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PERFECTION OF GLUED GRAPHS OF PERFECT ORIGINAL GRAPHS

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กราฟ G เป็น *กราฟสมบูรณ์* ก็ต่อเมื่อ ทุกๆ กราฟย่อยชักนำของ G มีรงคเลขและจำนวนคลีกเท่ากัน *กราฟปะติด* คือกราฟที่ได้จากการรวมกราฟสองกราฟที่ไม่มีจุดยอดร่วมกันโดยการปะติดจุดยอดและเส้นเชื่อมของกราฟย่อยเชื่อมโยงที่มีเส้นเชื่อมอย่างน้อยหนึ่งเส้นของทั้งสองกราฟนั้น ซึ่งเรียกกราฟย่อยที่กล่าวมาว่า *กราฟโคลน* และเรียกกราฟสองกราฟที่ไม่มีจุดยอดร่วมกันว่า *กราฟต้นฉบับ*

ผลลัพธ์หลักเกี่ยวข้องกับความสมบูรณ์ของกราฟปะติดเมื่อกราฟต้นฉบับเป็นกราฟสมบูรณ์ เราหาเงื่อนไขจำเป็นและหรือเงื่อนไขเพียงพอสำหรับความสมบูรณ์ของกราฟปะติด นอกจากนั้นเราศึกษารงคเลขและจำนวนคลีกของกราฟปะติดในพจน์ของตัวแปรเหล่านี้ของกราฟต้นฉบับ เราสนใจเฉพาะกราฟโคลนของกราฟปะติด เช่นกราฟย่อยชักนำของกราฟต้นฉบับทั้งสอง หรือกราฟบริบูรณ์ และกราฟต้นฉบับของกราฟปะติด เช่นกราฟสองส่วน กราฟบริบูรณ์ หรือกราฟป่าไม้

ภาควิชา............คณิตศาสตร์..........
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A graph G is *perfect* if the chromatic number and the clique number have the same value for every of its induced subgraph. A *glued graph* results from combining two vertex-disjoint graphs by identifying nontrivial connected isomorphic subgraphs of both graphs. Such subgraphs are referred to as the *clones*. The two vertex-disjoint graphs are referred to the *original graphs*.

The main results involve in the perfection of glued graphs whose original graphs are perfect. We find necessary and/or sufficient conditions for the perfection of glued graphs. We also study the chromatic numbers and the clique numbers of glued graphs in terms of these parameters of their original graphs. Only some specified clones and original graphs are investigated:- clones such as induced subgraphs of both original graphs and complete graphs; original graphs such as bipartite graphs, complete graphs and forests.

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Field of Study : ....Mathematics....		Advisor's Signature : ....
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INTRODUCTION

1.1 Perfect Graphs

A *k-coloring* of a graph G is a labeling $f : V(G) \to S$, where $|S| = k$. The labels are *colors*. A k-coloring is *proper* if adjacent vertices have different colors. The *chromatic number* of graph G, written $\chi(G)$, is the minimum number k such that G has a proper k-coloring. A *clique* of a graph G is a complete subgraph of G. The *clique number* of a graph G, written $\omega(G)$, is the order of the largest clique of G. For any graph G, it is always true that $\chi(G) \geq \omega(G)$, because vertices of a clique need different colors. The gap between the two parameters $\chi(G)$ and $\omega(G)$ can be arbitrarily large. A graph G is *perfect* if $\chi(F) = \omega(F)$ for every induced subgraph F of G, and a graph is *imperfect* if it is not perfect.

Now, we give trivial examples of perfect graphs as follows: complete graph, bipartite graph, chordal graph, interval graph[15], etc. Nontrivial examples of perfect graphs are as follows: bull-free Berge graph[7], planar Berge graph[13], degenerate Berge graph[1], etc. Other examples are compiled by Hougardy[8].

Example 1.1.1. Some imperfect graphs:

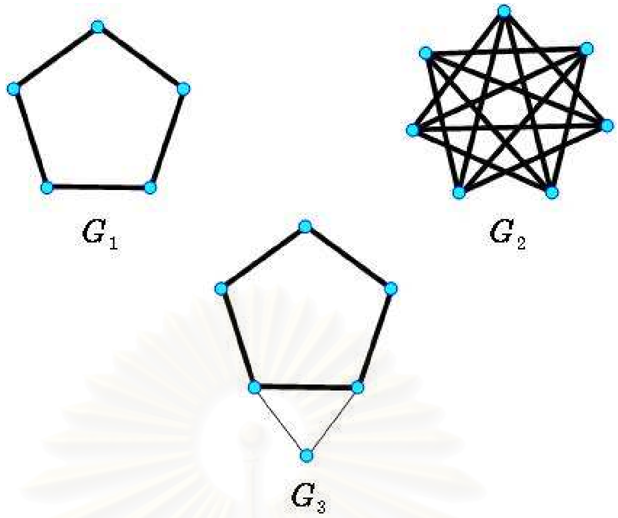

Figure 1.1.1: Some imperfect graphs

In Figure 1.1.1, all graphs are imperfect graphs. We observe that G_1 and G_2 are induced subgraphs of itself such that $\chi(G_1) \neq \omega(G_1)$ and $\chi(G_2) \neq \omega(G_2)$. But G_3 contains a proper induced subgraph whose chromatic number and clique number are unequal, while $\chi(G_3) = \omega(G_3)$. $\square$

Berge[2] defined perfect graphs in 1961. Research on perfect graphs has centered around two questions. The first is: "Which graphs are perfect?", the second is: "Do perfect graphs have special structures which allow us to find the chromatic number and the clique number quickly?". Research into these questions originally centered around the Strong Perfect Graph Conjecture, due to Berge:

A graph G is perfect if and only if

neither G nor $\overline{G}$ contains an odd induced cycle of length at least five.

An *odd hole* of a graph G is an induced subgraph of G which is an odd cycle of length at least 5. An *odd antihole* of a graph G is an induced subgraph of G whose complement is an odd hole of $\overline{G}$. A graph having no odd hole and no odd antihole is called a *Berge graph*. Thus, the Strong Perfect Graph Conjecture is equivalent to the statement that:

A graph is perfect if and only if it is a Berge graph.

Note that C_{2k+1}, $k \geq 2$, is an odd hole of itself. For all $k \geq 2$, we call graphs C_{2k+1} and $\overline{C_{2k+1}}$ an odd hole and an odd antihole, respectively. In Figure 1.1.1, G_1 and G_3 contain odd holes, and G_2 contains an odd antihole which are shown as bold edges. Also, G_1 is an odd hole and G_2 is an odd antihole.

A *minimal imperfect graph* is an imperfect graph whose proper induced subgraphs are all perfect. The Strong Perfect Graph Conjecture is equivalent to the following statement:

A graph is minimal imperfect if and only if
it is an odd hole or an odd antihole.

The Strong Perfect Graph Conjecture has led to the definitions and study of many new classes of graphs for which the correctness of this conjecture has been verified.

A weaker conjecture was proved by Lovász in 1972, named the Perfect Graph Theorem. This theorem only shows that the class of perfect graphs is closed under complementation. So, many authors attempt to find an other characterization of a perfect graph. Recently(2006), Chudnovsky et al.[5] were able to prove the Strong Perfect Graph Conjecture in its full generality. After remaining unsolved for more than 40 years it can now be called the Strong Perfect Graph Theorem.

However, the proof was very long (179 pages), recently(2009), Chudnovsky and Seymour[6] replaced the final 55 pages with a new much shorter proof.

The Perfect Graph Theorem and the Strong Perfect Graph Theorem are our major tools to verify the perfection of our glued graphs. We conclude them here for future references.

Theorem 1.1.2. (**Perfect Graph Theorem**)[9]

A graph is perfect if and only if its complement is perfect.

Theorem 1.1.3. (**Strong Perfect Graph Theorem**)[5]

A graph is perfect if and only if it contains no odd hole and no odd antihole.

1.2 Glued graphs

Let G_1 and G_2 be two nontrivial vertex-disjoint graphs. Let H_1 and H_2 be nontrivial connected subgraphs of G_1 and G_2, respectively, such that $H_1 \cong H_2$ with an isomorphism f, then *the glued graph of G_1 and G_2 at H_1 and H_2 with respect to f*, denoted by $G_1 \underset{H_1 \cong_f H_2}{\triangleleft\!\triangleright} G_2$, is the graph that results from combining G_1 with G_2 by identifying H_1 and H_2 with respect to the isomorphism f between H_1 and H_2. Let H be the copy of H_1 and H_2 in the glued graph $G_1 \underset{H_1 \cong_f H_2}{\triangleleft\!\triangleright} G_2$. We refer to H, H_1 and H_2 as the *clones* of the glued graph, G_1 and G_2, respectively, and refer to G_1 and G_2 as the *original graphs* of the glued graph.

The glued graph of G_1 and G_2 at the clone H, written $G_1 \underset{H}{\triangleleft\!\triangleright} G_2$, means that there exist subgraph H_1 of G_1 and subgraph H_2 of G_2 and isomorphism f between H_1 and H_2 such that $G_1 \underset{H}{\triangleleft\!\triangleright} G_2 = G_1 \underset{H_1 \cong_f H_2}{\triangleleft\!\triangleright} G_2$ and H is the copy of H_1 and H_2 in the resulting graph.

We use $G_1 \triangleleft\!\triangleright G_2$ to denote an arbitrary graph resulting from gluing graphs G_1 and G_2 at any isomorphic subgraph $H_1 \cong H_2$ with respect to any of their isomorphism.

The glue operator is a mathematical operator defined by Uiyyasathian[14] since 2003. In 2006, Promsakon[11] studied colorability of the glued graphs. Bounds of the chromatic numbers of the glued graphs in terms of the chromatic numbers of its original graphs were obtained in [12]. Later, the subject of total colorings of glued graphs was studied by Charoenpanitseri[4] and Pimpasalee[10] studied clique covering of glued graphs. Some results are useful for our study. More details regarding glued graphs can be explored in Promsakon[11].

Throughout the thesis, G_1 and G_2 are nontrivial graphs with disjoint vertex sets and the clone H is a nontrivial connected graph. For a glued graph $G_1 \underset{H_1 \cong_f H_2}{\triangleleft\!\triangleright} G_2$, we use $u \equiv v$ to denote the vertex in a glued graph $G_1 \underset{H_1 \cong_f H_2}{\triangleleft\!\triangleright} G_2$, where $u \in V(H_1)$ and $v \in V(H_2)$ and $f(u) = v$. Moreover, we use the following symbols for convenience:

$G(u_1, u_2, ..., u_n)$	:	a graph G on the vertex set $\{u_1, u_2, ..., u_n\}$;
$\overline{G}(u_1, u_2, ..., u_n)$	:	the complement of $G(u_1, u_2, ..., u_n)$;
$K_n(u_1, u_2, ..., u_n)$	:	a complete graph on the vertex set $\{u_1, u_2, ..., u_n\}$;
$P_n(u_1, u_2, ..., u_n)$	:	a path on the vertex set $\{u_1, u_2, ..., u_n\}$ and the edge set $\{u_1u_2, u_2u_3, ..., u_{n-1}u_n\}$;
$C_n(u_1, u_2, ..., u_n)$	:	a cycle on the vertex set $\{u_1, u_2, ..., u_n\}$ and the edge set $\{u_1u_2, u_2u_3, ..., u_{n-1}u_n, u_nu_1\}$.

For other terminologies and notations, see West[15].

Now, we show examples of different glued graphs even if they have the same pair of original graphs.

Example 1.2.1. Let G_1 and G_2 be graphs as shown in Figure 1.2.1.

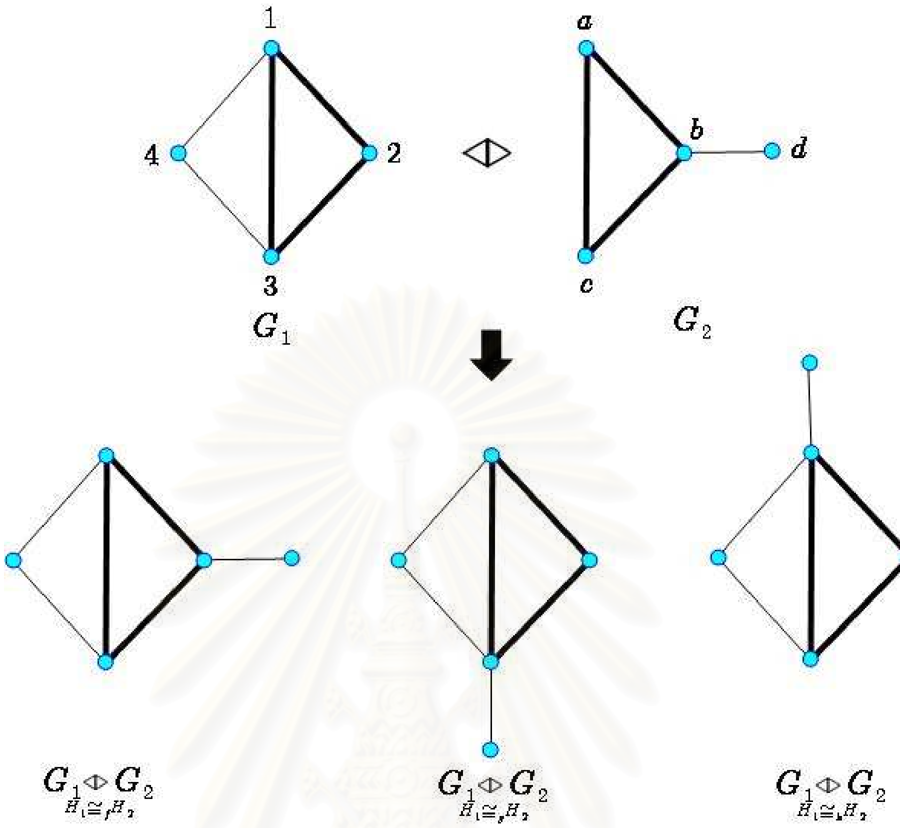

Figure 1.2.1: Glued graphs with different isomorphisms

Let $H_1 = K_3(1, 2, 3) \subseteq G_1$ and $H_2 = K_3(a, b, c) \subseteq G_2$. Consider three isomorphisms f, g and h between H_1 and H_2 as follows:

$$f(1) = a,\ f(2) = b,\ f(3) = c;$$

$$g(1) = c,\ g(2) = a,\ g(3) = b;$$

$$h(1) = b,\ h(2) = c,\ h(3) = a.$$

Glued graphs of G_1 and G_2 with respect to f, g and h are shown in Figure 1.2.1. $\square$

It is possible that a glued graph of simple graphs has multiple edges. This illustrates in Example 1.2.2.

