j)+f(x_i):i=2,4,...,n-1\}=&\{\frac{4n+4}{2},\frac{4n+6}{2},...,\frac{5n+1}{2}\}\cup\\ &\{\frac{6n+4}{2},\frac{6n+6}{2},...,\frac{7n+1}{2}\}\cup\cdots\cup\\ &\{\frac{2mn+2n+4}{2},\frac{2mn+2n+6}{2},...,\frac{2mn+3n+1}{2}\}.\end{aligned}$$

Then $S = \{\frac{n+5}{2}, \frac{n+7}{2}, \frac{n+9}{2}, \ldots \frac{2mn+3n+1}{2}\}$ is a set of $mn+n-1$ consecutive integers. By Theorem 2.1.1, f extends to a super edge-magic labeling of G. Therefore $\mu_s(mK_1 \vee P_n) \leq \frac{(2m-1)(n-1)}{2}$ when n is odd.

Case 2. n is even.

Define a vertex labeling $g : V(G) \to \{1, 2, \ldots m+n+s\}$ by

$$g(u) = \begin{cases} \frac{i+1}{2}, & \text{if } u = x_i, i \text{ is odd;} \\ \frac{n+i}{2}, & \text{if } u = x_i, i \text{ is even;} \\ \frac{3n}{2} + (j-1)n, & \text{if } u = y_j. \end{cases}$$

and

$$\begin{aligned} \{g(w_k) : k = 1, 2, ..., s\} =& \{n+1, n+2, ..., \frac{3n-1}{2}\} \cup \{\frac{3n+2}{2}, \frac{3n+4}{2}, \ldots, \frac{5n-2}{2}\} \cup \\ & \{\frac{5n+2}{2}, \frac{5n+4}{2}, \ldots, \frac{7n-2}{2}\} \cup \{\frac{7n+2}{2}, \frac{7n+4}{2}, \ldots, \frac{9n-2}{2}\} \\ & \cup \cdots \cup \{\frac{2mn-n+2}{2}, \frac{2mn-n+4}{2}, \ldots, \frac{2mn+n-2}{2}\} \\ =& \{n+1, n+2, ..., \frac{3n-2}{2}\} \cup \bigcup_{a=2}^{m} (\bigcup_{b=2}^{n} \{\frac{(2a-1)n+(2b-2)}{2}\}). \end{aligned}$$

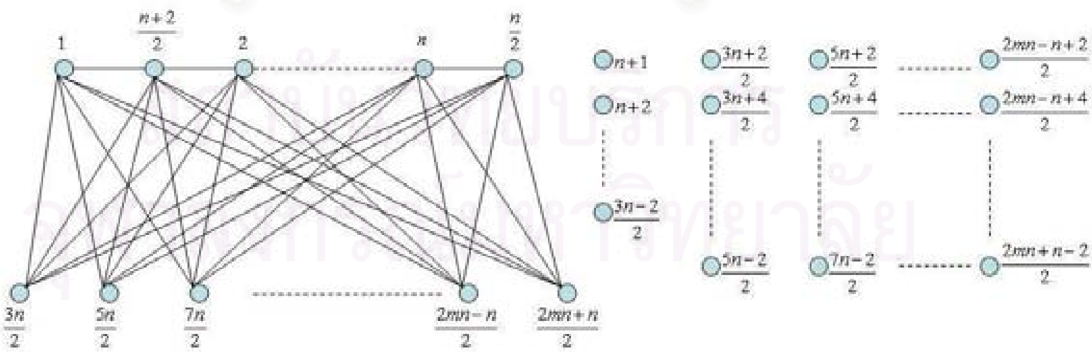


Figure 4.5: A vertex labeling of $(mK_1 \vee P_n) \cup sK_1$ when n is even.

Similarly, we can verify that $g(V(G)) = \{1,\ 2,\ 3, \ldots, m+n+s\}$ and $\{g(x)+g(y) : xy \in G\} = \{\frac{n+4}{2}, \frac{n+6}{2}, \frac{n+8}{2}, \ldots, \frac{2mn+3n}{2}\}$ is a set of

$mn + n - 1$ consecutive integers. By Theorem 2.1.1, g extends to a super edge-magic labeling of G. Therefore $\mu_s(mK_1 \vee P_n) \leq \dfrac{(2m-1)(n-1)-1}{2}$ when n is even. $\square$

Example 4.10. $8 \leq \mu_s(4K_1 \vee P_7) \leq 21$.

