การออกแบบระบบควบคุมป้อนกลับที่มีความไม่เป็นเชิงเส้น
แบบเซกเตอร์มีขอบเขตโดยใช้กรอบงานของซาเกียน

นายวันซี ไม

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิศวกรรมศาสตรมหาบัณฑิต
สาขาวิชาวิศวกรรมไฟฟ้า ภาควิชาวิศวกรรมไฟฟ้า
คณะวิศวกรรมศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย
ปีการศึกษา 2553

DESIGN OF FEEDBACK CONTROL SYSTEMS WITH A SECTOR-BOUNDED NONLINEARITY USING ZAKIAN'S FRAMEWORK

Mr. Van Sy MAI

A Thesis Submitted in Partial Fulfillment of the Requirements
for the Degree of Master of Engineering Program in Electrical Engineering
Department of Electrical Engineering
Faculty of Engineering
Chulalongkorn University
Academic Year 2010

Thesis Title	DESIGN OF FEEDBACK CONTROL SYSTEMS WITH A SECTOR-BOUNDED NONLINEARITY USING ZAKIAN'S FRAMEWORK
By	Mr. Van Sy Mai
Field of Study	Electrical Engineering
Thesis Advisor	Assistant Professor Suchin Arunsawatwong, Ph.D.

Accepted by the Faculty of Engineering, Chulalongkorn University in Partial Fulfillment of the Requirements for the Master's Degree

.... Dean of the Faculty of Engineering
(Associate Professor Boonsom Lerdhirunwong, Dr. Ing.)

THESIS COMMITTEE

.... Chairman
(Professor David Banjerdpongchai, Ph.D.)

.... Thesis Advisor
(Assistant Professor Suchin Arunsawatwong, Ph.D.)

.... Examiner
(Somboon Sangwongwanich, D. Eng.)

.... External Examiner
(Associate Professor Varaporn Jaovisidha, Ph.D.)

วันซี ไม: การออกแบบระบบควบคุมป้อนกลับที่มีความไม่เป็นเชิงเส้นแบบเซกเตอร์มีขอบเขตโดยใช้กรอบงานของซาเกียน. (DESIGN OF FEEDBACK CONTROL SYSTEMS WITH A SECTOR-BOUNDED NONLINEARITY USING ZAKIAN'S FRAMEWORK), อ. ที่ปรึกษาวิทยานิพนธ์หลัก: ผศ.ดร. สุชิน อรุณสวัสดิ์วงศ์, 78 หน้า.

จนถึงปัจจุบันการออกแบบระบบควบคุมด้วยหลักการเข้าคู่ของซาเกียน (Zakian) ได้ถูกศึกษาวิจัยอย่างกว้างขวางสำหรับระบบเชิงเส้นไม่แปรเปลี่ยนไปตามเวลาเท่านั้น สำหรับระบบไม่เชิงเส้นทั่วไปแล้วหัวข้อวิจัยดังกล่าวยังคงเป็นปัญหาเปิดอยู่ ด้วยเหตุนี้วิทยานิพนธ์ฉบับนี้จึงพัฒนาวิธีการออกแบบเชิงปฏิบัติสำหรับระบบควบคุมป้อนกลับจำพวกหนึ่ง ซึ่งประกอบด้วยพลานต์ที่เป็นเชิงเส้นไม่แปรเปลี่ยนไปตามเวลา (อาจมีความไม่แน่นอนด้วย) ต่อเชื่อมกับความไม่เป็นเชิงเส้นแบบสถิตไร้ความจำ วัตถุประสงค์การออกแบบที่พิจารณาในวิทยานิพนธ์ฉบับนี้ คือรับประกันว่าฟังก์ชันค่าคลาดเคลื่อนและสัญญาณขาออกของตัวควบคุมจะอยู่ภายในขอบเขตที่กำหนดตลอดเวลา สำหรับสัญญาณขาเข้าที่เป็นไปได้ใดๆ งานวิจัยที่นำเสนอในวิทยานิพนธ์ฉบับนี้ประกอบด้วยสองส่วน ส่วนแรกพิจารณาเสถียรภาพของระบบลูเร (Lur'e) ในความหมายที่ว่า สัญญาณขาออกมีขอบเขตเสมอเมื่อทั้งขนาดและความชันของสัญญาณขาเข้ามีขอบเขต จากการขยายความงานวิจัยก่อนหน้านี้อย่างตรงไปตรงมา เราได้แสดงให้เห็นว่าถ้าระบบสอดคล้องเงื่อนไขโปปอฟ (Popov) แล้วระบบจะมีเสถียรภาพในความหมายที่พิจารณาเสมอสำหรับความไม่เป็นเชิงเส้นใดๆ ซึ่งอยู่ภายในเซกเตอร์ที่มีขอบเขต ผลดังกล่าวได้นำไปใช้พัฒนาอสมการสำหรับคำนวณจุดเสถียรภาพโดยวิธีการเชิงตัวเลข ในส่วนที่สองนั้น เนื่องจากเกณฑ์การออกแบบต้นกำเนิดไม่สามารถนำมาใช้คำนวณได้ เราจึงได้ประดิษฐ์อสมการออกแบบที่สามารถใช้หาตัวควบคุม ซึ่งสอดคล้องวัตถุประสงค์การออกแบบข้างต้นในรูปของเกณฑ์การออกแบบแทนที่ ตัวอย่างเชิงเลขถูกนำเสนอและแสดงให้เห็นอย่างชัดเจนถึงประสิทธิภาพของแนวทางการออกแบบอย่างเป็นระบบที่พัฒนาขึ้นในวิทยานิพนธ์ฉบับนี้

ภาควิชา ......วิศวกรรมไฟฟ้า......	ลายมือชื่อนิสิต ....
สาขาวิชา ......วิศวกรรมไฟฟ้า......	ลายมือชื่อ อ.ที่ปรึกษาวิทยานิพนธ์หลัก ....
ปีการศึกษา ......2553......	

##517 06916 21: MAJOR ELECTRICAL ENGINEERING

KEYWORDS: NONLINEAR SYSTEMS / INPUT-OUTPUT STABILITY / LUR'E SYSTEMS / SECTOR-BOUNDED NONLINEARITY / POPOV CRITERION / PEAK OUTPUT / WORST CASE PERFORMANCE / CONTROL SYSTEMS DESIGN / PRINCIPLE OF MATCHING / METHOD OF INEQUALITIES

VAN SY MAI: DESIGN OF FEEDBACK CONTROL SYSTEMS WITH A SECTOR-BOUNDED NONLINEARITY USING ZAKIAN'S FRAMEWORK. THESIS ADVISOR: ASST. PROF. SUCHIN ARUNSAWATWONG, Ph.D., 78 pp.

So far control systems design by Zakian's principle of matching has been investigated extensively for linear time-invariant systems. For general nonlinear systems, this is still an open problem. In this regard, this thesis develops a practical method for designing a class of feedback control systems where the plant is a linear time-invariant (possibly uncertain) subsystem in cascade connection with a static memoryless nonlinearity. The design objective considered here is to ensure that the error function and the controller output stay within respective bounds for all time and for all possible inputs. The research conducted in the thesis comprises two parts. Part I considers the stability of Lur'e systems in the sense that the outputs are bounded whenever the magnitude and the slope of the input are bounded. It is shown by a straightforward extension of known results that if the Popov condition is satisfied, then the system is stable in the above sense for any nonlinearity lying in a sector bound. Based on this result, an inequality for determining stability points by numerical methods is developed. In Part II, since the original design criteria are computationally intractable, design inequalities that can be used for determining a controller satisfying the design objective are derived, thereby providing surrogate design criteria. The numerical examples are carried out and clearly illustrate the effectiveness of the systematic design approach developed here.

Department: Electrical Engineering Student's Signature: ....
Field of Study: Electrical Engineering Advisor's Signature: ....
Academic Year: 2010

Acknowledgements

I would like to express my deepest gratitude to my thesis supervisor, Assistant Professor Suchin Arunsawatwong, for his careful guidance and considerable support from the very beginning of my study at Chulalongkorn University. This thesis would not have been possible without his great encouragement and constant supervision. He always makes himself available for having insightful conversations, giving trenchant critiques and making useful suggestions not only on doing research but also on many aspects of life, for which I am extremely grateful.

I am thankful to Chulalongkorn University, in particular the Department of Electrical Engineering, for making an excellent and professional working environment. I am indebted to Assistant Professor Suchin Arunsawatwong, Professor David Banjerdpongchai, Assistant Professor Manop Wongsaisuwan and Associate Professor Watcharapong Khovidhungij, who are the professors in the Control Systems Research Laboratory, for providing me the fundamental background in the field of Control Systems. The knowledge that I gained from their lectures is particularly useful not only for the work in this thesis but also for my future research. In addition, I am also grateful to Assistant Professor Suchin Arunsawatwong, Professor David Banjerdpongchai, Associate Professor Varaporn Jaovisidha and Dr. Somboon Sangwongwanich for agreeing to be the committee members of my thesis. Their useful comments and constructive advices are absolutely invaluable. Thanks also go to all my friends at Chulalongkorn University, especially students of the Control Systems Research Laboratory for their great friendship and support.