Example 1.2.2. Let G_1 and G_2 be graphs as shown in Figure 1.2.2.

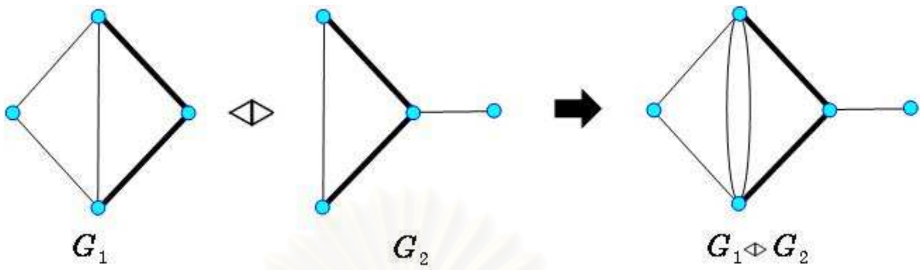

Figure 1.2.2: A glued graph containing multiple edges

However, multiple edges of a graph do not affect both the chromatic number and the clique number of such graph because multiple edges have the same pair of endpoints. Hence, we allow our glued graphs to have multiple edges. $\square$

We first observe some basic properties of glued graphs in the following remark.

Remark 1.2.3.

1. The original graphs are subgraphs of their glued graph.
2. The graph gluing does not create or destroy an edge.
3. A glued graph of disconnected graphs is also disconnected and a glued graph of connected graphs is also connected.
4. If $u \in V(G_1) - V(H)$ and $v \in V(G_2) - V(H)$ where G_1 and G_2 are graphs and H is a clone of $G_1 \underset{H}{\lhd\!\rhd} G_2$, then u and v are not adjacent in $G_1 \underset{H}{\lhd\!\rhd} G_2$.

Next, we introduce definitions of a new edge and a new clique.

Definition 1.2.4. An edge $e = ab$ in any glued graph $G_1 \lhd\!\rhd G_2$ is a **new edge** for an original graph G_i, i=1 or 2 if the corresponding vertices of a and b in G_i are not adjacent. A clique Q in any glued graph $G_1 \lhd\!\rhd G_2$ is a **new clique** for an original graph G_i, i=1 or 2 if the corresponding vertices of $V(Q)$ in G_i do not form a clique.

Example 1.2.5. Let $G_1 \underset{H}{\diamond} G_2$ be the glued graph of original graphs G_1 and G_2 whose clone H is shown as bold edges in Figure 1.2.3.

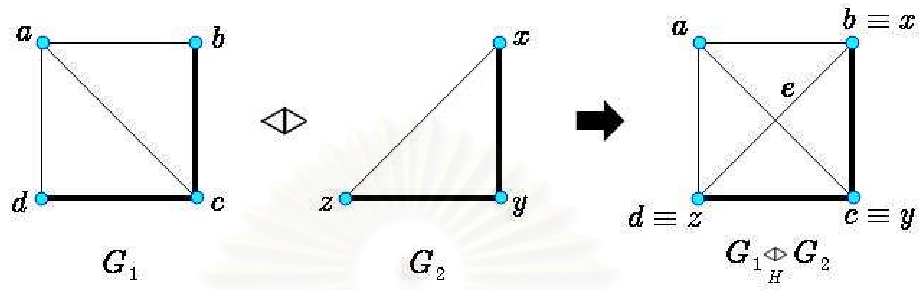

Figure 1.2.3: A glued graph containing new cliques

In Figure 1.2.3. It is easily seen that $G_1 \underset{H}{\diamond} G_2$ contains an edge e but vertices b and d are not adjacent in G_1. Thus, an edge e is a new edge for G_1. Moreover, $K_3(a, b \equiv x, d \equiv z)$ is a clique in $G_1 \underset{H}{\diamond} G_2$ but G_1 does not contain a clique $K_3(a, b, d)$, so $K_3(a, b \equiv x, d \equiv z)$ is a new clique for G_1. Similarly, $K_4(a, b \equiv x, c \equiv y, d \equiv z)$ is a new clique for G_1. □

Example 1.2.5 suggests some basic properties of a new edge and a new clique of a glued graph as shown in the following remark.

Remark 1.2.6.

1. If a glued graph $G_1 \diamond G_2$ has a new clique for G_i, i=1 or 2, then $G_1 \diamond G_2$ has a new edge for G_i.
2. Any new edge of a glued graph cannot be a new edge for both original graphs at the same time.
3. Both endpoints of a new edge of a glued graph must lie in the clone.

1.3 Thesis overview

Our purpose in this thesis is to study the perfection of glued graphs whose original graphs are perfect. It is possible that a glued graph of perfect graphs is imperfect; shown by an example in Section 2.1. We then look for conditions to obtain the perfection of glued graphs whose original graphs are perfect. We separate this study into many chapters as follows:

Chapter 2 contains all possibilities of perfect glued graphs and imperfect glued graphs. Also, the perfection of glued graphs whose one of original graphs is a minimal imperfect graph is investigated.

Chapter 3 gives a condition to guarantee the perfection of a glued graph whose original graphs are perfect. Our main results reveal that the clone of the glued graph must be a complete graph in order to get the desired result. To show such fact, we investigate values of the chromatic numbers and the clique numbers of glued graphs at complete clones in terms of these parameters of their original graphs. Later, we utilize them to prove our main theorem. Beyond that, we introduce the simplicial set elimination ordering as a generalization of the simplicial elimination ordering which is a characterization of chordal graphs.

While Chapter 3 studies the perfection of glued graphs at complete clones for arbitrary original graphs, Chapter 4 studies the perfection of glued graphs at arbitrary clones with some specified original graphs. The specified original graphs are such as bipartite graphs, complete graphs and forests. We also investigate values of the chromatic numbers and the clique numbers of glued graphs at arbitrary clones in terms of these parameters of their original graphs. Later, we utilize them to prove our main theorems.

Finally, we conclude main results of this work and give some open problems for future work in Chapter 5.

CHAPTER II

THE PERFECTION AND THE IMPERFECTION OF GLUED GRAPHS

This chapter contains all possibilities of perfect glued graphs and imperfect glued graphs. Also, the perfection of glued graphs whose one of original graphs is a minimal imperfect graph is investigated.

2.1 The Possibility of Perfect Glued Graphs and Imperfect Glued Graphs

First, we consider all six possibilities of perfect glued graphs and imperfect glued graphs, namely:

$$\left.\begin{array}{llcl} \text{Case 1.} & \text{imperfect} & \triangleleft\!\triangleright & \text{imperfect} \\ \text{Case 2.} & \text{imperfect} & \triangleleft\!\triangleright & \text{perfect} \\ \text{Case 3.} & \text{perfect} & \triangleleft\!\triangleright & \text{perfect} \end{array}\right\} \text{is imperfect.}$$

$$\left.\begin{array}{llcl} \text{Case 4.} & \text{imperfect} & \triangleleft\!\triangleright & \text{imperfect} \\ \text{Case 5.} & \text{imperfect} & \triangleleft\!\triangleright & \text{perfect} \\ \text{Case 6.} & \text{perfect} & \triangleleft\!\triangleright & \text{perfect} \end{array}\right\} \text{is perfect.}$$

Next, we introduce a new definition for convenience:

Definition 2.1.1. For graphs G_1 and G_2, a clone H of a glued graph $G_1 \underset{H}{\triangleleft\!\triangleright} G_2$ is called an **induced clone** of $G_1 \underset{H}{\triangleleft\!\triangleright} G_2$ if H is an induced subgraph of both G_1 and G_2.

If an original graph is imperfect, its glued graph at an induced clone is always imperfect. This can be conclued here:

Proposition 2.1.2. *Let G_1 and G_2 be graphs containing H as their induced subgraph. If G_1 or G_2 is an imperfect graph, then $G_1 \underset{H}{\diamond} G_2$ is imperfect.*

Proof. Let G_1 and G_2 be graphs containing H as their induced subgraph. Assume that G_1 or G_2 is an imperfect graph. Without loss of generality, assume G_1 is an imperfect graph. By the Strong Perfect Graph Theorem, G_1 contains an odd hole or an odd antihole. Since H is an induced subgraph of both G_1 and G_2, it follows that every induced subgraph of G_1 is also an induced subgraph of $G_1 \underset{H}{\diamond} G_2$. Hence, $G_1 \underset{H}{\diamond} G_2$ contains an odd hole or an odd antihole. Therefore, $G_1 \underset{H}{\diamond} G_2$ is imperfect. □

Proposition 2.1.2 is an example of Cases 1 and 2.

Now, we show examples for each remaining possibility. Examples 2.1.3, 2.1.4 and 2.1.5 for Cases 3, 4 and 5, respectively.

Example 2.1.3. Imperfect glued graphs of perfect graphs:

Let $G_1 = C_{2n}$, $G_2 = K_1 \vee P_{2n-2}$ and $H = P_{2n-2}$ where $n \geq 3$. Both G_1 and G_2 are Berge graphs. Observe that $G_1 \underset{H}{\diamond} G_2$ contains C_5 as an induced subgraph, so it is not a Berge graph. By the Strong Perfect Graph Theorem, G_1 and G_2 are perfect but $G_1 \underset{H}{\diamond} G_2$ is imperfect. For $n = 3$, G_1 and G_2 are illustrated in Figure 2.1.1. □

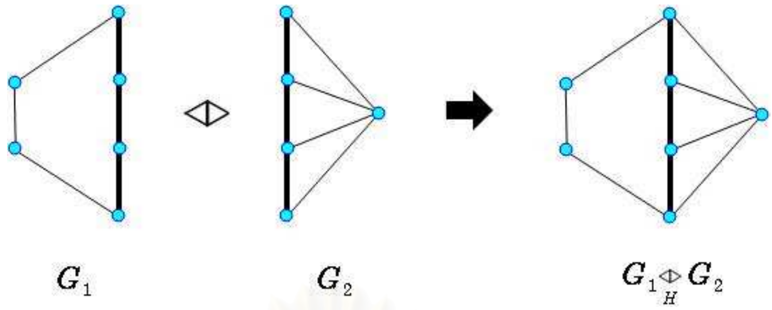

Figure 2.1.1: An imperfect glued graph of perfect graphs

Example 2.1.4. Perfect glued graphs of imperfect graphs:

Let $G_1 = \overline{C_{2n+1}}$, $G_2 = K_1 \vee C_{2n-3}$ and $H = K_{1,2n-3}$ where $n \geq 4$. Since G_1 is an odd antihole and G_2 contains an odd hole, both G_1 and G_2 are not Berge graphs. By the Strong Perfect Graph Theorem, they are imperfect. Observe that $\overline{G_1 \underset{H}{\Diamond} G_2} \cong P_6 + \overline{K_{2n-5}}$ which is perfect. By the Perfect Graph Theorem, we have that $G_1 \underset{H}{\Diamond} G_2$ is perfect. For $n = 4$, G_1 and G_2 are illustrated in Figure 2.1.2. $\square$

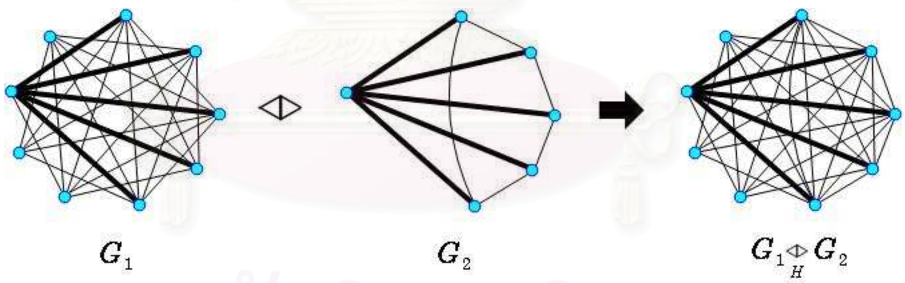

Figure 2.1.2: A perfect glued graph of imperfect graphs

Example 2.1.5. Perfect glued graphs of imperfect graphs and perfect graphs:

Let $G_1 = \overline{C_{2n+1}}$, $G_2 = K_{2n-1}$ and $H = K_{1,2n-2}$ where $n \geq 2$. Then G_1 is not a Berge graph but G_2 is a Berge graph. By the Strong Perfect Graph Theorem, G_1 is imperfect but G_2 is perfect. Observe that $\overline{G_1 \underset{H}{\Diamond} G_2} \cong P_5 + \overline{K_{2n-4}}$ which is perfect, by the Perfect Graph Theorem, $G_1 \underset{H}{\Diamond} G_2$ is perfect. For $n = 3$, G_1 and G_2 are illustrated in Figure 2.1.3. $\square$

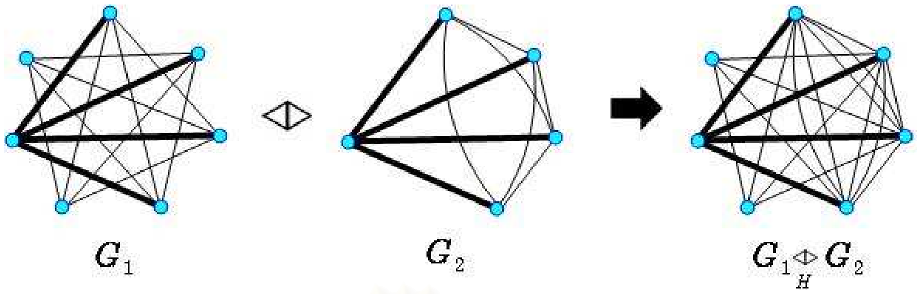

Figure 2.1.3: A perfect glued graph of an imperfect graph and a perfect graph

Example 2.1.3 shows that it is possible that a glued graph of perfect graphs is not perfect. We give conditions for the perfection of glued graphs whose original graphs are perfect in Chapters 3 and 4 where many examples of Case 6 are presented.

2.2 The Perfection of Glued Graphs of Minimal Imperfect Graphs and Perfect Graphs

In Proposition 2.1.2, we know that a glued graph of an imperfect graph and a perfect graph may be imperfect. In this section, we give conditions for the perfection of a glued graph of an imperfect graph and a perfect graph. Since every imperfect graph contains a minimal imperfect graph as an induced subgraph, it is enough to consider a glued graph of a minimal imperfect graph and a perfect graph. We recall the equivalent form of the Strong Perfect Graph Theorem that a graph is minimal imperfect if and only if it is an odd hole or an odd antihole. Then we study only the case that one of original graphs of a glued graph is an odd hole or an odd antihole.

