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FIXED POINT THEOREM OF HALF-CONTINUOUS MULTIVALUED MAPPINGS

Mr. Thanatkrit Kaewtem

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By	Mr. Thanatkrit Kaewtem
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Thesis Advisor	Associate Professor Imchit Termwuttipong, Ph.D.

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S. Hannongbua ....Dean of the Faculty of Science

(Professor Supot Hannongbua, Dr.rer.nat.)

THESIS COMMITTEE

Phichet Chaoha .... Chairman

(Associate Professor Phichet Chaoha, Ph.D.)

Imchit Termwuttipong .... Thesis Advisor

(Associate Professor Imchit Termwuttipong, Ph.D.)

.... Examiner

(Assistant Professor Wacharin Wichiramala, Ph.D.)

.... External Examiner

(Assistant Professor Attapol Kaewkhao, Ph.D.)

ธนัชกฤศ แก้วเต็ม : ทฤษฎีบทจุดตรึงของการส่งหลายค่าแบบครึ่งต่อเนื่อง (FIXED POINT THEOREM OF HALF-CONTINUOUS MULTIVALUED MAPPINGS) อ.ที่ปรึกษาวิทยานิพนธ์หลัก : รศ. ดร. อิ่มจิตต์ เติมวุฒิพงษ์ , 36 หน้า

กำหนดให้ E เป็นปริภูมิเวกเตอร์เชิงทอพอโลยีบนสนามของจำนวนจริง และ C เป็นเซตย่อยไม่ว่างของ E การส่ง F จาก C ไปยังเซตของเซตย่อยไม่ว่างของ E ได้ชื่อว่าเป็นการส่ง *แบบครึ่งต่อเนื่อง* ถ้าสำหรับแต่ละเวกเตอร์ $x \in C$ ที่ $x \notin F(x)$ จะมีฟังก์ชันนัลเชิงเส้นแบบต่อเนื่อง $p \in E^*$ และมีย่านใกล้เคียง W ของ x ใน C ซึ่ง สำหรับแต่ละเวกเตอร์ $y \in W$ ถ้า $y \notin F(y)$ แล้ว $p(z-y) > 0$ ทุกเวกเตอร์ $z \in F(y)$

ในงานนี้เราพิสูจน์ว่า ถ้า E เป็นปริภูมิเฮาส์ดอฟฟ์ซึ่งมีสมบัติคอนเวกซ์เฉพาะที่ และ C เป็นเซตย่อยไม่ว่างที่กระชับและคอนเวกซ์ของ E แล้วทุกการส่ง F แบบครึ่งต่อเนื่องบน C ไปยังเซตของเซตย่อยของ C ที่ไม่เป็นเซตว่างย่อมมีจุด x_0 ใน C ซึ่ง $x_0 \in F(x_0)$ นั่นคือ x_0 เป็นจุดตรึงของ F

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Let E be a topological vector space over $\mathbb{R}$ and let C be a nonempty subset of E. A mapping F from C into the set of nonempty subsets of E, is said to be *half-continuous* if for each $x \in C$ with $x \notin F(x)$ there exists a (nonzero) continuous linear functional $p \in E^*$ and a neighborhood W of x in C such that if $y \in W$, such that $y \notin F(y)$, then for every $z \in F(y)$, $p(z - y) > 0$.

In this work, we prove that if E is locally convex Hausdorff and C is a nonempty compact convex subset of E, then every half-continuous mapping F on C into the set of nonempty subsets of C there exists a point x_0 in C such that $x_0 \in F(x_0)$, that is x_0 is a fixed point of F.

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CONTENTS

CHAPTER I

INTRODUCTION

Almost a century ago, L.E.J. Brouwer proved a famous theorem in fixed point theory, that any continuous mapping from the closed unit ball of the Euclidean space $\mathbb{R}^n$ to itself has a fixed point. Later in 1930, J. Schauder extended Brouwer's theorem to Banach spaces (see [10]).

In 2008, P.J.J. Hering, et al. ([11]) proposed a new type of mapping which is possibly discontinuous. They called such mappings *locally gross direction preserving* and proved that every locally gross direction preserving mapping defined on a nonempty polytope (the convex hull of a finite subset of $\mathbb{R}^n$) has a fixed point. Their work both allows discontinuities of mappings and generalizes Brouwer's theorem.

Later, P. Bich ([3]) extended the work of Hering, et al., to an arbitrary nonempty compact convex subset of $\mathbb{R}^n$. Moreover, in [2], Bich established a new class of mappings which contains the class of locally gross direction preserving mappings. He called the mappings in that class *half-continuous* and proved that if C is a nonempty compact convex subset of a Banach space and $f : C \to C$ is half-continuous, then f has a fixed point. Furthermore, in the same work, Bich extended the notion of half-continuity to multivalued mappings and proved fixed point theorems which generalize several well-known results.

In this thesis, we prove that some results of Bich are also valid in locally convex Hausdorff topological vector spaces, and also show that several well-known theorems can be obtained from our results.

CHAPTER II

PRELIMINARIES

In this chapter, we review some notations, terminologies, and fundamental facts that will be used throughout our work.

Definition 2.0.1. Let X be a set and $A \subseteq X$. A point $a \in A$ is called a **fixed point** of a mapping $f : A \to X$ if $a = f(a)$.

Definition 2.0.2. A mapping F from a set X into 2^Y (the set of nonempty subsets of a set Y) is called a **multivalued** mapping from X into Y. A **fibers** of F at $y \in Y$ is the set $F^{-}(y) = \{x \in X : y \in F(x)\}$. For a multivalued mapping F from X into Y, a mapping $f : X \to Y$ is called a **selection** of F if $f(x) \in F(x)$ for all $x \in X$.

Definition 2.0.3. Let X be a set and $A \subseteq X$. A point $a \in A$ is called a **fixed point** of a mapping $F : A \to 2^X$ if $a \in F(a)$.

2.1 Topological Spaces

Definition 2.1.1. A **topology** τ on a set X is a collection of subsets of X such that

1. $\varnothing$ and X belong to τ.

2. Any union of elements of τ belongs to τ.

3. Any finite intersection of elements of τ belongs to τ.

By a **topological space** we mean a nonempty set X together with a topology τ on it, usually denoted by (X, τ) or simply by X. The elements of τ are called **open sets** (of X). A set $F \subseteq X$ is said to be **closed** in X if its complement is open in X. A subcollection $\mathcal{B}$ of τ is said to be a **basis** of τ if for every G in τ and $x \in G$ there is B_x in $\mathcal{B}$ such that $x \in B_x \subseteq G$.

Definition 2.1.2. Let X be a set and $\mathcal{S}$ a collection of subsets of X. $\mathcal{S}$ is said to be a **subbasis** for a topology on X if $\bigcup \mathcal{S} = X$. The topology generated by $\mathcal{S}$ (as a subbasis) is the collection of all unions of finite intersections of elements of $\mathcal{S}$.

Definition 2.1.3. Let (X, τ) be a topological space and $S \subseteq X$. Then $\tau_S = \{G \cap S : G \in \tau\}$ is a topology on S and it is called the **relative topology** on S. The set S equipped with the relative topology is called a **subspace** of X.

Definition 2.1.4. Let X and Y be topological spaces. The **product topology** on $X \times Y$ is the topology having as basis the collection of all sets of the form $U \times V$, where U is an open subset of X and V is an open subset of Y.

Definition 2.1.5. Let X be a topological space. A **neighborhood** of a point $x \in X$ is any open set that contains x.

Definition 2.1.6. Let X and Y be topological spaces. Let $f : X \to Y$ and $x \in X$. Then f is **continuous at** x if and only if for every neighborhood W of $f(x)$, there is a neighborhood V of x such that $f(V) \subseteq W$.

If f is continuous at every point in X, then f is said to be **continuous** (on X). If f is bijective and both f and f^{-1} are continuous, then f is said to be a **homeomorphism**.

Proposition 2.1.7. (see [8], p.119) *Let X, Y be topological spaces and $f : X \to Y$. Then the following statements are equivalent:*

(1) *f is continuous;*

(2) *If V is open in Y, then $f^{-1}(V)$ is open in X;*

(3) *If K is closed in Y, then $f^{-1}(K)$ is closed in X.*

Definition 2.1.8. Let X be a topological space, $x \in X$ and $f : X \to \mathbb{R}$. Then f is said to be **lower semicontinuous** (respectively **upper semicontinuous**) at x if for any $\epsilon > 0$ there exists a neighborhood U of x such that $f(y) > f(x) - \epsilon$ (respectively $f(y) < f(x) + \epsilon$) for all $y \in U$.

Remark 2.1.9. Clearly, a mapping f is lower semicontinuous if and only if $-f$ is upper semicontinuous.

Definition 2.1.10. Let X and Y be topological spaces. A mapping $F : X \to 2^Y$ is called **upper semicontinuous** if for each $x_0 \in X$ and neighborhood V of $F(x_0)$ in Y, there exists a neighborhood U of x_0 in X such that $F(x) \subseteq V$ for all $x \in U$.

Definition 2.1.11. A topological space X is said to be **Hausdorff** if any two distinct points in X have disjoint neighborhoods.

Proposition 2.1.12. (see [8], p.120) *If X and Y are Hausdorff topological spaces, then so is $X \times Y$.*

Definition 2.1.13. Let X be a topological space and $K \subseteq X$. A **cover** of K is a collection $\mathcal{G}$ of sets in X whose union contains K; that is,

$$K \subseteq \bigcup\{G : G \in \mathcal{G}\}.$$

If each $G \in \mathcal{G}$ is open, we call $\mathcal{G}$ an **open cover** of K. If $\mathcal{G}'$ is a subcollection of a cover $\mathcal{G}$ which is also a cover of K, then $\mathcal{G}'$ is called a **subcover** of $\mathcal{G}$. If $\mathcal{G}'$ consists of finitely many sets, then we call $\mathcal{G}'$ a **finite subcover** of $\mathcal{G}$. X is said to be **compact** if every open cover of X has a finite subcover. And K is said to be **compact** if K is compact with respect to the subspace topology.