I gratefully acknowledge the full financial support from AUN/SEED-Net collaborative research program for my master's degree in Thailand. The financial support from the International School of Engineering, Chulalongkorn University, for my trip to Taiwan in August 2010 to attend the SICE annual conference is also highly appreciated.

Finally, but most importantly, I would like to dedicate this thesis to my parents and my brother for their endless love and support.

Contents

CHAPTER Page

List of Tables

TABLE Page

List of Figures

FIGURE Page

FIGURE Page

List of Notations

Symbols

$\mathcal{A}$	convolution algebra
$\mathbb{C}$	field of complex numbers
L_n	space of function $f : \mathbb{R}_+ \to \mathbb{R}$ such that $\|f\|_n < \infty$; $n \in [1, \infty]$.
$\mathbb{R}$	field of real numbers
$\mathbb{R}_+$	field of nonnegative real numbers
$\|f\|_n$	n-norm of a function $f \in L_n$

Acronyms

SISO	Single Input Single Output
BIBO	Bounded Input Bounded Output
MBP	Moving Boundaries Process
MoI	Method of Inequalities
PoM	Principle of Matching

CHAPTER I

INTRODUCTION

1.1 Introduction

In recent decades, advances in the design of control systems have been made. With significant developments of computing facilities, designers can focus more on the problem formulations and let all computational tasks be carried out by computers using efficient numerical algorithms. As a consequence, the design problem can be formulated in a more realistic manner, which reflects more accurately the nature of the control requirements. In this direction, the framework proposed by Zakian [1,2], which consists of the Principle of Matching (PoM) and the Method of Inequalities (MoI), has been extensively used for designing control systems (see, for example, [1–16] and the references therein).

Following Zakian's framework [1, 2], it is readily appreciated that a principal aim in control systems design is to guarantee that a response (or an output) v of the system stays within a prescribed bound in the presence of all *possible inputs* (that is, inputs that can happen or are likely to happen in practice). Accordingly, the design criterion can be expressed as

$$|v(t, f)| \leq \varepsilon, \quad \forall f \quad \forall t \in \mathbb{R}, \tag{1.1}$$

where $v(t, f)$ is the value of v at time t in response to a possible input f, and ε is the largest value of $|v(t, f)|$ that can be accepted. Criterion (1.1) is frequently employed in practice by engineers to monitor the performances of the control systems and has long been investigated by a number of researchers (see, for example, [1, 2, 7, 14, 17, 18] and the references therein) with various sets of f. Furthermore, criterion (1.1) is particularly useful in the design of *critical systems* [2, 19] (see also [20]), in which any violation of the bound ε may result in an unacceptable operation. See [8, 10, 12] for examples of critical systems.

This work considers the design of a feedback control system shown in Figure 1.1, where $G_c(s, \mathbf{p})$ is the transfer function of the controller with the design parameter $\mathbf{p} \in \mathbb{R}^n$ and the input f is known only to the extent that it belongs to a *possible set* $\mathcal{P}$ (that is, a set that contains all possible inputs). The system is assumed to be at rest for $t \leq 0$.

Following previous works (see, for example, [1, 2, 7, 9, 10, 13, 14, 18]), it is readily appreciated that $\mathcal{P}$ should contain all inputs satisfying bounding conditions on both magnitude and slope. Therefore, in this work, we consider the following two possible sets.

$$\mathcal{P}_2 \triangleq \left\{ f \mid f \in L_2, \; \dot{f} \in L_2 \right\} \tag{1.2}$$

$$\mathcal{P}_\infty \triangleq \left\{ f \mid f \in L_\infty, \; \dot{f} \in L_\infty \right\} \tag{1.3}$$

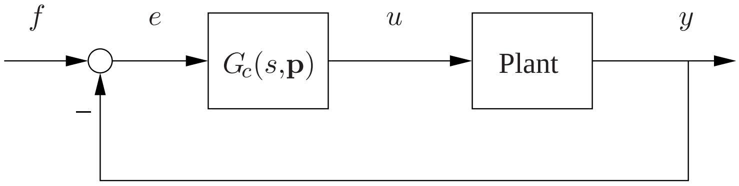


Figure 1.1: A feedback control system.

where L_n $(n = 2, \infty)$ denotes the set of all functions $f : \mathbb{R}_+ \to \mathbb{R}$ satisfying $\|f\|_n < \infty$. As usual,

$$\|f\|_2 \triangleq \left\{ \int_0^\infty |f(t)|^2 \, dt \right\}^{1/2} \tag{1.4}$$

$$\|f\|_\infty \triangleq \sup\{|f(t)| : t \in \mathbb{R}_+\}. \tag{1.5}$$

The set $\mathcal{P}_\infty$ is suitable for characterizing *persistent inputs* (that is, inputs that vary persistently for all time) whereas $\mathcal{P}_2$ is suitable for characterizing *transient inputs* (that is, those that do not). In comparison with the sets L_2 and L_∞, using $\mathcal{P}_2$ and $\mathcal{P}_\infty$ as possible sets make the design formulation more realistic and more appropriate because inputs in L_∞ may have stepwise discontinuities and those in L_2 may have unbounded magnitudes, and hence these features may not reflect accurately the physical behaviour of possible inputs. For further discussion, see [1,2].

In connection with (1.1), the design problem is to find $G_c(s, \mathbf{p})$ satisfying

$$\left.\begin{aligned} |e(f,t)| &\le E_{\max} \\ |u(f,t)| &\le U_{\max} \end{aligned}\right\} \quad \forall f \in \mathcal{P} \quad \forall t \ge 0, \tag{1.6}$$

where $\mathcal{P}$ can be either $\mathcal{P}_2$ or $\mathcal{P}_\infty$. Obviously, inequalities (1.6) are equivalent to

$$\begin{aligned} \hat{e} \le E_{\max}, \quad & \hat{e} \triangleq \sup_{f\in\mathcal{P}} \sup_{t\ge 0} |e(f,t)|, \\ \hat{u} \le U_{\max}, \quad & \hat{u} \triangleq \sup_{f\in\mathcal{P}} \sup_{t\ge 0} |u(f,t)|, \end{aligned} \tag{1.7}$$

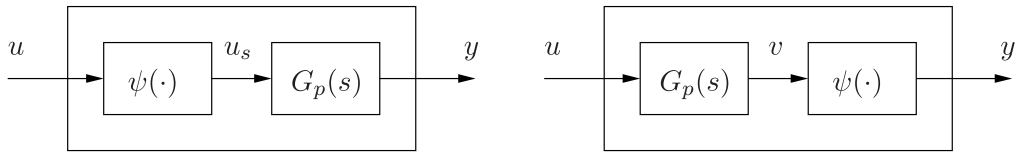
where the performance measures $\hat{e}$ and $\hat{u}$ are sometimes called the *peak values* of e and u, respectively. Evidently, inequalities (1.7) become useful design criteria once $\hat{e}$ and $\hat{u}$ can be computed in practice. In case it is difficult to compute the peaks $\hat{e}$ and $\hat{u}$, one may replace $\hat{e}$ and $\hat{u}$ in the design inequalities (1.7) by upper bounds $\tilde{e}$ and $\tilde{u}$ that are readily computable.

In searching for a solution of (1.7) in $\mathbb{R}^n$, it is necessary that a search algorithm should start from a *stability point*, that is, a point $\mathbf{p}$ for which

$$\hat{e}(\mathbf{p}) < \infty \quad \text{and} \quad \hat{u}(\mathbf{p}) < \infty. \tag{1.8}$$

From the above, it is seen clearly that (1.8) is a necessary condition for the satisfaction of the design criteria (1.7). For more discussion on this, see, for example, [2] and the references cited therein.

So far, in connection with the possible sets $\mathcal{P}_2$ and $\mathcal{P}_\infty$, the problem of solving (1.7) has been developed only for the case of linear time-invariant plants. See [14] for the latest review



a) b)

Figure 1.2: Nonlinear plant models: a) with an input nonlinearity, b) with an output nonlinearity.

and also the references therein. Therefore, it is our aim in this thesis to extend the framework for cases of nonlinear systems. Specifically, in this thesis, we assume that the plant is represented as a cascade connection of a transfer function $G_p(s)$ and a nonlinearity ψ. See Figure 1.2. Note that nonlinear systems of this type, where the nonlinearity is introduced to simulate the effect of either actuator or sensor nonlinear characteristics, constitute an important class of systems found in practice [21].