First, we consider a glued graph of an odd hole and any cycle. It is possible that the glued graph is not perfect. We illustrate this in the following example.

Example 2.2.1. Imperfect glued graphs of odd holes and cycles:

Let $G_1 = C_{2n+1}$, $G_2 = C_{2n-2}$ and $H = P_{2n-2}$ where $n \geq 3$. Observe that $G_1 \underset{H}{\diamond} G_2$ contains C_5 as an induced subgraph, so it is not a Berge graph. By the Strong Perfect Graph Theorem, $G_1 \underset{H}{\diamond} G_2$ is not perfect. For $n = 3$, G_1 and G_2 are illustrated in Figure 2.2.1. $\square$

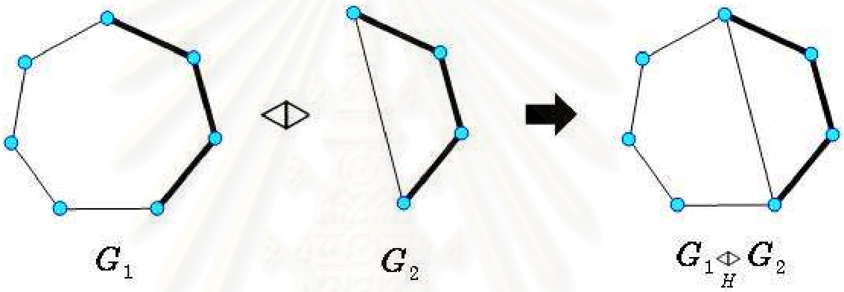

Figure 2.2.1: An imperfect glued graph of an odd hole and a cycle

Now, note that all cycles in G_1 and G_2 are in $G_1 \diamond G_2$. However, it is possible that $G_1 \diamond G_2$ contains a new cycle. We illustrate this in the following example.

Example 2.2.2. Let G_1 and G_2 be graphs in Figure 2.2.2.

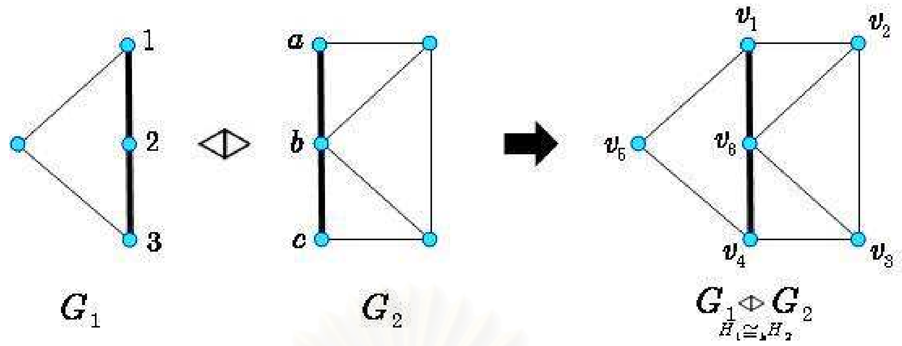

Figure 2.2.2: Created cycles

Let $H_1 = P_3(1, 2, 3) \subseteq G_1$ and $H_2 = P_3(a, b, c) \subseteq G_2$. Define $f : H_1 \to H_2$ by $f(1) = a$, $f(2) = b$ and $f(3) = c$. Then we get that $G_1 \underset{H_1 \cong_f H_2}{\diamondsuit} G_2$ contains an cycle $C_6(v_1, v_2, v_6, v_3, v_4, v_5)$ but $C_6(v_1, v_2, v_6, v_3, v_4, v_5)$ is not a cycle in G_1 and G_2. Also, we get that $G_1 \underset{H_1 \cong_f H_2}{\diamondsuit} G_2$ contains an induced cycle $C_5(v_1, v_2, ..., v_5)$ but $C_5(v_1, v_2, ..., v_5)$ is not a cycle in G_1 and G_2. $\square$

Example 2.2.2 shows that the graph gluing can create a new cycle. We call such new cycles as **created cycles** and all cycles in the original graphs as **original cycles**.

In Theorem 2.2.3, we give a condition for a cycle to obtain the perfection of a glued graph of an odd hole and the cycle. Alike, we characterize a cycle with perfect glued graph whose one of original graphs is an odd hole.

Theorem 2.2.3. *Let n be a positive integer and m an odd positive integer such that $m \geq \max\{5, n\}$. Then*
$C_m \diamondsuit C_n$ is a perfect graph if and only if its clone is P_n, and $n = 3$ or $n = m - 1$.

Proof. Let n be a positive integer and m an odd positive integer such that $m \geq \max\{5, n\}$. Let H be the clone of $C_m \underset{H}{\diamondsuit} C_n$.

Necessity. Assume that $C_m \underset{H}{\diamondsuit} C_n$ is a perfect graph. Since H is a connected subgraph of cycles C_m and C_n, H is a cycle or a path. If H is a cycle, $C_m \underset{H}{\diamondsuit} C_n \cong$

C_m which is not perfect. This is a contradiction. Thus, $H = P_l$ for some $l \le n$. Since C_m is not perfect but $C_m \underset{H}{\Diamond} C_n$ is perfect, $C_m \underset{H}{\Diamond} C_n$ has a new edge for C_m. Thus, the clone H is not an induced subgraph of C_n. Hence, $l = n$. That is, $H = P_n$. Since m is odd, $C_m \underset{H}{\Diamond} C_n$ contains exactly one even induced cycle and one odd induced cycle. But $C_m \underset{H}{\Diamond} C_n$ is perfect, the odd induced cycle has length 3. Since the even cycle or the odd cycle is created, $n = 3$ or $n = m - 1$, respectively.

Sufficiency. Assume that $H = P_n$, and $n = 3$ or $n = m - 1$. Note that $C_m \underset{P_3}{\Diamond} C_3 \cong C_m \underset{P_{m-1}}{\Diamond} C_{m-1}$. Without loss of generality, assume $n = 3$. We observe that $C_m \underset{P_n}{\Diamond} C_n$ contains exactly one even induced cycle and one odd induced cycle. Since $n = 3$, the odd induced cycle has length 3. Thus, these two cycles is neither an odd hole nor an odd antihole with 5 vertices. Since an odd antihole with 7 vertices has more than 2 cycles but $C_m \underset{P_n}{\Diamond} C_n$ contains exactly two cycles, an odd antihole with 7 vertices is not contained in $C_m \underset{P_n}{\Diamond} C_n$. Thus, $C_m \underset{P_n}{\Diamond} C_n$ contains no odd hole and no odd antihole. By the Strong Perfect Graph Theorem, $C_m \underset{P_n}{\Diamond} C_n$ is perfect. □

Next, we consider a glued graph of an odd hole and any perfect graph. It is possible that the glued graph is not perfect. We illustrate this in the following example.

Example 2.2.4. Imperfect glued graphs of odd holes and perfect graphs:

Let $G_1 = C_{2n+1}$, $G_2 = K_4 - e$ and $H = P_4$ where $n \ge 3$. Then G_2 is a perfect graph. Observe that $G_1 \underset{H}{\Diamond} G_2$ contains C_{2n-1} as an induced subgraph, so it is not a Berge graph. By the Strong Perfect Graph Theorem, $G_1 \underset{H}{\Diamond} G_2$ is not perfect. For $n = 3$, G_1 and G_2 are illustrated in Figure 2.2.3. □

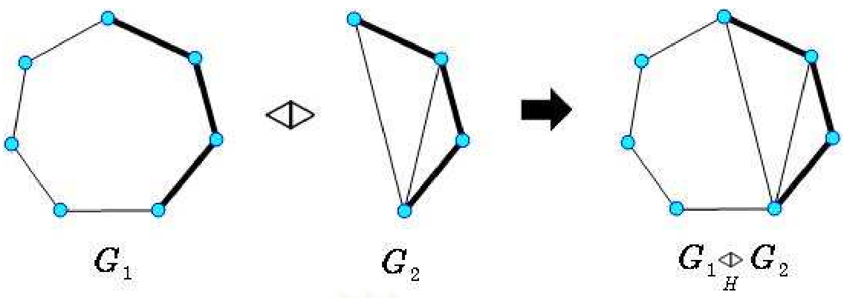

Figure 2.2.3: An imperfect glued graph of an odd hole and a perfect graph

In Theorem 2.2.5, we give a condition for a perfect graph to obtain the perfection of a glued graph of an odd hole and the perfect graph.

Theorem 2.2.5. *Let G be a perfect graph and m an odd positive integer such that $m \geq 5$. If G contains a cycle C_n, where n is odd or $n = m-1$, then $C_m \underset{C_n - e}{\diamond} G$ is perfect.*

Proof. Let G be a perfect graph and m an odd positive integer such that $m \geq 5$. Assume that G contains a cycle C_n, where n is odd or $n = m-1$. We observe that $C_m \underset{C_n - e}{\diamond} G$ has only one created induced cycle, namely C_{m-n+2}. Since m is odd and n is odd or $m-1$, $m-n+2$ is even or 3, respectively. Thus, C_{m-n+2} is not an odd hole. Since G is perfect, G contains no odd hole and no odd antihole. Besides, vertices in $V(G)$ and $V(C_m) - V(H)$ are not adjacent, $C_m \underset{C_n - e}{\diamond} G$ contains no odd hole and no odd antihole. By the Strong Perfect Graph Theorem, $C_m \underset{C_n - e}{\diamond} G$ is perfect. $\square$

Next, we recall a definition of a hamiltonian graph.

Definition 2.2.6. A **Hamiltonian graph** G is a graph containing a cycle with the vertex set $V(G)$.

Corollary 2.2.7. *Let G_n be the n-vertex hamiltonian perfect graph and m an odd positive integer such that $m \geq \max\{5, n\}$. Then $C_m \underset{P_n}{\diamond} G_n$ is a perfect graph if and only if n is odd or $n = m-1$.*

Proof. Let G_n be the n-vertex hamiltonian perfect graph and m an odd positive integer such that $m \geq \max\{5, n\}$.

Necessity. Assume that $C_m \underset{P_n}{\Leftrightarrow} G_n$ is a perfect graph. $C_m \underset{P_n}{\Leftrightarrow} G_n$ has only one created induced cycle, namely C_{m-n+2}. Since C_{m-n+2} is not an odd hole, $m-n+2$ is even or $m-n+2=3$. Since m is odd, n is odd or $n = m-1$.

Sufficiency follows from Theorem 2.2.5. □

Next, we consider a glued graph of an odd antihole and any perfect graph. It is possible that the glued graph is not perfect. We illustrate this in the following example.

Example 2.2.8. Imperfect glued graphs of odd antiholes and perfect graphs:

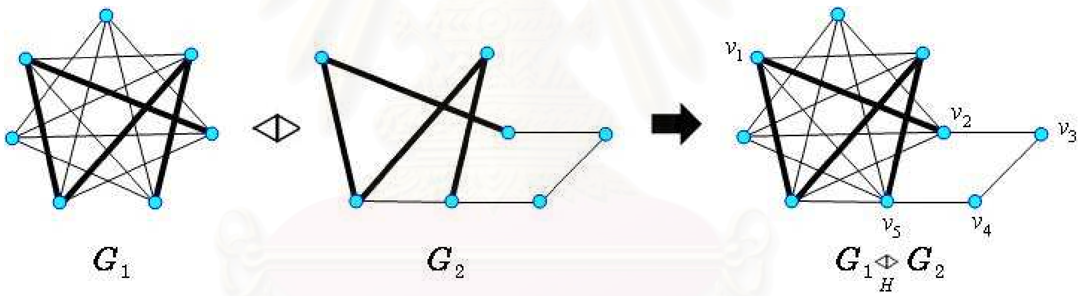

Figure 2.2.4: An imperfect glued graph of an odd antihole and a perfect graph

It is easily seen in Figure 2.2.4 that G_1 is an odd antihole and G_2 is a perfect graph. Observe that $G_1 \underset{H}{\Leftrightarrow} G_2$ contains $C_5(v_1, v_2, ..., v_5)$ as an induced subgraph, so it is not a Berge graph. By the Strong Perfect Graph Theorem, $G_1 \underset{H}{\Leftrightarrow} G_2$ is not perfect. □

Theorem 2.2.9. *Let G be a perfect graph and m an odd positive integer such that $m \geq 5$. If $\overline{C_m} \Leftrightarrow G$ is a perfect graph, then its clone contains P_n and G contains a cycle $P_n + uv$, where $n \leq m$ and $uv \in E(C_m)$.*

Proof. Let G be a perfect graph and m an odd positive integer such that $m \geq 5$. Let H be the clone of $\overline{C_m} \underset{H}{\Diamond} G$. Assume that $\overline{C_m} \underset{H}{\Diamond} G$ is a perfect graph. Then $\overline{C_m} \underset{H}{\Diamond} G$ contains no odd antihole, so $\overline{C_m} \underset{H}{\Diamond} G$ has a new edge for $\overline{C_m}$, say uv. Thus, $uv \in E(C_m)$. Note that u, v must lie in the clone H, and u is not adjacent to v in $\overline{C_m}$ but u is adjacent to v in G. Since H is connected, there is an u, v-path in $H \subseteq \overline{C_m}$. That is, H contains P_n, where $n \leq m$. Since $H \subseteq G$, G contains P_n. Together with the edge uv in G, we have G contains a cycle $P_n + uv$. $\square$

The converse of Theorem 2.2.9 is not true. This is confirmed by Example 2.2.8 because H contains P_3 and G_2 contains $P_3 + e$, but $G_1 \underset{H}{\Diamond} G_2$ is not perfect.

CHAPTER III

THE PERFECTION OF GLUED GRAPHS AT COMPLETE CLONES

This chapter gives a condition to guarantee the perfection of a glued graph whose original graphs are perfect. Our main results reveal that the clone of the glued graph must be a complete graph in order to get the desired result. To show such fact, we investigate values of the chromatic numbers and the clique numbers of glued graphs at complete clones in terms of these parameters of their original graphs. Later, we utilize them to prove our main theorem. Beyond that, we introduce the simplicial set elimination ordering as a generalization of the simplicial elimination ordering which is a characterization of chordal graphs.

3.1 The Perfection of Glued Graphs at Induced Clones

In this section, we give remarks and results about a glued graph whose clone is an induced subgraph of both original graphs.