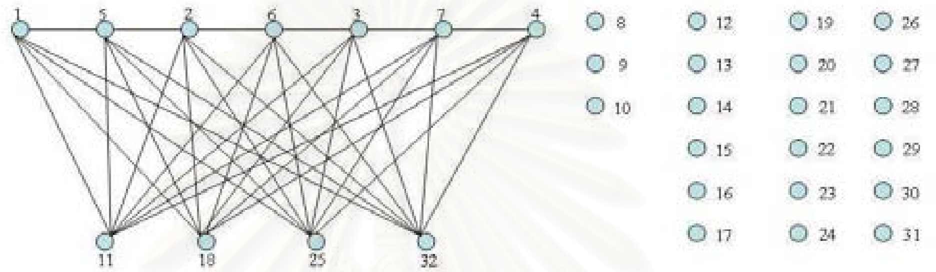


Figure 4.6: A vertex labeling of $(4K_1 \vee P_7) \cup 21K_1$.

Example 4.11. $6 \leq \mu_s(4K_1 \vee P_6) \leq 17$.

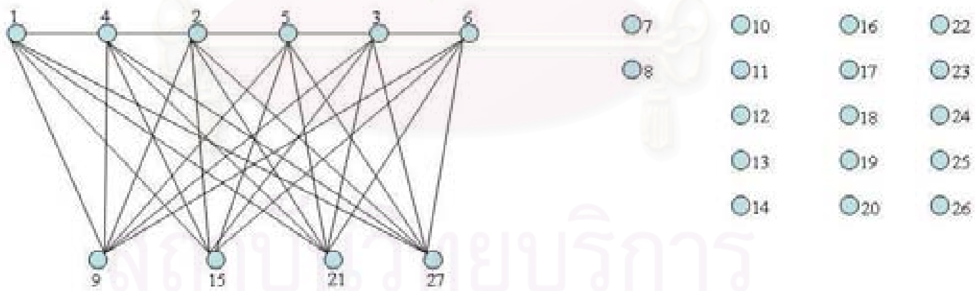


Figure 4.7: A vertex labeling of $(4K_1 \vee P_6) \cup 17K_1$.

We investigate a lower bound and an upper bound for the super edge-magic deficiency of a specific tripartite graph.

Theorem 4.12. *For all integers m, n and $m, n \geq 2$,*

$$\mu_s(K_{m,n,1}) \geq \frac{(m-1)(n-1)}{2}.$$

Proof. Let G be the tripartite graph $K_{m,n,1}$ with

$|V(G)| = m + n + 1$ and $|E(G)| = mn + m + n$.

Thus

$$\begin{aligned} |E(G)| = mn + m + n &= [(m-1)(n-1) + m + n - 1] + m + n \\ &= (m-1)(n-1) + 2m + 2n - 1 > 2m + 2n - 1 = 2(m+n+1) - 3 \\ &= 2|V(G)| - 3. \end{aligned}$$

By Theorem 2.1.3, G is not super edge-magic.

Let k be a positive integer such that $G \cup kK_1$ is super edge-magic.

By Theorem 2.1.3, $|E(G \cup kK_1)| \leq 2|V(G \cup kK_1)| - 3$.

Thus $mn + m + n \leq 2(m + n + 1 + k) - 3$, then $k \geq \dfrac{(m-1)(n-1)}{2}$.

Hence $\mu_s(G) \geq \dfrac{(m-1)(n-1)}{2}$. □

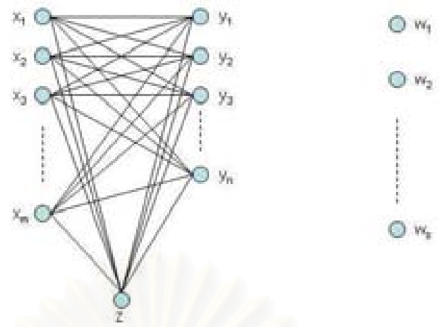
Theorem 4.13. *For all positive integers m, n and $m \geq n$,*

$$\mu_s(K_{m,n,1}) \leq m(n-1).$$

Proof. Let $s = m(n-1)$ and $G \cong K_{m,n,1} \cup sK_1$ be the graph with

$V(G) = \{x_i : 1 \leq i \leq m\} \cup \{y_j : 1 \leq j \leq n\} \cup \{z\} \cup \{w_k : 1 \leq k \leq s\}$ and

$E(G) = \{x_i y_j : 1 \leq i \leq m, 1 \leq j \leq n\} \cup \{z x_i : 1 \leq i \leq m\} \cup \{z y_j : 1 \leq j \leq n\}$.



