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COHEN-EGYPTIAN FRACTION AND ENGEL SERIES EXPANSIONS

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จิตตินาถ รัตนมุง : การกระจายอนุกรมแบบเศษส่วนอียิปต์ของโคเฮนและแบบเองเกล (COHEN-EGYPTIAN FRACTION AND ENGEL SERIES EXPANSIONS) อ.ที่ปรึกษาวิทยานิพนธ์หลัก: อ.ดร. ตวงรัตน์ ไชยชนะ, 42 หน้า.

ในปี ค.ศ. 1951 เพอร์รองได้เขียนในหนังสือของเขาว่าจำนวนจริงที่ไม่ใช่ศูนย์ สามารถเขียนแทนด้วยการกระจายอนุกรมแบบเองเกลและเขียนได้หนึ่งเดียว การกระจายอนุกรมนี้จะแทนจำนวนตรรกยะที่ไม่ใช่ศูนย์ ก็ต่อเมื่อ ตัวเลขหนึ่งที่ปรากฏในตัวส่วน ณ ตำแหน่งหนึ่งเป็นต้นไปมีค่าเท่ากัน ในปี ค.ศ. 1973 โคเฮนได้สร้างขั้นตอนวิธีเพื่อเขียนแทนจำนวนจริงที่ไม่ใช่ศูนย์ ด้วยผลรวมของเศษส่วนอียิปต์และเขียนได้หนึ่งเดียว ซึ่งเราขอเรียกว่า การกระจายอนุกรมแบบเศษส่วนอียิปต์ของโคเฮน และโคเฮนได้ให้ลักษณะเฉพาะของจำนวนตรรกยะว่าเป็นจำนวนที่มีการกระจายอนุกรมแบบเศษส่วนอียิปต์ของโคเฮนเป็นอนุกรมจำกัด

ในวิทยานิพนธ์ฉบับนี้ เราได้ขยายงานดังกล่าวในฟีลด์บริบูรณ์เทียบกับแวลูเอชันแบบนอน-อาร์คีมีเดียนและวิยุต ได้แก่ ฟีลด์ของจำนวนพี-แอดิก และ ฟีลด์ฟังก์ชันสองฟีลด์ คือ ฟีลด์บริบูรณ์เทียบกับแวลูเอชันดีกรี และฟีลด์บริบูรณ์เทียบกับแวลูเอชันไพร์มแอดิก เราแสดงขั้นตอนวิธีการสำหรับการเขียนแทนสมาชิกในฟีลด์บริบูรณ์เหล่านี้ด้วยการกระจายอนุกรมข้างต้น นอกจากนี้เรายังสร้างและพิสูจน์เกณฑ์การตรวจสอบความเป็นตรรกยะโดยใช้การกระจายเหล่านี้

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It was shown in the book written by Perron in 1951 that each nonzero real number can be uniquely written as an Engel series expansion. This series expansion represents a nonzero rational number if and only if each digit in such expansion is identical from certain point onward. In 1973, Cohen devised an algorithm to uniquely represent each nonzero real number as a sum of Egyptian fractions, which we refer to as its Cohen-Egyptian fraction expansion. Cohen also characterized the real rational numbers as those with finite Cohen-Egyptian fraction expansions.

In this thesis, we extend their work to three complete fields with respect to discrete non-archimedean valuations, namely, the p-adic number field and two kinds of function fields (the one completed with respect to the degree valuation and the one completed with respect to a prime-adic valuation). We present algorithms for constructing these two types of series representations of elements in these fields and establish rationality criteria through the use of these expansions.

In the last part, we analyze the relationship between these two expansions in all fields involved.

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CONTENTS

CHAPTER I

PRELIMINARIES

It is well-known that each real number is representable as a series expansion in many different shapes. For example, representations of real numbers by Cantor's series, Lüroth series and Sylvester series, appeared in the Lecture Notes in Mathematics; Representations of Real Numbers by Infinite Series, [5], and by series of reciprocals of odd integers, by Oppenheim, [10]. Two kinds of expansions referred to here as the Engel series (or ES) expansion and the Cohen-Egyptian fraction (or CEF) expansion are considered.

It was proved, [11], that each nonzero real number A can be uniquely written as an infinite Engel series expansion of the form

$$A = a_0 + \sum_{i=1}^{\infty} \frac{1}{a_1 a_2 \cdots a_i},$$

where $a_1 \geq 2, \quad a_{i+1} \geq a_i$ for all $i \geq 1$. Also well-known is the fact that an Engel series expansion represents a nonzero rational number if and only if each digit in the expansion is identical from certain point onward. In 1973, Cohen, [4], devised an algorithm to uniquely represent each nonzero real number A as a sum of Egyptian fractions, which we refer to as its Cohen-Egyptian fraction expansion,

$$A = n_0 + \sum_{k=1}^{\infty} \frac{1}{n_1 n_2 \cdots n_k},$$

where n_0 is a nonnegative integer, $(n_1,\ n_2, \ldots)$ is a non-decreasing sequence of positive integers with $n_1 \geq 2$, and no term of the sequence appears infinitely often. Recently, it has come to our attention that the shapes of both of these expansions for real numbers seem remarkably similar yet are not exactly identical. This naturally leads to a question whether the two expansions are related in any meaningful way.

The work of this thesis centers around these 3 topics; algorithms of CEF and ES expansions, characterization of rationality and possible relationships between CEF and ES expansions. In the first part of this work, the algorithms for constructing CEF and ES expansions are given in discrete-valued non-archimedean fields in the same spirit as that of the real case.

Regarding the relationship problem, we show in the last chapter that ES and CEF expansions are indeed related. In the case of real numbers, we describe the similarity and the distinction between ES and CEF expansions in Theorem 4.1. In the non-archimedean situation, we show that the two series representations are identical, their use to characterize rational elements depend significatly on the underlying nature of each specific field. We end this thesis by providing criteria for rationality in our three different non-archimedean fields.

We begin with basic definitions and results, given mainly without proofs, and give brief background meterials needed in the work of this thesis ([9] and [1]). A principal result is Theorem 1.12, which shows how to represent elements in the complete discrete non-archimedean valued fields as formal Laurent series.

Throughout, we denote by $K^\times$ the set of nonzero elements in a field K.

Definition 1.1. A **valuation** on K is a map $|\cdot| : K \to \mathbb{R}$ with the following properties:

(i) $\forall \alpha \in K$, $|\alpha| \geq 0$ and $|\alpha| = 0$ if and only if $\alpha = 0$,

(ii) $\forall \alpha, \beta \in K$, $|\alpha\beta| = |\alpha||\beta|$,

(iii) $\forall \alpha, \beta \in K,\ |\alpha + \beta| \leq |\alpha| + |\beta|$.

There is always at least one valuation on K, namely, that given by setting $|\alpha| = 1$ if $\alpha \in K^{\times}$ and $|0| = 0$. This valuation is called the **trivial valuation** on K.

Definition 1.2. A valuation $|\cdot|$ on K is called **non-archimedean** if the condition (iii) in Definition 1.1 is replaced by a stronger condition, called the **strong triangle inequality**

$$\forall \alpha, \beta \in K,\ |\alpha + \beta| \leq \max\{|\alpha|, |\beta|\}.$$

Any other valuation on K is called **archimedean**.

A **valued field** $(K, |\cdot|)$ is a field K together with a prescribed valuation $|\cdot|$. If the valuation is non-archimedean, then K is called a **non-archimedean valued field**.

Examples of non-archimedean valuation are as follows:

Example 1.3.

(1) For $K = \mathbb{Q}$, the usual absolute value $|\cdot|$ is an archimedean valuation.

(2) For $K = \mathbb{Q}$, let p be a prime number. By the fundamental theorem of arithmetic, each $\alpha \in \mathbb{Q}^{\times}$ can be written uniquely in the form

$$\alpha = p^{\nu_p(\alpha)} \frac{a}{b}$$

where $\nu_p(\alpha) \in \mathbb{Z},\ a, b \in \mathbb{Z}\ (b > 0)$, $(a, b) = 1$ and $p \nmid ab$.

Define $|\cdot|_p : \mathbb{Q} \to \mathbb{R}$ by

$$|\alpha|_p = \left(\tfrac{1}{p}\right)^{\nu_p(\alpha)} \text{ if } \alpha \neq 0 \text{ and } |0|_p = 0.$$

Then $|\cdot|_p$ is a non-archimedean valuation on $\mathbb{Q}$ and called the ***p*-adic valuation**.

(3) Consider the field $F(x)$ of rational functions over a field F.

Let $\pi(x)$ be an irreducible polynomial in $F[x]$. Any $\alpha \in F(x)^\times$ can be written uniquely as

$$\alpha = \pi(x)^{\nu_\pi(\alpha)} \frac{a(x)}{b(x)}$$

where $\nu_\pi(\alpha) \in \mathbb{Z}$, $a(x)$ and $b(x)$ are relatively prime elements of $F[x]$, $b(x)$ is a nonzero monic polynomial and $\pi(x) \nmid a(x)b(x)$.

Define $|\cdot|_\pi : F(x) \to \mathbb{R}$ by

$$|\alpha|_\pi = c^{\nu_\pi(\alpha)} \text{ where } 0 < c < 1 \text{ if } \alpha \neq 0 \text{ and } |0|_\pi = 0.$$

Then $|\cdot|_\pi$ is a non-archimedean valuation on $F(x)$ and called the $\pi(x)$**-adic valuation**.