Remark 3.1.1. For graphs G_1 and G_2, if the clone of $G_1 \lhd\!\rhd G_2$ is an induced subgraph of G_1 and G_2, then

1. $G_1 \lhd\!\rhd G_2$ has no new edge for any original graphs and

2. $G_1 \lhd\!\rhd G_2$ has no new clique for any original graphs.

Proof. Let G_1 and G_2 be graphs. Let H be the clone of $G_1 \underset{H}{\lhd\!\rhd} G_2$ which is an induced subgraph of G_1 and G_2. By Remark 1.2.6(3), The endpoints of a new edge must lie in the clone H but H is an induced subgraph of G_1 and G_2. Hence $G_1 \underset{H}{\lhd\!\rhd} G_2$ has no new edge for any original graphs. By Remark 1.2.6(1), $G_1 \underset{H}{\lhd\!\rhd} G_2$ has no new clique for any original graphs. □

The lower bounds of the chromatic numbers and the clique numbers of glued graphs at arbitrary clones in terms of these parameters of their original graphs are shown in the following remark.

Remark 3.1.2. For graphs G_1 and G_2, we have

1. $\chi(G_1 \lhd\!\rhd G_2) \geq \max\{\chi(G_1), \chi(G_2)\}$ and
2. $\omega(G_1 \lhd\!\rhd G_2) \geq \max\{\omega(G_1), \omega(G_2)\}$.

This holds because G_1 and G_2 are subgraphs of $G_1 \lhd\!\rhd G_2$.

In general, $\chi(G_1 \lhd\!\rhd G_2) \leq \chi(G_1)\chi(G_2)$. This upper bound was shown in [12] along with its sharpness.

Unlike the chromatic number, we have not had an upper bound of the clique number of glued graphs in terms of the clique numbers of their original graphs. Promsakon conjectured in [11] that $\omega(G_1 \lhd\!\rhd G_2) \leq \omega(G_1)\omega(G_2)$.

Theorem 3.1.3. *For graphs G_1 and G_2, if the clone of a glued graph $G_1 \lhd\!\rhd G_2$ is an induced subgraph of both G_1 and G_2, then $\omega(G_1 \lhd\!\rhd G_2) = \max\{\omega(G_1), \omega(G_2)\}$.*

Proof. Let G_1 and G_2 be graphs. Let H be the clone of $G_1 \underset{H}{\lhd\!\rhd} G_2$ which is an induced subgraph of G_1 and G_2. By Remark 3.1.2(2), we have $\omega(G_1 \underset{H}{\lhd\!\rhd} G_2) \geq \max\{\omega(G_1), \omega(G_2)\}$. Since H is an induced subgraph of G_1 and G_2, by Remark 3.1.1(2), $G_1 \underset{H}{\lhd\!\rhd} G_2$ has no new clique for any original graphs. Hence, $\omega(G_1 \underset{H}{\lhd\!\rhd} G_2) = \max\{\omega(G_1), \omega(G_2)\}$. □

For a bipartite graph G_1 and a nontrivial graph G_2, we have $\chi(G_1) = 2$, $\chi(G_2) \geq 2$ and $\chi(G_1 \lhd\!\rhd G_2) \geq \max\{\chi(G_1), \chi(G_2)\}$.

It is possible that $\chi(G_1 \Diamond G_2) > \max\{\chi(G_1), \chi(G_2)\}$ even if $\chi(G_1) = 2$. We illustrate this in the following example.

Example 3.1.4. Let G_1 and G_2 be graphs as shown in Figure 3.1.1.

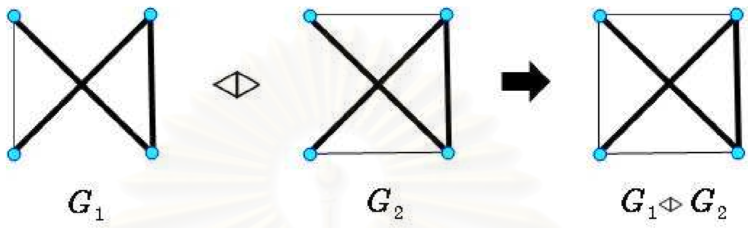

Figure 3.1.1: A glued graph of a bipartite graph G_1 and a graph G_2 such that $\chi(G_1 \Diamond G_2) > \max\{\chi(G_1), \chi(G_2)\}$.

It is easily seen that G_1 is a bipartite graph that is $\chi(G_1) = 2$ and G_2 is a graph such that $\chi(G_1) = 3$. But $\chi(G_1 \Diamond G_2) = 4 > 3 = \max\{\chi(G_1), \chi(G_2)\}$. $\square$

Theorem 3.1.5 shows that the chromatic number of a glued graph of a bipartite graph and any graph at an induced clone does not exceed the chromatic number of its original graphs.

Theorem 3.1.5. *For graphs G_1 and G_2, if G_1 or G_2 is a bipartite graph and the clone of a glued graph $G_1 \Diamond G_2$ is an induced subgraph of both G_1 and G_2, then* $\chi(G_1 \Diamond G_2) = \max\{\chi(G_1), \chi(G_2)\}$.

Proof. Let G_1 and G_2 be graphs. Let H be the clone of $G_1 \underset{H}{\Diamond} G_2$ which is an induced subgraph of both G_1 and G_2. Assume that G_1 or G_2 is a bipartite graph. Without loss of generality, assume G_2 is a bipartite graph. There are proper colorings $f : V(G_1) \rightarrow S_1$ and $g : V(G_2) \rightarrow S_2$ of G_1 and G_2, respectively, where $|S_1| = \chi(G_1) \geq 2$, $|S_2| = \chi(G_2) = 2$ and $S_1 \cap S_2 = \varnothing$. By Remark 3.1.2, $\chi(G_1 \underset{H}{\Diamond} G_2) \geq \max\{\chi(G_1), \chi(G_2)\} = \chi(G_1)$. It suffices to show that $\chi(G_1 \underset{H}{\Diamond} G_2) \leq \chi(G_1)$, that is $G_1 \underset{H}{\Diamond} G_2$ has a proper $\chi(G_1)$-coloring. Let a_1b_1 and a_2b_2 be edges in the copies of H in G_1 and G_2, respectively, such that $a_1 \equiv a_2$ and $b_1 \equiv b_2$. Let $h : V(G_1 \underset{H}{\triangleleft\!\triangleright} G_2) \to S_1$ be defined by

$$h(v) = \begin{cases} f(v) & \text{, if } v \in V(G_1); \\ f(a_1) & \text{, if } v \in V(G_2) - V(H) \text{ and } g(v) = g(a_2); \\ f(b_1) & \text{, if } v \in V(G_2) - V(H) \text{ and } g(v) = g(b_2). \end{cases}$$

To show that h is proper, let u and v be vertices in $G_1 \underset{H}{\triangleleft\!\triangleright} G_2$ such that u and v are adjacent. Since H is an induced subgraph of both G_1 and G_2, G_1 and G_2 are induced subgraphs of $G_1 \underset{H}{\triangleleft\!\triangleright} G_2$. If $u, v \in V(G_1)$, then u and v are adjacent in G_1, so $h(u) = f(u) \neq f(v) = h(v)$. Assume that $u, v \in V(G_2)$. Then u and v are adjacent in G_2. Without loss of generality, assume $g(u) = g(a_2)$ and $g(v) = g(b_2)$. Thus, $h(u) = f(a_1) \neq f(b_1) = h(v)$. Besides, vertices in $V(G_1) - V(H)$ and $V(G_2) - V(H)$ are not adjacent. Thus, h is proper. Hence, $\chi(G_1 \underset{H}{\triangleleft\!\triangleright} G_2) \leq \chi(G_1)$. □

Next, we study the perfection of glued graphs at induced clones.

Proposition 3.1.6. *Let G_1 and G_2 be graphs containing H as their induced subgraph. If $G_1 \underset{H}{\triangleleft\!\triangleright} G_2$ is a perfect graph, then both G_1 and G_2 are perfect.*

Proof. This follows from Proposition 2.1.2. □

The converse of Proposition 3.1.6 is not true. Namely, if H is not a complete graph, one can find perfect graphs G_1 and G_2 containing H as their induced subgraph while $G_1 \underset{H}{\triangleleft\!\triangleright} G_2$ is not perfect.

Theorem 3.1.7. *Let H be a connected incomplete graph. If H is a perfect graph, then there exist perfect graphs G_1 and G_2 containing H as their induced subgraph such that $G_1 \underset{H}{\triangleleft\!\triangleright} G_2$ is not perfect.*

Proof. Let H be a connected incomplete graph. Assume that H is a perfect graph. Let $|V(H)| = r$. Let $H_1(u_1, u_2, ..., u_r)$ and $H_2(v_1, v_2, ..., v_r)$ be the copies of H with an isomorphism $f : V(H_1) \to V(H_2)$ which is defined by $f(u_i) = v_i$ for all $i \in \{1, 2, ..., r\}$. Let $P_l(u_1, u_2, ..., u_l)$ and $P_l(v_1, v_2, ..., v_l)$ be the longest induced paths of H_1 and H_2, respectively. Since H_1 and H_2 are not complete graphs, $l \geq 3$. Choose $G_1 = H_1 \vee K_1(z)$; a join graph between H_1 and a new vertex z, and choose $G_2 = (H_2 \vee K_2(x, y)) - \{xv_l, yv_1\}$, where x and y are distinct new vertices. Then G_1 and G_2 are perfect. Consider $G_1 \underset{H_1 \cong_f H_2}{\Diamond} G_2$, it is easily seen that the corresponding vertices of v_1, x, y, v_l, z in $G_1 \underset{H_1 \cong_f H_2}{\Diamond} G_2$ form an induced cycle C_5. By the Strong Perfect Graph Theorem, $G_1 \underset{H_1 \cong_f H_2}{\Diamond} G_2$ is not perfect. □

Example 3.1.8. We illustrate a glued graph in Theorem 3.1.7. We construct 4-vertex graphs $H_1(u_1, u_2, u_3, u_4)$ and $H_2(v_1, v_2, v_3, v_4)$ having the longest induced paths $P_4(u_1, u_2, u_3, u_4)$ and $P_4(v_1, v_2, v_3, v_4)$, respectively. Choose G_1 to be $H_1 \vee K_1(z)$ and G_2 to be $H_2 \vee K_2(x, y) - \{xv_4, yv_1\}$. Define $f : H_1 \to H_2$ by $f(u_i) = v_i$ for all $i \in \{1, 2, ..., 4\}$. Then $G_1 \underset{H_1 \cong_f H_2}{\Diamond} G_2$ contains an induced cycle $C_5(w_1, x', y', w_4, z')$. Hence, $G_1 \underset{H_1 \cong_f H_2}{\Diamond} G_2$ is not perfect. We have G_1 and G_2 as graphs in Figure 3.1.2.

□

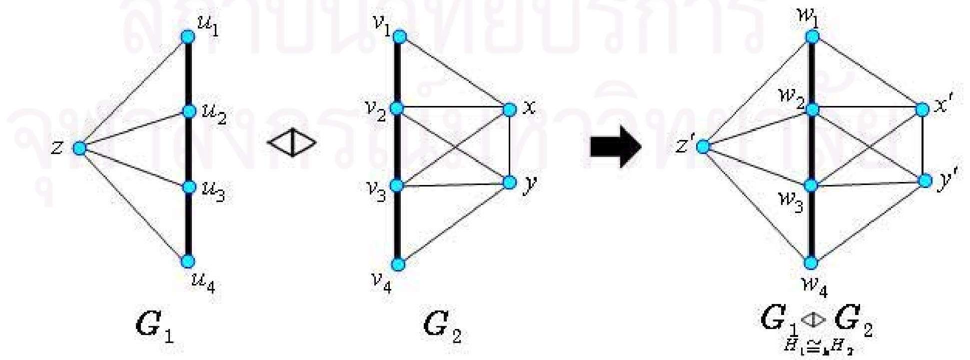

Figure 3.1.2: An imperfect glued graph of perfect graphs at an induced clone

3.2 The Perfection of Glued Graphs at Complete Clones

First, we introduce a definition for convenience:

Definition 3.2.1. For graphs G_1 and G_2, a clone H of a glued graph $G_1 \underset{H}{\lhd\!\rhd} G_2$ is called a **complete clone** of $G_1 \underset{H}{\lhd\!\rhd} G_2$ if H is a complete graph; otherwise, it is called an **incomplete clone**.

Note that a glued graphs of perfect graphs at an induced clone may be imperfect. We know that a complete clone of any glued graph is an induced subgraph of both original graphs. Now, we consider glued graphs of perfect original graphs at complete clones.

When the clone is a complete graph, the chromatic numbers of glued graphs do not exceed the chromatic numbers of their original graphs, see Lemma 3.2.2

For a positive integer r, a *glued graph at a complete clone*, $G_1 \underset{K_r}{\lhd\!\rhd} G_2$, denotes an arbitrary glued graph of graphs G_1 and G_2 at any clone which is isomorphic to K_r.

Throughout the rest of the thesis, K_r in our proofs always means the clone of the glued graph $G_1 \underset{K_r}{\lhd\!\rhd} G_2$, not arbitrary subgraph K_r in the glued graph.