In the context of the Principle of Matching, the main objective of this thesis is to develop a systematic and practical method for designing the feedback control system shown in Figure 1.1, where the plant takes the form of either Figure 1.2a or Figure 1.2b, so as to guarantee that the criteria (1.7) are satisfied. To this end, the stability problem (1.8) needs to be resolved first. That is to say, it is necessary to establish practical conditions for determining stability points of the system. Once such a point is obtained, the design criteria (1.7) are suitable for solutions by numerical methods, provided that $\hat{e}$ and $\hat{u}$ can be computed in practice.

1.2 Literature Review

In this section, the design framework, which consists of the Principle of Matching (PoM) and the Method of Inequalities (MoI) is briefly described.

1.2.1 The Method of Inequalities

The Method of Inequalities [2,3,18,22] is a design method that expresses constrains and design specifications of a control system as a set of inequalities, that is,

$$\phi_i(\mathbf{p}) \leq \epsilon_i, \qquad i = 1,\, 2, \ldots\, m, \tag{1.9}$$

where $\phi_i(\mathbf{p})$ is a performance measure that characterizes a particular behavior of the system, $\mathbf{p} \in \mathbb{R}^n$, as usual, denotes the design parameter, and ϵ_i is the maximum value of $\phi_i(\mathbf{p})$ that can be accepted. The design solution is any point $\mathbf{p}$ that satisfies (1.9).

In practice, these inequalities are usually solved numerically using search algorithms in the space of design parameters. Throughout this work, an algorithm called the Moving Boundaries Process (MBP) [3] is used. Alternatively, other algorithms for solving (1.9) may be used (see, for example, [2] and the references therein).

1.2.2 The Principle of Matching

In the systems design by the Principle of Matching [1, 2, 4], designers are concerned with the relation between the system and its environment. Specifically, the environment affects the system by some signals. If, using appropriate and well-defined criteria, the resulted responses are acceptable, then these signals are call tolerable inputs, and the system and its environment are said to be matched. As far as the system performances are taken into account, a practical model of the environment is required and usually characterized by a possible set $\mathcal{P}$ (see, Section 1.1). Let $\mathcal{T}$ denote a set containing all tolerable inputs. Consequently, the main objective of a match is to ensure that

$$\mathcal{P} \subseteq \mathcal{T}. \tag{1.10}$$

For the problem considered in this thesis, condition (1.10) is equivalent to (1.7).

In a practical design, to compare the sets $\mathcal{P}$ and $\mathcal{T}$, practical criteria (in forms of inequalities) are required (see, for example, [1, 2] and also Section 1.1). Once the criteria are obtained, the design problem can be solved by using the formulation described in Section 1.2.1.

1.3 Objectives

The purpose of this thesis is fourfold.

1. Study the input-output stability properties of Lur'e feedback systems to ensure that the outputs are bounded for any input in the set $\mathcal{P}_2$ or $\mathcal{P}_\infty$, where the linear subsystem belongs to a large subclass of convolution systems.

2. Based on the obtained results, develop a useful inequality for determining stability points of the system by numerical methods.

3. By using Zakian's Principle of Matching [1, 2], develop a practical method for designing feedback control systems where the plant takes the form of either Figure 1.2a or Figure 1.2b, so as to ensure that the error function e and the controller output u stay within respective bounds for all time and for all possible inputs.

4. Design controllers for some practical applications to illustrate the effectiveness of the developed method.

1.4 Scope of Thesis

The scope of this research work is specified as follows. Consider the nonlinear feedback system as shown in Figure 1.1.

1. Provide stability conditions, and then develop a computationally tractable inequality for determining stability points in connection with the possible sets $\mathcal{P}_2$ and $\mathcal{P}_\infty$.

2. Develop a practical method for designing the system where the plant is a linear time-invariant subsystem and is subject to a nonlinearity in its input or output channels (see Figure 1.2), so as to ensure the satisfaction of the design criteria (1.7).

3. Design controllers for some SISO systems whose the plant may be infinite-dimensional by using the developed method.

1.5 Methodology

This thesis extends Zakian's framework [2] for designing a feedback control system shown in Figure 1.1.

First, stability points can be obtained efficiently using the stability results developed in Chapter 2.

Second, it is suggested [23] that, by the decomposition technique used in [24, 25], the nonlinearity ψ can be replaced by a constant gain and an equivalent disturbance, thus resulting in a linear system subject to two inputs.

Next, by using the Schauder fixed point theorem (see, for example, [26, 27]), sufficient conditions for the satisfaction of (1.7) are derived from the resultant linear system.

Finally, practical sufficient conditions for ensuring (1.7) are developed, providing surrogate design criteria that are in keeping with the MoI and suitable for solutions by numerical methods.

1.6 Expected Outcomes

The outcome of this research work is expected to include

1. A numerical procedure for determining stability points of a class of nonlinear systems.

2. A practical method for designing the feedback control system shown in Figure 1.1 using Zakian's framework, which is considered as the most significant contribution of the thesis.

3. Numerical examples showing the usefulness of the developed method.

1.7 Thesis Outline

The rest of the thesis is organized as follows. Chapter 2 considers the input-output stability of Lur'e systems in the above sense. The stability results and a numerical method for stabilizing the system are presented. Chapter 3 develops a practical method for designing feedback control systems with input nonlinearity. To illustrate the usefulness of the method, a numerical design of a hydraulic force control system is carried out. The developed method is extended to the case of systems with output nonlinearity in Chapter 4. The design of a heat-conduction process shows that the control problem of an infinite-dimensional system can be solved efficiently using the proposed approach. Chapter 5 considers the design problem of a class of uncertain nonlinear systems. Finally, we conclude the thesis in Chapter 6.

CHAPTER II

STABILITY CONDITIONS AND NUMERICAL STABILIZATION

This chapter considers the stability of Lur'e systems in the sense that the outputs are bounded for any input in the sets $\mathcal{P}_2$ and $\mathcal{P}_\infty$, where the linear subsystem belongs to a large subclass of convolution systems. It is shown that if the well-known Popov criterion is satisfied, then the system is stable in the above sense for any nonlinearity lying in a sector bound. Based on the Popov criterion, a practical inequality for determining stability points by numerical methods is then developed. The usefulness of the obtained results is illustrated by a numerical example, in which the plant is a nonrational transfer function.

2.1 Lur'e Systems

Consider the single-input, single-output feedback connection as shown in Figure 2.1, where ψ is a nonlinear element, G is a linear subsystem, and input f_1 together with f_2 belong to either $\mathcal{P}_2$ or $\mathcal{P}_\infty$.

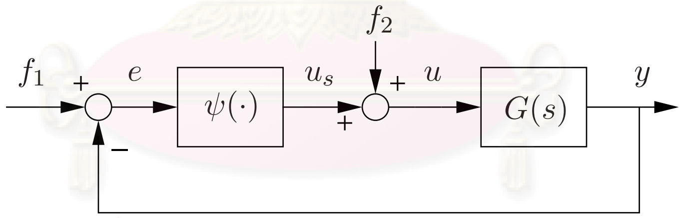


Figure 2.1: Lur'e system.

Assumption 2.1. *The nonlinear function $\psi : \mathbb{R} \to \mathbb{R}$ is continuous, time-invariant and memoryless, and satisfies $\psi(0) = 0$.*

The nonlinearity ψ is said to lie in the $\text{sector}\,[k_1,\, k_2]$, denoted by $\psi \in \text{sector}\,[k_1,\, k_2]$, if

$$\psi(0) = 0 \quad \text{and} \quad k_1 \leq \frac{\psi(\sigma)}{\sigma} \leq k_2 \quad \forall \sigma \neq 0. \tag{2.1}$$

See [28, p.2] for equivalent forms of (2.1) and see Figure 2.2 for its graphical description.

Assumption 2.2. *The linear part is a time-invariant and non-anticipative subsystem with zero initial conditions, and is characterized by a transfer function $G(s)$. The input u and the output y are related by the convolution integral*

$$y(t) = g * u\,(t) \triangleq \int_0^t g(t-\tau)u(\tau)d\tau, \qquad t \geq 0, \tag{2.2}$$

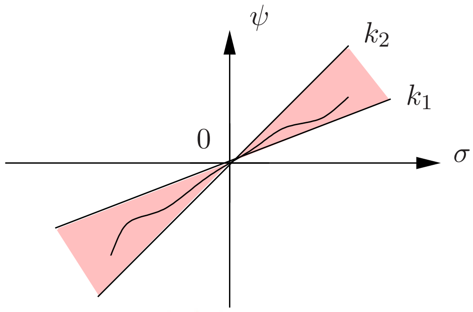


Figure 2.2: The nonlinearity ψ lying in a sector bound.

where g is the impulse response of $G(s)$.