Define a vertex labeling $f : V(G) \to \{1, 2, \ldots mn + n + 1\}$ by

$$f(u) = \begin{cases} i + 1, & \text{if } u = x_i; \\ (m + 1)j + 1, & \text{if } u = y_j; \\ 1, & \text{if } u = z. \end{cases}$$

and

$$\begin{aligned} \{f(w_k) : k = 1, 2, ..., s\} =& \{m + 3, m + 4, ..., 2m + 2\} \cup \{2m + 4, 2m + 5, \ldots, 3m + 3\} \\ & \cup \cdots \cup \{mn - n + m + 1, mn - n + m + 2, \ldots, mn + n\} \\ =& \bigcup_{a=1}^{n-1} \left(\bigcup_{b=1}^{m} \{am + (a + 1) + b\}\right). \end{aligned}$$

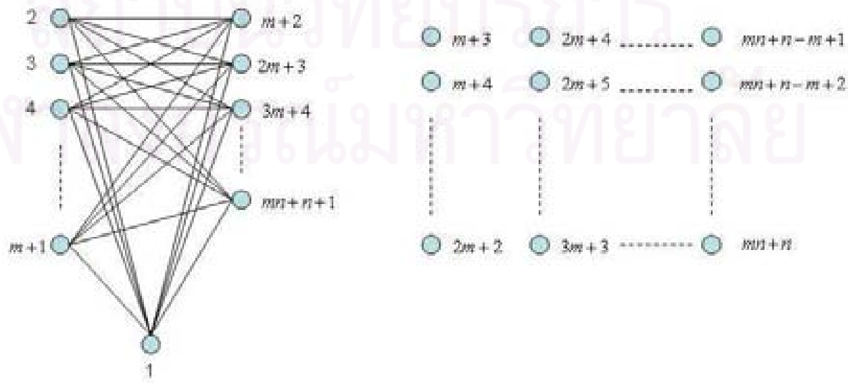


Figure 4.8: A vertex labeling of $K_{m,n,1} \cup sK_1$.

In order to show that f extends to a super edge-magic labeling of G, it suffices to verify by Theorem 2.1.1:

a) $f(V(G)) = \{1,\ 2,\ 3, \ldots, mn+n+1\}$

b) $S = \{f(x)+f(y) : xy \in E(G)\}$ consists of $mn+n+m$ consecutive integers.

To show that $f(V(G)) = \{1,\ 2,\ 3, \ldots, mn+n+1\}$, we consider the labels of vertices as follows:

Vertex z is labeled by numbers 1 and $x_1, x_2, x_3, ..., x_m$ are labeled by numbers $2, 3, 4, ..., m+1$, respectively and $y_1, y_2, y_3, ..., y_n$ are labeled by numbers $m+2, 2m+3, 3m+4, ..., mn+n+1$, respectively and $w_1, w_2, ..., w_s$ are labeled by remaining numbers. Hence $f(V(G)) = \{1,\ 2,\ 3, \ldots, mn+n+1\}$.

To show that S consists of $mn+n+m$ consecutive integers, we consider $f(x)+f(y)$ for all edges xy in G.

For edge zx_i :$i = 1, 2, 3, \ldots, m$,

$f(z)+f(x_i) = 1+(i+1) = i+2.$

For edge zy_j :$i = 1, 2, 3, \ldots, n$,

$f(z)+f(y_j) = 1+(m+1)j+1 = (m+1)j+2.$

For edge x_iy_j :$i = 1, 2, 3, \ldots, m,\quad j = 1, 2, ..., n$,

$f(x_i)+f(y_j) = (i+1)+(m+1)j+1 = (m+1)j+i+2.$

We note that

$$\begin{aligned} S =& \{f(x)+f(y) : xy \in E(G)\} \\ =& \{f(z)+f(x_i) : i = 1, 2, ..., m\} \cup \{f(z)+f(y_j) : j = 1, 2, ..., m\} \cup \\ & \bigcup_{j=1}^{n} \{f(x_i)+f(y_j) : i = 1, 2, ..., m\} \end{aligned}$$

and

$$\begin{aligned}
\{f(z)+f(x_i)\} &= \{3,4,5,...,m+2\} \\
\{f(z)+f(y_j): i=1,2,...,m\} &= \{m+3, 2m+4, 3m+5, ..., mn+n+2\} \\
\bigcup_{j=1}^{n}\{f(x_i)+f(y_j): i=1,2,...,m\} &= \{m+4, m+5, ..., 2m+3\}\cup \\
&\quad \{2m+5, 2m+6, ..., 3m+4\}\cup\cdots\cup \\
&\quad \{mn+n+3, mn+n+4, ..., mn+n+m+2\}.
\end{aligned}$$