Definition 2.1.14. Let X be a topological space. The intersection of all closed sets in X containing F is called the **closure** of F in X and is denoted by $\overline{F}$.

Proposition 2.1.15. ([8]) *Let X be a topological space.*

(1) *A set $F \subseteq X$ is closed if and only if $F = \overline{F}$.*

(2) *A closed subset of a compact space is compact.*

(3) *Every compact subset of a Hausdorff space is closed.*

Definition 2.1.16. A topological space X is said to be **locally compact** at x if there is a compact subset of X containing a neighborhood of x. If X is locally compact at each of its points, we simply say that X is **locally compact**.

Remark 2.1.17. A compact space is always locally compact.

Definition 2.1.18. An open cover $\mathcal{U}$ of a topological space X is said to be **locally finite** if each $x \in X$ has a neighborhood that intersects only finitely many members of $\mathcal{U}$. If $\mathcal{U}$ and $\mathcal{V}$ are open covers of X, then $\mathcal{V}$ is a **refinement** of $\mathcal{U}$ if for each $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ with $V \subseteq U$. X is said to be **paracompact** if every open cover of X has a locally finite refinement.

Remark 2.1.19. It is known that any compact space is paracompact and the real line $\mathbb{R}$ is paracompact but not compact (see [12]).

For the topological spaces X and Y, let Y^X denote the set of all mappings from X to Y and $\mathfrak{C}[X, Y]$ the set of all continuous mappings from X to Y.

Definition 2.1.20. Let X and Y be topological spaces. For a compact subset K of X and an open subset U of Y , let

$$\mathfrak{S}(K, U) = \{f \in Y^X : f(K) \subseteq U\}.$$

The **compact-open topology** on Y^X is the topology generated by the sets $\mathfrak{S}(K, U)$ as a subbasis.

Definition 2.1.21. Let X be a topological space and (Y, d) a metric space. For each $f \in Y^X$, a compact subset K of X and $\epsilon > 0$, let

$$\mathfrak{B}_K(f, \epsilon) = \{g \in Y^X : \sup\{d(f(x), g(x)) : x \in K\} < \epsilon\}.$$

The set $\mathfrak{B}_K(f, \epsilon)$ form a basis for a topology on Y^X. It is called the **topology of compact convergence**.

Remark 2.1.22. If X is a topological space and Y a metric space, then, on $\mathfrak{C}[X, Y]$, the topology of compact convergence and the compact-open topology coincide (see [12]).

Let A and B be sets. Then A is **isomorphic** to B if there is a bijection from A onto B.

Theorem 2.1.23. (see [12]) *Let X, Y and Z be topological spaces where Y is locally compact Hausdorff. Suppose $\mathfrak{C}[Y, Z]$ has the compact-open topology, then $\mathfrak{C}[X \times Y, Z]$ and $\mathfrak{C}[X, \mathfrak{C}[Y, Z]]$ are isomorphic as sets.*

Definition 2.1.24. A **directed set** is a pair $\{D, \preceq\}$, where D is a set and $\preceq$ is a relation on D such that

1. $\alpha \preceq \alpha$ for every $\alpha \in D$.
2. For any $\alpha, \beta, \gamma \in D$, if $\alpha \preceq \beta$ and $\beta \preceq \gamma$, then $\alpha \preceq \gamma$.
3. For any $\alpha, \beta \in D$, there is some $\gamma \in D$ such that $\alpha \preceq \gamma$ and $\beta \preceq \gamma$.

Definition 2.1.25. Let X be a set. A **net** in X is a function from a directed set $\{D, \preceq\}$ into X. The net $\{(\alpha, x_\alpha) : \alpha \in D\}$ will be denoted by $(x_\alpha)_{\alpha \in D}$.

Definition 2.1.26. A net $(x_\alpha)_{\alpha \in D}$ in a topological space X is said to **converges** to a point x in X if for any neighborhood W of x, there is an element $\alpha_0 \in D$ such that for any $\alpha \in D, \alpha_0 \preceq \alpha$ implies $x_\alpha \in W$. We call x a **limit** of the net $(x_\alpha)_{\alpha \in D}$ and write $x_\alpha \to x$.

Definition 2.1.27. Let X be a set. Let $f : D \to X$ be a net and let $f(\alpha) = x_\alpha$. If M is a directed set and $g : M \to D$ is a mapping such that

1. $\mu_1 \preceq \mu_2$ implies $g(\mu_1) \preceq g(\mu_2)$,

2. for each $\alpha \in D$, there is some $\mu \in M$ such that $\alpha \preceq g(\mu)$,

then the composition $f \circ g : M \to X$ is called a **subnet** of $(x_\alpha)_{\alpha\in D}$. For $\mu \in M$, the net $f \circ g(\mu)$ is denoted by $(x_{\alpha_\mu})_{\mu\in M}$.

Proposition 2.1.28. ([8]) *Let X and Y be topological spaces.*

(1) *The space X is Hausdorff if and only if every net in X converges to at most one limit.*

(2) *Let $A \subseteq X$ and $x \in X$. Then $x \in \overline{A}$ if and only if there is a net in A which converges to x.*

(3) *A mapping $f : X \to Y$ is continuous at $x \in X$ if and only if for any net $(x_\alpha)_{\alpha\in D}$ in X, $(x_\alpha)_{\alpha\in D}$ converges to x implies $(f(x_\alpha))_{\alpha\in D}$ converges to $f(x)$.*

(4) *If the net converges to a point $x \in X$, then so does any of its subnet.*

(5) *The space X is compact if and only if every net in X has a subnet converging to some point of X.*

Proposition 2.1.29. (see [12], p.188) *Let X and Y be topological spaces, $(x_\alpha)_{\alpha\in D}$ a net in X and $(y_\alpha)_{\alpha\in D}$ a net in Y. If $(x_\alpha)_{\alpha\in D}$ converges to a point x in X and $(y_\alpha)_{\alpha\in D}$ converges to a point y in Y, then (x_α, y_α) converges to (x, y) in $X \times Y$.*

Definition 2.1.30. Let X and Y be topological spaces and let F be a multivalued mapping from X into Y. The **graph** of F is the set

$$G_F = \{(x, y) \in X \times Y : y \in F(x)\}.$$

We say that F has a **closed graph** if G_F is a closed subset of $X \times Y$.

Theorem 2.1.31. (see [15]) *Let X be a topological space, Y a compact Hausdorff space and $F : X \to 2^Y$ a multivalued mapping with nonempty closed values. Then the following statements are equivalent.*

(1) *F is upper semicontinuous.*

(2) *The graph of F is closed in $X \times Y$.*

(3) *For each $x \in X, y \in Y$, net (x_α) in X and net (y_α) in Y, if $x_\alpha \to x, y_\alpha \to y$ and $y_\alpha \in F(x_\alpha)$, then $y \in F(x)$.*

2.2 Topological Vector Spaces

Definition 2.2.1. A **topological vector space** is a vector space E over a field $\mathbb{F}$ equipped with a topology such that the mapping $(x, y) \mapsto x + y$ of $E \times E$ to E and $(\lambda, x) \mapsto \lambda x$ of $\mathbb{F} \times E$ to E (with the usual topology on $\mathbb{F}$) are continuous.

In this research, by a topological vector space we mean a topological vector space over $\mathbb{R}$. For each $x \in E$ and $\alpha \in \mathbb{R} \setminus \{0\}$, the **translation operator** T_x and the **multiplication operator** M_α are defined by

$$T_x(y) = x + y \quad \text{and} \quad M_\alpha(y) = \alpha y$$

for $y \in E$. We note that T_x and M_α are homeomorphism of E onto E ([14], p.8).

Proposition 2.2.2. *Let E be a topological vector space. Let $(x_\alpha)_{\alpha \in D}$ and $(y_\alpha)_{\alpha \in D}$ be nets in E and $x, y \in E$. If $x_\alpha \to x$ and $y_\alpha \to y$, then $x_\alpha + y_\alpha \to x + y$.*

Proof. Assume that $x_\alpha \to x$ and $y_\alpha \to y$. By Proposition 2.1.29, $(x_\alpha, y_\alpha) \to (x, y)$. Since the addition is continuous, by Proposition 2.1.28(3), $x_\alpha + y_\alpha \to x + y$. □

Definition 2.2.3. Let E be a vector space. A set $[x, y] = \{\lambda x + (1 - \lambda) y : 0 \leq \lambda \leq 1\}$ is called the **line segment** joining vectors x and y in E. A set $C \subseteq E$ is said to be **convex** if every pair of points x, y in C, $[x, y] \subseteq C$.

Definition 2.2.4. Let S be a nonempty subset of a topological vector space. The **convex hull** of S is defined to be the intersection of all convex sets containing S. We denote it by co S.

Note that the convex hull of S consists of all the points which are expressible in the form

$$\alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n,$$

where $x_1, x_2, ..., x_n$ are any points of S, $\alpha_k \geq 0$ for each k and $\sum_{k=1}^{n} \alpha_k = 1$.