Note, in passing, that (2.2) is a very general representation which includes rational systems, infinite-dimensional systems and systems with delays (also see Section 2.4).

Let $\mathcal{A}$ denote the convolution algebra whose elements take the form

$$g(t) = \begin{cases} g_a(t) + \sum_{i=0}^{\infty} g_i \delta(t - t_i), & t \geq 0 \\ 0, & t < 0 \end{cases}, \tag{2.3}$$

where $\delta(\cdot)$ is the Dirac delta function, $0 = t_0 < t_1 < t_2 \ldots$ are constants and

$$\sum_{i=0}^{\infty} |g_i| < \infty \quad \text{and} \quad \int_0^{\infty} |g_a(t)| \, dt < \infty. \tag{2.4}$$

Accordingly, $\mathcal{A}$ can be seen as a set containing impulse responses of all BIBO stable linear-time invariant systems. For details on this, see, for example, [28] and [29].

2.2 Stability Conditions for Lur'e Systems

This section presents two stability conditions for ensuring the boundedness of the system outputs for the case of the inputs in $\mathcal{P}_2$ (Theorem 2.1), and for the case of the inputs in $\mathcal{P}_\infty$ (Theorem 2.2).

2.2.1 Boundedness of output with respect to $\mathcal{P}_2$

When the system inputs are restricted in the two-norms of their magnitude and slope, the following theorem, which is essentially an extension of the result in [30] (see also [28]), can be used.

Theorem 2.1. *Let Assumptions 2.1 and 2.2 be satisfied. Suppose that $f_1, f_2 \in \mathcal{P}_2$ and that $g, \dot{g} \in \mathcal{A}$. The responses e, u and y are bounded for any $\psi \in \mathrm{sector}\,[0, k]$ if there exist $q \in \mathbb{R}$ and $\beta \in \mathbb{R}$ such that*

$$\mathrm{Re}\,[(1 + qj\omega)\, G\,(j\omega)] + \frac{1}{k} \geq \beta > 0, \quad \forall \omega \geq 0. \tag{2.5}$$

Proof. By using the linearity of $G(s)$, the input f_2 can be replaced by an equivalent input r, at f_1. See Figure 2.3, where $y_s = y - r$.

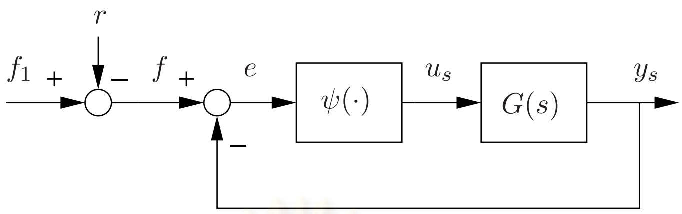


Figure 2.3: Equivalent closed-loop system.

Clearly,

$$r = g * f_2, \quad \dot{r} = \dot{g} * f_2 + g(0) f_2. \tag{2.6}$$

Since $g \in \mathcal{A}$ and $\dot{g} \in \mathcal{A}$, it follows that g does not contain any impulse. This can be seen by the fact that if g contains any impulse, then $\dot{g}$ would have higher order impulses and thus would not belong to $\mathcal{A}$. Moreover, $g(t) = 0$ for all $t < 0$ by the definition (2.3). Therefore, $g(0) < \infty$. As a consequence, equations (2.6) and condition $f_2 \in L_2$ imply that $r \in \mathcal{P}_2$. Hence, $f \in \mathcal{P}_2$.

By employing the method in [30], it can be proved that $y_s \in L_\infty$. In the following, the proof [28] that is different from the one in [30] is described.

From the equivalent system in Figure 2.3, it follows that

$$f = e + g * \psi(e). \tag{2.7}$$

By differentiation, we have

$$\dot{f} = \dot{e} + \dot{g} * \psi(e) + g(0)\psi(e). \tag{2.8}$$

Thus, for any positive T, we have

$$\begin{aligned} \int_0^T (f + q\dot{f})\psi(e)dt \;&=\; \int_0^T [e - \frac{\psi(e)}{k}]\psi(e)dt + q\int_0^T \dot{e}\psi(e)dt \\ &\quad + \int_0^T \{[g + q\dot{g} + qg(0) + \frac{1}{k}] * \psi(e)\}\psi(e)dt. \end{aligned} \tag{2.9}$$

Using Cauchy–Schwarz inequality and the sector condition yields

$$\int_0^T (f + q\dot{f})\psi(e)dt \;\le\; (\|f\|_2 + q\|\dot{f}\|_2)\|\psi(e)\|_{2T}, \tag{2.10}$$

$$\left[e - \frac{\psi(e)}{k}\right]\psi(e) \ge 0, \tag{2.11}$$

where $\|\psi(e)\|_{2T}^2 \triangleq \int_0^T [\psi(e)]^2 dt$. Moreover, by virtue of Lemma A.1 in Appendix A, we can assume that $q \ge 0$. Thus,

$$q\int_0^T \dot{e}\psi(e)dt \;=\; q\int_{e(0)}^{e(t)} \psi(e)de \ge -q\int_0^{e(0)} \psi(e)de. \tag{2.12}$$

In connection with (2.5) and from the above, we obtain

$$(\|f\|_2 + q\|\dot{f}\|_2)\|\psi(e)\|_{2T} \geq \beta\|\psi(e)\|_{2T}^2 - q\int_0^{e(0)} \psi(e)de. \tag{2.13}$$

Inequality (2.13) implies that there exists a real number $C > 0$ such that

$$\|\psi(e)\|_{2T} < C. \tag{2.14}$$

It is easy to see that C does not depend on T. Hence, $\psi(e) \in L_2$. Then $e \in \mathcal{P}_2$ by (2.7) and (2.8).

It is shown ([2, p. 59], see also Lemma A.3 in Appendix A) that if $f \in \mathcal{P}_2$, then $f \in L_\infty$. As a result, $e,\ u_s,\ y_s \in L_\infty$. Since $f_2,\ r \in L_\infty$ and $y = y_s + r$, it readily follows that $u,\ y \in L_\infty$. □

Condition (2.5) is the well-known Popov criterion (see, for example, [21, 29–31]). Traditionally, the criterion has been used to guarantee the global asymptotic stability of Lur'e systems for any $\psi \in \text{sector}\,[0,\ k]$ (known as absolute stability). It is interesting to note that the concept of absolute stability is defined for unforced systems. In this work, however, the Popov criterion is used to ensure the input-output stability of forced systems in connection with the sets $\mathcal{P}_\infty$ and $\mathcal{P}_2$.

Notice that, in connection with Theorem 2.1, the class of G includes all stable systems whose impulse response g does not contain the Dirac delta function, for example, systems with strictly proper rational transfer functions, and some infinite-dimensional systems such as heat-conduction processes or systems with time-delays.

It may be noted that if one needs to ensure the boundedness of only e and y, then the condition that $f_2 \in \mathcal{P}_2$ can be relaxed and replaced with $f_2 \in L_2$. In this case, the input f_2 and hence u may not be bounded, but e and y are always bounded.

Remark 2.1. *From Theorem 2.1, it follows that when $k \to \infty$, (2.5) becomes*

$$\text{Re}\,[(1 + qj\omega)\,G(j\omega)] \geq \beta > 0, \quad \forall\omega \geq 0. \tag{2.15}$$

Remark 2.2. *For $\psi \in \text{sector}\,[k_1,\ k_2]$, the use of a loop transformation*

$$\tilde{\psi}(e) = \psi(e) - k_1 e, \quad \tilde{G}(s) = \frac{G(s)}{1 + k_1 G(s)}, \tag{2.16}$$

yields $\tilde{\psi} \in \text{sector}\,[0,\ k_2 - k_1]$. Hence, inequality (2.5) becomes

$$\text{Re}\left[(1 + qj\omega)\,\tilde{G}(j\omega)\right] + \frac{1}{k_2 - k_1} > 0, \quad \forall\omega \geq 0. \tag{2.17}$$

Remark 2.3. *It is noted that Theorem 2.1 requires $g \in \mathcal{A}$. However, it is also applicable for the case of systems which have one pole at the origin. In this case, all the assumptions are the same as those in Theorem 2.1 except that the nonlinearity $\psi \in \text{sector}\,[\epsilon,\ k + \epsilon]$ for a sufficiently small $\epsilon > 0$ and that g can be decomposed as*

$$g(t) = c + g_1(t), \quad t \geq 0, \tag{2.18}$$

where $c > 0, g_1 \in \mathcal{A}$ and $\dot{g}_1 \in \mathcal{A}$.