Lemma 3.2.2. *For graphs G_1 and G_2, $\chi(G_1 \underset{K_r}{\lhd\!\rhd} G_2) = \max\{\chi(G_1), \chi(G_2)\}$.*

Proof. Let G_1 and G_2 be graphs. Let $\chi(G_1) = m$ and $\chi(G_2) = n$. Assume $m \geq n$. By Remark 3.1.2(1), we have $\chi(G_1 \underset{K_r}{\lhd\!\rhd} G_2) \geq \max\{\chi(G_1), \chi(G_2)\} = m$. It suffices to show that $\chi(G_1 \underset{K_r}{\lhd\!\rhd} G_2) \leq m$, that is $G_1 \underset{K_r}{\lhd\!\rhd} G_2$ has a proper m-coloring. Let f be an m-coloring of G_1 with colors $a_1, a_2, ..., a_m$ and g an n-coloring with colors $b_1, b_2, ..., b_n$. Note that any pair of vertices in K_r must have different colors. Without loss of generality, for $i \in \{1, 2, ..., r\}$, let a_i and b_i be colors of the corresponding vertices of K_r in G_1 and G_2, respectively. Let $h : V(G_1 \underset{K_r}{\lhd\!\rhd} G_2) \to \{a_1, a_2, ..., a_m\}$ defined by

$$h(v) = \begin{cases} f(v) & \text{, if } v \in V(G_1); \\ a_i & \text{, if } v \in V(G_2) - V(K_r) \text{ and } g(v) = b_i. \end{cases}$$

Since the clone is a complete graph, h is well-defined. To show that h is proper, let u and v be vertices in $G_1 \underset{K_r}{\oplus} G_2$ such that u and v are adjacent. If $u, v \in V(G_1)$, then $h(u) = f(u) \neq f(v) = h(v)$. If $u, v \in V(G_2)$, then $g(u) = b_i$ and $g(v) = b_j$ for some $i \neq j$, so $h(u) = a_i \neq a_j = h(v)$. Besides, vertices in $V(G_1) - V(K_r)$ and $V(G_2) - V(K_r)$ are not adjacent. Hence, h is proper. That is, $G_1 \underset{K_r}{\oplus} G_2$ has a proper m-coloring. Therefore, $\chi(G_1 \underset{K_r}{\oplus} G_2) \leq m$. $\square$

Lemma 3.2.3. *For graphs G_1 and G_2, $\omega(G_1 \underset{K_r}{\oplus} G_2) = \max\{\omega(G_1), \omega(G_2)\}$.*

Proof. This follows from Theorem 3.1.3. $\square$

The condition in Lemmas 3.2.2 and 3.2.3 that the clone must be a complete graph is necessary. This is confirmed by Theorem 3.2.5.

Theorem 3.2.4. [3] *For every graph G, $\chi(G) \leq \Delta(G) + 1$. If G is a connected graph other than a complete graph or an odd cycle, then $\chi(G) \leq \Delta(G)$.*

Theorem 3.2.5. *Let H be a connected graph. If H is not a complete graph, then there exist graphs G_1 and G_2 such that $\chi(G_1 \underset{H}{\oplus} G_2) > \max\{\chi(G_1), \chi(G_2)\}$ and $\omega(G_1 \underset{H}{\oplus} G_2) > \max\{\omega(G_1), \omega(G_2)\}$.*

Proof. Let H be a connected graph. Assume that H is not a complete graph. Let $|V(H)| = r$, so $r \geq 3$. Choose $G_1 = K_r$ and choose $G_2 = H \vee K_1$. Then $G_1 \underset{H}{\oplus} G_2 = K_{r+1}$. If H is an odd cycle of length at least 5, $\chi(H) = 3 < r - 1$. Otherwise, H is not an odd cycle of length at least 5, by Theorem 3.2.4, $\chi(H) \leq \Delta(H) \leq r - 1$. Now, we have $\chi(H) \leq r - 1$, so $\chi(G_2) = \chi(H \vee K_1) \leq (r-1) + 1 = r$. Hence, $\chi(G_1 \underset{H}{\oplus} G_2) = r + 1 > r = \max\{\chi(G_1), \chi(G_2)\}$. Since H is not a complete graph and $|V(H)| = r$, we get $\omega(H) \le r-1$. So, $\omega(G_2) = \omega(H \vee K_1) \le (r-1)+1 = r$. Therefore, $\omega(G_1 \underset{H}{\Diamond} G_2) = r+1 > r = \max\{\omega(G_1), \omega(G_2)\}$. $\square$

Example 3.2.6. We illustrate an example of graphs in Theorem 3.2.5. We construct a 4-vertex graph H, choose G_1 to be K_4 and choose G_2 to be $H \vee K_1$. Observe that $G_1 \underset{H}{\Diamond} G_2 \cong K_5$ such that $\chi(G_1 \underset{H}{\Diamond} G_2) = 5 > 4 = \max\{\chi(G_1), \chi(G_2)\}$ and $\omega(G_1 \underset{H}{\Diamond} G_2) = 5 > 4 = \max\{\omega(G_1), \omega(G_2)\}$. we have G_1 and G_2 as graphs in Figure 3.2.1.

$\square$

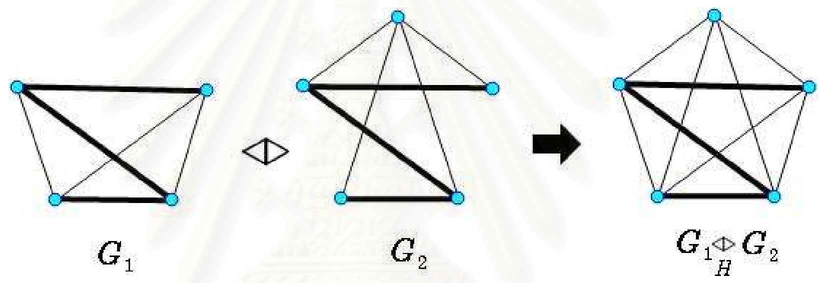

Figure 3.2.1: A glued graph with
$\chi(G_1 \underset{H}{\Diamond} G_2) > \max\{\chi(G_1), \chi(G_2)\}$ and $\omega(G_1 \underset{H}{\Diamond} G_2) > \max\{\omega(G_1), \omega(G_2)\}$

To show the perfection of a graph G, if an induced subgraph F of G is disconnected, we consider the perfection of each component of F. Thus, throughout the thesis, it suffices to consider the perfection of every connected induced subgraph of G.

For graphs G_1 and G_2, $G_1 \cap G_2$ denotes the graph on the vertex set $V(G_1) \cap V(G_2)$ and the edge set $E(G_1) \cap E(G_2)$.

Lemma 3.2.7. *Let G_1 and G_2 be perfect graphs containing H as a nontrivial connected subgraph. Let F be a connected induced subgraph of $G_1 \underset{H}{\Diamond} G_2$. If F has at most one vertex in H, then $\chi(F) = \omega(F)$.*

Proof. Let G_1 and G_2 be perfect graphs containing H as a nontrivial connected subgraph. Let F be a connected induced subgraph of $G_1 \underset{H}{\diamond} G_2$. If F has no vertex in H, then F is an induced subgraph of either G_1 or G_2. Since G_1 and G_2 are perfect, $\chi(F) = \omega(F)$. Otherwise, F has exactly one vertex in H. Let $F_1 = F \cap G_1$ and $F_2 = F \cap G_2$. Since F_1 is an induced subgraph of a perfect graph G_1, $\chi(F_1) = \omega(F_1)$. Similarly, $\chi(F_2) = \omega(F_2)$. We have that F is the union of F_1 and F_2 with one common vertex. We can verify that $\chi(F) = \max\{\chi(F_1), \chi(F_2)\}$ and $\omega(F) = \max\{\omega(F_1), \omega(F_2)\}$. Therefore, $\chi(F) = \omega(F)$. $\square$

The graph gluing at a complete clone preserves the perfection. Theorem 3.2.8 illustrates this fact and it is yielded by Lemmas 3.2.2 and 3.2.3.

Theorem 3.2.8. *For graphs G_1 and G_2,*

$G_1 \underset{K_r}{\diamond} G_2$ is a perfect graph if and only if both G_1 and G_2 are perfect.

Furthermore, $\chi(G_1 \underset{K_r}{\diamond} G_2) = \omega(G_1 \underset{K_r}{\diamond} G_2) = \max\{\omega(G_1), \omega(G_2)\}$.

Proof. Let G_1 and G_2 be graphs.

Necessity follows from Proposition 3.1.6.

For sufficiency, assume that G_1 and G_2 are perfect graphs. We will show that $\chi(F) = \omega(F)$ for every induced subgraph F of $G_1 \underset{K_r}{\diamond} G_2$. Let F be an induced subgraph of $G_1 \underset{K_r}{\diamond} G_2$. If F is disconnected, we consider the perfection of each component of F. We may assume that F is connected. By Lemma 3.2.7, it is suffices to prove only case that F has at least two vertices in K_r. Assume that F has at least two vertices in K_r. Let $F_1 = F \cap G_1$ and $F_2 = F \cap G_2$. Since F_1 is an induced subgraph of a perfect graph G_1, we get $\chi(F_1) = \omega(F_1)$. Similarly, $\chi(F_2) = \omega(F_2)$. Now, let $F_r = F \cap K_r$. Then F_r is a complete graph. We have that $F = F_1 \underset{F_r}{\diamond} F_2$. By Lemmas 3.2.2 and 3.2.3, $\chi(F) = \max\{\chi(F_1), \chi(F_2)\}$ and $\omega(F) = \max\{\omega(F_1), \omega(F_2)\}$, respectively. Hence, $\chi(F) = \omega(F)$. Therefore, $G_1 \underset{K_r}{\diamond} G_2$ is perfect. Furthermore, $\chi(G_1 \underset{K_r}{\diamond} G_2) = \omega(G_1 \underset{K_r}{\diamond} G_2) = \max\{\omega(G_1), \omega(G_2)\}$. $\square$

If the clone is not a complete graph, it fails to be concluded the perfection of glued graphs of perfect graphs. It is illustrated by Theorems 3.2.9 and 3.2.11.

Theorem 3.2.9. *Let H be a connected graph. If H is not a complete graph, then there exist a perfect graph G_1 and an imperfect graph G_2 such that $G_1 \underset{H}{\diamond} G_2$ is perfect.*

Proof. Let H be a connected graph. Assume that H is not a complete graph. Let $|V(H)| = r$, so $r \geq 3$. Let $H_1(u_1, u_2, ..., u_r)$ and $H_2(v_1, v_2, ..., v_r)$ be the copies of H with an isomorphism $f : V(H_1) \to V(H_2)$ which is defined by $f(u_i) = v_i$ for all $i \in \{1, 2, ..., r\}$. Since H_2 is not a complete graph, there are at least 2 non-adjacent vertices , say v_1 and v_r. Choose $G_1 = K_r(u_1, u_2, ..., u_r)$. Choose $G_2 = \overline{C_{2r-1}}(v_1, x_1, ..., v_{r-1}, x_{r-1}, v_r)$, where $x_1, x_2, ..., x_{r-1}$ are distinct new vertices. Then G_1 is perfect but G_2 is not perfect. Since H_1 and H_2 are not complete graphs, $H_1 \subseteq K_r(u_1, u_2, ..., u_r)$ and $H_2 \subseteq K_r(v_1, v_2, ..., v_r) - v_1v_r \subseteq \overline{C_{2r-1}}(v_1, x_1, ..., v_{r-1}, x_{r-1}, v_r)$. We can verify that $G_1 \underset{H}{\diamond} G_2 \cong G_1 \underset{H_1 \cong_f H_2}{\diamond} G_2 \cong \overline{C_{2r-1}}(v_1, x_1, ..., v_{r-1}, x_{r-1}, v_r) + v_1v_r$, consequently, $\overline{G_1 \underset{H}{\diamond} G_2} \cong P_{2r-1}$. Since P_{2r-1} is perfect, by the Perfect Graph Theorem, $G_1 \underset{H}{\diamond} G_2$ is perfect. $\square$

Example 3.2.10. We illustrate an example of graphs in Theorem 3.2.9. We construct 3-vertex graphs $H_1(u_1, u_2, u_3)$ and $H_2(v_1, v_2, v_3)$. Choose G_1 to be $K_3(u_1, u_2, u_3)$ and choose G_2 to be $\overline{C_5}(v_1, x_1, v_2, x_2, v_3)$. Then G_1 is a perfect graph but G_2 is an imperfect graph. Observe that $\overline{G_1 \underset{H}{\diamond} G_2} \cong P_5$. Since P_5 is perfect, by the Perfect Graph Theorem, $G_1 \underset{H}{\diamond} G_2$ is perfect. We have G_1 and G_2 as graphs in Figure 3.2.2.

$\square$

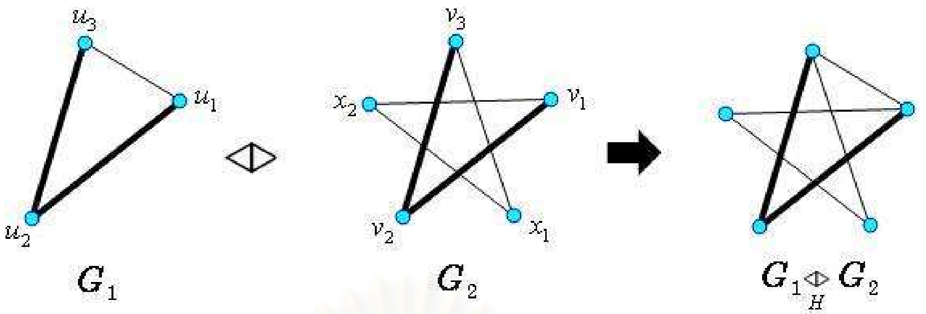

Figure 3.2.2: A perfect glued graph of a perfect graph and an imperfect graph at an incomplete clone

Theorem 3.2.11. *Let H be a connected graph. If H is not a complete graph, then there exist perfect graphs G_1 and G_2 such that $G_1 \underset{H}{\Diamond} G_2$ is not perfect.*

Proof. Let H be a connected graph. Assume that H is not a complete graph. Let $|V(H)| = r$, so $r \geq 3$. Let $H_1(u_1, u_2, ..., u_r)$ and $H_2(v_1, v_2, ..., v_r)$ be the copies of H with an isomorphism $f : V(H_1) \to V(H_2)$ which is defined by $f(u_i) = v_i$ for all $i \in \{1, 2, ..., r\}$. Since H_2 is not a complete graph, there are at least 2 non-adjacent vertices, say v_1 and v_2. Choose $G_1 = K_r(u_1, u_2, ..., u_r)$. Choose $G_2 = \overline{C_{2r+1}}(v_1, x_1, ..., v_r, x_r, x_{r+1}) - v_1v_2$, where $x_1, x_2, ..., x_{r+1}$ are distinct new vertices. Then G_1 and G_2 are perfect. Since H_1 and H_2 are not complete graphs, $H_1 \subseteq K_r(u_1, u_2, ..., u_r)$ and $H_2 \subseteq K_r(v_1, v_2, ..., v_r) - v_1v_2 \subseteq \overline{C_{2r+1}}(v_1, x_1, ..., v_r, x_r, x_{r+1}) - v_1v_2$. We can verify that $G_1 \underset{H}{\Diamond} G_2 \cong G_1 \underset{H_1 \cong_f H_2}{\Diamond} G_2 \cong \overline{C_{2r+1}}$. Thus, $G_1 \underset{H}{\Diamond} G_2$ is not a Berge graph. By the Strong Perfect Graph Theorem, $G_1 \underset{H}{\Diamond} G_2$ is not perfect. □

Example 3.2.12. We illustrate an example of graphs in Theorem 3.2.11. We construct 3-vertex graphs $H_1(u_1, u_2, u_3)$ and $H_2(v_1, v_2, v_3)$. Choose G_1 to be $K_3(u_1, u_2, u_3)$ and choose G_2 to be $\overline{C_7}(v_1, x_1, v_2, x_2, v_3, x_3, x_4) - v_1v_2$. Then G_1 and G_2 are perfect graphs. Since $G_1 \underset{H}{\Diamond} G_2 \cong \overline{C_7}$, by the Strong Perfect Graph Theorem, $G_1 \underset{H}{\Diamond} G_2$ is not perfect. We have G_1 and G_2 as graphs in Figure 3.2.3.