(4) Define $|\cdot|_\infty$ on $F(x)$ by, for all $f(x), g(x) \in F[x] \smallsetminus \{0\}$,

$$\left|\frac{f(x)}{g(x)}\right|_\infty = c^{\deg g(x) - \deg f(x)} \text{ where } 0 < c < 1 \text{ and } |0|_\infty = 0.$$

Then $|\cdot|_\infty$ is a non-archimedean valuation on $F(x)$ and called the **degree valuation**.

Theorem 1.4. *Let $(K, |\cdot|)$ be a non-archimedean valued field and $\alpha, \beta \in K$. If $|\alpha| \neq |\beta|$, then*

$$|\alpha + \beta| = \max\{|\alpha|, |\beta|\}.$$

Let b be a real number greater than one. From a non-archimedean valuation $|\cdot|$, we define $\nu : K \to \mathbb{R} \cup \{\infty\}$ by

$$\nu(\alpha) = -\log_b|\alpha| \quad \text{if } \alpha \neq 0 \quad \text{and} \quad \nu(0) = \infty.$$

With the convention $\infty + a = \infty = a + \infty$ for all $a \in \mathbb{R} \cup \{\infty\}$ and $\infty > a$ for all $a \in \mathbb{R}$, the properties of $|\cdot|$ translate to

$(i)'$ $\forall \alpha \in K,\ \ \nu(\alpha) \in \mathbb{R} \cup \{\infty\}$ and $\nu(\alpha) = \infty$ if and only if $\alpha = 0$,

$(ii)'$ $\forall \alpha, \beta \in K$, $\nu(\alpha\beta) = \nu(\alpha) + \nu(\beta)$,

$(iii)'$ $\forall \alpha, \beta \in K$, $\nu(\alpha + \beta) \geq \min\{\nu(\alpha), \nu(\beta)\}$ with equality when $\nu(\alpha) \neq \nu(\beta)$.

A mapping $\nu : K \to \mathbb{R} \cup \{\infty\}$ satisfies $(i)' - (iii)'$ is called an **exponential valuation** of K corresponding to the valuation $|\cdot|$.

Definition 1.5. A non-archimedean valuation $|\cdot|$ is called a **discrete valuation** if $\nu(K^{\times})$ is a discrete subgroup of the additive group of real numbers, i.e., $\nu(K^{\times}) = \{0\}$ or $\nu(K^{\times})$ is an infinite cyclic subgroup of $(\mathbb{R}, +)$.

Two kinds of examples of discrete valuations are as follows:

Example 1.6.

(1) The p-adic valuation, $|\cdot|_p$, is a discrete non-archimedean valuation on $\mathbb{Q}$.

(2) The $\pi(x)$-adic valuation, $|\cdot|_\pi$, and the degree valuation, $|\cdot|_\infty$, are discrete non-archimedean valuations on $F(x)$.

The concepts of convergence and completeness of our mentioned fields are defined in the usual ways.

Definition 1.7. Let $(K, |\cdot|)$ be a valued field. A sequence $\{a_n\}$ of elements of K **converges** to α in K if $\forall \varepsilon > 0\ \exists N$ such that $\forall n > N,\ |a_n - \alpha| < \varepsilon$.

Definition 1.8. The field K is called **complete** with respect to the valuation $|\cdot|$ if every Cauchy sequence in K, with respect to $|\cdot|$, has a limit in K.

Definition 1.9. A field $\widehat{K}$ with valuation $\widehat{|\cdot|}$ is a **completion** of K with $|\cdot|$ if

(1) $\widehat{K}$ is an extension of K,

(2) $\widehat{K}$ is complete with respect to $\widehat{|\cdot|}$ which is a prolongation of $|\cdot|$ over K,

(3) every element of $\widehat{K}$ is a limit of some Cauchy sequence in K.

Example 1.10.

(1) In the case of $\mathbb{Q}$, with the usual absolute value, its completion is the field $\mathbb{R}$ of real numbers.

(2) In the case of $(\mathbb{Q}, |\cdot|_p)$, its completion is the p-adic number field $(\mathbb{Q}_p, |\cdot|_p)$.

(3) In the case of $(F(x), |\cdot|_\pi)$ its completion is $(F((\pi(x))), |\cdot|_\pi)$ the field of formal Laurent series in $\pi(x)$.

(4) In the case of $(F(x), |\cdot|_\infty)$, its completion is $(F((1/x)), |\cdot|_\infty)$ the field of formal Laurent series in $1/x$.

Definition 1.11. Let $(K, |\cdot|)$ be a non-archimedean valued field.

(1) The set $\mathcal{O} := \{\alpha \in K : |\alpha| \leq 1\}$ is a ring, called the **valuation ring** of $(K, |\cdot|)$.

(2) The set $\mathcal{P} := \{\alpha \in K : |\alpha| < 1\}$ is the unique maximal ideal of $\mathcal{O}$.

(3) The field $\mathcal{O}/\mathcal{P}$ is called the **residue class field** of $(K, |\cdot|)$.

A representative of elements in a complete field is in the next theorem, see e.g. [9].

Theorem 1.12. *Let K be a complete field with respect to a discrete non-archimedean valuation $|\cdot|$. For each integer m let π_m be an element of K such that $\nu(\pi_m) = m$. Let $\mathcal{A}$ be a complete set of representatives in $\mathcal{O}$ of the elements of $\mathcal{O}/\mathcal{P}$, that is, $\mathcal{A}$ consists of exactly one element from each of the residue classes of $\mathcal{P}$ in $\mathcal{O}$. Then every $\alpha \in K^\times$ can be written uniquely in the form*

$$\alpha = \sum_{i=r}^{\infty} a_i \pi_i,$$

where $r = \nu(\alpha)$, $a_i \in \mathcal{A}$ for each i, and $a_r \notin \mathcal{P}$.

Example 1.13.

(1) In case $\pi_m = p^m, \ m \in \mathbb{Z}, \ p$ is a prime number and $\mathcal{A} = \mathbb{Z}/p\mathbb{Z}$, we have a unique representation of any element in the p-adic number field $\mathbb{Q}_p$ of the form

$$\sum_{i=r}^{\infty} a_i p^i,$$

where $r \in \mathbb{Z}, \quad a_i \in \mathbb{Z}/p\mathbb{Z}$ for each i and $a_r \neq 0$.

(2) An element $\pi_m = x^{-m}$ in $F((1/x))$ and the set $\mathcal{A} = F$ give a representation of an element in $F((1/x))$ of the form

$$\sum_{i=r}^{\infty} a_i x^{-i},$$

where $r \in \mathbb{Z}, \quad a_i \in F$ for each i and $a_r \neq 0$.

(3) An element $\pi_m = x^m$ in $F((x))$ and the set $\mathcal{A} = F$ give a representation of an element in $F((x))$ of the form

$$\sum_{i=r}^{\infty} a_i x^i,$$

where $r \in \mathbb{Z}, \quad a_i \in F$ for each i and $a_r \neq 0$.

CHAPTER II

COHEN-EGYPTIAN FRACTION EXPANSIONS

In this chapter, an algorithm is given in the first section to construct series representations of nonzero real numbers. We give detailed proofs of its convergence, uniqueness and characterization of rational numbers. In the second section, we treat the case of complete discrete non-archimedean valued fields.

2.1 The case of real numbers

According to the Egyptian legend ([3]), the evil god Seth damaged the eye of Horus, son of Isis and Osiris. The Eye of Horus had mystical siginificance, as each of its parts was associated with a fraction of the form $1/2^n$. Thoth, the benevolent ibis-headed god, is credited with restoring the eye 'by the touch of his finger' making it whole. This is interpreted as reference to the geometric sum

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}.$$

This sum is made whole (i.e., it sums to 1) by the addition of one more counting unit, one more finger, $1/64$. Frational expressions of this sort ocurred naturally within the Egyptian system of arithmetic. The mathematician-scribes of dynastic Egypt denoted rational numbers by strings of unit fractions (fractions whose numerators are 1), which has since been referred to as Egyptian fractions.

There has been a good deal of works about Egyptian fraction expansions, see e.g. [2], [3], [4] and [12]. We are here interested in the result of Cohen, see [4], where an algorithm to uniquely represent each nonzero real number as a sum of Egyptian fractions is obtained.

To construct such expansion, we proceed as in Cohen, [4], making use of the following lemma.

Lemma 2.1. *For any $y \in (0,1)$, there exist a unique integer $n \geq 2$ and a unique $r \in \mathbb{R}$ such that*

$$1 = ny - r \quad \text{and} \quad 0 \leq r < y.$$

Proof. Let $y \in (0,1)$. Define $n = \lceil 1/y \rceil \in \mathbb{N}$ and $r = ny - 1$. Put $\langle 1/y \rangle := n - 1/y \in [0,1)$ and so

$$r = ny - 1 = y \left\langle \frac{1}{y} \right\rangle \in [0, y).$$

To prove the uniqueness, assume there exist integer $m \geq 2$ and $s \in \mathbb{R}$ such that

$$1 = my - s \quad \text{and} \quad 0 \leq s < y.$$

From $ny - r = 1 = my - s$, we get

$$1 + \frac{1}{y} > n = \frac{1+r}{y} \geq \frac{1}{y} \quad \text{and} \quad 1 + \frac{1}{y} > m = \frac{1+s}{y} \geq \frac{1}{y}.$$

Since there is only one integer with this property, we conclude that $n = m$ and consequently, $r = s$ proving the lemma. $\square$

Theorem 2.2. *Each $A \in \mathbb{R}^{\times}$ is uniquely representable as a series expansion of the form*

$$A = n_0 + \sum_{k=1}^{\infty} \frac{1}{n_1 n_2 \cdots n_k},$$

with

$$n_0 = \lfloor A \rfloor, \ n_1 \geq 2, \quad n_{k+1} \geq n_k \quad (k \geq 1) \tag{2.1}$$

and no term of the sequence appears infinitely often.