2.2.2 Boundedness of output with respect to $\mathcal{P}_\infty$

When the system inputs are persistent, the boundedness of the system outputs can be ensured by the following theorem, in which an additional assumption on g is required. Note that Bergen [32] considered a similar problem for rational systems with only input $f_1 \in \mathcal{P}_\infty$.

Theorem 2.2. *Let Assumptions 2.1 and 2.2 be satisfied. Suppose that* $f_1,\ f_2 \in \mathcal{P}_\infty$ *and that* $g,\ \dot{g} \in \mathcal{A}$. *The outputs* e, u *and* y *are bounded for any* $\psi \in \mathrm{sector}\,[0,\ k]$ *if there exist* $q \in \mathbb{R}$, $\beta \in \mathbb{R}$ *and* $\alpha > 0$ *such that* (2.5) *is satisfied and*

$$\int_0^\infty \mathrm{e}^{2\alpha t} g^2(t)dt < \infty. \tag{2.19}$$

Proof. Consider the equivalent closed-loop system shown in Figure 2.3, where r is defined by (2.6) and $f_1,\ f_2 \in \mathcal{P}_\infty$. It is easy to show that $r \in \mathcal{P}_\infty$, and hence $f \in \mathcal{P}_\infty$. The rest of the proof readily follows the technique used in [32] and, for the sake of completeness, the outline is given as follows.

According to [31, 32], Theorem 2.2 only needs to be proved for the nonlinearity ψ in the reduced sector $[\epsilon, k-\epsilon]$ where $\epsilon > 0$ is arbitrarily small. Also, without loss of generality, we can assume that $q \geq 0$ (see Lemma A.2 in Appendix A).

Note that

$$e(t) = f(t) - \int_0^t \mathrm{e}^{\alpha(t-\tau)} g(t-\tau)\mathrm{e}^{-\alpha(t-\tau)} u_s(\tau)d\tau. \tag{2.20}$$

Using the triangle inequality and Cauchy-Schwarz inequality (a special case of Hölder inequality) yields

$$|e(t)| \leq \quad |f(t)| + \left[\int_0^t \mathrm{e}^{2\alpha x} g^2(x)dx\right]^{1/2} \mathrm{e}^{-\alpha t} \left[\int_0^t \mathrm{e}^{2\alpha\tau} u_s^2(\tau)d\tau\right]^{1/2}. \tag{2.21}$$

By Lemma A.5 (see Appendix A), it follows that if all the conditions of Theorem 2.2 are satisfied, then the following inequality holds for sufficiently small $\alpha > 0$

$$\int_0^t \mathrm{e}^{2\alpha\tau} u_s^2(\tau)d\tau \leq \int_0^t \frac{\mathrm{e}^{2\alpha\tau}}{\beta^2}\left[f(\tau) + q\dot{f}(\tau)\right]^2 d\tau + \frac{2q}{\beta}\int_0^{e(0)} \psi(e)de, \quad \forall t \geq 0. \tag{2.22}$$

From (2.21) and (2.22), we arrive at the following inequality

$$\begin{aligned} |e(t)| \leq |f(t)| + \left[\int_0^t \mathrm{e}^{2\alpha x} g^2(x)dx\right]^{1/2} & \left\{\frac{1}{\beta^2}\int_0^t \mathrm{e}^{-2\alpha(t-\tau)}\left[f(\tau) + q\dot{f}(\tau)\right]^2 d\tau \right. \\ & \left. + \frac{2q}{\beta}\mathrm{e}^{-2\alpha t}\int_0^{e(0)} \psi(e)de\right\}^{1/2}. \end{aligned} \tag{2.23}$$

According to the conditions of Theorem 2.2, it is easy to show that the right-hand side of inequality (2.23) is bounded for all $t \geq 0$. It follows immediately that e is bounded. As a consequence, u, u_s and y are also bounded. □

Obviously, the class of $G(s)$ in Theorem 2.2 is a subset of that in Theorem 2.1. One can see that if g decays to zero exponentially, then condition (2.19) is always satisfied. Therefore, Theorem 2.2 is applicable to an important subclass of convolution systems (comprising, for example, rational systems, retarded delay differential systems and feedback systems with a heat equation).

Note, in passing, that condition (2.19) arises as a consequence of using the exponential weighting technique (see, for example, [28, 32]) in proving Theorem 2.2.

It is also worth noting that Remarks 2.1 and 2.2 in the preceding subsection are also valid in this case. An extension of Theorem 2.2 to the case that $G(s)$ contains one integrator is presented in Remark 2.4.

Remark 2.4. *For the case $G(s)$ has one pole at the origin, all the assumptions are the same as those in Theorem 2.2 except that $\psi \in \text{sector}\,[\epsilon,\ k+\epsilon]$ for a sufficiently small $\epsilon > 0$ and*

$$g(t) = c + g_1(t), \quad t \geq 0, \tag{2.24}$$

where $c > 0$, $g_1, \dot{g}_1 \in \mathcal{A}$ and there exists $\alpha > 0$ such that

$$\int_0^\infty \mathrm{e}^{2\alpha t} g_1^2(t)dt < \infty. \tag{2.25}$$

Furthermore, the condition $f_2 \in \mathcal{P}_\infty$ can be relaxed to $f_2 \in L_\infty$ without having to change the theorem's results. In addition, it is shown [32] that for $q = 0$, the system is L_∞ stable. That is to say, the bounding conditions on the slopes of the inputs are not necessary. If this is the case, then the nonlinear element may be time-varying as long as it satisfies the sector bound condition (2.1).

2.3 Numerical Stabilization

This section develops a practical inequality for obtaining design parameters (which are usually coefficients in controller transfer functions) such that the system responses are guaranteed to be bounded with respect to the set $\mathcal{P}_2$ or $\mathcal{P}_\infty$ and for any nonlinearity lying in a sector bound.

Theorems 2.1 and 2.2 provide sufficient conditions to ensure the boundedness of the outputs of the closed-loop system in Figure 2.1. Both theorems share the same condition (2.5) that can be tested graphically based on the analysis of the Popov plot of $G(j\omega)$, which is the plot of $\omega\text{Im}\,[G(j\omega)]$ versus $\text{Re}\,[G(j\omega)]$. Since $\beta > 0$ can be arbitrarily small, inequality (2.5) is equivalent to the following: *the Popov plot lies to the right and is bounded away from the straight line that has a slope $1/q$ and passes through the point $K \triangleq (-1/k,\, 0)$.* This line is called the Popov line. See Figure 2.4.

Define Ω as the convex hull of the Popov plot (that is, the minimal convex set containing the plot). Then, the relation between the Popov plot and its convex hull is stated in the following proposition.

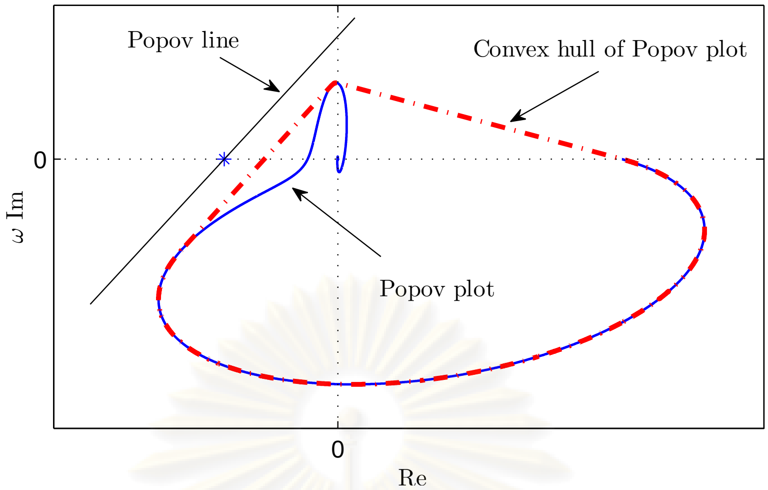


Figure 2.4: The Popov plot lying to the right of the Popov line.

Proposition 2.1. *The Popov plot lies to the right of the Popov line if and only if so does the convex hull* Ω*.*

Proof. By virtue of Proposition B.1 (see Appendix B), the proof readily follows. □

Let $K_m \triangleq (-1/k_m,\ 0)$ denote the point in Ω that is furthest to the left on the negative real axis. If such a point does not exist, then K_m is at the origin (that is, $k_m = \infty$). Then it can be shown that Ω *lies to the right and is bounded away from the Popov line if and only if the point* K *lies to the left of the point* K_m. That is to say, for a given sector $[0,\ k]$, condition (2.5) is satisfied if and only if

$$k < k_m. \tag{2.26}$$

Note that, for a given Popov plot $\{\mathrm{Re}\, G(j\omega) + j\omega \mathrm{Im}\, G(j\omega)\ :\ \omega \in [0, \infty]\}$, the convex hull Ω can be computed efficiently by using available methods. In this thesis, the method given in [33] is used. Once Ω is obtained, the point K_m can be determined easily. In this connection, the algorithm for evaluating k_m is outlined in Table 2.1.