Then $S = \{3,4,5,\ldots,mn+n+m+2\}$ is a set of $mn+n+m$ consecutive integers.

By Theorem 2.1.1, f extends to a super edge-magic labeling of G.

Therefore $\mu_s(K_{m,n,1}) \leq m(n-1)$. □

Example 4.14. $1 \leq \mu_s(K_{3,2,1}) \leq 3$.

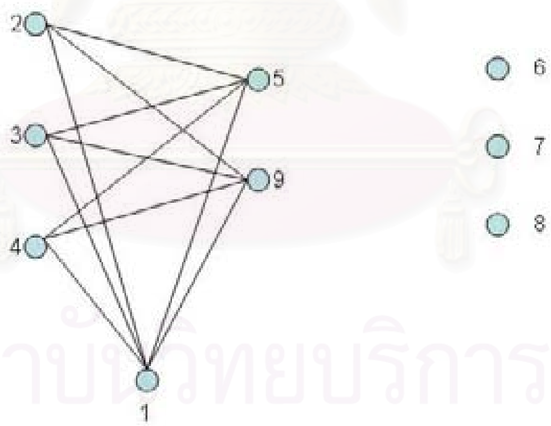


Figure 4.9: A vertex labeling of $K_{3,2,1} \cup 3K_1$.

CHAPTER V

SUPER EDGE-MAGIC REDUNDENCY OF SOME GRAPHS

In contrast with the super edge-magic deficiency of a graph, we define the super edge-magic redundency of a graph as follows.

Definition 5.1. The *super edge-magic redundency* of a graph G, $\eta_s(G)$, is the smallest number of edges which are removed from the graph G and the remaining graph is super edge-magic.

Example 5.2. Since cycle C_4 is not super edge-magic, $\eta_s(G) \geq 1$. Deleting one edge from C_4, the resulting graph is path P_3 which is super edge-magic. Then $\eta_s(G) = 1$.

Figure 5.1: Path P_3 is a super edge-magic subgraph of cycle C_4 with magic constant 11.

Theorem 5.3. *Let G be a $(p,\ q)$-graph. If G contains a super edge-magic spanning subgraph $(p,\ 2p-3)$-graph, then $\eta_s(G) = q - 2p + 3$.*

Proof. Let H be the super edge-magic spanning subgraph with p vertices and $2p-3$ edges. Since $E(H) = 2p-3$, by Theorem 2.1.3, there is no super edge-magic subgraph in G which contains H. Hence $\eta_s(G) = q - 2p + 3$. □

Corollary 5.4. *Let G be a $(p,\, q)$-graph. If G contains the square of path P_p, then $\eta_s(G) = q - 2p + 3$.*

Proof. Since $|E(P_p^2)| = (p-1) + (p-2) = 2p-3$, by Theorem 5.3, $\eta_s(G) = q - 2p + 3$. □

Theorem 5.5. *Let G be a $(p,\, q)$-graph. If G has a Hamiltonian path, then $\eta_s(G) \leqslant q - p + 1$.*

Proof. Let P be Hamiltonian path of G. Since P is a path of p vertices and a path is always super edge-magic, P is super edge-magic subgraph of G. Hence $\eta_s(G) \leqslant q - p + 1$. □

Theorem 5.6. *Let G be a $(p,\, q)$-graph. If G is Hamiltonian and p is odd, then $\eta_s(G) \leqslant q - p$.*

Proof. Since a Hamiltonian cycle in G is a cycle of length p, it is a super edge-magic subgraph of G. Thus $\eta_s(G) \leqslant q - p$. □

Theorem 5.7. [4] *If G is a super edge-magic bipartite or tripartite graph and m is odd, then mG is super edge-magic.*

Theorem 5.8. *If a (p,q)-graph G is bipartite or tripartite graph and $\eta_s(G) = k$ for some positive integer k, then $\eta_s(mG) \leq mk$ for m is odd.*