Moreover, each expansion terminates if and only if it represents a nonzero rational number.

Proof. Let $A \in \mathbb{R}^{\times}$ and $n_0 = \lfloor A \rfloor$. Define

$$r_0 = A - n_0 \in [0, 1).$$

If $r_0 = 0$, then the process stops and we write $A = n_0$. When $r_0 \neq 0$, by Lemma 2.1, there are unique $n_1 \in \mathbb{N}$ and $r_1 \in \mathbb{R}$ such that

$$1 = n_1 r_0 - r_1, \quad 0 \leq r_1 < r_0, \quad n_1 \geq 2.$$

Thus,

$$A = n_0 + r_0 = n_0 + \frac{1}{n_1} + \frac{r_1}{n_1}.$$

If $r_1 = 0$, then the process stops and we write $A = n_0 + 1/n_1$. When $r_1 \neq 0$, by Lemma 2.1, there are unique $n_2 \in \mathbb{N}$ and $r_2 \in \mathbb{R}$ such that

$$1 = n_2 r_1 - r_2, \quad 0 \leq r_2 < r_1, \quad n_2 \geq n_1;$$

the last inequality being followed from $n_1 = \lceil 1/r_0 \rceil$, $n_2 = \lceil 1/r_1 \rceil$ and $r_1 < r_0$.

Observe also that

$$A = n_0 + \frac{1}{n_1} + \frac{1}{n_1 n_2} + \frac{r_2}{n_1 n_2}.$$

Continuing this process, we get

$$A = n_0 + \frac{1}{n_1} + \frac{1}{n_1 n_2} + \cdots + \frac{1}{n_1 n_2 \cdots n_k} + \frac{r_k}{n_1 n_2 \cdots n_k},$$

with

$$1 = n_i r_{i-1} - r_i, \quad 1 > r_{i-1} > r_i \geq 0 \ \text{ and } \ 2 \leq n_i \leq n_{i+1} \ \text{ for all } \ i = 1, 2, \ldots. \tag{2.2}$$

If some $r_k = 0$, then the process stops, otherwise the series convergence follows at once from

$$\left| \frac{r_k}{n_1 n_2 \cdots n_k} \right| \to 0 \quad \text{as} \quad k \to \infty.$$

To prove the uniqueness, let

$$n_0 + \sum_{k=1}^{\infty} \frac{1}{n_1 n_2 \cdots n_k} = A = m_0 + \sum_{k=1}^{\infty} \frac{1}{m_1 m_2 \cdots m_k}, \tag{2.3}$$

with the restictions (2.1) on both digits n_i and m_j. Now

$$\sum_{k \geq 1} \frac{1}{n_1 n_2 \cdots n_k} \leq \sum_{k \geq 1} \frac{1}{2^k} = 1. \tag{2.4}$$

It is clear that the restrictions (2.1) imply the strict inequality in (2.4). This also applies to the rightmost summand in (2.3). Equating integer and fractional parts in (2.3), we get

$$n_0 = m_0, \quad \sum_{k=1}^{\infty} \frac{1}{n_1 n_2 \cdots n_k} = \sum_{k=1}^{\infty} \frac{1}{m_1 m_2 \cdots m_k} =: w, \quad \text{say}.$$

Since $n_{k+1} \geq n_k$,

$$n_1 w - 1 = \frac{1}{n_2} + \frac{1}{n_2 n_3} + \frac{1}{n_2 n_3 n_4} + \cdots \leq \frac{1}{n_1} + \frac{1}{n_1 n_2} + \frac{1}{n_1 n_2 n_3} + \cdots = w,$$

so $0 < n_1 - 1/w \leq 1$. But there is exactly one integer n_1 satisfying these restrictions. Then $n_1 = m_1$ and

$$\sum_{k \geq 2} \frac{1}{n_2 \cdots n_k} = \sum_{k \geq 2} \frac{1}{m_2 \cdots m_k}.$$

Proceeding in the same manner, we conclude that $n_i = m_i$ for all i.

Finally, we look at its rationality characterization. If $A \in \mathbb{Q}^{\times}$, then $r_0 \in \mathbb{Q}$, say $r_0 := p/q$, where $p, q \in \mathbb{N}$. From (2.2), we see that each r_i is a rational number whose denominator is q. Using this fact and the second inequality condition in (2.2), we deduce that $r_j = 0$ for some $j \leq p$, i.e., the expansion terminates. On the other hand, it is clear that each terminating series expansion represents a rational number. Now suppose that A is irrational and there are a j and an integer n such that $n_i = n$ for all $i \geq j$. Then

$$A = n_0 + \sum_{k=1}^{j} \frac{1}{n_1 n_2 \cdots n_k} + \frac{1}{n_1 n_2 \cdots n_j} \sum_{k=1}^{\infty} \frac{1}{n^k}.$$

Since $\sum_{k \geq 1} 1/n^k = 1/(n-1)$, it follows that A is rational, which is impossible. $\square$

Definition 2.3. A series expression as in Theorem 2.2 is called a **Cohen-Egyptian fraction expansion** and abbreviated by a **CEF expansion**.

The following examples illustrate the CEF expansions of $\frac{34}{13}$ and e, respectively.

Example 2.4.

(1) We construct the CEF expansion of $\dfrac{34}{13}$ as follows.

Observe that $\dfrac{34}{13} = 2 + \dfrac{8}{13}$. Since $\left\lceil \dfrac{13}{8} \right\rceil = 2$, we write

$$1 = 2 \cdot \frac{8}{13} - \frac{3}{13},$$

so

$$\frac{34}{13} = 2 + \frac{1}{2} + \frac{3/13}{2}.$$

From $\left\lceil \dfrac{13}{3} \right\rceil = 5$, we obtain

$$1 = 5 \cdot \frac{3}{13} - \frac{2}{13},$$

and hence

$$\frac{34}{13} = 2 + \frac{1}{2} + \frac{1}{2 \cdot 5} + \frac{2/13}{2 \cdot 5}.$$

Next, since $\left\lceil \dfrac{13}{2} \right\rceil = 7$, we get

$$1 = 7 \cdot \frac{2}{13} - \frac{1}{13}$$

and so

$$\frac{34}{13} = 2 + \frac{1}{2} + \frac{1}{2 \cdot 5} + \frac{1}{2 \cdot 5 \cdot 7} + \frac{1/13}{2 \cdot 5 \cdot 7}.$$

Therefore, the CEF expansion of $\dfrac{34}{13}$ is

$$\frac{34}{13} = 2 + \frac{1}{2} + \frac{1}{2 \cdot 5} + \frac{1}{2 \cdot 5 \cdot 7} + \frac{1}{2 \cdot 5 \cdot 7 \cdot 13}.$$

(2) By the same algorithm , Cohen ([4]) gave

$$e = 2 + \frac{1}{2} + \frac{1}{2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \cdots + \frac{1}{n!} + \cdots .$$

2.2 The non-archimedean case

In this section, our algorithm of constructing CEF expansions of elements in complete discrete non-archimedean valued fields is given.

Let K be a complete field with respect to a discrete non-archimedean valuation $|\cdot|$ and $\mathcal{A} \subseteq \mathcal{O}$ be a set of representatives of $\mathcal{O}/\mathcal{P}$. Let $\alpha \in K^{\times}$. By Theorem 1.12, α can be uniquely represented as

$$\alpha = \sum_{i=r}^{\infty} a_i \pi^i, \quad (a_r \neq 0)$$

where $r \in \mathbb{Z}$, $a_i \in \mathcal{A}$ and $\pi \in K$ is a prime element which is usually normalized so that $|\pi| = 2^{-1}$. Define the exponential valuation $\nu(\alpha)$ of α by

$$|\alpha| = 2^{-\nu(\alpha)} = 2^{-r} \quad \text{and} \quad \nu(0) := \infty.$$

The **head part** $\langle\alpha\rangle$ of α is defined as the finite series

$$\langle\alpha\rangle := \sum_{i=\nu(\alpha)}^{0} a_i \pi^i \ \text{ if } \ \nu(\alpha) \leq 0 \ \text{ and } \ 0 \text{ otherwise.}$$

Denote the set of all head parts by

$$S := \left\{ \langle\alpha\rangle : \ \alpha \in K^{\times} \right\}.$$

To construct a Cohen-Egyption fraction, CEF, expansion, similar to Lemma 2.1, we start with the following lemma.