From the above, it is easy to see that the convex hull Ω, and hence the value k_m, depend only on $G(j\omega)$ and can be obtained numerically. It should be noted that when the Popov plot of $G(j\omega)$ has a complex shape (see Figure 2.4), it is easier to determine k_m from Ω than from the Popov plot. Therefore, inequality (2.26) provides a computationally tractable test for checking the satisfaction of Theorems 2.1 and 2.2.

Furthermore, condition (2.26) can be used to develop a useful inequality for stabilizing the system in conjunction with the method of inequalities. To this end, let $\mathbf{p} \in \mathbb{R}^n$ be a vector of design parameters in $G(s, \mathbf{p})$. Also let $\phi(\mathbf{p}) \triangleq k - k_m(\mathbf{p})$ and replace (2.26) by

$$\phi(\mathbf{p}) \leq -\gamma, \tag{2.27}$$

input: Popov plot
output: k_m
begin
  compute Ω;
  $P \triangleq \{(x,\, y) \mid (x,\, y) \in \Omega,\ x < 0,\ y = 0\}$;
  if $P = \{\,\}$,
    $k_m = \infty$;
  else
    $k_m = -1/\min\limits_{(x,y)\in P} x\,$;
  end
end

Table 2.1: Algorithm for determining k_m using the convex hull of the Popov plot.

where γ is a small positive number. Clearly, a stability point is obtained by solving (2.27). Since $\phi(\mathbf{p})$ can readily be computed in practice, it follows that inequality (2.27) provides a practical condition for obtaining a stability point $\mathbf{p}$. The condition is in keeping with the method of inequalities [3] and is always soluble by numerical methods.

It should be noted that the convex hull Ω can be obtained numerically for $\omega \in [0, \infty]$ if it lies entirely in the finite plane, that is, the Popov plot is in the finite plane. This requirement can be guaranteed by the following proposition.

Proposition 2.2. *If the impulse response g of a transfer function $G(s)$ can be decomposed as $g(t) \triangleq c + g_1(t),\ t \geq 0$, where $g_1, \dot{g}_1 \in \mathcal{A}$ and $|c| < \infty$, then the Popov plot of $G(s)$ lies in the finite plane.*

Proof. First, notice that

$$G(j\omega) = \frac{c}{j\omega} + G_1(j\omega). \tag{2.28}$$

Hence,

$$\begin{aligned} \mathrm{Re}\,[G(j\omega)] &= \mathrm{Re}\,[G_1(j\omega)] \\ \mathrm{Im}\,[G(j\omega)] &= \mathrm{Im}\,[G_1(j\omega)] - \frac{c}{\omega}. \end{aligned} \tag{2.29}$$

Second, it can be shown [28, 29] that if $g_1 \in \mathcal{A}$, then the function $\omega \mapsto G_1(j\omega)$ is continuous and bounded on $\mathbb{R}$. That is to say,

$$|\mathrm{Re}\,[G_1(j\omega)]| < \infty, \quad \forall \omega \in \mathbb{R}. \tag{2.30}$$

Similarly, condition $\dot{g}_1 \in \mathcal{A}$ implies that the function $\omega \mapsto j\omega G_1(j\omega)$ is bounded on $\mathbb{R}$. Thus,

$$|\mathrm{Re}\ \{j\omega[G_1(j\omega)]\}| = |\omega \mathrm{Im}\,[G_1(j\omega)]| < \infty, \quad \forall \omega \in \mathbb{R}. \tag{2.31}$$

It follows from (2.29), (2.30) and (2.31) that $\mathrm{Re}\,[G(j\omega)]$ and $\omega \mathrm{Im}\,[G(j\omega)]$ are bounded on $\mathbb{R}$. This completes the proof. □

Proposition 2.2 is stated and proved for cases where $G(s)$ may have one pole at the origin. Therefore, it is applicable to the class of $G(s)$ in Theorems 2.1, 2.2 and Remarks 2.3, 2.4.

2.4 Numerical Example

2.4.1 Stability Conditions of Systems with Input Nonlinearity

The aim of this subsection is to demonstrate that the stability results for Lur'e systems developed in Section 2.2 can easily be applied to systems with a nonlinearity at the input channel of the plant.

Consider the system shown in Figure 2.5, where $G_p(s)$ and $G_c(s,\mathbf{p})$ are the transfer functions of the plant and the controller with a design parameter $\mathbf{p} \in \mathbb{R}^n$, respectively. Clearly, this system can be transformed into the form of Lur'e system shown in Figure 2.1, where $G(s,\mathbf{p}) \triangleq G_c(s,\mathbf{p})G_p(s)$. Let g_c, g_p and g denote the impulse responses of $G_p(s)$, $G_c(s)$ and $G(s)$, respectively.

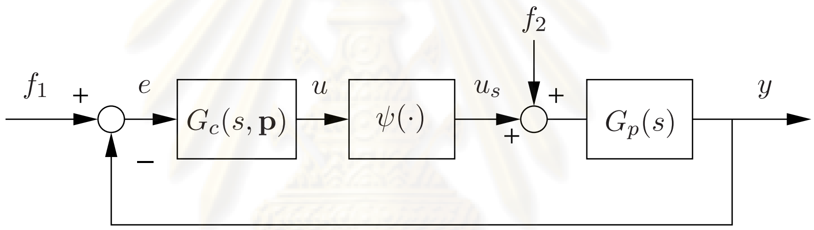


Figure 2.5: A feedback control system with an input nonlinearity and with two inputs.

Proposition 2.3. *Consider the system in Figure 2.5. If inequality* (2.27) *holds and if* $g_c,\ g_p,\ \dot{g} \in \mathcal{A}$*, then the system responses* $e,\ u,\ u_s$ *and* y *are bounded with respect to* $\mathcal{P}_2$ *for any* $\psi \in \mathrm{sector}\,[0,\ k]$. *In addition, if condition* (2.19) *is also satisfied, then the system responses are bounded with respect to* $\mathcal{P}_\infty$ *and* $\mathcal{P}_2$.

Proof. By straightforward manipulations, the proof readily follows from Theorems 2.1 and 2.2. □

In case the composite transfer function $G(s)$ has an integrator, by using appropriate loop transformations, it can be shown that the system responses are always bounded. For details on this, see Appendix C.

2.4.2 Stabilization of a Heat-Conduction Process

In order to illustrate that the stability conditions in Theorem 2.1 and 2.2 are applicable to a wide range of linear time-invariant subsystems, consider a heat-conduction process in a metallic rod; see Figure 2.6. The rod has length L, cross-sectional area A, and is made of material with density ρ, heat capacity C and thermal conductivity σ.

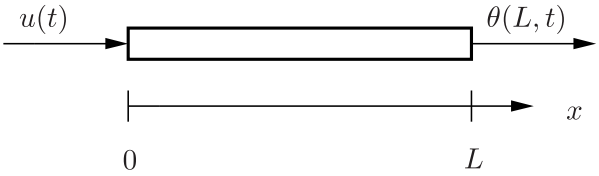


Figure 2.6: The heating metallic rod.

The control signal $u(t)$ is the heat flow injected at $x = 0$. The output $y(t) = \theta(L,t)$ is the temperature measured at $x = L$. It is shown [34] that with appropriate boundary conditions, the transfer function of the process is nonrational and given by

$$G_p(s) = \frac{a}{\sqrt{\lambda s}\,\sinh(\sqrt{\lambda s})}, \tag{2.32}$$

where $a \triangleq 1/(\sigma A)$ and $\lambda \triangleq C\rho/\sigma$. Suppose that with a suitable setting, we have

$$G_p(s) = \frac{20}{\sqrt{s}\,\sinh(\sqrt{s})}. \tag{2.33}$$

It is known (see, for example, [35]) that the transfer function $G_p(s)$ has one pole at the origin and the others on the negative real axis. The impulse response g_p is given by

$$g_p(t) = 20 + 40\sum_{n=1}^{\infty}(-1)^n \mathrm{e}^{-n^2\pi^2 t}, \quad t > 0. \tag{2.34}$$

For the case $G_c(s) = 1$, the impulse response g_p satisfies Remarks 2.3 and 2.4, and therefore both Theorems 2.1 and 2.2 are applicable. The Popov test gives $k_m = 0.8899$, ensuring that the responses e, u, u_s and y are bounded for any $\psi \in \text{sector}\,(0,\,k_m)$.