□

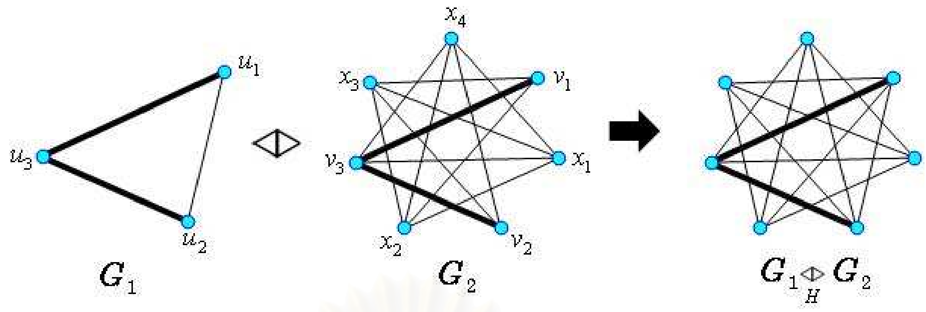

Figure 3.2.3: An imperfect glued graph of perfect graphs at an incomplete clone

Now, we conclude that for a connected graph H, if H is not a complete graph, then there exist graphs G_1 and G_2 such that

1. $\chi(G_1 \underset{H}{\diamondplus} G_2) > \max\{\chi(G_1), \chi(G_2)\}$ and $\omega(G_1 \underset{H}{\diamondplus} G_2) > \max\{\omega(G_1), \omega(G_2)\}$. Moreover, $\chi(G_1 \underset{H}{\diamondplus} G_2) = \omega(G_1 \underset{H}{\diamondplus} G_2)$. (Theorem 3.2.5)

2. $\chi(G_1 \underset{H}{\diamondplus} G_2) = \max\{\chi(G_1), \chi(G_2)\}$ and $\omega(G_1 \underset{H}{\diamondplus} G_2) = \max\{\omega(G_1), \omega(G_2)\}$. Moreover, $\chi(G_1 \underset{H}{\diamondplus} G_2) = \omega(G_1 \underset{H}{\diamondplus} G_2)$. (Theorem 3.2.9)

3. $\chi(G_1 \underset{H}{\diamondplus} G_2) > \max\{\chi(G_1), \chi(G_2)\}$ and $\omega(G_1 \underset{H}{\diamondplus} G_2) = \max\{\omega(G_1), \omega(G_2)\}$. Moreover, $\chi(G_1 \underset{H}{\diamondplus} G_2) > \omega(G_1 \underset{H}{\diamondplus} G_2)$. (Theorem 3.2.11)

Next, we recall a definition of a chordal graph and the **simplicial (perfect) elimination ordering**. It is well-known that the simplicial elimination ordering characterizes a subclass of perfect graphs, namely the chordal graphs.

Definition 3.2.13. A graph is **chordal** if it is simple and has no induced cycle of length at least four.

Definition 3.2.14. [15] A vertex of G is **simplicial** if its neighborhood in G forms a clique. A **simplicial elimination ordering** of a graph G is an ordering $v_n, ..., v_1$ for deletion of vertices so that each vertex v_i is a simplicial vertex of the remaining graph induced by $\{v_1, ..., v_i\}$. (These orderings are also called **perfect elimination orderings.**)

Theorem 3.2.15. [15] *A simple graph has a simplicial elimination ordering if and only if it is a chordal graph.*

For a subset S of $V(G)$, a *open neighborhood* of S in G, written $N_G(S)$ or $N(S)$, is the set of vertices in $V(G) - S$ which are adjacent to vertices in S. We use $G[S]$ and $G - S$ for the induced subgraph of G on the vertex set S and $V(G) - S$, respectively.

We now extend the **simplicial (perfect) elimination ordering** to a new definition as follows:

Definition 3.2.16. A subset V_i of $V(G)$ is **simplicial** if its *open neighborhood* in G forms a clique. A **simplicial set elimination ordering** of a graph G is an ordering $V_1, ..., V_k$ for deletion of nonempty vertex subsets so that each V_i is a simplicial vertex subset of the remaining graph induced by $\bigcup_{t=i}^{k} V_t$ with $|V_i| = 1$ or $|V_i| = 2$ for all $i \in \{1, 2, ..., k\}$, and $V_1, ..., V_k$ partitions $V(G)$.

Note that a simplicial elimination ordering of a graph G is a simplicial set elimination ordering of G with $|V_i| = 1$ for all $i \in \{1, 2, ..., n(G)\}$.

Remark 3.2.17. For a simple graph G, let $V_1, ..., V_k$ be a partition of $V(G)$. Let $G_1 = G$, and for each $i \in \{2, 3, ..., k\}$, let $G_i = G - \bigcup_{t=1}^{i-1} V_t$. If $V_1, ..., V_k$ is a simplicial set elimination ordering of G, then for each $i \in \{1, 2, ..., k-1\}$, G_i is a glued graph of $G_i[V_i \cup N(V_i)]$ and G_{i+1} at a complete clone $G_i[N(V_i)]$
(a clone can be a vertex).

Theorem 3.2.18. *A simple graph with a simplicial set elimination ordering is a perfect graph.*

Proof. Let G be a simple graph. Assume that G has a simplicial set elimination ordering $V_1, ..., V_k$. Let $G_1 = G$, and for each $i \in \{2, 3, ..., k\}$, let $G_i = G - \bigcup_{t=1}^{i-1} V_t$.

Since $|V_k| = 1$ or 2, $G_k \cong K_1$ or K_2, so G_k is perfect. By Remark 3.2.17, G_i is a glued graph of $G_i[V_i \cup N(V_i)]$ and G_{i+1} at a complete clone $G_i[N(V_i)]$, it is enough to claim that $G_i[V_i \cup N(V_i)]$ is perfect for all $i \in \{1, 2, ..., k-1\}$. Let $i \in \{1, 2, ..., k-1\}$. Let C be an induced cycle in $G_i[V_i \cup N(V_i)]$. Since $N_{G_i}(V_i)$ forms a clique, at most 2 vertices in $N_{G_i}(V_i)$ can be in C. Together with vertices in V_i, C has length at most 4. Again, since $N_{G_i}(V_i)$ forms a clique, 2 vertices in $N_{G_i}(V_i)$ cannot be adjacent in the complement of $G_i[V_i \cup N(V_i)]$. Besides, each vertex in $N_{G_i}(V_i)$ must be adjacent to at least one vertex in V_i, so it can be adjacent to at most one vertex of V_i in the complement of $G_i[V_i \cup N(V_i)]$. Thus, there is no cycle in the complement of $G_i[V_i \cup N(V_i)]$. Hence, $G_i[V_i \cup N(V_i)]$ contains no odd hole and no odd antihole. By the Strong Perfect Graph Theorem, we get $G_i[V_i \cup N(V_i)]$ is perfect. By Theorem 3.2.8, G_i is perfect for all $i \in \{1, 2, ..., k-1\}$. $\square$

The converse of the theorem is not true, for instance, C_{2n}, where $n \geq 3$, is perfect while it has no simplicial set elimination ordering.

CHAPTER IV

THE PERFECTION OF GLUED GRAPHS AT ARBITRARY CLONES

While, Chapter 3 studies the perfection of glued graphs at complete clones for arbitrary original graphs, this chapter studies the perfection of glued graphs at arbitrary clones with some specified original graphs. We also investigate values of the chromatic numbers and the clique numbers of glued graphs at arbitrary clones in terms of these parameters of their original graphs. Later, we utilize them to prove our main theorems.

4.1 Background

First, we introduce new definitions for convenience:

Definition 4.1.1. An odd hole C of any glued graph $G_1 \Leftrightarrow G_2$ is a **created odd hole** if both G_1 and G_2 do not have the corresponding odd hole of C. An odd antihole $\overline{C}$ of any glued graph $G_1 \Leftrightarrow G_2$ is a **created odd antihole** if both G_1 and G_2 do not have the corresponding odd antihole of $\overline{C}$.

The following theorem is directly obtained from the Strong Perfect Graph Theorem.

Theorem 4.1.2. *A glued graph of perfect graphs is perfect if and only if it contains no created odd hole and no created odd antihole.*

Proof. Let $G_1 \Leftrightarrow G_2$ be the glued graph of perfect graphs G_1 and G_2.

Necessity. If $G_1 \triangleleft\!\triangleright G_2$ contains a created odd hole or created odd antihole, by the Strong Perfect Graph Theorem, it is not perfect.

Sufficiency. Assume that $G_1 \triangleleft\!\triangleright G_2$ contains no created odd hole and no created odd antihole. Since G_1 and G_2 are perfect, by the Strong Perfect Graph Theorem, they contain no odd hole and no odd antihole. Thus, $G_1 \triangleleft\!\triangleright G_2$ contains no odd hole and no original odd antihole. Hence, it is perfect. $\square$

This theorem helps us to verify the perfection of our glued graphs.

We next consider a glued graph of bipartite graphs at arbitrary clones.

Theorem 4.1.3. [15] *A graph is bipartite if and only if it has no odd cycle.*

Theorem 4.1.4. [12] *A glued graph $G_1 \triangleleft\!\triangleright G_2$ is a bipartite graph if and only if G_1 and G_2 are bipartite.*

Corollary 4.1.5. *If G_1 and G_2 are bipartite graphs , then $G_1 \triangleleft\!\triangleright G_2$ is perfect.*

Proof. Let G_1 and G_2 be bipartite graphs. By Theorem 4.1.4, $G_1 \triangleleft\!\triangleright G_2$ is bipartite graph and hence $G_1 \triangleleft\!\triangleright G_2$ is perfect. $\square$

Theorem 4.1.6. *A triangle-free graph G is a perfect graph if and only if G is bipartite.*

Proof. Necessity. It follows directly from the Strong Perfect Graph Theorem and Theorem 4.1.3.

Sufficiency. By Theorem 4.1.3, a bipartite graph has no triangle as an induced subgraph. Also, it is perfect. $\square$

Corollary 4.1.7. *A triangle-free glued graph $G_1 \triangleleft\!\triangleright G_2$ is a perfect graph if and only if G_1 and G_2 are bipartite.*

Proof. This follows from Theorems 4.1.6 and 4.1.4. $\square$

4.2 The Perfection of Glued Graphs at Arbitrary Clones

In this section, we study the perfection of glued graphs at arbitrary clones with some specified original graphs.

First, we consider glued graphs of complete graphs at arbitrary clones. Lemmas 4.2.1 and 4.2.2 show that the chromatic numbers and the clique numbers of glued graphs do not exceed the chromatic numbers and the clique numbers of their original graphs, respectively.

Lemma 4.2.1. $\chi(K_m \lhd\!\!\rhd K_n) = \max\{m, n\}$.

Proof. Let H be the clone of a glued graph $K_m \underset{H}{\lhd\!\!\rhd} K_n$ and $|V(H)| = r$. Note that $\chi(K_m) = m$ and $\chi(K_n) = n$. Assume $m \geq n$. By Remark 3.1.2(1), we have $\chi(K_m \underset{H}{\lhd\!\!\rhd} K_n) \geq \max\{\chi(K_m), \chi(K_n)\} = m$. It suffices to show that $\chi(K_m \underset{H}{\lhd\!\!\rhd} K_n) \leq m$, that is $K_m \underset{H}{\lhd\!\!\rhd} K_n$ has a proper m-coloring. Let f be an m-coloring of K_m with colors $a_1, a_2, ..., a_m$ and g an n-coloring of K_n with colors $b_1, b_2, ..., b_n$. Note that any pair of vertices in H must have different colors. Without loss of generality, for $i \in \{1, 2, ..., r\}$, let a_i and b_i be colors of the corresponding vertices of H in K_m and K_n, respectively. Let $h : V(K_m \underset{H}{\lhd\!\!\rhd} K_n) \to \{a_1, a_2, ..., a_m\}$ defined by

$$h(v) = \begin{cases} f(v) & \text{, if } v \in V(K_m); \\ a_i & \text{, if } v \in V(K_n) - V(H) \text{ and } g(v) = b_i. \end{cases}$$

To show that h is proper, let u and v be vertices in $K_m \underset{H}{\lhd\!\!\rhd} K_n$ such that u and v are adjacent. If $u, v \in V(K_m)$, then $h(u) = f(u) \neq f(v) = h(v)$. If $u, v \in V(K_n)$, then $g(u) = b_i$ and $g(v) = b_j$ for some $i \neq j$, so $h(u) = a_i \neq a_j = h(v)$. Besides, vertices in $V(K_m) - V(H)$ and $V(K_n) - V(H)$ are not adjacent. Hence, h is proper. That is, $K_m \underset{H}{\lhd\!\!\rhd} K_n$ has a proper m-coloring. Therefore, $\chi(K_m \underset{H}{\lhd\!\!\rhd} K_n) \leq m$. $\square$

Lemma 4.2.2. $\omega(K_m \diamond K_n) = \max\{m, n\}$.

Proof. Let H be the clone of a glued graph $K_m \underset{H}{\diamond} K_n$. By Remark 3.1.2(2), we have $\omega(K_m \underset{H}{\diamond} K_n) \geq \max\{\omega(K_m), \omega(K_n)\}$. Since vertices in the clone H are adjacent to all other vertices and vertices in $V(K_m) - V(H)$ and $V(K_n) - V(H)$ are not adjacent, this graph gluing does not create a new clique. So, $\omega(K_m \underset{H}{\diamond} K_n) \leq \omega(K_m)$ and $\omega(K_m \underset{H}{\diamond} K_n) \leq \omega(K_n)$. Hence, $\omega(K_m \underset{H}{\diamond} K_n) \leq \max\{\omega(K_m), \omega(K_n)\}$. Therefore, $\omega(K_m \underset{H}{\diamond} K_n) = \max\{\omega(K_m), \omega(K_n)\}$. □

Theorem 4.2.3 shows the perfection of a glued graph of complete graphs at arbitrary clones.