Lemma 2.5. *For any $\alpha \in K^{\times}$ such that $\nu(\alpha) \geq 1$, there exist a unique $n \in S$ such that $\nu(n) \leq -1$ and a unique $r \in K$ such that*

$$1 = n\alpha - r \quad \textit{and} \quad \nu(r) \geq \nu(\alpha) + 1 \quad (\textit{i.e.}, 0 \leq |r| < |\alpha|).$$

Proof. Let $\alpha \in K^{\times}$ such that $\nu(\alpha) \geq 1$. Define $n = \langle 1/\alpha \rangle$. Then

$$\nu(n) = \nu\left(1/\alpha\right) = -\nu(\alpha) \leq -1.$$

Putting $r = n\alpha - 1$, we show now that $\nu(r) \geq \nu(\alpha) + 1$. Since $n = \langle 1/\alpha \rangle$, we have

$$\frac{1}{\alpha} = n + \sum_{k \geq 1} c_k \pi^k,$$

where $c_k \in \mathcal{A}$, and so

$$n\alpha - 1 = -\alpha \sum_{k \geq 1} c_k \pi^k.$$

Thus

$$\nu(r) = \nu(n\alpha - 1) = \nu(-\alpha) + \nu\left(\sum_{k \geq 1} c_k \pi^k\right) \geq \nu(\alpha) + 1 > \nu(\alpha).$$

To prove the uniqueness, assume there exist $n_1 \in S$ such that $\nu\left(n_1\right) \leq -1$ and $r_1 \in K$ such that

$$1 = n_1\alpha - r_1, \quad 0 \leq \left|r_1\right| < \left|\alpha\right|.$$

From $n\alpha - r = 1 = n_1\alpha - r_1$, we get $(n - n_1)\alpha = r - r_1$. If $n \neq n_1$, we have $|n - n_1| \geq 1$ since $n, n_1 \in S$. Using $|\alpha| > |r - r_1|$, we deduce that

$$|r - r_1| < |\alpha| \leq |n - n_1||\alpha| = |r - r_1|,$$

which is a contradiction. Thus, $n = n_1$ and so $r = r_1$. $\square$

We now prove that every nonzero element of a complete discrete non-archimedean valued field has a CEF expansion. Furthermore, our proof is constructive and gives an algorithm to construct such an expansion.

Theorem 2.6. *Each $\alpha \in K^{\times}$ has a CEF expansion of the form*

$$\alpha = n_0 + \sum_{k=1}^{\infty} \frac{1}{n_1 n_2 \cdots n_k},$$

where

$$n_k \in S,\ \nu(n_k) \le -k \ \text{ and } \ \nu(n_{k+1}) \le \nu(n_k) - 1 \ \ (k \ge 1). \tag{2.5}$$

This series representation is unique subject to the digit condition (2.5).

Proof. Let $\alpha \in K^{\times}$. Define $n_0 = \langle \alpha \rangle$ and $r_0 = \alpha - n_0$. Then $\nu(r_0) \ge 1$. If $r_0 = 0$, then the process stops and we write $\alpha = n_0$. When $r_0 \ne 0$, by Lemma 2.5, there are $n_1 \in S$ and $r_1 \in K$ such that

$$n_1 = \left\langle \frac{1}{r_0} \right\rangle, \qquad r_1 = n_1 r_0 - 1,$$

where $\nu(n_1) \le -1$ and $\nu(r_1) \ge \nu(r_0) + 1$. So

$$\alpha = n_0 + r_0 = n_0 + \frac{1}{n_1} + \frac{r_1}{n_1}.$$

If $r_1 = 0$, then the process stops and we write $\alpha = n_0 + 1/n_1$. When $r_1 \ne 0$, by Lemma 2.5, there are $n_2 \in S$ and $r_2 \in K$ such that

$$n_2 = \left\langle \frac{1}{r_1} \right\rangle, \qquad r_2 = n_2 r_1 - 1,$$

where $\nu(n_2) \leq -1$ and $\nu(r_2) \geq \nu(r_1) + 1$. So

$$\alpha = n_0 + \frac{1}{n_1} + \frac{1}{n_1 n_2} + \frac{r_2}{n_1 n_2}.$$

Continuing this process, we generally obtain

$$n_k = \left\langle \frac{1}{r_{k-1}} \right\rangle, \quad r_k = n_k r_{k-1} - 1$$
$$\alpha = n_0 + \frac{1}{n_1} + \frac{1}{n_1 n_2} + \cdots + \frac{1}{n_1 n_2 \cdots n_k} + \frac{r_k}{n_1 n_2 \cdots n_k},$$

where

$$n_k \in S, \quad \nu(n_k) \leq -1, \quad \nu(r_k) \geq \nu(r_{k-1}) + 1 \quad \text{for all} \quad k \geq 1.$$

Thus,

$$\nu(n_{k+1}) = -\nu(r_k) \leq -\nu(r_{k-1}) - 1 = \nu(n_k) - 1 \quad \text{for all} \quad k \geq 1.$$

We observe that the process terminates if $r_k = 0$.

Next, we show that $\nu(n_k) \leq -k$ for all $k \geq 1$. By construction, we have $\nu(n_1) \leq -1$. Assume that $\nu(n_k) \leq -k$. Then

$$\nu(n_{k+1}) \leq \nu(n_k) - 1 \leq -k - 1.$$

Regarding convergence, we consider

$$\begin{aligned}
\nu\left(\frac{r_k}{n_1 n_2 \cdots n_k}\right) &= -\nu(n_1) - \nu(n_2) - \cdots - \nu(n_k) + \nu(r_k) \\
&= -\nu(n_1) - \nu(n_2) - \cdots - \nu(n_k) - \nu(n_{k+1}) \\
&\geq 1 + 2 + \cdots + k + (k+1) \to \infty \quad \text{as} \quad k \to \infty.
\end{aligned}$$

It remains to prove the uniqueness. Suppose that α has two such expansions

$$n_0 + \sum_j \frac{1}{n_1 n_2 \cdots n_j} = \alpha = m_0 + \sum_i \frac{1}{m_1 m_2 \cdots m_i}.$$

Since $\nu\left(\sum_j 1/n_1 n_2 \cdots n_j\right) = \nu\left(1/n_1\right) \geq 1$ and $n_0 \in S$, we have $n_0 = \langle \alpha \rangle$. Similarly, we obtain $m_0 = \langle \alpha \rangle$. These give $n_0 = m_0$ and so $\sum_{j\geq 1} 1/n_1 n_2 \cdots n_j = \sum_{i \geq 1} 1/m_1 m_2 \cdots m_i$. Putting

$$\omega := \sum_{j\geq 1} \frac{1}{n_1 n_2 \cdots n_j} = \sum_{i\geq 1} \frac{1}{m_1 m_2 \cdots m_i},$$

we have $n_1 \omega = 1 + \sum_{j \geq 2} 1/n_2 \cdots n_j$ and hence

$$1 = n_1 \omega - \sum_{j \geq 2} \frac{1}{n_2 \cdots n_j}.$$

Thus

$$\nu\left(\sum_{j\geq 2} \frac{1}{n_2 \cdots n_j}\right) = \nu\left(\frac{1}{n_2}\right) = -\nu\left(n_2\right) \geq -\nu\left(n_1\right) + 1 = \nu\left(\omega\right) + 1.$$

By Lemma 2.5, since n_1 is the unique element in S with such property, we deduce $n_1 = m_1$. Continuing in the same manner, we conclude that the two expansions are identical. $\square$

Example 2.7.

(1) Consider the CEF expansion of $\dfrac{x^5 + x^4 + x^3 + x^2 + 1}{x^3 + x + 1}$ in $\mathbb{Q}((1/x))$ with respect to the degree valuation, by the division algorithm, we have

$$\frac{x^5 + x^4 + x^3 + x^2 + 1}{x^3 + x + 1} = x^2 + x + \frac{-x^2 - x + 1}{x^3 + x + 1}.$$

Since $\left\langle \dfrac{x^3+x+1}{-x^2-x+1} \right\rangle = -x+1$, we write

$$1 = (-x+1)\left(\frac{-x^2-x+1}{x^3+x+1}\right) - \frac{-3x}{x^3+x+1},$$

so

$$\frac{x^5+x^4+x^3+x^2+1}{x^3+x+1} = x^2+x+\frac{1}{-x+1}+\frac{(-3x)/(x^3+x+1)}{-x+1}.$$

Next, since $\left\langle \dfrac{x^3+x+1}{-3x} \right\rangle = -\dfrac{1}{3}x^2 - \dfrac{1}{3}$, we get

$$1 = \left(-\frac{1}{3}x^2-\frac{1}{3}\right)\left(\frac{-3x}{x^3+x+1}\right) - \frac{-1}{x^3+x+1}$$

and so

$$\begin{aligned}\frac{x^5+x^4+x^3+x^2+1}{x^3+x+1} &= x^2+x+\frac{1}{-x+1}+\frac{1}{(-x+1)\left(-\frac{1}{3}x^2-\frac{1}{3}\right)}\\ &\quad + \frac{(-1)/(x^3+x+1)}{(-x+1)\left(-\frac{1}{3}x^2-\frac{1}{3}\right)}.\end{aligned}$$

Hence the CEF expansion of $\dfrac{x^5+x^4+x^3+x^2+1}{x^3+x+1}$ is

$$\begin{aligned}\frac{x^5+x^4+x^3+x^2+1}{x^3+x+1} &= x^2+x+\frac{1}{-x+1}+\frac{1}{(-x+1)\left(-\frac{1}{3}x^2-\frac{1}{3}\right)}\\ &\quad + \frac{1}{(-x+1)\left(-\frac{1}{3}x^2-\frac{1}{3}\right)(-x^3-x-1)}.\end{aligned}$$