Now suppose that we wish to design a controller $G_c(s, \mathbf{p})$ so that the outputs of the system are bounded with respect to $\mathcal{P}_2$, or $\mathcal{P}_\infty$, for any nonlinearity lying in a wider sector bound, for example $\psi \in \text{sector}\,(0,\,2]$. To this end, the structure of the controller is chosen as

$$G_c(s, \mathbf{p}) = \frac{s + p_1}{s + p_2}, \tag{2.35}$$

where $\mathbf{p} = [p_1,\,p_2]^T$ is the design parameter. Note that if $p_2 > 0$, then the impulse response of $G(s)$ satisfies Remarks 2.3 and 2.4.

In this thesis, inequality (2.27) is solved by using the MBP algorithm [3]. From a starting point $\mathbf{p}_0 = [1,\,1]^T$, a stability point $\mathbf{p} = [0.35,\,13.10]^T$ is located within 10 iterations for which $\phi(\mathbf{p}) = -0.5553$ (the corresponding $k_m = 2.5553$). The Popov plots of $G_p(s)$ and $G(s)$ are displayed in Figure 2.7.

2.5 Conclusions and Discussion

This chapter has considered input-output stability properties of Lur'e systems, in which the linear subsystem is allowed to be a nonrational transfer function belonging to a subclass of $\mathcal{A}$.

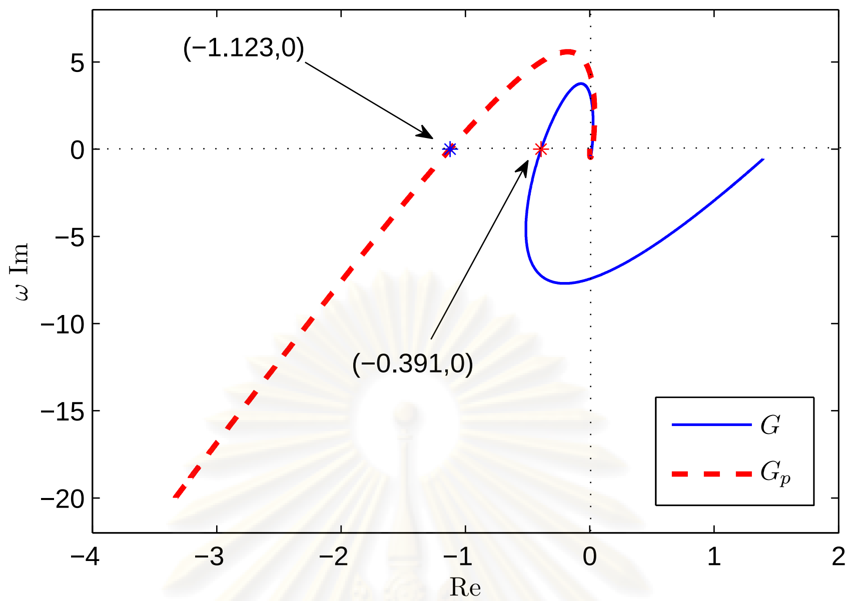


Figure 2.7: The Popov plots of $G_p(s)$ and $G(s)$.

It is shown that if the inputs have either bounded two norms, or bounded infinity norms, on both their magnitude and slope, then the well-known Popov criterion ensures the boundedness of the outputs for any nonlinearity lying in a given sector bound. Based on the obtained results, this chapter develops a practical condition for obtaining stability points that is readily soluble by numerical methods. The merit of the contribution has been demonstrated by the numerical example, where the plant is governed by a heat-conduction equation.

In control systems design using Zakian's framework, the problem of finding stability points arises in the following way. A chief design objective is to ensure that the output v always stays within a desired bound during operation for any input $f \in \mathcal{P}$, that is to say,

$$\hat{v} \leq \varepsilon, \quad \hat{v} \triangleq \sup_{f \in \mathcal{P}} \sup_{t \geq 0} |v(f,t)|, \tag{2.36}$$

where $\varepsilon > 0$ is given. In searching for a solution in the space $\mathbb{R}^n$, it is necessary that a search algorithm should start from a stability point, that is, a point $\mathbf{p}$ for which

$$\hat{v}(\mathbf{p}) < \infty. \tag{2.37}$$

It is important to note [2, 36] that in general, inequality (2.37) cannot be solved by numerical methods using only the performance function $\hat{v}$. Therefore, it is necessary to replace (2.37) by a practical (either equivalent or sufficient) condition of the form

$$\phi(\mathbf{p}) \leq C, \tag{2.38}$$

where $\phi(\mathbf{p})$ is always finite and can be computed in practice, and C is a specified bound (see, for example, inequality (2.27)). Accordingly, condition (2.38) provides a useful inequality for obtaining stability points by numerical methods. For further discussion, see [36] and [2].

In connection with the results obtained in this chapter for determining stability points, the design problem for Lur'e systems based on the criterion of the form (2.36) is soluble by numerical methods if the peak output $\hat{v}$ can be computed in practice.

CHAPTER III

DESIGN OF FEEDBACK SYSTEMS WITH INPUT NONLINEARITY

This chapter develops a practical method for designing a feedback control system comprising a static memoryless nonlinearity and linear time-invariant convolution subsystems so as to ensure that the error function and the controller output stay within prescribed bounds for all time and for all inputs having bounded magnitude and bounded slope. Since the original design criteria are computationally intractable, we derive practical sufficient conditions for ensuring them. The conditions provide surrogate design criteria that are in keeping with the method of inequalities. Essentially, the nonlinearity is replaced with a fixed gain and an equivalent disturbance; thus, the nominal system used during the design process becomes linear and the associated performance measures are readily obtainable by known methods. A design example of a hydraulic force control system is carried out to demonstrate the usefulness of the method.

3.1 Introduction

This chapter considers the design of a feedback control system shown in Figure 3.1, where $\psi(\cdot)$ is a continuous, time-invariant and memoryless nonlinear function, $G_p(s)$ and $G_c(s, \mathbf{p})$ are the transfer functions of the plant and the controller with the design parameter $\mathbf{p} \in \mathbb{R}^n$, respectively. The system is assumed to be at rest for $t \leq 0$. The input f is known only to the extent that it belongs to a possible set $\mathcal{P}$ described by

$$\mathcal{P} \triangleq \left\{ f \in L_\infty \ | \ \|f\|_\infty \leq M, \ \|\dot{f}\|_\infty \leq D \right\}, \tag{3.1}$$

where the bounds M and D are given.

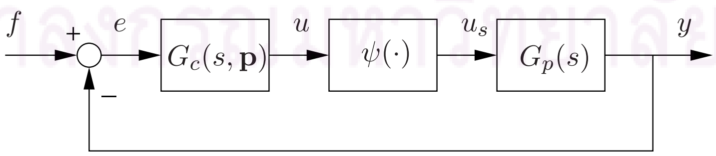


Figure 3.1: A feedback control system with an input nonlinearity.

The design problem considered in the chapter is to determine a controller transfer function $G_c(s, \mathbf{p})$ such that design objectives (1.7) are satisfied, that is,

$$\begin{aligned} \hat{e} &\leq E_{\max}, & \hat{e} &\triangleq \sup_{f \in \mathcal{P}} \|e\|_\infty, \\ \hat{u} &\leq U_{\max}, & \hat{u} &\triangleq \sup_{f \in \mathcal{P}} \|u\|_\infty, \end{aligned} \tag{3.2}$$

where $E_{\max}$ and $U_{\max}$ are the given bounds. Recall that $\hat{e}$ and $\hat{u}$ are the peak values of e and u with respect to the possible set $\mathcal{P}$, respectively. Evidently, once $\hat{e}$ and $\hat{u}$ can be computed in practice, inequalities (3.2) become useful design criteria that can be solved by numerical methods.

So far, the problem of computing $\hat{e}$ and $\hat{u}$, and hence that of solving the design criteria (3.2), have been investigated only for cases of linear time-invariant systems (see, for example, [6, 13, 14, 17] and also the references therein). In general, computing the peaks $\hat{e}$ and $\hat{u}$ for general nonlinear systems are extremely difficult, since the optimization problems defined in (3.2) are non-convex and infinite-dimensional.

The purpose of this chapter is to develop a practical method for designing the controller $G_c(s)$ satisfying the design criteria (3.2) for $G_p(s)$ representing a lumped- or distributed-parameter system. To this end, we derive sufficient conditions of the form

$$\tilde{e} \leq E_{\max} \quad \text{and} \quad \tilde{u} \leq U_{\max} \tag{3.3}$$

to ensure the satisfaction of (3.2), where $\tilde{e}$ and $\tilde{u}$ are readily computable upper bounds of $\hat{e}$ and $\hat{u}$. As a consequence, inequalities (3.3) are more tractable and suitable for solution by numerical methods for a wide range of $G_p(s)$.