Theorem 4.2.3. *A glued graph $K_m \diamond K_n$ is perfect.*

Proof. Let H be the clone of $K_m \underset{H}{\diamond} K_n$. We will show that $\chi(F) = \omega(F)$ for every induced subgraph F of $K_m \underset{H}{\diamond} K_n$. Let F be a connected induced subgraph of $K_m \underset{H}{\diamond} K_n$. By Lemma 3.2.7, it is suffices to prove only case that F has at least two vertices in H. Assume that F has at least two vertices in H. Let $F_1 = F \cap K_m$ and $F_2 = F \cap K_n$. Since F_1 is an induced subgraph of K_m, F_1 is a complete graph. Also, $\chi(F_1) = \omega(F_1)$. Similarly, F_2 is a complete graph and hence $\chi(F_2) = \omega(F_2)$. Now, let $F_H = F \cap H$. We have that $F = F_1 \underset{F_H}{\diamond} F_2$. By Lemmas 4.2.1 and 4.2.2, $\chi(F) = \max\{\chi(F_1), \chi(F_2)\}$ and $\omega(F) = \max\{\omega(F_1), \omega(F_2)\}$, respectively. Therefore, $\chi(F) = \omega(F)$. □

Notation $G - e$ denotes a subgraph of G on the vertex set $V(G)$ and the edge set $E(G) - \{e\}$ where $e \in E(G)$, and $G - E$ denotes a subgraph of G on the vertex set $V(G)$ and the edge set $E(G) - E$ where $E \subseteq E(G)$.

We consider a glued graph of $K_m - E_1$ and $K_n - E_2$. It is possible that a glued graph of $K_m - E_1$ and $K_n - E_2$ is not perfect. We illustrate this in the following example.

Example 4.2.4. Let G_1 and G_2 be graphs in Figure 4.2.1.

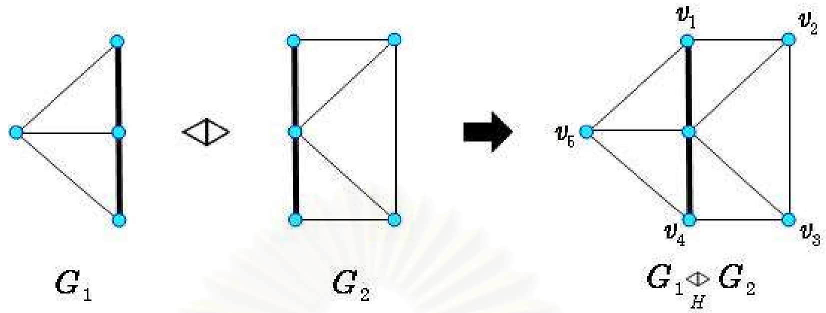

Figure 4.2.1: An imperfect glued graph of

$K_4 - e$ and $K_5 - E$ where $|E| = 3$

It is easily seen that $G_1 = K_4 - e$ and $G_2 = K_5 - E$ where $|E| = 3$, both G_1 and G_2 are perfect graphs but $G_1 \underset{H}{\Diamond} G_2$ contains an induced cycle $C_5(v_1, v_2, v_3, v_4, v_5)$. By the Strong Perfect Graph Theorem, $G_1 \underset{H}{\Diamond} G_2$ is not perfect. $\square$

In Theorem 4.2.5, we give a condition for two edge sets E_1 and E_2 to obtain the perfection of a glued graph of $K_m - E_1$ and $K_n - E_2$.

Theorem 4.2.5. *Let m and n be positive integers. If $G_1 = K_m - e$ or $K_m - \{e_1, e_2\}$ and $G_2 = K_n - e$ or $K_n - \{e_1, e_2\}$, then $G_1 \Diamond G_2$ is perfect.*

Proof. Let m and n be positive integers. Let $G_1 = K_m - e$ or $K_m - \{e_1, e_2\}$ and $G_2 = K_n - e$ or $K_n - \{e_1, e_2\}$. We will show that $G_1 \Diamond G_2$ contains no odd hole and no odd antihole. Let C be a cycle in $G_1 \Diamond G_2$. Since at most 4 edges which does not have both two endpoints in $V(C)$, C has length at most 4. Similarly, if C is a cycle in the complement of $G_1 \Diamond G_2$, then C has length at most 4. Hence, $G_1 \Diamond G_2$ contains no odd hole and no odd antihole. By the Strong Perfect Graph Theorem, $G_1 \Diamond G_2$ is perfect. $\square$

The converse of Theorem 4.2.5 is not true. This is confirmed by Theorem 4.2.6.

Theorem 4.2.6. *Let m and n be positive integers. Then there exist graphs G_1 and G_2 such that $G_1 \triangleleft\!\triangleright G_2$ is a perfect graph but $G_1 = K_m - E_1$, $G_2 = K_n - E_2$ where $|E_1| \geq 3$ or $|E_2| \geq 3$.*

Proof. Let m and n be positive integers. Note that original graphs of any glued graph have at least one edge. If $m \leq 3$ or $n \leq 3$, then $K_m - E_1$ or $K_n - E_2$ has no edge, where $|E_1| \geq 3$ or $|E_2| \geq 3$. Thus, $m \geq 4$ and $n \geq 4$. Choose $G_1 = C_m - e$ and choose $G_2 = C_n - e$. Then $G_1 = K_m - E_1$ and $G_2 = K_n - E_2$, where $|E_1| \geq 3$ and $|E_2| \geq 3$. Also, G_1 and G_2 are bipartite graphs. By Corollary 4.1.5, $G_1 \triangleleft\!\triangleright G_2$ is perfect. □

Example 4.2.7. We illustrate an example of graphs in Theorem 4.2.6. We construct graphs $G_1 = C_4 - e$ and $G_2 = C_5 - e$. Then $G_1 \triangleleft\!\triangleright G_2$ is a bipartite graph, so $G_1 \triangleleft\!\triangleright G_2$ is perfect. We have G_1 and G_2 as graphs in Figure 4.2.2.

□

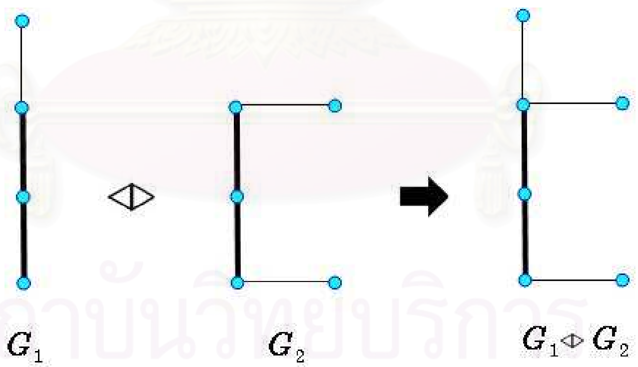

Figure 4.2.2: A perfect glued graph of

$C_4 - e$ and $C_5 - e$

The condition in Theorem 4.2.5 that $G_1 = K_m - e$ or $K_m - \{e_1, e_2\}$ and $G_2 = K_n - e$ or $K_n - \{e_1, e_2\}$ is necessary. This is confirmed by Theorem 4.2.8.

Theorem 4.2.8. *Let m and n be positive integers. Then there exist graphs $G_1 = K_m - E_1$ and $G_2 = K_n - E_2$, where $|E_1| \geq 3$ or $|E_2| \geq 3$ but $G_1 \triangleleft\!\triangleright G_2$ is not perfect.*

Proof. Let m and n be positive integers. Assume that $n \geq m-1$. If $m \leq 4$ or $n \leq 4$, then $K_m - E_1$ or $K_n - E_2$ are bipartite graphs, where $|E_1| \geq 3$ or $|E_2| \geq 3$. So, a glued graph of $K_m - E_1$ and $K_n - E_2$ is perfect. Thus, we may assume that $m \geq 5$ and $n \geq 5$. Let k be a positive integer such that $k \geq 3$. Consider $K_m(u_1, u_2, ..., u_m)$. Note that K_m contains a hamiltonian cycle C_m. Let $e_1 = u_1u_{m-1}, e_2 = u_1u_{m-2}, e_3 = u_{m-2}u_m$ and $e_4, ..., e_k \in E(K_m) - E(C_m) - \{e_1, e_2, e_3\}$. Let $E_1 = \{e_1, e_2, ..., e_k\}$. Choose $G_1 = K_m - E_1$. Consider $P_{m-2}(v_1, v_2, ..., v_{m-2})$. Let $E_2 = \{v_iv_j : |i-j| > 1 \text{ for all } i, j = 1, 2, ..., m-2\}$, so $|E_2| \geq 1$. Note that $P_{m-2} = K_{m-2} - E_2$. Choose $G_2 = P_{m-2} \vee K_{n-m+2}$. Then $G_2 = K_n - E_2$. Let $H_1 = P_{m-2}(u_1, u_2, ..., u_{m-2}) \subseteq G_1$ and $H_2 = P_{m-2}(v_1, v_2, ..., v_{m-2}) \subseteq G_2$. Define $f : H_1 \to H_2$ by $f(u_i) = v_i$ for all $i \in \{1, 2, ..., m-2\}$. Let x be a vertex of a subgraph K_{n-m+2} in G_2. We observe that the corresponding vertices of $x, u_1, u_m, u_{m-1}, u_{m-2}$ in $\underset{H_1 \cong_f H_2}{G_1 \diamond G_2}$ form an induced cycle C_5. Therefore, $\underset{H_1 \cong_f H_2}{G_1 \diamond G_2}$ is not perfect. $\square$

Example 4.2.9. We illustrate an example of graphs in Theorem 4.2.8.

We construct graphs $G_1 = K_6(u_1, u_2, ..., u_6,) - \{u_1u_5, u_1u_4, u_4u_6\}$ and $G_2 = K_5(v_1, v_2, v_3, v_4, x) - \{v_1v_4, v_1v_3, v_2v_4\}$. Let $H_1 = P_4(u_1, u_2, u_3, u_4) \subseteq G_1$ and $H_2 = P_4(v_1, v_2, v_3, v_4) \subseteq G_2$. Define $f : H_1 \to H_2$ by $f(u_i) = v_i$ for all $i \in \{1, 2, 3, 4\}$. Then $\underset{H_1 \cong_f H_2}{G_1 \diamond G_2}$ contains an induced cycle $C_5(x', w_1, w_6, w_5, w_4)$. Hence, $\underset{H_1 \cong_f H_2}{G_1 \diamond G_2}$ is not perfect. We have G_1 and G_2 as graphs in Figure 4.2.3.

$\square$

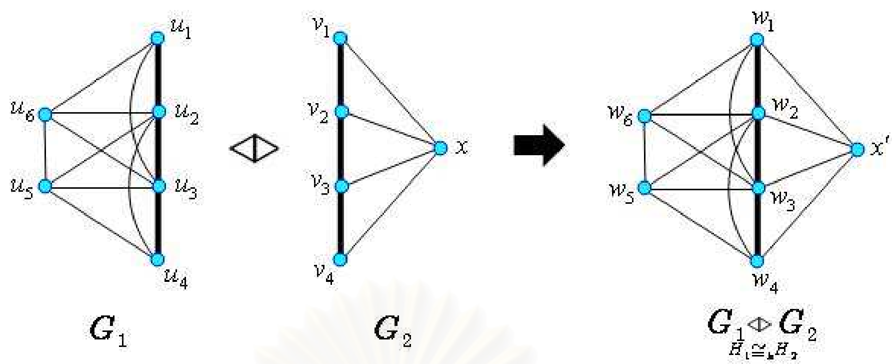

Figure 4.2.3: An imperfect glued graph of

$K_6 - E_1$ and $K_5 - E_2$, where $|E_1| \geq 3$ or $|E_2| \geq 3$

Here, we focus a perfect graph G with $\omega(G) = 2$. Note that G is a perfect graph with $\omega(G) = 2$ if and only if G is a bipartite graph. Thus, we consider a glued graph of a bipartite graph and a perfect graph. It is possible that the glued graph is not perfect. We illustrate this in the following example.

Example 4.2.10. Let G_1 and G_2 be graphs as shown in Figure 4.2.4.

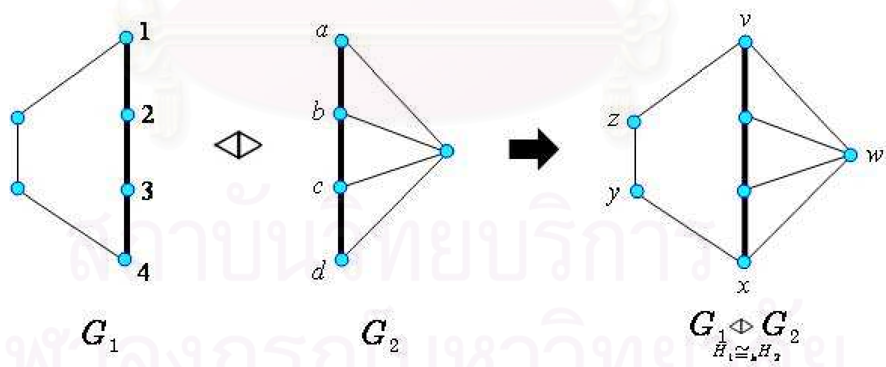

Figure 4.2.4: An imperfect glued graph of a bipartite graph and a perfect graph

Let $H_1 = P_3(1, 2, 3, 4) \subseteq G_1$ and $H_2 = P_3(a, b, c, d) \subseteq G_2$. Define $f : H_1 \to H_2$ by $f(1) = a$, $f(2) = b$, $f(3) = c$ and $f(4) = d$. It can be easily seen that $\underset{H_1\cong_f H_2}{G_1 \diamond G_2}$ contains an induced cycle $C_5(v, w, x, y, z)$. Hence, $\underset{H_1\cong_f H_2}{G_1 \diamond G_2}$ is not perfect. $\square$

In example 4.2.10, it is easily seen that a glued graph of a bipartite graph and a perfect graph may be imperfect. Note that a bipartite graph can have a cycle which affect appearance of a created odd hole. Thus, we give a condition for the perfection of the glued graph that one of original graphs contains no cycle.

Next, we recall a definition of a forest.

Definition 4.2.11. A graph with no cycle is *acyclic*. A **forest** is an acyclic graph. A **tree** is a connected acyclic graph.