(2) In order to find the CEF expansion of -1 in $\mathbb{Q}_5$ with respect to the 5-adic valuation, by direct calculation, we write

$$-1 = 4 + 4 \cdot 5 + 4 \cdot 5^2 + 4 \cdot 5^3 + \cdots .$$

Since $\left\langle \dfrac{1}{4 \cdot 5 + 4 \cdot 5^2 + 4 \cdot 5^3 + \cdots} \right\rangle = \dfrac{4}{5} + 4$, we write

$$1 = \left(\frac{4}{5} + 4\right)\left(4 \cdot 5 + 4 \cdot 5^2 + 4 \cdot 5^3 + \cdots\right) - \left(4 \cdot 5^2 + 4 \cdot 5^3 + 4 \cdot 5^4 + \cdots\right),$$

so

$$-1 = 4 + \frac{1}{\frac{4}{5} + 4} + \frac{4 \cdot 5^2 + 4 \cdot 5^3 + 4 \cdot 5^4 + \cdots}{\frac{4}{5} + 4}.$$

Form $\left\langle \dfrac{1}{4 \cdot 5^2 + 4 \cdot 5^3 + 4 \cdot 5^4 + \cdots} \right\rangle = \dfrac{4}{5^2} + \dfrac{4}{5} + 4$, we obtain

$$1 = \left(\frac{4}{5^2} + \frac{4}{5} + 4\right)\left(4 \cdot 5^2 + 4 \cdot 5^3 + 4 \cdot 5^4 + \cdots\right) - \left(4 \cdot 5^3 + 4 \cdot 5^4 + 4 \cdot 5^4 + \cdots\right),$$

and hence

$$-1 = 4 + \frac{1}{\frac{4}{5} + 4} + \frac{1}{\left(\frac{4}{5} + 4\right)\left(\frac{4}{5^2} + \frac{4}{5} + 4\right)} + \frac{4 \cdot 5^3 + 4 \cdot 5^4 + 4 \cdot 5^4 + \cdots}{\left(\frac{4}{5} + 4\right)\left(\frac{4}{5^2} + \frac{4}{5} + 4\right)}.$$

Next, since $\left\langle \dfrac{1}{4 \cdot 5^3 + 4 \cdot 5^4 + 4 \cdot 5^4 + \cdots} \right\rangle = \dfrac{4}{5^3} + \dfrac{4}{5^2} + \dfrac{4}{5} + 4$, we get

$$1 = \left(\frac{4}{5^3} + \frac{4}{5^2} + \frac{4}{5} + 4\right)\left(4 \cdot 5^3 + 4 \cdot 5^4 + 4 \cdot 5^4 + \cdots\right) - \left(4 \cdot 5^4 + 4 \cdot 5^5 + 4 \cdot 5^6 + \cdots\right),$$

and so

$$\begin{aligned}-1 =&4+\frac{1}{\frac{4}{5}+4}+\frac{1}{\left(\frac{4}{5}+4\right)\left(\frac{4}{5^2}+\frac{4}{5}+4\right)}\\&+\frac{1}{\left(\frac{4}{5}+4\right)\left(\frac{4}{5^2}+\frac{4}{5}+4\right)\left(\frac{4}{5^3}+\frac{4}{5^2}+\frac{4}{5}+4\right)}\\&+\frac{4\cdot 5^4+4\cdot 5^5+4\cdot 5^6+\cdots}{\left(\frac{4}{5}+4\right)\left(\frac{4}{5^2}+\frac{4}{5}+4\right)\left(\frac{4}{5^3}+\frac{4}{5^2}+\frac{4}{5}+4\right)}.\end{aligned}$$

Repeating in the same manner, we obtain the CEF expansion of -1 as follows

$$\begin{aligned}-1 =&4+\frac{1}{\frac{4}{5}+4}+\frac{1}{\left(\frac{4}{5}+4\right)\left(\frac{4}{5^2}+\frac{4}{5}+4\right)}\\&+\frac{1}{\left(\frac{4}{5}+4\right)\left(\frac{4}{5^2}+\frac{4}{5}+4\right)\left(\frac{4}{5^3}+\frac{4}{5^2}+\frac{4}{5}+4\right)}\\&+\frac{1}{\left(\frac{4}{5}+4\right)\left(\frac{4}{5^2}+\frac{4}{5}+4\right)\left(\frac{4}{5^3}+\frac{4}{5^2}+\frac{4}{5}+4\right)\left(\frac{4}{5^4}+\frac{4}{5^3}+\frac{4}{5^2}+\frac{4}{5}+4\right)}+\cdots.\end{aligned}$$

CHAPTER III

ENGEL SERIES EXPANSIONS

We begin this chapter by a construction in the real case.

3.1 The case of real numbers

Recall that following result, see e.g. Kapitel IV of [11], which asserts that each nonzero real number can be uniquely represented as an infinite series expansion described by the following theorem.

Theorem 3.1. *Each $A \in \mathbb{R}^{\times}$ is uniquely representable as an infinite series expansion of the form*

$$A = a_0 + \sum_{i=1}^{\infty} \frac{1}{a_1 a_2 \cdots a_i},$$

where

$$a_0 = \begin{cases} \lfloor A \rfloor & \text{if } A \notin \mathbb{Z} \\ A - 1 & \text{if } A \in \mathbb{Z}, \end{cases} \qquad a_1 \geq 2, \quad a_{i+1} \geq a_i \quad (i \geq 1). \tag{3.1}$$

Moreover, $A \in \mathbb{Q}^{\times}$ if and only if $a_{i+1} = a_i$ (≥ 2) for all sufficiently large i.

Proof. Let $A \in \mathbb{R}^{\times}$. Define $A_1 = A - a_0$. Then $0 < A_1 \leq 1$. If $A_i \neq 0$ for all $i \geq 1$ is already defined, put

$$a_i = 1 + \left\lfloor \frac{1}{A_i} \right\rfloor \tag{3.2}$$

and

$$A_{i+1} = a_i A_i - 1. \tag{3.3}$$

Observe that a_i is the least integer $> 1/A_i$ and

$$\frac{1}{a_i} < A_i \leq \frac{1}{a_i - 1}.$$

We now claim that

$$0 < \ldots \leq A_{i+1} \leq A_i \leq \ldots \leq A_2 \leq A_1 \leq 1. \tag{3.4}$$

First, we show that $A_i > 0$ for all $i \geq 1$ by induction. If $i = 1$, then we have seen that $A_1 > 0$. Assume now that $A_i > 0$ for $i \geq 1$. By (3.2), we see that $a_i \in \mathbb{N}$. Since

$$A_{i+1} = a_i A_i - 1 = \left(A_i - \frac{1}{a_i}\right) a_i$$

and $1/a_i < A_i$, we have $A_{i+1} > 0$. If there exists $m \in \mathbb{N}$ such that $A_{m+1} > A_m$, then

$$a_m A_m - 1 = A_{m+1} > A_m$$

and so $a_m - 1 > 1/A_m$, contradicting the minimal property of a_i and the claim is proved.

From (3.2) and (3.4), we deduce that $a_1 \geq 2$ and $a_{i+1} \geq a_i$ for all $i \geq 1$. Iterating (3.3), we get

$$A_1 = \frac{1}{a_1} + \frac{1}{a_1 a_2} + \cdots + \frac{1}{a_1 a_2 \cdots a_i} + \frac{A_{i+1}}{a_1 a_2 \cdots a_i}.$$

To establish the convergence, let

$$B_i = \frac{1}{a_1} + \frac{1}{a_1 a_2} + \cdots + \frac{1}{a_1 a_2 \cdots a_i} \quad \text{for all} \quad i \geq 1.$$

Since $A_i > 0$ and $a_i \in \mathbb{N}$ for all $i \geq 1$, the sequence of real numbers (B_i) is increasing and bounded above by A_1. Thus, $\lim_{i\to\infty} B_i$ exists and so

$$\frac{1}{a_1a_2\cdots a_i} \to 0 \ \text{ as } \ i \to \infty.$$

By (3.4),

$$0 < \frac{A_{i+1}}{a_1a_2\cdots a_i} \leq \frac{1}{a_1a_2\cdots a_i} \to 0 \ \text{ as } \ i \to \infty,$$

showing that any real number has an infinite series expansion. To prove the uniqueness, we assume that A has two infinite such expansions

$$a_0 + \sum_{i=1}^{\infty} \frac{1}{a_1a_2\cdots a_i} = A = b_0 + \sum_{i=1}^{\infty} \frac{1}{b_1b_2\cdots b_i}, \tag{3.5}$$

with the restrictions $a_0 \in \mathbb{Z},\ a_1 \geq 2,\ a_{i+1} \geq a_i$ for all $i \geq 1$ and the same restrictions also for the b_i's. From the restrictions, we note that

$$0 < A_1 := \sum_{i=1}^{\infty} \frac{1}{a_1a_2\cdots a_i} \leq 1.$$

If $A_1 = 1$, then by (3.5) we also have $\sum_{i\geq 1} 1/b_1b_2\cdots b_i = 1$, forcing $a_0 = b_0$. If $0 < A_1 < 1$, then (3.5) shows that $0 < \sum_{i\geq 1} 1/b_1b_2\cdots b_i < 1$, forcing again $a_0 = b_0$. In either case, cancelling out the terms a_0, b_0 in (3.5) we get

$$A_1 := \sum_{i=1}^{\infty} \frac{1}{a_1a_2\cdots a_i} = \sum_{i=1}^{\infty} \frac{1}{b_1b_2\cdots b_i}. \tag{3.6}$$

Since $a_{i+1} \geq a_i$ for all $i \geq 1$,

$$a_1A_1 - 1 = \frac{1}{a_2} + \frac{1}{a_2a_3} + \frac{1}{a_2a_3a_4} + \cdots \leq \frac{1}{a_1} + \frac{1}{a_1a_2} + \frac{1}{a_1a_2a_3} + \cdots = A_1,$$

so $0 < a_1 - 1/A_1 \leq 1$. But there is exactly one integer a_1 satisfying these restrictions.