Proposition 2.2.5. *Let G, H be subsets of a topological vector space and α a nonzero real number. If G is open, then $H + G$, αG and co G are open.*

Proof. Assume that G is open. We note that $x + G = T_x(G)$ for all $x \in H$. Since T_x is open for all $x \in H$,

$$H + G = \bigcup \{x + G : x \in H\}$$

is open. And since M_α is an open map, $\alpha G = M_\alpha(G)$ is open. Next, for each $n \in \mathbb{N}$, let

$$K_n = \left\{ \Lambda : \{1, 2, ..., n\} \to [0, 1] : \sum_{i=1}^{n} \Lambda(i) = 1 \right\}. \tag{2.1}$$

For each $n \in \mathbb{N}$ and $\Lambda \in K_n$, let

$$G_{n,\Lambda} = \left\{ x \in E : \text{ there exists } x_1, x_2, ..., x_n \in G \text{ such that } x = \sum_{i=1}^{n} \Lambda(i) x_i \right\} \tag{2.2}$$

Then co $G = \bigcup_{n \in \mathbb{N}} \bigcup_{\Lambda \in K_n} G_{n,\Lambda}$. We complete the proof by showing that $G_{n,\Lambda}$ is open for all $n \in \mathbb{N}$ and $\Lambda \in K_n$. Let $n \in \mathbb{N}$ and $\Lambda \in K_n$. It is easy to see that

$$G_{n,\Lambda} = \Lambda(1)G + \Lambda(2)G + ... + \Lambda(n)G. \tag{2.3}$$

Notice that $\Lambda(j) > 0$ for some $j \in \{1, 2, ..., n\}$. Consequently, $G_{n,\Lambda}$ is open. □

Definition 2.2.6. A topological vector space E is said to be **locally convex** if every neighborhood of 0 contains a convex neighborhood of 0.

Example 2.2.7. Any normed space is a locally convex Hausdorff topological vector space (see [8], p.166).

Definition 2.2.8. Let E be a topological vector space over $\mathbb{R}$. By a **linear functional** we mean a linear mapping $f : E \to \mathbb{R}$. The **dual space** of E is the vector space E^* whose elements are the continuous linear functional on E.

Clearly, E^* is a vector space with the operations defined by

$$(p+q)(x) = p(x) + q(x) \text{ and } (\alpha p)(x) = \alpha p(x)$$

for $p, q \in E^*$, $x \in E$ and $\alpha \in \mathbb{R}$. In this work, we consider E^* equipped with the topology of compact convergence, so E^* becomes a topological vector space. To see this, let $(p, q) \in E^* \times E^*$ and W a neighborhood of $p+q$ in E^*. There exists a compact set K in E and $\epsilon > 0$ such that

$$\mathfrak{B}_K(p+q, \epsilon) = \{f \in E^* : \sup_{x \in K} |\langle (p+q) - f, x\rangle| < \epsilon\} \subseteq W.$$

Let $f \in \mathfrak{B}_K(p, \frac{\epsilon}{4})$ and $g \in \mathfrak{B}_K(q, \frac{\epsilon}{4})$. Then, for each $x \in K$,

$$|\langle (p+q) - (f+g), x\rangle| \leq |\langle p - f, x\rangle| + |\langle q - g, x\rangle| < \frac{\epsilon}{2}.$$

Thus, $\sup_{x \in K} |\langle (p+q) - (f+g), x\rangle| \leq \frac{\epsilon}{2} < \epsilon$; i.e., $f + g \in \mathfrak{B}_K(p+q, \epsilon) \subseteq W$. Hence, the addition is continuous.

Next, let $(t, p) \in \mathbb{R} \times E^*$ and W a neighborhood of tp in E^*. There exists a compact set K in E and $\epsilon > 0$ such that

$$\mathfrak{B}_K(tp, \epsilon) = \{f \in E^* : \sup_{x \in K} |\langle tp - f, x\rangle| < \epsilon\} \subseteq W.$$

Since K is compact and $p \in E^*$, there exists $m \in \mathbb{R}$ such that $|\langle p, x\rangle| \leq m$ for every $x \in K$. Let $f \in \mathfrak{B}_K(p, \frac{\epsilon}{4(|t|+1)})$ and $s \in \mathbb{R}$ be such that $|t - s| < \frac{\epsilon(|t|+1)}{4(\epsilon + m(|t|+1))}$.

Then, for each $x \in K$,

$$\begin{aligned}
|\langle sf - tp, x\rangle| &\leq |\langle sf - tf, x\rangle| + |\langle tf - tp, x\rangle| \\
&= |\langle f, x\rangle|\,|s-t| + |t|\,|\langle f - p, x\rangle| \\
&< \left(\frac{\epsilon}{|t|+1} + |\langle p, x\rangle|\right)\frac{\epsilon\,(|t|+1)}{4\,(\epsilon + m\,(|t|+1))} + |t|\,\frac{\epsilon}{4\,(|t|+1)} \\
&\leq \left(\frac{\epsilon}{|t|+1} + m\right)\frac{\epsilon\,(|t|+1)}{4\,(\epsilon + m\,(|t|+1))} + |t|\,\frac{\epsilon}{4\,(|t|+1)} \\
&= \frac{\epsilon}{2}.
\end{aligned}$$

Thus, $\sup_{x\in K} |\langle sf - tp, x\rangle| \leq \frac{\epsilon}{2} < \epsilon$; i.e., $sf \in \mathfrak{B}_K(tp, \epsilon) \subseteq W$. Hence, the scalar multiplication is continuous. Therefore, E^* is a topological vector space as desired.

For $p \in E^*$ and $x \in E$, it is often convenient to write $\langle p, x\rangle$ instead of $p(x)$. The reason for this is that oftenly the vector x or the continuous linear functional p may be given in a notation containing parentheses or other complicated form.

Definition 2.2.9. Let E be a topological vector space. We say that E^* **separates points** on E if $\langle p, x_1\rangle \neq \langle p, x_2\rangle$ for some $p \in E^*$, whenever x_1 and x_2 are distinct points of E.

Remark 2.2.10. The property "E^* separates points on E "guarantees that $E^* \neq \{0\}$ if $E \neq \{0\}$.

The next separation theorem in topological vector space is useful.

Theorem 2.2.11. (see [14], p.58) *Let E be a Hausdorff topological vector space and let A, B be disjoint nonempty convex subsets of E. If A is open, there exists $p \in E^*$ and $\gamma \in \mathbb{R}$ such that $\langle p, x\rangle < \gamma \leq \langle p, y\rangle$ for every $x \in A$ and $y \in B$.*

If E is a locally convex Hausdorff topological vector space, then E^* separates points on E (see [14], p.59). Note that the converse is not true as shown in the following example. It appeared as an exercise in [14]. We give the proof here.

Example 2.2.12. Let $0 < p < 1$. Consider the topological vector space

$$\ell^p = \{(x_n)_{n=1}^{\infty} : \sum_{n=1}^{\infty} |x_n|^p < \infty\}$$

metrized by

$$d(x, y) = \sum_{n=1}^{\infty} |x_n - y_n|^p ,$$

where $x = (x_n), y = (y_n)$. Then, ℓ^p is not locally convex but $(\ell^p)^*$ separates points on ℓ^p.

Proof. Suppose ℓ^p is locally convex. Let $r > 0$ be arbitrary. By local convexity, there exists a convex neighborhood U of 0 such that $U \subseteq B(0; r)$. So $B(0; \epsilon) \subseteq U$ for some $\epsilon > 0$. Then

$$\text{co } B(0; \epsilon) \subseteq U \subseteq B(0; r). \tag{2.4}$$

Put $\delta = \epsilon + 1$. For each $n \in \mathbb{N}$, let $x_n = (\underbrace{0, 0, ..., 0, \delta}_{n \text{ terms}}, 0, ...)$. Since $\delta^p < \delta$, $x_n \in B(0; \delta)$ and so

$$z := \frac{1}{n}(x_1 + x_2 + ... + x_n) \in \text{co } B(0; \epsilon) \tag{2.5}$$

for all $n \in \mathbb{N}$. From (2.4) and (2.5) we have

$$n\left(\frac{\delta}{n}\right)^p = d(z, 0) < r$$

for all $n \in \mathbb{N}$. This means local convexity requires that, given $r > 0$, there is $\delta > 1$ such that

$$n^{1-p}\delta^p < r \tag{2.6}$$

for all $n \in \mathbb{N}$. This is impossible because n in (2.6) can be chosen so that $n^{1-p}\delta^p$ is arbitrarily large. Hence, ℓ^p is not locally convex.

Next, we will show that $(\ell^p)^*$ separates points on ℓ^p. Let $x = (x_n)$, $y = (y_n)$ belong to ℓ^p be such that $x \neq y$. Then $x_m \neq y_m$ for some $m \in \mathbb{N}$. Suppose that $y \in \text{co } B(x; \epsilon)$ for all $\epsilon > 0$. Let $\epsilon > 0$ be given. Then

$$y = \sum_{j=1}^{k} \alpha_j x^{(j)}, \tag{2.7}$$

where $\sum_{j=1}^{k}\alpha_j = 1,\ \alpha_j \geq 0$ and $x^{(1)}, x^{(2)}, ..., x^{(k)} \in B(x;\epsilon)$. We note that

$$\left|x_m - x_m^{(j)}\right|^p \leq \sum_{n=1}^{\infty}\left|x_n - x_n^{(j)}\right|^p < \epsilon \tag{2.8}$$

for all $j = 1, 2, ..., k$. Since $\epsilon > 0$ is arbitrary, we have $x_m = x_m^{(j)}$ for all $j = 1, 2, ..., k$. Hence,

$$y_m = \sum_{j=1}^{k}\alpha_j x_m^{(j)} = \sum_{j=1}^{k}\alpha_j x_m = x_m\left(\sum_{j=1}^{k}\alpha_j\right) = x_m \ ,$$

which is a contradiction. So, $y \notin \text{co}\ B(x;\epsilon)$ for some $\epsilon > 0$. By Proposition 2.2.5 and Theorem 2.2.11, there exists $\Lambda \in (\ell^p)^*$ such that $\langle \Lambda, x\rangle \neq \langle \Lambda, y\rangle$. □

Definition 2.2.13. Let E be a topological vector space whose dual E^* separates points on E. The smallest topology on E that makes every $p \in E^*$ continuous is called the **weak topology** of E.