Theorem 4.2.12. *Let G_1 and G_2 are perfect graphs. If G_1 or G_2 is a forest, then $\chi(G_1 \diamond G_2) = \max\{\chi(G_1), \chi(G_2)\}$*

Proof. Let G_1 and G_2 be perfect graphs. Assume that G_1 or G_2 is a forest. Without loss of generality, assume G_2 is a forest. Let H be the clone of a glued graph $G_1 \underset{H}{\diamond} G_2$. Note that $\chi(G_1) \geq 2$ and $\chi(G_2) = 2$. By Remark 3.1.2(1), we have $\chi(G_1 \underset{H}{\diamond} G_2) \geq \max\{\chi(G_1), \chi(G_2)\} = \chi(G_1)$. It suffices to show that $\chi(G_1 \underset{H}{\diamond} G_2) \leq \chi(G_1)$, that is $G_1 \underset{H}{\diamond} G_2$ has a proper $\chi(G_1)$-coloring. There are proper colorings $f : V(G_1) \to S_1$ and $g : V(G_2) \to S_2$ of G_1 and G_2, respectively, where $|S_1| = \chi(G_1) \geq 2$, $|S_2| = \chi(G_2) = 2$ and $S_1 \cap S_2 = \varnothing$. Let a_1b_1 and a_2b_2 be edges in the copies of H in G_1 and G_2, respectively, such that $a_1 \equiv a_2$ and $b_1 \equiv b_2$. Let $h : V(G_1 \underset{H}{\diamond} G_2) \to S_1$ defined by

$$h(v) = \begin{cases} f(v) & , \text{ if } v \in V(G_1); \\ f(a_1) & , \text{ if } v \in V(G_2) - V(H) \text{ and } g(v) = g(a_2); \\ f(b_1) & , \text{ if } v \in V(G_2) - V(H) \text{ and } g(v) = g(b_2). \end{cases}$$

To show that h is proper, let u and v be vertices in $G_1 \underset{H}{\diamond} G_2$ such that u and v are adjacent. If $u, v \in V(G_1)$, then $h(u) = f(u) \neq f(v) = h(v)$. Assume that $u, v \in V(G_2)$. If u and v are adjacent in G_2, without loss of generality, assume $g(u) = (a_2)$ and $g(v) = g(b_2)$, so $h(u) = f(a_1) \neq f(b_1) = h(v)$. Since G_2 is forest and H is connected, $G_1 \underset{H}{\Diamond} G_2$ has no new edge for G_1 but it can have a new edge for G_2. If uv is a new edge for G_2, then u and v are adjacent in G_1, so $h(u) = f(u) \neq f(v) = h(v)$.

Clearly, vertices in $V(G_1) - V(H)$ and $V(G_2) - V(H)$ are not adjacent. Thus, h is proper. Hence, $\chi(G_1 \underset{H}{\Diamond} G_2) \leq \chi(G_1)$. $\square$

Theorem 4.2.13. *Let G_1 and G_2 be perfect graphs. If G_1 or G_2 is a forest, then $\omega(G_1 \Diamond G_2) = \max\{\omega(G_1), \omega(G_2)\}$*

Proof. Let G_1 and G_2 be perfect graphs. Assume that G_1 or G_2 is a forest. Without loss of generality, assume G_2 is a forest. Let H be the clone of a glued graph $G_1 \underset{H}{\Diamond} G_2$. Note that $\omega(G_1) \geq 2$ and $\omega(G_2) = 2$. By Remark 3.1.2(2), we have $\omega(G_1 \underset{H}{\Diamond} G_2) \geq \max\{\omega(G_1), \omega(G_2)\}$. Since G_2 is a forest and H is connected, $G_1 \underset{H}{\Diamond} G_2$ has no new edge for G_1. Thus, $G_1 \underset{H}{\Diamond} G_2$ has no new clique for G_1 but it can have a new clique for G_2. Assume that $G_1 \underset{H}{\Diamond} G_2$ has a new clique for G_2. Then all vertices in the new clique are in cycles of length 3 but G_2 has no cycle, so all cycle of length 3 are in G_1. Thus, all new cliques of $G_1 \underset{H}{\Diamond} G_2$ are cliques in G_1. So, $\omega(G_1 \underset{H}{\Diamond} G_2) \leq \omega(G_i)$ for $i = 1, 2$. Hence, $\omega(G_1 \underset{H}{\Diamond} G_2) \leq \max\{\omega(G_1), \omega(G_2)\}$. Therefore, $\omega(G_1 \underset{H}{\Diamond} G_2) = \max\{\omega(G_1), \omega(G_2)\}$. $\square$

For perfect graphs G_1 and G_2, we conclude from Theorems 4.2.12 and 4.2.13 that if G_1 or G_2 is a forest, then $\chi(G_1 \Diamond G_2) = \omega(G_1 \Diamond G_2)$. However, we need more work to show that $G_1 \Diamond G_2$ is perfect. Theorem 4.2.14 concludes this.

Theorem 4.2.14. *Let G_1 and G_2 be perfect graphs. If G_1 or G_2 is a forest, then $G_1 \Diamond G_2$ is perfect.*

Proof. Let G_1 and G_2 be perfect graphs. Assume that G_1 or G_2 is a forest. Without loss of generality, assume G_2 is a forest. Let H be the clone of a glued graph $G_1 \underset{H}{\Diamond} G_2$. We will show that $G_1 \underset{H}{\Diamond} G_2$ contains no created odd hole and no created odd antihole. Note that all vertices of an odd antihole are in cycles. Since G_2 is a forest and H is connected, $G_1 \underset{H}{\diamond} G_2$ has no new edge for G_1 but it can have a new edge for G_2. Thus, all cycles of $G_1 \underset{H}{\diamond} G_2$ are cycles in a perfect graph G_1. Hence, $G_1 \underset{H}{\diamond} G_2$ contains no created odd hole and no created odd antihole. By Corollary 4.1.2, $G_1 \diamond G_2$ is perfect. $\square$

CHAPTER V

CONCLUSION AND OPEN PROBLEMS

We conclude main results of this work and give some open problems for future work in this chapter.

5.1 Conclusion

We have obtained necessary and/or sufficient conditions for the perfection of the glued graphs. We also have investigated values of the chromatic numbers and the clique numbers of glued graphs in terms of these parameters of their original graphs. The results are as follows:

The perfection of glued graphs at induced clones:

1. Let G_1 and G_2 be graphs containing H as their induced subgraph.
   - $\omega(G_1 \underset{H}{\Diamond} G_2) = \max\{\omega(G_1), \omega(G_2)\}$.
   - If G_1 or G_2 is a bipartite graph, then $\chi(G_1 \underset{H}{\Diamond} G_2) = \max\{\chi(G_1), \chi(G_2)\}$.
   - If $G_1 \underset{H}{\Diamond} G_2$ is a perfect graph, then both G_1 and G_2 are perfect.
2. Let H be a connected incomplete graph. If H is a perfect graph, then there exist perfect graphs G_1 and G_2 containing H as their induced subgraph such that $G_1 \underset{H}{\Diamond} G_2$ is not perfect.

The perfection of glued graphs at complete clones:

1. For graphs G_1 and G_2,

   - $\chi(G_1 \underset{K_r}{\triangleleft\!\triangleright} G_2) = \max\{\chi(G_1), \chi(G_2)\}$.
   - $\omega(G_1 \underset{K_r}{\triangleleft\!\triangleright} G_2) = \max\{\omega(G_1), \omega(G_2)\}$.
   - $G_1 \underset{K_r}{\triangleleft\!\triangleright} G_2$ is a perfect graph if and only if both G_1 and G_2 are perfect. Furthermore, $\chi(G_1 \underset{K_r}{\triangleleft\!\triangleright} G_2) = \omega(G_1 \underset{K_r}{\triangleleft\!\triangleright} G_2) = \max\{\omega(G_1), \omega(G_2)\}$.

2. Let H be a connected graph. If H is not a complete graph, then

   - there exist graphs G_1 and G_2 such that $\chi(G_1 \underset{H}{\triangleleft\!\triangleright} G_2) > \max\{\chi(G_1), \chi(G_2)\}$ and $\omega(G_1 \underset{H}{\triangleleft\!\triangleright} G_2) > \max\{\omega(G_1), \omega(G_2)\}$;
   - there exist a perfect graph G_1 and an imperfect graph G_2 such that $G_1 \underset{H}{\triangleleft\!\triangleright} G_2$ is perfect;
   - there exist perfect graphs G_1 and G_2 such that $G_1 \underset{H}{\triangleleft\!\triangleright} G_2$ is not perfect.

3. A simple graph with a simplicial set elimination ordering is a perfect graph.

The perfection of glued graphs at arbitrary clones:

1. A glued graph of perfect graphs is perfect if and only if it contains no created odd hole and no created odd antihole.

2. A triangle-free glued graph $G_1 \triangleleft\!\triangleright G_2$ is a perfect graph if and only if G_1 and G_2 are bipartite.

3. Let m and n be positive integers.

   - $\chi(K_m \diamond K_n) = \max\{m, n\}$.
   - $\omega(K_m \diamond K_n) = \max\{m, n\}$.
   - A glued graph $K_m \diamond K_n$ is perfect.

4. Let m and n be positive integers.

   - If $G_1 = K_m - e$ or $K_m - \{e_1, e_2\}$ and $G_2 = K_n - e$ or $K_n - \{e_1, e_2\}$, then $G_1 \diamond G_2$ is perfect.
   - There exist graphs G_1 and G_2 such that $G_1 \diamond G_2$ is a perfect graph but $G_1 = K_m - E_1$, $G_2 = K_n - E_2$ where $|E_1| \geq 3$ or $|E_2| \geq 3$.
   - There exist graphs $G_1 = K_m - E_1$ and $G_2 = K_n - E_2$, where $|E_1| \geq 3$ or $|E_2| \geq 3$ but $G_1 \diamond G_2$ is not perfect.

5. Let G_1 and G_2 be perfect graphs. If G_1 or G_2 is a forest, then

   - $\chi(G_1 \diamond G_2) = \max\{\chi(G_1), \chi(G_2)\}$.
   - $\omega(G_1 \diamond G_2) = \max\{\omega(G_1), \omega(G_2)\}$.
   - $G_1 \diamond G_2$ is perfect.

5.2 Open Problems

We have some open problems for future work as follows:

1. In Section 2.2, we have already obtained a condition of a perfect graph for the perfection of a glued graph of an odd hole and the perfect graph. It is an open problem to find a condition of a perfect graph for the perfection of a glued graph of an odd antihole and the perfect graph.

2. For the subject of induced clones, an open problem is to find specified original graphs to obtain the perfection of glued graphs at induced clones.

3. In Theorem 3.2.18, we have that if G has a simplicial set elimination ordering, then G is a perfect graph. Also, there is an example of a perfect graph having no simplicial set elimination ordering. Thus, it is interested to find a subclass of perfect graphs which is characterized by simplicial set elimination ordering.

4. In Section 4.2, we have already obtained specified original graphs such as complete graphs and forests, for the perfection of glued graphs at arbitrary clones. However, a forest has no cycle, an interested question is: "Are there any other original graphs having a cycle(except complete graphs), to obtain the perfection of glued graphs at arbitrary clones?".

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APPENDIX

A *graph* G is a triple consisting of a *vertex set* $V(G)$, an *edge set* $E(G)$, and a relation that associates with each edge two vertices (not necessary to be distinct) called its *endpoints*. The number of elements in $V(G)$ is represented by $n(G)$ and the number of elements in $E(G)$ is represented by $e(G)$.

A *loop* is an edge whose endpoints are equal. An *multiple edges* are edges having the same pair of endpoints. A *simple graph* is a graph having no loops and no multiple edges.

A graph is *trivial* if it has no edge; otherwise it is *nontrivial*.

The *degree* of a vertex v in a graph G, written $d_G(v)$ or $d(v)$, is the number of edges incident to v, except that each loop at v counts twice. The *maximum degree* is $\Delta(G)$ and the *minimum degree* is $\delta(G)$.

The *neighborhood* of v, written $N_G(v)$ or $N(v)$, is the set of vertices adjacent to v.

A *subgraph* F of a graph G is a graph F such that $V(F) \subseteq V(G)$ and $E(F) \subseteq E(G)$ and the assignment of endpoints to edges in F is the same as in G. We then write $F \subseteq G$. If $F \neq G$, then it is called a *proper subgraph*. A subgraph F of a graph G is an *induced subgraph* of G if whenever u and v are vertices of F and uv is an edge of G, then uv is an edge of F.

A *complete graph* is a simple graph whose vertices are pairwise adjacent; the complete graph with n vertices is denoted K_n.

The *complement* $\overline{G}$ of a simple graph G is the simple graph with vertex set $V(G)$ defined by $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$.

A graph G is *bipartite* if $V(G)$ is the union of two disjoint (possible empty) independent sets called *partite set* of G.

A *path* is a simple graph whose vertices can be ordered so that two vertices are adjacent if and only if they are consecutive in the list. A *cycle* is a graph with an equal number of vertices and edges whose vertices can be placed around a circle so that two vertices are adjacent if and only if they appear consecutive along the circle. The *length* of a path or cycle is its number of edges.

A graph G is *connected* if it has a u, v-path whenever $u, v \in V(G)$. Otherwise, G is *disconnected*.

An *interval representation* of a graph is a family of intervals assigned to the vertices so that vertices are adjacent if and only if the corresponding intervals intersect. A graph having such a representation is an *interval graph*.

A *bull* is the graph with vertices a, b, c, d, e and edges ab, ac, bc, bd, ce. A Berge graph having no a bull as an induced subgraph is called a *bull-free Berge graph*.

A *planar Berge graph* is a Berge graph which has a drawing without crossings.

A *degenerate Berge graph* is a Berge graph such that every induced subgraph F has a vertex of degree at most $\omega(F) + 1$.

A *triangle-free graph* is a graph having no triangle(K_3) as an induced subgraph.

The *union* of graphs $G_1, ..., G_k$, written $G_1 \cup ... \cup G_k$, is the graph with vertex set $V(G_1) \cup ... \cup V(G_k)$ and edge set $E(G_1) \cup ... \cup E(G_k)$.

The graph obtained by taking the union of graphs G_1 and G_2 with disjoint vertex sets is the *disjoint union* or *sum*, written $G_1 + G_2$.

The *join* of simple graphs G_1 and G_2, written $G_1 \vee G_2$, is the graph obtained from the disjoint union $G_1 + G_2$ by adding the edges $\{uv : u \in V(G_1), v \in V(G_2)\}$.

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