Thus, $a_1 = b_1$. Cancelling out the terms a_1 and b_1 in (3.6) and repeat the arguments we see that $a_i = b_i$ for all i.

Finally, we prove the rationality characterization. If its series expansion is infinite periodic of period 1, then it clearly represents a rational number. To prove its converse, let $A = a/b \in \mathbb{Q}^{\times}$. Since

$$A_1 = A - a_0 = \frac{a - ba_0}{b},$$

we see that A_1 is a rational number in the interval $(0, 1]$ whose denominator is b. In general, from (3.3), we deduce that for each $i \geq 1$, A_i is a rational number in the interval $(0, 1]$ whose denominator is b. But the number of rational numbers in the interval $(0, 1]$ whose denominator is b is finite. This implies that there are two least indices $h, k \in \mathbb{N}$ such that $A_{h+k} = A_h$. Thus, by (3.2), we have $a_{h+k} = a_h$. From (3.1), we know that the sequence $\{a_i\}$ is increasing. We must then have $k = 1$ and the assertion follows. $\square$

Definition 3.2. A series expression as in Theorem 3.1 is called a **Engel series expansion** and abbreviated by a **ES expansion**.

Example 3.3. We give two examples of the ES expansions of $\dfrac{4}{73}$ and $\sqrt{2}$, respectively.

(1) We construct the ES expansion of $\dfrac{4}{73}$ as follows.

Observe that $\dfrac{4}{73} \notin \mathbb{Z}$, so we have $a_0 = \left\lfloor \dfrac{4}{73} \right\rfloor = 0$ and $A_1 = \dfrac{4}{73}$.

Since $a_1 = 1 + \left\lfloor \dfrac{1}{A_1} \right\rfloor = 1 + \left\lfloor \dfrac{73}{4} \right\rfloor = 19$, we get

$$A_2 = 19 \cdot \frac{4}{73} - 1 = \frac{3}{73},$$

so

$$\frac{4}{73} = 0 + \frac{1}{19} + \frac{3/73}{19}.$$

From $a_2 = 1 + \left\lfloor \frac{1}{A_2} \right\rfloor = 1 + \left\lfloor \frac{73}{3} \right\rfloor = 25$, we obtain

$$A_3 = 25 \cdot \frac{3}{73} - 1 = \frac{2}{73},$$

and hence

$$\frac{4}{73} = 0 + \frac{1}{19} + \frac{1}{19 \cdot 25} + \frac{2/73}{19 \cdot 25}.$$

Since $a_3 = 1 + \left\lfloor \frac{1}{A_3} \right\rfloor = 1 + \left\lfloor \frac{73}{2} \right\rfloor = 37$, we get

$$A_4 = 37 \cdot \frac{2}{73} - 1 = \frac{1}{73},$$

so

$$\frac{4}{73} = 0 + \frac{1}{19} + \frac{1}{19 \cdot 25} + \frac{1}{19 \cdot 25 \cdot 37} + \frac{1/73}{19 \cdot 25 \cdot 37}.$$

From $a_4 = 1 + \left\lfloor \frac{1}{A_4} \right\rfloor = 1 + \left\lfloor \frac{73}{1} \right\rfloor = 74$, we obtain

$$A_5 = 74 \cdot \frac{1}{73} - 1 = \frac{1}{73},$$

and hence

$$\frac{4}{73} = 0 + \frac{1}{19} + \frac{1}{19 \cdot 25} + \frac{1}{19 \cdot 25 \cdot 37} + \frac{1}{19 \cdot 25 \cdot 37 \cdot 74} + \frac{1/73}{19 \cdot 25 \cdot 37 \cdot 74}.$$

Next, since $a_5 = 1 + \left\lfloor \frac{1}{A_5} \right\rfloor = 1 + \left\lfloor \frac{73}{1} \right\rfloor = 74$, we get

$$A_6 = 74 \cdot \frac{1}{73} - 1 = \frac{1}{73}$$

and so

$$\frac{4}{73} = 0+\frac{1}{19}+\frac{1}{19\cdot 25}+\frac{1}{19\cdot 25\cdot 37}+\frac{1}{19\cdot 25\cdot 37\cdot 74}+\frac{1}{19\cdot 25\cdot 37\cdot 74^2}+\frac{1/73}{19\cdot 25\cdot 37\cdot 74^2}.$$

Note that $a_i = a_4 = 74$ for all $i \geq 5$.

Therefore, the ES expansion of $\dfrac{4}{73}$ is

$$\frac{4}{73} = 0 + \frac{1}{19} + \frac{1}{19\cdot 25} + \frac{1}{19\cdot 25\cdot 37} + \frac{1}{19\cdot 25\cdot 37\cdot 74} + \frac{1}{19\cdot 25\cdot 37\cdot 74^2} + \frac{1}{19\cdot 25\cdot 37\cdot 74^3} + \cdots .$$

(2) By the same algorithm we have

$$\sqrt{2} = 1 + \frac{1}{3} + \frac{1}{3\cdot 5} + \frac{1}{3\cdot 5\cdot 5} + \frac{1}{3\cdot 5\cdot 5\cdot 16} + \frac{1}{3\cdot 5\cdot 5\cdot 16\cdot 18} + \cdots .$$

Remarks. In passing, we make the following observations.

(a) For $i \geq 1$, we have

$$a_{i+1} = a_i \Longleftarrow A_{i+1} = A_i \Longleftrightarrow a_iA_i - 1 = A_i \Longleftrightarrow a_i = 1 + \frac{1}{A_i} \Longleftrightarrow \frac{1}{A_i} \in \mathbb{Z}\setminus\{0\}.$$

(b) If $A \in \mathbb{R}\setminus\mathbb{Q}$, then $A_i \in \mathbb{R}\setminus\mathbb{Q}$ and so $1/A_i \in \mathbb{R}\setminus\mathbb{Q}$ for all $i \geq 1$.

(c) If $A \in \mathbb{Z}\setminus\{0\}$, then its ES expansion is

$$A = A - 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots .$$

3.2 The non-archimedean case

Using the same notation as in Section 2.2, we recall the Knopfmachers' series expansion algorithm for series expansions in $K^\times$, [7]. For $\alpha \in K^\times$, let

$$a_0 := \langle \alpha \rangle \in S.$$

Define

$$A_1 := \alpha - a_0.$$

If $A_i \neq 0$ for all $i \geq 1$ is already defined, put

$$a_i = \left\langle \frac{1}{A_i} \right\rangle, \quad A_{i+1} = \left(A_i - \frac{1}{a_i} \right) \frac{s_i}{r_i}$$

if $a_i \neq 0$, where r_i and $s_i \in K^\times$ which may depend on $a_1, \ldots, a_i$. Then for $i \geq 1$

$$\alpha = a_0 + A_1 = \cdots = a_0 + \frac{1}{a_1} + \frac{r_1}{s_1}\frac{1}{a_2} + \cdots + \frac{r_1 \cdots r_{i-1}}{s_1 \cdots s_{i-1}}\frac{1}{a_i} + \frac{r_1 \cdots r_i}{s_1 \cdots s_i} A_{i+1}. \qquad (3.7)$$

The process ends in a finite expansion if some $A_{i+1} = 0$. If some $a_i = 0$, then A_{i+1} is not defined. To take care of this difficulty, we impose the condition

$$\nu(s_i) - \nu(r_i) \geq 2\nu(a_i) - 1.$$

Thus

$$\alpha = a_0 + \frac{1}{a_1} + \sum_{i=1}^{\infty} \frac{r_1 \cdots r_i}{s_1 \cdots s_i} \cdot \frac{1}{a_{i+1}}.$$

When $r_i = 1$ and $s_i = a_i$, the algorithm yields a well-defined (with respect to the valuation) and unique series expansion, termed ES expansion. Summing up, we have

Theorem 3.4. *Every $\alpha \in K^{\times}$ has a finite or an infinite convergent ES expansion of the form*

$$\alpha = a_0 + \sum_{i=1}^{\infty} \frac{1}{a_1 a_2 \cdots a_i},$$

where the digits a_i are subject to the restrictions

$$a_0 = \langle \alpha \rangle \in S\ , \quad a_i \in S, \quad \nu(a_i) \leq -i, \quad \nu(a_{i+1}) \leq \nu(a_i) - 1 \quad (i \geq 1).$$

Example 3.5.

(1) Consider the ES expansion of $\dfrac{x^5+x^4+x^3+x^2+1}{x^3+x+1}$ in $\mathbb{Q}((1/x))$ with respect to the degree valuation. Observe that

$$\frac{x^5+x^4+x^3+x^2+1}{x^3+x+1} = x^2 + x + \frac{-x^2-x+1}{x^3+x+1},$$

so $a_0 = x^2 + x$ and $A_1 = \dfrac{-x^2-x+1}{x^3+x+1}$.