Proof. Since $\eta_s(G) = k$, G contains a super edge-magic spanning subgraph H with p vertices and $q-k$ edges. Since G is bipartite(or tripartite), H is also bipartite(or tripartite). From Theorem 5.7, mH is super edge-magic. Thus the graph mH is a super edge-magic subgraph of mG. Hence

$\eta_s(mG) \leq |E(mG)| - |E(mH)| = mq - m(q-k) = mk.$ $\square$

Theorem 5.9. [2] *A wheel W_n is not super edge-magic.*

Theorem 5.10. $\eta_s(W_n) = 1$ *when* $1 \leq n \leq 6$.

Proof. By Theorem 5.9, $\eta_s(W_n) \geq 1$. By Table 1, $F_n \cong K_1 \vee P_n$ is super edge-magic when $1 \leq n \leq 6$ and F_n is a subgraph of W_n, thus $\eta_s(W_n) = 1$. $\square$

Theorem 5.11. [5] *The disjoint union of stars $K_{1,m}$ and $K_{1,n}$ is super edge-magic if and only if m is multiple of $n+1$ or n is multiple of $m+1$.*

Lemma 5.12. *The disjoint union of stars $K_{1,m}$ and $K_{1,n}$ and an isolated vertex K_1 is super edge-magic.*

Proof. Let $G \cong K_{1,m} \cup K_{1,n} \cup K_1$ with $V(G = \{v_i : i = 1,2,...,m+n+3\})$ and $E(G) = \{v_2v_i : i = 3,4,5,...,m+2\} \cup \{v_1v_i : i = m+4, m+5, m+6, ..., m+n+3\}$
Define a vertex labeling $f : V(G) \rightarrow \{1,2,...,m+n+3\}$ by $f(v_i) = i$.
It can be verified that $f(V(G)) = \{1,2,...,m+n+3\}$.
For edge v_2v_i, $i = 3,4,...,m+2$,
$f(v_2) + f(v_i) = 2 + i$.
For edge v_1v_i, $i = m+4, m+5, ..., m+n+3$,
$f(v_2) + f(v_i) = 1 + i$.
Then $\{f(x) + f(y) : xy \in E(G)\} = \{5,6,...,m+4\} \cup \{m+5, m+6, ..., m+n+4\}$ is a set of $m+n$ consecutive integers. From Theorem 2.1.1, f extends to a super edge-magic labeling of G. $\square$

Theorem 5.13.

$$\eta_s(K_{1,m} \cup K_{1,n}) = \begin{cases} 0, & \textit{either } m \textit{ is a multiple of } n+1 \textit{ or } n \textit{ is multiple of } m+1; \\ 1, & \textit{otherwise.} \end{cases}$$

Proof. Let G be the disjoint union of stars $K_{1,m}$ and $K_{1,n}$.

If m is a multiple of $n+1$ or n is a multiple of $m+1$, by Theorem 5.11, G is super edge-magic. Thus $\eta_s(G) = 0$.

If m is not a multiple of $n+1$ and n is not a multiple of $m+1$, by Theorem 5.11, G is not super edge-magic. Deleting one leaf from G, the resulting graph is the disjoint union of two star and K_1. By Lemma 5.12, the resulting graph is super edge-magic. Hence $\eta_s(G) = 1$. □

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APPENDIX

Definition 1. A graph G consists of a finite nonempty set $V(G)$ of elements, called vertices, and the set $E(G)$ of 2-elment subsets of $V(G)$, called edges. We call $V(G)$ as the vertex-set of G and $E(G)$ as the edge-set of G. If $\{x, y\}$ is an edge in a graph G, then an edge $\{x, y\}$ joins x and y , or x and y are *adjacent* and are *neighbors*, or an edge $\{x, y\}$ is *incident* with x(or y). We usually write $\{x, y\}$ as xy.

Definition 2. A subgraph of a graph G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A spanning subgraph of a graph G is a subgraph with vertex set $V(G)$.

Definition 3. A u, v*-path* in a graph G is a finite sequence of distinct vertices and edges of the form $u = v_{i_0}, e_{i_1}, v_{i_1}, e_{i_2}, \ldots e_{i_n}, v_{i_n} = v$ where $e_{i_1} = v_{i_0}v_{i_1}, e_{i_2} = v_{i_1}v_{i_2}, \ldots e_{i_n} = v_{i_{n-1}}v_{i_n}$.
The length of a path is its number of edges.