Remark 2.2.14. A set W is open in the weak topology if and only if for each $x \in W$ there are $p_1, p_2, ..., p_n \in E^*$ and positive real numbers $\epsilon_1, \epsilon_2, ..., \epsilon_n$ such that

$$\{y \in E : |\langle p_i, y - x\rangle| < \epsilon_i,\ i = 1, 2, ..., n\} \subseteq W.$$

Proposition 2.2.15. *Let E be a topological vector space. If E^* separates points on E, then the weak topology is Hausdorff.*

Proof. Let $x_1,\ x_2 \in E$ be such that $x_1 \neq x_2$. Then there exists $p \in E^*$ such that $\langle p, x_1\rangle \neq \langle p, x_2\rangle$. Let $\epsilon = |\langle p, x_1 - x_2\rangle|$. Then $\{z \in E : |\langle p, z - x_1\rangle| < \epsilon/2\}$ and $\{z \in E : |\langle p, z - x_2\rangle| < \epsilon/2\}$ are disjoint neighborhoods of x_1 and x_2 in weak topology, respectively. □

Remark 2.2.16. If E^* separates points on E, then the (strong) topology on E is also Hausdorff.

Definition 2.2.17. Let E be a topological vector space. An **open half space** H in E is the set of the form $H = \{x \in E : p(x) > \alpha\}$, where $p \in E^* \setminus \{0\}$ and $\alpha \in \mathbb{R}$. Let C be a topological space and $F : C \to 2^E$. A mapping F is called **upper demicontinuous** if for each $x_0 \in C$ and any open half space H in E containing $F(x_0)$, there exists a neighborhood U of x_0 in C such that $F(x) \subseteq H$ for all $x \in U$.

It is clear that an upper semicontinuous multivalued mapping is upper demicontinuous but the converse is not true (see [1]).

In the following, we list some important theorems which will be used in our work.

Theorem 2.2.18. (Browder, see [4]) *Let C be a nonempty compact convex subset of a Hausdorff topological vector space E. Suppose $T : C \to 2^C$ is a multivalued mapping having nonempty convex values and open fibers, then T has a fixed point.*

Theorem 2.2.19. (Ben-El-Mechaiekh et al., see [10]) *Let X be a paracompact Hausdorff space and Y a convex subset of a topological vector space. Suppose $\Phi : X \to 2^Y$ is a multivalued mapping having nonempty convex values and open fibers, then there exists a continuous selection $\varphi : X \to Y$ of Φ.*

Theorem 2.2.20. (see [14], p.58) *Let A and B be disjoint nonempty convex subsets of a locally convex Hausdorff topological vector space E. If A is compact and B is closed, then there exists $p \in E^*$ and $\alpha_1, \alpha_2 \in \mathbb{R}$ such that $\langle p, x\rangle < \alpha_1 < \alpha_2 < \langle p, y\rangle$ for all $x \in A$ and $y \in B$.*

Theorem 2.2.21. (see [14], p.70) *Let E be a topological vector space on which E^* separates points. Suppose that A and B are disjoint nonempty compact convex sets in E. Then there exists $p \in E^*$ such that*

$$\sup_{x \in A} \langle p, x\rangle < \inf_{y \in B} \langle p, y\rangle.$$

CHAPTER III

HALF-CONTINUITY AND FIXED POINT THEOREMS

3.1 Half-continuous Mappings

Now, we introduce the notion of half-continuity in topological vector spaces, and investigate some of their properties.

Definition 3.1.1. Let E be a topological vector space and let C be a nonempty subset of E. A mapping $f : C \to E$ is said to be **half-continuous** if for each $x \in C$ with $x \neq f(x)$ there exists a (nonzero) continuous linear functional $p \in E^*$ and a neighborhood W of x in C such that

$$\langle p, f(y) - y \rangle > 0 \tag{3.1}$$

for all $y \in W$ with $y \neq f(y)$.

By the name "half-continuous", it induces us to think that continuous mappings should be half-continuous. The following theorem tells us that the statement is affirmative, if E^* separates points on E.

Theorem 3.1.2. *Let E be a topological vector space whose E^* separates points and let C be a nonempty subset of E. Then every continuous mapping $f : C \to E$ is half-continuous.*

Proof. Assume that $f : C \to E$ is continuous. Let $x \in C$ be such that $x \neq f(x)$. Since E^* separates points on E, we may assume that, $\langle p, f(x) - x \rangle > 0$ for some $p \in E^* \setminus \{0\}$.

Define $\psi : C \to \mathbb{R}$ by

$$\psi(z) = \langle p, f(z) - z \rangle.$$

for all $z \in C$. Since f and p are continuous, so is ψ, and thus there exists a neighborhood W of x in C such that $\psi(W) \subseteq (0, \infty)$. Then, for each $y \in W$ with $y \neq f(y)$, we have $\langle p, f(y) - y \rangle = \psi(y) > 0$. Therefore, f is half-continuous. $\square$

The hypothesis that E^* separates points on E cannot be relaxed as will be shown in the following examples. To see this, we will first prove the following useful result.

Lemma 3.1.3. *If E is a topological vector space whose only convex open subsets are the empty set and E itself, then E^* contains only the zero functional.*

Proof. Assume that only convex open subsets of E are the empty set and E itself. Let $\Lambda \in E^*$. Suppose that $\Lambda(x) \neq 0$ for some $x \in E$. If $\Lambda(x) > 0$, then $x \in \Lambda^{-1}(0, \infty)$. Thus, $\Lambda^{-1}(0, \infty)$ is a nonempty convex open subset of E, so $\Lambda^{-1}(0, \infty) = E$. This is impossible because $0 \notin \Lambda^{-1}(0, \infty)$. Similarly, for the case $\Lambda(x) < 0$. Hence, $\Lambda(x) = 0$ for all $x \in E$. Consequently, $E^* = \{0\}$. $\square$

Example 3.1.4. Let E be a nontrivial vector space. Then the topology $\{\varnothing, E\}$ makes E into a locally convex topological vector space that is not Hausdorff. By Lemma 3.1.3, $E^* = \{0\}$, so E^* does not separate points on E. Consequently, every continuous self-mapping on E which is not the identity, is not half-continuous.

Example 3.1.5. (see [14], p.35) For $0 < p < 1$, let $L^p[0, 1]$ be the collection of equivalence classes of Lebesgue measurable functions $f : [0, 1] \to \mathbb{R}$ such that

$$\Delta(f) = \int_0^1 |f(t)|^p \, dt < \infty.$$

Then, $L^p[0, 1]$ is a Hausdorff topological vector space such that $(L^p[0, 1])^* = \{0\}$ (further example see [5]). To prove this, we need the following lemma.

Lemma 3.1.6. *For each* $\alpha,\ \beta \geq 0$ *and* $0 < p < 1,\ (\alpha+\beta)^p \leq \alpha^p + \beta^p$.

Proof. Let $\alpha, \beta \geq 0$ and $0 < p < 1$. If $\alpha = \beta = 0$, then the result is obvious. Assume that $\alpha \geq \beta > 0$. Since the map $x \mapsto x^p$ is continuous on $[\alpha, \alpha+\beta]$ and differentiable on $(\alpha, \alpha+\beta)$, by Mean Value Theorem, there exists $\xi \in (\alpha, \alpha+\beta)$ such that

$$(\alpha+\beta)^p - \alpha^p = p\xi^{p-1}\beta. \tag{3.2}$$

Since $p - 1 < 0$,

$$p\xi^{p-1}\beta < p\alpha^{p-1}\beta \leq p\beta^{p-1}\beta < \beta^p. \tag{3.3}$$

From (3.2) and (3.3), we have $(\alpha+\beta)^p < \alpha^p + \beta^p$. □

Proof. (Example 3.1.5) By using Lemma 3.1.6, we can see that $d(f,g) = \Delta(f-g)$ defines a metric on $L^p[0,1]$.

Let G be a nonempty open convex subset of $L^p[0,1]$. By translation, we may assume that $0 \in G$. Then $B(0;R) \subseteq G$ for some $R > 0$. Let $f \in L^p[0,1]$. Choose $n \in \mathbb{N}$ be such that

$$n^{p-1}\Delta(f) < R. \tag{3.4}$$

Define $\varphi : [0,1] \to \mathbb{R}$ by

$$\varphi(x) = \int_0^x |f(t)|^p\, dt$$

for all $x \in [0,1]$. Since $|f(t)|^p$ is nonnegative and $\int_0^1 |f(t)|^p\, dt < \infty$, $|f(t)|^p$ is integrable over $[0,1]$ and hence φ is continuous on $[0,1]$ (see [13], p.88 and p.105). For each $i \in \{1, 2, ..., n\}$, since $\frac{i\Delta(f)}{n} \in [\varphi(0), \varphi(1)]$, by Intermediate Value Theorem, there exists $x_i \in [0,1]$ such that

$$\int_0^{x_i} |f(t)|^p\, dt = \varphi(x_i) = \frac{i\Delta(f)}{n}. \tag{3.5}$$

Notice that $0 = x_0 < x_1 < x_2 < ... < x_n = 1$ and for any $i \in \{1, 2, ..., n\}$,

$$\begin{aligned}\int_{x_{i-1}}^{x_i} |f(t)|^p \, dt &= \int_0^{x_i} |f(t)|^p \, dt - \int_0^{x_{i-1}} |f(t)|^p \, dt \\ &= \frac{i\Delta(f)}{n} - \frac{(i-1)\Delta(f)}{n} \\ &= \frac{\Delta(f)}{n}.\end{aligned}$$

For each $i \in \{1, 2, ..., n\}$, define $g_i : [0,1] \to \mathbb{R}$ by

$$g_i(t) = \begin{cases} f(0) & \text{if } t = 0; \\ nf(t) & \text{if } x_{i-1} < t \leq x_i; \\ 0 & \text{otherwise.} \end{cases}$$

Then, for $i \in \{1, 2, ..., n\}$,

$$\Delta(g_i) = \int_0^1 |g_i(t)|^p \, dt = \int_{x_{i-1}}^{x_i} |nf(t)|^p \, dt = n^p \frac{\Delta(f)}{n} = n^{p-1}\Delta(f) < R. \tag{3.6}$$

Hence, $g_i \in B(0, R)$ for all $i \in \{1, 2, ..., n\}$. This implies that

$$f = \frac{1}{n}(g_1 + ... + g_n) \in \text{ co } B(0; R) \subseteq G.$$

So, $L^p[0,1] \subseteq G$, and hence $L^p[0,1] = G$. By Lemma 3.1.3, $(L^p[0,1])^* = \{0\}$. □

Remark 3.1.7. There is a half-continuous mapping which is not continuous.