From $a_1 = \left\langle \dfrac{1}{A_1} \right\rangle = \left\langle \dfrac{x^3+x+1}{-x^2-x+1} \right\rangle = -x+1$, we write

$$A_2 = \left(\frac{-x^2-x+1}{x^3+x+1} - \frac{1}{-x+1} \right)(-x+1) = \frac{-3x}{x^3+x+1},$$

so

$$\frac{x^5+x^4+x^3+x^2+1}{x^3+x+1} = x^2 + x + \frac{1}{-x+1} + \frac{-3x}{(-x+1)(x^3+x+1)}.$$

Next, since $a_2 = \left\langle \dfrac{1}{A_2} \right\rangle = \left\langle \dfrac{x^3+x+1}{-3x} \right\rangle = -\dfrac{1}{3}x^2 - \dfrac{1}{3}$, we get

$$A_3 = \left(\frac{-3x}{x^3+x+1} - \frac{-3}{x^2+1} \right)\left(-\frac{1}{3}x^2 - \frac{1}{3} \right) = \frac{-1}{x^3+x+1},$$

and hence

$$\frac{x^5+x^4+x^3+x^2+1}{x^3+x+1} = x^2+x+\frac{1}{-x+1}+\frac{1}{(-x+1)\left(-\frac{1}{3}x^2-\frac{1}{3}\right)}$$
$$+\frac{-1}{(-x+1)\left(-\frac{1}{3}x^2-\frac{1}{3}\right)(x^3+x+1)}.$$

Therefore, the ES expansion of $\dfrac{x^5+x^4+x^3+x^2+1}{x^3+x+1}$ is

$$\frac{x^5+x^4+x^3+x^2+1}{x^3+x+1} = x^2+x+\frac{1}{-x+1}+\frac{1}{(-x+1)\left(-\frac{1}{3}x^2-\frac{1}{3}\right)}$$
$$+\frac{1}{(-x+1)\left(-\frac{1}{3}x^2-\frac{1}{3}\right)(-x^3-x-1)}.$$

(2) To construct the ES expansion of -1 in $\mathbb{Q}_5$ with respect to the 5-adic valuation, by elementary calculation, we write

$$-1 = 4+4\cdot 5+4\cdot 5^2+4\cdot 5^3+\cdots,$$

so $a_0 = 4$ and $A_1 = 4\cdot 5+4\cdot 5^2+4\cdot 5^3+\cdots$.

From $a_1 = \left\langle \dfrac{1}{A_1}\right\rangle = \left\langle \dfrac{1}{4\cdot 5+4\cdot 5^2+4\cdot 5^3+\cdots}\right\rangle = \dfrac{4}{5}+4$, we write

$$\begin{aligned} A_2 &= \left((4\cdot 5+4\cdot 5^2+4\cdot 5^3+\cdots)-\frac{1}{\frac{4}{5}+4}\right)\left(\frac{4}{5}+4\right) \\ &= 4\cdot 5^2+4\cdot 5^3+4\cdot 5^4+\cdots, \end{aligned}$$

so

$$-1 = 4 + \frac{1}{\frac{4}{5}+4} + \frac{4\cdot 5^2 + 4\cdot 5^3 + 4\cdot 5^4 + \cdots}{\frac{4}{5}+4}.$$

Next, since $a_2 = \left\langle \frac{1}{A_2} \right\rangle = \left\langle \frac{1}{4\cdot 5^2 + 4\cdot 5^3 + 4\cdot 5^4 + \cdots} \right\rangle = \frac{4}{5^2} + \frac{4}{5} + 4$, we get

$$\begin{aligned} A_3 &= \left(\left(4\cdot 5^2 + 4\cdot 5^3 + 4\cdot 5^4 + \cdots\right) - \frac{1}{\frac{4}{5^2}+\frac{4}{5}+4} \right) \left(\frac{4}{5^2} + \frac{4}{5} + 4 \right) \\ &= 4\cdot 5^3 + 4\cdot 5^4 + 4\cdot 5^4 + \cdots, \end{aligned}$$

and hence

$$-1 = 4 + \frac{1}{\frac{4}{5}+4} + \frac{1}{\left(\frac{4}{5}+4\right)\left(\frac{4}{5^2}+\frac{4}{5}+4\right)} + \frac{4\cdot 5^3 + 4\cdot 5^4 + 4\cdot 5^4 + \cdots}{\left(\frac{4}{5}+4\right)\left(\frac{4}{5^2}+\frac{4}{5}+4\right)}.$$

By so doing, we have the general form of A_m and a_m as follows:

$$\begin{aligned} A_m &= 4\cdot 5^m + 4\cdot 5^{m+1} + 4\cdot 5^{m+2} + \cdots, \quad && m = 1,2,3,\ldots, \\ a_m &= \frac{4}{5^m} + \frac{4}{5^{m-1}} + \cdots + \frac{4}{5} + 4, && m = 1,2,3,\ldots, \end{aligned}$$

and so the ES expansion of -1 is

$$\begin{aligned} -1 =& 4 + \frac{1}{\frac{4}{5}+4} + \frac{1}{\left(\frac{4}{5}+4\right)\left(\frac{4}{5^2}+\frac{4}{5}+4\right)} \\ &+ \frac{1}{\left(\frac{4}{5}+4\right)\left(\frac{4}{5^2}+\frac{4}{5}+4\right)\left(\frac{4}{5^3}+\frac{4}{5^2}+\frac{4}{5}+4\right)} \\ &+ \frac{1}{\left(\frac{4}{5}+4\right)\left(\frac{4}{5^2}+\frac{4}{5}+4\right)\left(\frac{4}{5^3}+\frac{4}{5^2}+\frac{4}{5}+4\right)\left(\frac{4}{5^4}+\frac{4}{5^3}+\frac{4}{5^2}+\frac{4}{5}+4\right)} + \cdots. \end{aligned}$$

Observe that the digits a_m are of the form

$$a_m = \frac{5^{m+1} - 1}{5^m} \quad (m = 0, 1, 2, \ldots).$$

Based on the work of Grabner and Knopfmacher in [6], we will give more precise result in Section 4.2 describing the digits a_m appeared in the ES expansion of rational elements in the p-adic number field $\mathbb{Q}_p$.

CHAPTER IV

RATIONALITY CHARACTERIZATION IN THE NON-ARCHIMEDEAN CASE

4.1 Relationships between CEF and ES expansions

In the first section, we investigate possible relationships between CEF and ES expansions in the field of real numbers and in the complete discrete non-archimedean valued fields.

In the case of real numbers, we show that for irrational numbers both kinds of expansions are identical, while for rational numbers, their CEF expansions always terminate, but their ES expansions are infinite, periodic of period 1.

Theorem 4.1. *Let $A \in \mathbb{R}^{\times}$ and the notation be as set out in Theorems 2.2 and 3.1. I. If $A \in \mathbb{Q}^{\times}$, then its CEF expansion is finite, while its ES expansion is infinite periodic of period* 1*. More precisely, for $A \in \mathbb{Q} \setminus \mathbb{Z}$, let its CEF and ES expansions be, respectively,*

$$A = n_0 + \sum_{k=1}^{\infty} \frac{1}{n_1 n_2 \cdots n_k} = a_0 + \sum_{i=1}^{\infty} \frac{1}{a_1 a_2 \cdots a_i}. \tag{4.1}$$

If m is the least positive integer such that $1/A_m \in \mathbb{Z}$, then

$$a_0 = n_0,\ a_1 = n_1,\ \ldots,\ a_{m-1} = n_{m-1},\ a_m = n_m + 1,\ a_m = a_{m+j}\ (j \geq 1)$$

and the digits n_j terminate at n_m.

II. If $A \in \mathbb{R} \setminus \mathbb{Q}$, then its CEF and its ES expansions are identical.

Proof. Both assertions follow mostly from Theorems 2.2, 3.1 and Remark (b) except for the result related to the expansions in (4.1) which we show now.

Let $A \in \mathbb{Q} \setminus \mathbb{Z}$ and let m be the least positive integer such that $1/A_m \in \mathbb{Z}$. We treat two seperate cases.

Case $m = 1$. In this case, we have $1/A_1 \in \mathbb{Z}$ and $a_1 = 1 + \lfloor 1/A_1 \rfloor = 1 + 1/A_1$. Since $r_0 = A - n_0 = A - \lfloor A \rfloor = A - a_0 = A_1$, we get $n_1 = 1/A_1$ and so $a_1 = n_1 + 1$. We have $r_1 = n_1 r_0 - 1 = 0$, and so the CEF expansion terminates. On the other hand, by Remark (a) after Theorem 3.1, we have $a_1 = a_j$ for all $j \geq 2$.

Case $m > 1$. Thus, $1/A_1 \notin \mathbb{Z}$ and $A_1 = r_0$. By Lemma 2.1, we have $a_1 = n_1$. For $1 \leq j \leq m-2$, we assume that $A_j = r_{j-1}$ and $a_j = n_j$. Then

$$A_{j+1} = a_j A_j - 1 = n_j r_{j-1} - 1 = r_j.$$

Since $1/A_{j+1} \notin \mathbb{Z}$, again by Lemma 2.1, $a_{j+1} = n_{j+1}$. This shows that $a_1 = n_1, \ldots, a_{m-1} = n_{m-1}$. Since $1/A_m \in \mathbb{Z}$, we have $a_m = 1 + \lfloor 1/A_m \rfloor = 1 + 1/A_m$ and thus

$$A_m = a_{m-1} A_{m-1} - 1 = n_{m-1} r_{m-2} - 1 = r_{m-1}.$$

From the construction of CEF expansion, we know that $n_m = \lceil 1/r_{m-1} \rceil$. Thus, $n_m = 1/A_m$ showing that $a_m = n_m + 1$. Furthermore, $r_m = n_m r_{m-1} - 1 = 0$, implying that the CEF expansion terminates at n_m, and by Remark (a) after Theorem 3.1, $a_m = a_{m+j}$ for all $j \geq 1$. $\square$

For the non-archimedean case, from Section 2.2 and 3.2, after devising CEF and ES expansions for nonzero elements in these fields, it is clear that the constructions of CEF and ES expansions are identical which implies at once that the two representations are exactly the same in the non-archimedean case.