Definition 4. A graph G is *connected* if every pair of vertices is joined by a path and *disconnected* otherwise.

Definition 5. The *degree* of a vertex v in a graph G, denoted by *deg* v, is the number of edges incident with v.

Definition 6. Let G_1 and G_2 be graphs with disjoint vertex-sets $V(G_1)$ and $V(G_2)$ and edge-sets $E(G_1)$ and $E(G_2)$, respectively. The *join* of G_1 and G_2, denoted by $G_1 \vee G_2$, is a graph with the vertex-set $V(G_1) \cup V(G_2)$ and the edge-set $E(G_1) \cup E(G_2)$ and all edges joining vertices in $V(G_1)$ and $V(G_2)$.

Definition 7. A *path* P_n is a simple graph whose vertices can be ordered so that two vertices are adjacent if and only if they are consecutive in the list.

Definition 8. A *cycle* C_n is a graph with an equal number of vertices and edges whose vertices can be place around a circle so that two vertices are adjacent if and only if they appear consecutively along the circle.

Definition 9. The *square* of path P_n^2 with n vertices, $n \geq 3$, is a graph which is obtained from P_n by adding edges that join all vertices u and v if there exists a u, v-path of length 2 in P_n.

Definition 10. A *complete graph* K_n is a graph of n vertices which any two distinct vertices are adjacent.

Definition 11. The *wheel* W_n, $n \geq 3$, is the graph $K_1 \vee C_n$.

Definition 12. The *fan* F_n is the graph $K_1 \vee P_n$.

Definition 13. The *friendship* graph of n triangles, $n \geq 3$, is the graph obtained by taking n copies of the cycle C_3 with a vertex in common.

Definition 14. Let G_1 and G_2 be graphs with disjoint vertex-sets $V(G_1)$ and $V(G_2)$ and edge-sets $E(G_1)$ and $E(G_2)$ respectively. The *product* of G_1 and G_2, denoted by $G_1 \times G_2$, is a graph with the vertex-set $V(G_1) \times V(G_2)$ and specified by putting (u_1, u_2) adjacent to (v_1, v_2) if either $u_1 = v_1$ and $u_2v_2 \in E(G_2)$ or $u_2 = v_2$ and $u_1v_1 \in E(G_1)$.

Definition 15. A *tree* is a connected graph with n vertices and $n - 1$ edges.

Definition 16. A *rooted tree* is a tree with one vertex z chosen as root. For each vertex v, let $P(v)$ be the unique z, r-path. The *parent* of v is its neighbor on $P(v)$; its *children* are its other neighbors.

Definition 17. Let $G_1, G_2, \ldots, G_m$ be graphs with disjoint vertex-sets $V(G_1), V(G_2), \ldots, V(G_m)$ and the edge-sets $E(G_1), E(G_2), \ldots, E(G_m)$ respectively. The *disjoint* *union* of $G_1, G_2, \ldots, G_m$ denoted by $G_1 \cup G_2 \cup \ldots \cup G_m$, is a graph with the vertex-set $V(G_1) \cup V(G_2) \cup \ldots \cup V(G_m)$ and the edge-set $E(G_1) \cup E(G_2) \cup \ldots \cup E(G_m)$

If $G_1 = G_2 = \cdots = G_m = G$ then $G_1, G_2, \ldots, G_m$ is denoted by mG and is called the disjoint union of m copies of G.

Definition 18. The *corona product* $G_1 \odot G_2$ of two graphs G_1 and G_2 defined as the graph obtained by taking one copy of G_1(which has p_1 vertices) and p_1 copies of G_2, and then joining the i-th vertex of G_1 to every vertex of i-copy of G_2.

Definition 19. An *independent set* or *partite set* in a graph is a set of pairwise nonadjacent vertices.

Definition 20. A *complete bipartite graph* $K_{m,n}$ is a graph of $m+n$ vertices which is the union of two disjoint partite sets and two vertices are adjacent if and only if they are in the different partite sets.

Definition 21. A *complete tripartite graph* $K_{m,n,k}$ is a graph of $m+n+k$ vertices which is the union of three disjoint partite sets and two vertices are adjacent if and only if they are in the different partite sets.

Definition 22. A *Hamiltonian graph* is a graph with a spanning cycle.

Definition 23. A *Hamiltonian path* is a spanning path.

VITA

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