For example [2], let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 3 & \text{if } x \in [0,1); \\ 2 & \text{otherwise.} \end{cases}$$

Clearly, f is discontinuous.

To show that f is half-continuous, let $x \in \mathbb{R}$ be such that $x \neq f(x)$. This implies that $x \neq 2$.

Case $x < 2$. Define a continuous linear functional p on $\mathbb{R}$ by $\langle p, y\rangle = y$ for all $y \in \mathbb{R}$. Choose $\epsilon = 2 - x > 0$. Let $z \in (x - \epsilon, x + \epsilon)$ be such that $z \neq f(z)$. Then $z < 2$ and hence $\langle p, f(z) - z\rangle = f(z) - z > 0$.

Case $x > 2$. Define a continuous linear functional p on $\mathbb{R}$ by $\langle p, y\rangle = -y$ for all $y \in \mathbb{R}$. Choose $\epsilon = x - 2 > 0$. Let $z \in (x - \epsilon, x + \epsilon)$ be such that $z \neq f(z)$. Then $2 < z$ and hence $\langle p, f(z) - z\rangle = z - f(z) > 0$.

Moreover, half-continuity is not closed under the composition, the addition and the scalar multiplication. To see this consider a half-continuous mapping $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(x) = \begin{cases} 3 & \text{if } x \in [3, \infty); \\ 0 & \text{if } x \in (-\infty, 3). \end{cases}$$

Then, for $x \in \mathbb{R}$,

$$(g \circ f)(x) = \begin{cases} 3 & \text{if } x \in [0, 1); \\ 0 & \text{otherwise} \end{cases}$$

$$(g + f)(x) = \begin{cases} 5 & \text{if } x \in [3, \infty); \\ 3 & \text{if } x \in [0, 1) \\ 2 & \text{otherwise} \end{cases}$$

$$(2g)(x) = \begin{cases} 6 & \text{if } x \in [3, \infty); \\ 0 & \text{if } x \in (-\infty, 3). \end{cases}$$

It is easy to see that $g \circ f$, $g + f$ and $2g$ are not half-continuous.

Although half-continuity is not closed under the addition and the scalar multiplication, the following lemma gives us a surprise. The assertion is useful in proving the theorem in the final section.

Proposition 3.1.8. *Let E be a topological vector space whose E^* separates point. Let C a nonempty subset of E and $f : C \to E$. Then f is half-continuous if and only if for any $\beta \in \mathbb{R}$, the mapping $x \mapsto (1-\beta)x + \beta f(x)$ is half-continuous.*

Proof. The sufficiency is clear. To prove the necessity, let $\beta \in \mathbb{R}$ and let $g : C \to E$ be defined by $g(x) = (1-\beta)x + \beta f(x)$ for all $x \in C$. The case $\beta = 0$ is obvious. Let $\beta > 0$. To show that g is half-continuous, let $x \in C$ be such that $x \neq g(x)$. Then $x \neq f(x)$ and hence there exists $p \in E^*$ and a neighborhood W of x in C such that $\langle p, f(y) - y\rangle > 0$ for all $y \in W$ with $y \neq f(y)$. Then

$$\langle p, g(y) - y\rangle = \langle p, (1-\beta)y + \beta f(y) - y\rangle = \beta\langle p, f(y) - y\rangle > 0$$

for all $y \in W$ with $y \neq g(y)$. If $\beta < 0$, then consider $-p$ instead of p. □

Next, we give a sufficient condition for mappings on topological vector space to be half-continuous.

Theorem 3.1.9. *Let E be a topological vector space, C a nonempty subset of E and $f : C \to E$. Suppose that for each $x \in C$ with $x \neq f(x)$, there exists $p \in E^*$ such that $\langle p, f(x) - x\rangle > 0$ and $p \circ f$ is lower semicontinuous at x. Then f is half-continuous.*

Proof. Suppose $f : C \to E$ satisfies the property in the supposition of the theorem. To show that f is half-continuous, let $x \in C$ be such that $x \neq f(x)$. Then there exists $p \in E^*$ such that $\langle p, f(x) - x\rangle > 0$ and $p \circ f$ is lower semicontinuous at x. Let $\alpha \in \mathbb{R}$ be such that $\langle p, f(x) - x\rangle > \alpha > 0$. Since p is continuous at x, there exists a neighborhood V of x in E such that $|\langle p, x - z\rangle| < \alpha$ for all $z \in V$. Thus,

$$\beta := \inf_{z \in V}\langle p, x - z\rangle + \langle p, f(x) - x\rangle > \inf_{z \in V}\langle p, x - z\rangle + \alpha \geq 0. \tag{3.7}$$

Since $p \circ f$ is lower semicontinuous at x, there exists a neighborhood U of x in C such that

$$\langle p, f(y)\rangle > \langle p, f(x)\rangle - \beta \tag{3.8}$$

for all $y \in U$. Then, for each $y \in U \cap V$ with $y \neq f(y)$, we have from (3.7) and (3.8) that

$$\begin{aligned}\langle p, f(y) - y\rangle &= \langle p, f(y)\rangle + \langle p, -y\rangle \\ &> \langle p, f(x)\rangle - \beta + \langle p, -y\rangle \\ &= \langle p, f(x) - x\rangle - \beta + \langle p, x - y\rangle \\ &\geq \langle p, f(x) - x\rangle - \beta + \inf_{z\in V}\langle p, x - z\rangle \\ &= 0.\end{aligned}$$

Therefore, f is half-continuous. □

As a consequence of Theorem 3.1.9 and Remark 2.1.9, we have the following theorem

Theorem 3.1.10. *Let E be a topological vector space, C a nonempty subset of E and $f : C \to E$. Suppose that for each $x \in C$ with $x \neq f(x)$, there exists $p \in E^*$ such that $\langle p, f(x) - x\rangle < 0$ and $p \circ f$ is upper semicontinuous at x. Then f is half-continuous.*

Remark 3.1.11. If E is a Banach space, then Theorem 3.1.9 and 3.1.10 are Proposition 2.4 in [2].

By considering the mapping f in Remark 3.1.7, we note that the converse of Theorem 3.1.9 and Theorem 3.1.10 are not true (see [2]). We will show that at $x = 0$, f satisfies neither assumptions of Theorem 3.1.9 nor assumptions of Theorem 3.1.10. Let $p \in E^*$. Suppose that $\langle p, 3\rangle > 0$. Then $3\langle p, 1\rangle > 0$, so $\langle p, 1\rangle > 0$. Set $\epsilon = \langle p, \frac{1}{2}\rangle > 0$. Let U be any neighborhood of 0. Then there exists $\delta > 0$ such that $(-\delta, \delta) \subseteq U$. Choose $z = -\frac{\delta}{2} \in U$. Thus,

$$\langle p, f(z)\rangle = \langle p, 2\rangle < \langle p, \frac{5}{2}\rangle = \langle p, 3\rangle - \epsilon.$$

This means $p \circ f$ is not lower semicontinuous at 0. Similarly, we can show that f does not satisfy the assumption of Theorem 3.1.10.

The following lemmas are useful for our main theorem.

Lemma 3.1.12. *Let C be a nonempty compact subset of a Hausdorff topological vector space E. If $\varphi : C \to E^*$ is continuous, then the mapping $x \mapsto \langle \varphi(x), x \rangle$ is continuous.*

Proof. Let $\psi : C \to \mathbb{R}$ be defined by $\psi(x) = \langle \varphi(x), x \rangle$ for all $x \in C$. Define $\widehat{\varphi} : C \to \mathfrak{C}[C, \mathbb{R}]$ by $\widehat{\varphi}(x) = \varphi(x)_{|C}$ for all $x \in C$. It is clear that $\widehat{\varphi}$ is continuous. By Theorem 2.1.23, $\mathfrak{C}[C, \mathfrak{C}[C, \mathbb{R}]]$ is isomorphic to $\mathfrak{C}[C \times C, \mathbb{R}]$, there exists a bijective mapping Ψ from $\mathfrak{C}[C, \mathfrak{C}[C, \mathbb{R}]]$ onto $\mathfrak{C}[C \times C, \mathbb{R}]$ such that

$$\Psi(\widehat{\varphi})(x, y) = \langle \widehat{\varphi}(x), y \rangle = \langle \varphi(x), y \rangle$$

for every $x, y \in C$. Therefore, $\psi = \Psi(\widehat{\varphi}) \circ \Delta$, where Δ is a diagonal mapping from C onto $C \times C$. Since both $\Psi(\widehat{\varphi})$ and Δ are continuous, so is ψ. $\square$

Lemma 3.1.13. (Browder, see [4]) *Let C be a nonempty compact convex subset of a locally convex Hausdorff topological vector space E. If $\varphi : C \to E^*$ is continuous, then there exists $u_0 \in C$ such that*

$$\langle \varphi(u_0), v - u_0 \rangle \leq 0$$

for all $v \in C$.