4.2 Characterizations of rational elements in the non-archimedean fields

In this section, we characterize rational elements in three different non-archimedean valued fields, namely, the field of p-adic numbers and the two function fields, one completed with respect to the $\pi(x)$-adic valuation and the other with respect to the degree valuation.

The following characterization of rational elements by p-adic ES expansions is due to Grabner and Knopfmacher, [6].

Theorem 4.2. *Let $x \in p\mathbb{Z}_p \setminus \{0\}$. Then x is rational, $x = \alpha/\beta$, if and only if either the p-adic ES expansion of x is finite, or there exists an m and an $s \geq m$, such that*

$$a_{m+j} = \frac{p^{s+j+1} - \gamma}{p^{s+j}} \qquad (j = 0, 1, 2, \ldots),$$

where $\gamma \mid \beta$.

For the function fields, the word "rational elements" refers to elements in $F(x)$. Let F denote a field and $\pi(x)$ an irreducible polynomial of degree d over F. There are two types of valuations in the field of rational functions $F(x)$, namely, the $\pi(x)$-adic valuation $|\cdot|_\pi$, and the degree valuation $|\cdot|_\infty$ defined as follows: from the unique representation in $F(x)$,

$$\frac{f(x)}{g(x)} = \pi(x)^m \frac{r(x)}{s(x)}$$

where $f(x)$, $g(x) \in F[x] \setminus \{0\}$; $r(x)$ and $s(x)$ are relatively prime elements of $F[x]$; $s(x)$ is a nonzero monic polynomial; $\pi(x) \nmid r(x)s(x)$ and $m \in \mathbb{Z}$

set

$$|0|_\pi = 0, \quad \left|\frac{f(x)}{g(x)}\right|_\pi = 2^{-md}; \quad |0|_\infty = 0, \quad \left|\frac{f(x)}{g(x)}\right|_\infty = 2^{\deg f(x) - \deg g(x)}.$$

Recall from Example 1.10 that $F((\pi(x)))$ and $F((1/x))$ are the completions of $F(x)$, with respect to the $\pi(x)$-adic and the degree valuations, respectively. The extension of the valuations to $F((\pi(x)))$ and $F((1/x))$ are again denoted by $|\cdot|_\pi$ and $|\cdot|_\infty$.

For a characterization of rational elements, we prove:

Theorem 4.3. *The CEF expansion of nonzero α in $F((\pi(x)))$ or in $F((1/x))$ terminates if and only if $\alpha \in F(x)^\times$.*

Proof. Although the assertions in both fields $F((\pi(x)))$ and $F((1/x))$ are the same, their respective proofs are different. In fact, when the field F has finite characteristic, both results have already been shown in Laohakosol, [8], and the proof given here is basically the same.

We use the notation of Section 2.2 and 3.2 with added subscripts π or ∞ to distinguish their corresponding meanings.

If the CEF expansion of α in either field is finite, then α is clearly rational. It remains to prove the converse. We begin with the field $F((\pi(x)))$. Assume that $\alpha \in F(x)^\times$. Then α has the CEF expansion of the form,

$$\alpha = n_0 + \sum_{k=1}^{\infty} \frac{1}{n_1 n_2 \cdots n_k},$$

where

$$n_k \in S,\ \nu(n_k) \leq -k \ \text{ and } \ \nu(n_{k+1}) \leq \nu(n_k) - 1 \ \text{ for all } \ k \geq 1.$$

By construction, each $k \geq 1$, $r_k \in F(x)$ and so can be uniquely represented in the form

$$r_k = \pi(x)^{\nu(r_k)} \frac{p_k(x)}{q_k(x)}, \tag{4.2}$$

where $p_k(x)$ and $q_k(x)$ are relatively prime elements of $F[x]$, $q_k(x)$ is a nonzero monic polynomial and $\pi(x) \nmid p_k(x)q_k(x)$. Since $n_k = \langle 1/r_{k-1}\rangle \in S_\pi$ and $\nu(n_k) \leq -k$, it is of the form

$$\begin{aligned} n_k &= s_{\nu(n_k)}(x)\pi(x)^{\nu(n_k)} + s_{\nu(n_k)+1}(x)\pi(x)^{\nu(n_k)+1} + \cdots + s_{-1}(x)\pi(x)^{-1} + s_0(x) \\ &=: m_k(x)\pi(x)^{\nu(n_k)}, \end{aligned} \tag{4.3}$$

where $s_{\nu(n_k)}(x), \ldots, s_0(x)$ are polynomials over F, not all 0, of degree $< d$ and $m_k(x) \in F[x]$. Thus,

$$\begin{aligned} |n_k|_\infty &\leq \max\{\left|s_{\nu(n_k)}(x)\pi(x)^{\nu(n_k)}\right|_\infty, \left|s_{\nu(n_k)+1}(x)\pi(x)^{\nu(n_k)+1}\right|_\infty, \ldots \\ &\qquad \ldots, \left|s_{-1}(x)\pi(x)^{-1}\right|_\infty, |s_0(x)|_\infty\} \\ &\leq 2^{d-1}. \end{aligned}$$

This yields

$$|m_k(x)|_\infty \leq 2^{d-d\nu(n_k)-1}. \tag{4.4}$$

By construction, we have

$$r_k = n_k r_{k-1} - 1. \tag{4.5}$$

Substituting (4.2) and (4.3) into (4.5) and using $\nu(r_{k-1}) = -\nu(n_k)$ lead to

$$\pi(x)^{-\nu(n_{k+1})}p_k(x)q_{k-1}(x) = q_k(x)\left(m_k(x)p_{k-1}(x) - q_{k-1}(x)\right). \tag{4.6}$$

Since $\gcd\left(\pi(x)^{-\nu(n_{k+1})}p_k(x), q_k(x)\right) = 1$, it follows that $q_k(x) \mid q_{k-1}(x)$. Successively, we have

$$|q_k(x)|_\infty \leq |q_{k-1}(x)|_\infty \leq \cdots \leq |q_1(x)|_\infty.$$

Together with (4.6) yield

$$|p_k(x)|_\infty \le |\pi(x)|_\infty^{\nu(n_{k+1})} \max\left\{|m_k(x)p_{k-1}(x)|_\infty, |q_1(x)|_\infty\right\}.$$

Using (2.5) and (4.4), we consequently have

$$\begin{aligned} |p_k(x)|_\infty &\le 2^{d(\nu(n_k)-1)} \max\left\{2^{d-d\nu(n_k)-1}|p_{k-1}(x)|_\infty, |q_1(x)|_\infty\right\} \\ &\le \max\left\{\frac{1}{2}|p_{k-1}(x)|_\infty, \frac{|q_1(x)|_\infty}{2^{d(k+1)}}\right\}. \end{aligned}$$

This shows that $|p_k(x)|_\infty \le \frac{1}{2}|p_{k-1}(x)|_\infty$ for all large k which implies that from some k onwards, $p_k(x) = 0$, and so $r_k = 0$, i.e., the expansion terminates.

Finally, for the field $F((1/x))$, we assume that $\alpha = p(x)/q(x) \in F(x)^\times$. Without loss of generality, assume $\deg p(x) \ge \deg q(x)$. By the division algorithm, we have

$$\alpha = \frac{p(x)}{q(x)} = N_0(x) + \frac{R_0(x)}{q(x)} := n_0 + r_0,$$

where

$$n_0 := N_0(x) = \langle\alpha\rangle \in S_\infty,\ R_0(x) \in F[x],\ 0 \le \deg R_0 < \deg q,\ \text{and } r_0 = \frac{R_0(x)}{q(x)}.$$

From the division algorithm,

$$\frac{q(x)}{R_0(x)} = N_1(x) + \frac{R_1(x)}{R_0(x)};\ N_1(x), R_1(x) \in F[x];\ 0 \le \deg R_1 < \deg R_0 < \deg q,$$

which is, in the terminology of Lemma 2.5,

$$1 = r_0 N_1 + \frac{R_1}{q} = r_0 n_1 - r_1.$$

Again, from the division algorithm,

$$\frac{-q(x)}{R_1(x)} = N_2(x) + \frac{R_2(x)}{R_1(x)};\ N_2(x), R_2(x) \in F[x];\ 0 \le \deg R_2 < \deg R_1 < \deg R_0 < \deg q,$$

or equivalently in the terminology of Lemma 2.5,

$$1 = r_1 N_2 - \frac{R_2}{q} = r_1 n_2 - r_2.$$

Repeating in the same manner, in general we have

$$r_j = (-1)^j \frac{R_j}{q}, \quad 0 \le \deg R_j < \deg R_{j-1} < \cdots < \deg R_1 < \deg q.$$

There must then exist $k \in \mathbb{N}$ such that $\deg R_k = 0$, i.e., $R_k \in F^\times$. Thus, the CEF expansion of α is

$$\alpha = n_0 + \frac{1}{n_1} + \cdots + \frac{1}{n_1 \cdots n_k} + \frac{r_k}{n_1 \cdots n_k} = n_0 + \frac{1}{n_1} + \cdots + \frac{1}{n_1 \cdots n_k} + \frac{1}{n_1 \cdots n_k n_{k+1}},$$

where $n_{k+1} = (-1)^k R_k^{-1} q \in F[x]$, which is a terminating CEF expansion. $\square$

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