Proof. Assume that $\varphi : C \to E^*$ is continuous. Suppose that for each $u \in C$ there exists $v \in C$ such that $\langle \varphi(u), v - u \rangle > 0$. Define $\Phi : C \to 2^C$ by

$$\Phi(u) = \{v \in C : \langle \varphi(u), v - u \rangle > 0\}$$

for all $u \in C$. Then $\Phi(u) \neq \varnothing$ for every $u \in C$. Let $u \in C$, $u_1, u_2 \in \Phi(u)$ and $\lambda \in [0, 1]$. Then

$$\langle \varphi(u), \lambda u_1 + (1 - \lambda) u_2 - u \rangle = \lambda \langle \varphi(u), u_1 - u \rangle + (1 - \lambda) \langle \varphi(u), u_2 - u \rangle > 0.$$

This implies that Φ is convex valued.

Next, we will prove that $\Phi^{-}(v)$ is open in C for all $v \in C$. Let $v \in C$ and define a mapping $\psi : C \to \mathbb{R}$ by $\psi(u) = \langle \varphi(u), v - u \rangle$ for all $u \in C$. From Lemma 3.1.12 and the continuity of φ, we have that ψ is continuous. Consequently, $\Phi^{-}(v) = \psi^{-1}(0, \infty)$ is open in C. By Theorem 2.2.18, there exists $u_0 \in C$ such that $0 = \langle \varphi(u_0), u_0 - u_0 \rangle > 0$, which is a contradiction. $\square$

Lemma 3.1.14. *Let X be a compact topological space, Y a Hausdorff topological space and $g : X \to Y$ a bijective mapping. If g is continuous, then g is a homeomorphism.*

Proof. Assume that g is continuous. To show that $g^{-1} : Y \to X$ is continuous, let K be a closed subset of X. Then K is compact, and hence $g(K)$ is a compact subset of Y. Since Y is Hausdorff, $g(K)$ is closed. By Proposition 2.1.7, g^{-1} is continuous. $\square$

Let X and Y be sets. Let f and g be mappings from X to Y. The set $\mathcal{C}(f, g) = \{x \in X : f(x) = g(x)\}$ is called the **coincidence set** of f and g.

The next theorem is inspired by the idea of Theorem 3.1 in [2].

Theorem 3.1.15. *Let C be a nonempty compact convex subset of a locally convex Hausdorff topological vector space E and $g : C \to C$ a bijective continuous mapping. Suppose $f : C \to C$ satisfies the property that:*

for each $x \in C$ with $g(x) \neq f(x)$ there exists a (nonzero) continuous linear functional $p \in E^*$ and a neighborhood W of $g^{-1}(x)$ in C such that

$$\langle p, f(y) - g(y) \rangle > 0 \tag{3.9}$$

for all $y \in W$ with $g(y) \neq f(y)$. Then $\mathcal{C}(f, g)$ is nonempty.

Proof. Assume that $f : C \to C$ satisfies the property in the supposition of the theorem. Suppose that $\mathcal{C}(f,g) = \varnothing$. Define $\Phi : C \to 2^{E^*}$ by

$$\Phi(x) = \{p \in E^* : \text{there exists a neighborhood } W \text{ of } g^{-1}(x) \text{ in } C \text{ such that } \langle p, f(y) - g(y)\rangle > 0 \text{ for all } y \in W \text{ with } g(y) \neq f(y)\}$$

for all $x \in C$. Clearly, $\Phi(x)$ is nonempty for all $x \in C$. Let $x \in C$, $p, q \in \Phi(x)$ and $\lambda \in [0,1]$. There are neighborhoods W_1 and W_2 of $g^{-1}(x)$ in C such that

$$\forall y \in W_1,\ g(y) \neq f(y) \Rightarrow \langle p, f(y) - g(y)\rangle > 0 \tag{3.10}$$

and

$$\forall y \in W_2,\ g(y) \neq f(y) \Rightarrow \langle q, f(y) - g(y)\rangle > 0. \tag{3.11}$$

Since E^* is a vector space, $\lambda p + (1-\lambda)q \in E^*$. For each $y \in W_1 \cap W_2$ with $g(y) \neq f(y)$, we have

$$\langle \lambda p + (1-\lambda)q, f(y) - g(y)\rangle = \lambda\langle p, f(y) - g(y)\rangle + (1-\lambda)\langle q, f(y) - g(y)\rangle > 0.$$

Hence, $\lambda p + (1-\lambda)q \in \Phi(x)$. Consequently, Φ is convex valued.

Next, let $p \in E^*$ and $x \in \Phi^-(p)$. There exists a neighborhood W of $g^{-1}(x)$ in C such that $\langle p, f(y) - g(y)\rangle > 0$ for all $y \in W$ with $g(y) \neq f(y)$. Then $x \in g(W) \subseteq \Phi^-(p)$. By Lemma 3.1.14, g is open, hence $\Phi^-(p)$ is open in C. From Theorem 2.2.19 and Lemma 3.1.13, there exists a continuous selection $\varphi : C \to E^*$ of Φ and $x_0 \in C$ such that

$$\langle \varphi(x_0), y - x_0\rangle \leq 0 \tag{3.12}$$

for all $y \in C$. Since g is surjective, $x_0 = g(z_0)$ for some $z_0 \in C$. Since $f(z_0) \in C$,

$$\langle \varphi(g(z_0)), f(z_0) - g(z_0)\rangle \leq 0. \tag{3.13}$$

Also, since $\varphi(g(z_0)) \in \Phi(g(z_0))$, by the definition of Φ, we have $\langle \varphi(g(z_0)), f(z_0) - g(z_0)\rangle > 0$, which is a contradiction. □

If g in Theorem 3.1.15 is the identity mapping, then the following result is immediate.

Corollary 3.1.16. *Let C be a nonempty compact convex subset of a locally convex Hausdorff topological vector space E. If $f : C \to C$ is half-continuous, then f has a fixed point.*

Remark 3.1.17. If E is a Banach space, then the previous corollary is the Theorem 3.1 in [2].

The following result is immediately obtained from Theorem 3.1.2 and Theorem 3.1.16.

Corollary 3.1.18. (Brouwer-Schauder-Tychonoff, see [10])

Let C be a nonempty compact convex subset of a locally convex Hausdorff topological vector space E. Then every continuous mapping $f : C \to C$ has a fixed point.

The result in Corollary 3.1.18 was proved by Tychonoff in 1935. It includes the works of Brouwer(1912) and Schauder(1930), where the spaces considered are Euclidean space and Banach space, respectively.

3.2 Half-continuous Multivalued Mappings and Fixed Point Theorem

Now, we consider half-continuity of multivalued mappings, and prove that under a certain assumption they have fixed point property.

Definition 3.2.1. Let E be a topological vector space and C a nonempty subset of E. A mapping $F : C \to 2^E$ is said to be **half-continuous** if for each $x \in C$ with $x \notin F(x)$ there exists a (nonzero) continuous linear functional $p \in E^*$ and a neighborhood W of x in C such that

$$\forall y \in W, \quad y \notin F(y) \Rightarrow \forall z \in F(y), \ \langle p, z - y\rangle > 0. \tag{3.14}$$

The following theorems gives a sufficient condition for a multivalued mapping to be half-continuous.

Theorem 3.2.2. *Let C be a nonempty subset of a locally convex Hausdorff topological vector space E. If $F : C \to 2^E$ is an upper demicontinuous mapping with nonempty closed convex values, then F is half-continuous.*

Proof. Assume $F : C \to 2^E$ is upper demicontinuous with nonempty closed convex values. Let $x \in C$ be such that $x \notin F(x)$. Suppose F fails to be half-continuous. By Theorem 2.2.20, there exists $p \in E^*$ and $\alpha \in \mathbb{R}$ such that

$$\langle p, x\rangle < \alpha < \langle p, y\rangle \tag{3.15}$$

for all $y \in F(x)$. This implies that $F(x) \subseteq H := p^{-1}(\alpha, \infty)$. Since F is upper demicontinuous, there exists a neighborhood U of x in C such that $F(y) \subseteq H$ for all $y \in U$. Set $V = U \setminus \overline{H}$. Then V is a neighborhood of x in C. Indeed, if $x \in \overline{H} = \overline{p^{-1}(\alpha, \infty)} \subseteq p^{-1}\overline{(\alpha, \infty)} = p^{-1}[\alpha, \infty)$, then $\alpha \leq \langle p, x\rangle$, which is a contradiction. Since F is not half-continuous, there exists $x_V \in V \setminus F(x_V)$ and $z_V \in F(x_V)$ such that

$$\langle p, z_V - x_V\rangle \leq 0. \tag{3.16}$$

Since $x_V \in U$, $F(x_V) \subseteq H$, so $z_V \in H$. Then, by (3.16), $\alpha < \langle p, z_V\rangle \leq \langle p, x_V\rangle$. This means that $x_V \in H$, which is a contradiction. Therefore, F is half-continuous. $\square$

However, there is a half-continuous mapping which is not upper demicontinuous. To see this, consider the mapping $F : \mathbb{R} \to 2^{\mathbb{R}}$ defined by

$$F(x) = \begin{cases} [-1, 1] & \text{if } x \neq 0; \\ \{0\} & \text{if } x = 0 \end{cases}$$

for all $x \in \mathbb{R}$. To see that F is not upper demicontinuous at 0, let p be the identity mapping on $\mathbb{R}$ and $H = p^{-1}(-\frac{1}{2}, \infty)$. Then, H is an open half space in $\mathbb{R}$ containing $F(0)$. Let U be a neighborhood of 0 in $\mathbb{R}$. There is $\epsilon > 0$ such that $(-\epsilon, \epsilon) \subseteq U$. Then $F(\frac{\epsilon}{2}) = [-1, 1] \nsubseteq H$.

To show that F is half-continuous, let $x \in \mathbb{R}$ be such that $x \notin F(x)$. Then $x \notin [-1, 1]$.

Case $x > 1$. Let $p \in \mathbb{R}^*$ be define by $\langle p, y \rangle = -y$ for all $y \in \mathbb{R}$ and $\epsilon = x - 1$. Let $y \in (x - \epsilon, x + \epsilon) = (1, 2x - 1)$ be such that $y \notin F(y)$. For each $z \in F(y)$, $z \leq 1 < y$ and hence $\langle p, z - y \rangle = y - z > 0$.

Case $x < -1$. Let $p \in \mathbb{R}^*$ be define by $\langle p, y \rangle = y$ for all $y \in \mathbb{R}$ and $\epsilon = -1 - x$. Let $y \in (x - \epsilon, x + \epsilon) = (2x + 1, -1)$ be such that $y \notin F(y)$. For each $z \in F(y)$, $y < -1 \leq z$ and hence $\langle p, z - y \rangle = z - y > 0$.

In the case E is a topological vector space whose E^* separates points, we need more assumptions on the mapping as the following result.

Theorem 3.2.3. *Let E be a topological vector space whose E^* separates points and let C be a nonempty compact subset of E. If $F : C \to 2^C$ is upper semicontinuous with nonempty closed convex values, then F is half-continuous.*

Proof. Assume that $F : C \to 2^C$ is upper semicontinuous with nonempty closed convex values. Suppose that F is not half-continuous. Then there exists $x \in C$ with $x \notin F(x)$ such that for each $p \in E^* \setminus \{0\}$ and for each a neighborhood W

of x in C, there exists $x_W \in W \setminus F(x_W)$ and $z_W \in F(x_W)$ such that

$$\langle p, z_W - x_W \rangle \leq 0. \tag{3.17}$$

We note that $F(x)$ is compact. By Theorem 2.2.21, there exists $p \in E^* \setminus \{0\}$ such that

$$0 < \langle p, y - x \rangle \tag{3.18}$$

for all $y \in F(x)$. Let $\mathcal{D}$ be the set of all neighborhoods of x in C, which is directed by reverse inclusion, i.e., for $W_1, W_2 \in \mathcal{D}$,

$$W_1 \preceq W_2 \Leftrightarrow W_2 \subseteq W_1. \tag{3.19}$$

For each $W \in \mathcal{D}$, pick $x_W \in W \setminus F(x_W)$. Then x_W converges to x. Indeed, for a fix neighborhood U of x in C, then for each $W \in \mathcal{D}$ with $U \preceq W$, we have $x_W \in W \subseteq U$. Since C is compact, by Theorem 2.1.28 (5), we may choose $(z_W)_{W \in \mathcal{D}}$ to be a convergent net in C, say z_W converges to z. By Theorem 2.1.31, $z \in F(x)$. By Proposition 2.1.28 (4) and Proposition 2.2.2, $z_W - x_W$ converges to $z - x$. From (3.17), Proposition 2.1.28 (2,3) we have $\langle p, z - x \rangle \leq 0$. This contradicts (3.18). Hence, F is half-continuous. □

Next, we will prove the main result which guarantees the possessing of fixed points if the multivalued mapping is half-continuous. To do this, we need the following lemma.

Lemma 3.2.4. *Let C be a nonempty subset of a topological vector space E and $F : C \to 2^E$. If F is half-continuous, then F has a half-continuous selection.*

Proof. Assume that F is half-continuous. Let f be any selection of F. Define $\widetilde{f} : C \to E$ by

$$\widetilde{f}(x) = \begin{cases} x & \text{if } x \in F(x); \\ f(x) & \text{if } x \notin F(x). \end{cases}$$

Clearly, $\widetilde{f}$ is a selection of F. To show that $\widetilde{f}$ is half-continuous, let $x \in C$ be such that $x \neq \widetilde{f}(x)$. Then $x \notin F(x)$ and hence there exists $p \in E^*$ and a neighborhood W of x in C such that

$$\forall y \in W, \;\; y \notin F(y) \Rightarrow \forall z \in F(y), \; \langle p, z - y \rangle > 0.$$

Hence, $\langle p, \widetilde{f}(y) - y \rangle = \langle p, f(y) - y \rangle > 0$ for every $y \in W$ with $y \neq \widetilde{f}(y)$. □

Remark 3.2.5. There exists a multivalued mapping which is not half-continuous but some of its selection is half-continuous. For example, let $F : [0, 1] \to 2^{[0,1]}$ be defined by

$$F(x) = \begin{cases} (\frac{3}{4}, 1] \cup \{0\} & \text{if } \; x \in [0, \frac{1}{2}]; \\ \{\frac{3}{4}\} & \text{if } \; x \in (\frac{1}{2}, 1]. \end{cases}$$

Then F is not half-continuous since (3.14) fails for $x = \frac{1}{2}$. Nevertheless, a mapping $f : [0, 1] \to [0, 1]$ defined by

$$f(x) = \begin{cases} 1 & \text{if } \; x \in [0, \frac{1}{2}]; \\ \frac{3}{4} & \text{if } \; x \in (\frac{1}{2}, 1], \end{cases}$$

is a half-continuous selection of F.

Corollary 3.1.16 and Lemma 3.2.4 yield the following main result.

Theorem 3.2.6. *Let C be a nonempty compact subset of a locally convex Hausdorff topological vector space E. If $F : C \to 2^C$ is half-continuous, then F has a fixed point.*

The following result is immediately obtained from Theorem 3.2.6 and 3.2.2.

Corollary 3.2.7. *Let C be a nonempty compact convex subset of a locally convex Hausdorff topological vector space E. If $F : C \to 2^C$ is upper demicontinuous with nonempty closed convex values, then F has a fixed point.*

We note that if C is a subset of a topological space X and $F : C \to 2^X$ has closed graph, then the set of fixed points of F is closed in C. To see this, let A be the set of fixed points of F and $x \in C \setminus A$. Then $(x, x) \notin G_F$. There are open subsets U and V of C such that $(x, x) \in U \times V$ and $(U \times V) \cap G_F = \varnothing$. Put $W = U \cap V$. Then, W is an open subset of C that contains x and $W \cap A = \varnothing$. It follows that $C \setminus A$ is open, so A is closed as desired.

From Corollary 3.2.7 and Theorem 2.1.31, we have

Corollary 3.2.8. (Kakutani-Fan-Glicksberg, see [6, 9]) *Let C be a nonempty compact convex subset of a locally convex Hausdorff topological vector space E. If $F : C \to 2^C$ is upper semicontinuous with nonempty closed convex values, then the set of fixed points of F is nonempty and compact.*

3.3 Some consequences

In case that the half-continuous mapping f is not a self-mapping on C but f has some nice property, then f still posseses a fixed point in C. We state the results in the following theorems.

Theorem 3.3.1. *Let C be a nonempty compact convex subset of a locally convex Hausdorff topological vector space E. Suppose that $f : C \to E$ is half-continuous and for each $x \in C$ with $x \neq f(x)$ there exists $\lambda < 1$ such that $\lambda x + (1-\lambda) f(x) \in C$. Then f has a fixed point.*

Proof. Suppose that f has no fixed point. For each $x \in C$, let

$$\Lambda(x) = \{\lambda \in \mathbb{R} : \lambda < 1 \text{ and } \lambda x + (1-\lambda) f(x) \in C\}.$$

Define $F : C \to 2^C$ by

$$F(x) = \{\lambda x + (1-\lambda) f(x) : \lambda \in \Lambda(x)\}$$

for all $x \in C$. Then $F(x) \neq \varnothing$ for every $x \in C$. We will show that F is half-continuous. Let $x \in C$ be such that $x \notin F(x)$. Since f is half-continuous, there exists $p \in E^* \setminus \{0\}$ and a neighborhood W of x in C such that

$$\langle p, f(y) - y \rangle > 0 \tag{3.20}$$

for all $y \in W$ with $y \neq f(y)$. Let $y \in W$ be such that $y \notin F(y)$ and let $z \in F(y)$. Then there exists $\lambda \in \Lambda(y)$ such that $z = \lambda y + (1-\lambda)f(y)$. Thus,

$$\langle p, z - y \rangle = \langle p, \lambda y + (1-\lambda)f(y) - y \rangle = (1-\lambda)\langle p, f(y) - y \rangle > 0.$$

By Theorem 3.2.6, $x_0 \in F(x_0)$ for some $x_0 \in C$. Then there exists $\alpha \in \Lambda(x_0)$ such that $x_0 = \alpha x_0 + (1-\alpha)f(x_0)$. That is $x_0 = f(x_0)$, which is a contradiction. □

Remark 3.3.2. From Theorem 3.3.1, for $x \in C$ with $x \neq f(x)$, if there is $\lambda < 0$ such that $z := \lambda x + (1-\lambda)f(x) \in C$, then $f(x)$, in fact, is an element in C. Indeed, by setting $\mu = \frac{\lambda}{\lambda - 1}$, then $0 < \mu < 1$ and so, by convexity of C, $f(x) = \mu x + (1-\mu)z \in C$.

As a special case of Theorem 3.3.1 we obtain

Corollary 3.3.3. (Fan-Kaczynski, see [10]) *Let C be a nonempty compact convex subset of a locally convex Hausdorff topological vector space E. Suppose that $f : C \to E$ is continuous and for each $x \in C$ with $x \neq f(x)$ the line segment $[x, f(x)]$ contains at least two points of C. Then f has a fixed point.*

Next, we derive a generalization of a fixed point theorem due to F.E. Browder and B.R. Halpern. To do this, let us recall the definition of inward and outward mappings.

Definition 3.3.4. (see [10]) Let C be a subset of a vector space E. A mapping $f : C \to E$ is called **inward** (respectively **outward**) if for each $x \in C$ there exists $\lambda > 0$ (respectively $\lambda < 0$) satisfying $x + \lambda(f(x) - x) \in C$.

As a consequence of Theorem 3.3.1 and Proposition 3.1.8, we have fixed point theorem for nonself half-continuous mapping which is inward or outward, as follow.

Theorem 3.3.5. *Let C be a nonempty compact convex subset of a locally convex Hausdorff topological vector space E. Then every half-continuous inward (or outward) mapping $f : C \to E$ has a fixed point.*

Proof. Let $f : C \to E$ be a half-continuous mapping. First, we assume that f is inward. Let $x \in C$ be such that $x \neq f(x)$. Then $x + \lambda(f(x) - x) \in C$ for some $\lambda > 0$. Put $\beta = 1 - \lambda$. Then $\beta < 1$ and $\beta x + (1 - \beta)f(x) = x + \lambda(f(x) - x) \in C$. By Theorem 3.3.1, f has a fixed point.

Next, assume that f is outward. Define $g : C \to E$ by $g(x) = 2x - f(x)$ for all $x \in C$. By Proposition 3.1.8, g is half-continuous. Let $x \in C$ be arbitrary. Since f is outward, $x + \lambda(f(x) - x) \in C$ for some $\lambda < 0$. Thus $x + (-\lambda)(g(x) - x) = x + \lambda(f(x) - x) \in C$. This implies that g is inward. Hence, there is $x_0 \in C$ such that $x_0 = g(x_0) = 2x_0 - f(x_0)$. That is $x_0 = f(x_0)$. □

Remark 3.3.6. In Theorem 3.3.5, if f is a continuous inward (or outward) mapping, then Theorem 3.3.5 is the theorem proved by F.E. Browder (1967) and B.R. Halpern (1968) (see [10]).

In the final part, we prove the fixed points theorem for half-continuous inward and outward multivalued mappings.

Definition 3.3.7. (see [1]) Let C be a subset of a vector space E. A mapping $F : C \to 2^E$ is called **inward** (respectively **outward**) if for each $x \in C$ there exists $y \in F(x)$ and $\lambda > 0$ (respectively $\lambda < 0$) satisfying $x + \lambda(y - x) \in C$.

Theorem 3.3.8. *Let C be a nonempty compact convex subset of a locally convex Hausdorff topological vector space E. Then every half-continuous inward (or outward) mapping $F : C \to 2^E$ has a fixed point.*

Proof. Let $F : C \to 2^E$ be a half-continuous mapping. Suppose that F is inward but it has no fixed point. Define $G : C \to 2^C$ by

$$G(x) = \{u \in C : \text{ there exists } v \in F(x) \text{ and } \lambda > 0 \text{ such that } u = x + \lambda(v - x)\}$$

for all $x \in C$. Then, $G(x)$ is nonempty for all $x \in C$. Let $x \in C$ be such that $x \notin G(x)$. Since F is half-continuous, there exists $p \in E^*$ and a neighborhood W of x in C such that

$$\forall y \in W,\ y \notin F(y) \Rightarrow \forall z \in F(y),\ \langle p, z - y\rangle > 0.$$

Let $y \in W$ be such that $y \notin G(y)$ and let $z \in G(y)$. There exists $v \in F(y)$ and $\lambda > 0$ such that $z = y + \lambda(v - y)$. It follows that $\langle p, z - y\rangle = \lambda\langle p, v - y\rangle > 0$. Hence, G is half-continuous. By Theorem 3.2.6, there exists $x_0 \in C$ such that $x_0 \in G(x_0)$. Thus $x_0 = x_0 + \alpha(v - x_0)$ for some $v \in F(x_0)$ and $\alpha > 0$. That is $x_0 \in F(x_0)$, which is a contradiction.

Next, assume that F is outward. Define $H : C \to 2^E$ by $H(x) = 2x - F(x)$ for all $x \in C$. Let $x \in C$ be such that $x \notin H(x)$. Then $x \notin F(x)$ and so, by half-continuity of F, there exists $p \in E^*$ and a neighborhood W of x in C such that

$$\forall y \in W,\ y \notin F(y) \Rightarrow \forall z \in F(y),\ \langle p, z - y\rangle > 0.$$

Set $q = -p$. Let $y \in W$ be such that $y \notin H(y)$ and let $z \in H(y)$. Then $z = 2y - v$ for some $v \in F(y)$. Hence, $\langle q, z - y\rangle = \langle q, y - v\rangle = \langle p, v - y\rangle > 0$. This means H is half-continuous. Next, we will show that H is inward. Let $x \in C$ be arbitrary. Since F is outward, there exists $y \in F(x)$ and $\lambda < 0$ satisfying $x + \lambda(y - x) \in C$. Then $x + (-\lambda)(2x - y - x) = x + \lambda(y - x) \in C$. Since $2x - y \in H(x)$, we get the desire. Thus $x_0 \in H(x_0)$ for some $x_0 \in C$. By the definition of H, there exists $v \in F(x_0)$ such that $x_0 = 2x_0 - v$. That is $x_0 \in F(x_0)$. □

Any selection of half-continuous inward multivalued mappings may not be inward as shown in the following example.

Example 3.3.9. Let $F : [0, 1] \to 2^{\mathbb{R}}$ be defined by

$$F(x) = \begin{cases} [x+1, \infty) & \text{if } \ x \in [0,1); \\ \{0, 1, 2\} & \text{if } \ x = 1 \end{cases}$$

for all $x \in [0, 1]$. Clearly, F is half-continuous. To see that F is inward, let $x \in [0, 1]$. If $x = 1$, then $1 + (1)(0 - 1) = 0 \in [0, 1]$. Assume that $x \neq 1$. By letting $y = x + 1$ and $\lambda = 1 - x$, we have $x + \lambda(y - x) = 1 \in [0, 1]$. Hence, F is inward.

Now, consider a mapping $f : [0, 1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x+2 & \text{if } \ x \in [0,1); \\ 2 & \text{if } \ x = 1 \end{cases}$$

for all $x \in [0, 1]$. It is clear that f is a selection of F. For $\lambda > 0$, we have $1 + \lambda(2 - 1) = 1 + \lambda > 1$. This implies that f is not inward at 1.

Remark 3.3.10. If the half-continuity of F is replaced by upper demicontinuity, then Theorem 3.3.8 is the result of Halpern-Bergman(1968) (see [1]) and Fan(1969) (see [7]).

As an interesting special case of Theorem 3.3.8, we obtain

Corollary 3.3.11. *Let C be a nonempty compact convex subset of a locally convex Hausdorff topological vector space E. Suppose $F : C \to 2^E$ is half-continuous and for each $x \in C$, $F(x) \cap C$ is nonempty. Then F has a fixed point.*

REFERENCES

[1] Aliprantis, C.D. and Border, K.C. , *Infinite Dimensional Analysis : A Hitchhiker's Guide*, 3rd ed., Springer, New York, 2006.

[2] Bich, P., Some fixed point theorems for discontinuous mappings, *Cahiers de la Maison des Sciences Economiques*, **b06066** (2006), 1-10.

[3] Bich, P., An answer to a question by Hering et al., *Operations Research Letters*, **36** (2008), 525-526.

[4] Browder, F.E., The fixed point theory of multivalued mappings in topological vector spaces, *Math. Ann.* **177** (1968), 283-301.

[5] Cater, S., On a class of metric linear spaces which are not locally convex, *Math. Ann.*, **157** (1964), 210-214.

[6] Fan, K., Fixed-point and minimax theorem in locally convex topological linear spaces, *Proc. Nat. Acad. Sci. U.S.A.* **38** (1952), 121-126.

[7] Fan, K., Extension of two fixed point theorems of F.E. Browder, *Math. Z.*, **112** (1969), 234-240.

[8] Folland, G. B., *Real Analysis : Modern Techniques and Their Applications*, 2nd ed., John Wiley & Sons, New York, 1999.

[9] Glicksberg, I.L., A further generalization of Kakutani fixed point theorem, with application to Nash equilibrium points, *Proc. AMS* **3** (1952), 170-174.

[10] Granas, A. and Dugundji, J., *Fixed Point Theory*, Springer, New York, 2003.

[11] Herings, P.J.J., Laan, G., Talman, D. and Yang, Z. , A Fixed point theorem for discontinuous functions, *Operations Research Letters*, **36** (2008), 89-93.

[12] Munkres, J. R., *Topology: A First Course*, Prentice-Hall, New Delhi, 1975.

[13] Royden, H. L., *Real Analysis*, 3rd ed., Prentice-Hall, New Jersey, 1988.

[14] Rudin, W., *Functional Analysis*, McGraw-Hill, New York, 1973.

[15] Takahashi, W., *Nonlinear Functional Analysis : Fixed Point Theory and its Applications*, Yokohama Publishers, Japan, 2000.

VITA

Name	Mr. Thanatkrit Kaewtem
Date of Birth	25 June 1983
Place of Birth	Singburi, Thailand
Education	B.S. (Mathematics) with First Class Honours,
	Kasetsart University, 2006
	Grad Dip. (Teaching), Burapha University, 2007