The key ideas are as follows. First, by the decomposition technique used in [24], the nonlinearity ψ is replaced by a constant gain and an equivalent disturbance, thus resulting in a linear system subject to two inputs. Second, by using Schauder's theorem (see, for example, [37]), sufficient conditions for (3.2) are derived from the resultant linear system and then are used to develop practical design inequalities to achieve (3.2).

The organization of this chapter is as follows. Section 3.2 uses the decomposition technique to derive sufficient conditions for ensuring the satisfaction of the original design criteria (3.2); the main result is stated in Theorem 3.2. Section 3.3 derives sufficient conditions for (3.2) that are in the spirit of the Method of Inequalities. The stability condition to ensure the boundedness of the system outputs is given in Section 3.4. The developed method is illustrated with a design example of a hydraulic force control system in Section 3.5. Finally, conclusions and discussion are given in Section 3.6.

3.2 Main Results

This section derives the main results of the chapter by making use of the technique due to [24], in which the nonlinearity is replaced with a constant gain and an equivalent disturbance. The result is presented in Theorem 3.2, providing sufficient conditions for the satisfaction of the original design criteria (3.2). The conditions will be used in Section 3.3 to develop practical design inequalities for determining a controller $G_c(s)$ satisfying the design criteria (3.2).

Assumption 3.1. *For every input $f \in \mathcal{P}$, there are unique $e : \mathbb{R}_+ \to \mathbb{R}$ and $u : \mathbb{R}_+ \to \mathbb{R}$ that* *satisfy the following equations*

$$\begin{aligned} u &= g_c * e \\ e &= f - u_s * g_p = f - \psi(u) * g_p, \end{aligned} \tag{3.4}$$

where g_p and g_c are the impulse responses of the plant and the controller, and $g_c * e$ is the convolution of g_c and e, given by

$$g_c * e\,(t) \triangleq \int_0^t g_c(t-\tau)e(\tau)d\tau, \qquad t \geq 0. \tag{3.5}$$

Next, the decomposition technique is introduced. For a fixed value $K \in \mathbb{R}$, define a function $\phi : \mathbb{R} \to \mathbb{R}$ such that

$$\phi(x) \triangleq \psi(x) - Kx, \quad x \in \mathbb{R}. \tag{3.6}$$

Consequently, the nonlinearity can be represented as in Figure 3.2.

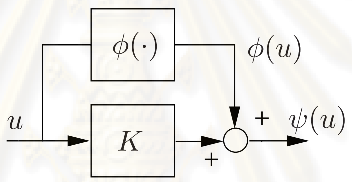


Figure 3.2: Decomposition of the nonlinearity ψ.

As a result, the nonlinear system (3.4) is equivalent to the system shown in Figure 3.3. Note that if $u \in L_\infty$, then so does $\phi(u)$.

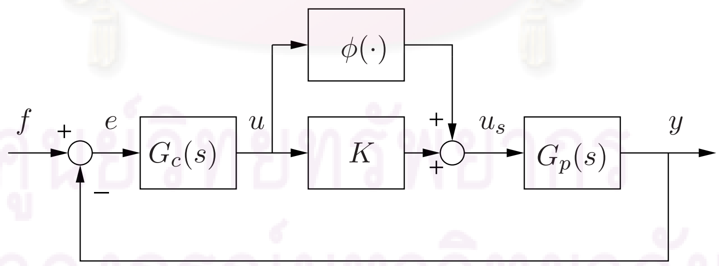


Figure 3.3: Equivalent system for the system (3.4).

Oldak, Baril and Gutman [24] used the decomposition (3.6) in connection with the design by quantitative feedback theory (see, for example, [37]) for feedback systems containing hard nonlinearities found in practice such as saturation, dead-zone, friction, etc. It is important to note that the design formulation considered here is very different from that in [24]. For example, the design objectives are different.

Now consider the auxiliary system shown in Figure 3.4, where $f \in \mathcal{P}$ and $w \in \mathcal{U}$, defined by

$$\mathcal{U} \triangleq \{x \in L_\infty \mid \|x\|_\infty \leq U_{\max}\}. \tag{3.7}$$

The system in Figure 3.4 is described by

$$\begin{aligned} u' &= g_c * e' \\ e' &= f - g_p * [Ku' + \phi(w)]\,. \end{aligned} \tag{3.8}$$

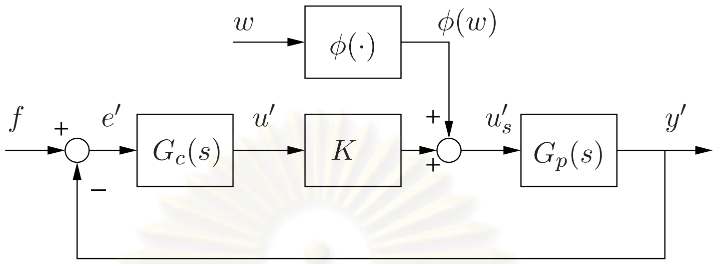


Figure 3.4: Auxiliary nonlinear system.

Let h be the impulse response of the transfer function

$$H(s) \triangleq \frac{G_p(s)G_c(s)}{1 + KG_p(s)G_c(s)}. \tag{3.9}$$

Assumption 3.2. *The impulse response h satisfies conditions that $h \in \mathcal{A}$ and $\dot{h} \in \mathcal{A}$.*

It should be noted that by virtue of the convolution representation, the plant transfer function $G_p(s)$ in (3.4) can represent a lumped- or distributed-parameter system as long as h satisfies Assumption 3.2. For example, the plant can be a system with time-delays or a heat conduction process.

In the following, the main result is stated and can be proved by using the technique due to [27], which is essentially an application of Schauder's theorem (see, for example, [26, 27]).

Theorem 3.1 (**Schauder Theorem** [26]). *Let Ω be a closed, bounded and convex subset in a Banach space*[1]*. Every compact mapping $\Phi : \Omega \to \Omega$ has a fixed point.*

For a function $x : \mathbb{R}_+ \to \mathbb{R}$ and for a fixed $T > 0$, define x_T as follows

$$x_T(t) = \begin{cases} x(t), & 0 \le t \le T \\ 0, & t > T. \end{cases}$$

Also, for a given $X \subset L_\infty$, define

$$X_T = \{x_T |\, x \in X\}\,.$$

Theorem 3.2. *Let Assumptions 3.1 and 3.2 be satisfied. The design criteria* (3.2) *for the system in Figure 3.1 are satisfied if the following conditions for the system in Figure 3.4 hold:*

$$\begin{aligned} \hat{e}' &\le E_{\max}, \quad \hat{e}' \triangleq \sup_{f\in\mathcal{P},\, w\in\mathcal{U}} \|e'\|_\infty \\ \hat{u}' &\le U_{\max}, \quad \hat{u}' \triangleq \sup_{f\in\mathcal{P},\, w\in\mathcal{U}} \|u'\|_\infty. \end{aligned} \tag{3.10}$$

[1] which is defined as a complete normed vector space (see, for example, [38, 39]).

Proof. Consider the system in Figure 3.4 with $f \in \mathcal{P}$, $w \in \mathcal{U}$. From (3.8), we have

$$u' = h_1 * \phi(w) + h_2 * f, \tag{3.11}$$

where h_1 and h_2 are the impulse responses of $H_1(s)$ and $H_2(s)$, respectively, given by

$$H_1(s) = -\frac{G_p(s)G_c(s)}{1 + KG_p(s)G_c(s)}, \quad H_2(s) = \frac{G_c(s)}{1 + KG_p(s)G_c(s)}. \tag{3.12}$$

Now, define

$$\mathcal{E} \triangleq \{x \in L_\infty \mid \|x\|_\infty \le E_{\max}\}. \tag{3.13}$$

Let (3.10) hold. Consequently, it follows that $e' \in \mathcal{E}$ and $u' \in \mathcal{U}$ for all $f \in \mathcal{P}$ and all $w \in \mathcal{U}$. Accordingly, for any $T \in [0, \infty)$ and for each $f \in \mathcal{P}_T$, equation (3.11) defines an operator $\Phi : \mathcal{U}_T \to \mathcal{U}_T$ such that

$$u'_T = \Phi(w_T). \tag{3.14}$$

Note that $\mathcal{U}_T$ is a bounded, closed and convex subset of the Banach space L_T for any $T \in [0, \infty)$.

Furthermore, by abuse of notation, $\phi(x)$ can be seen from (3.6) as a function generated by a mapping $\phi : L_\infty \to L_\infty$ such that

$$\phi(x) = \psi(x) - Kx, \quad x \in L_\infty. \tag{3.15}$$

Evidently, ϕ is continuous on L_∞ because of the continuity of ψ. Consequently, by virtue of Lemma D.1, it can be shown that if $h_1, \dot{h}_1 \in \mathcal{A}$, then the operator Φ is compact over $\mathcal{U}_T$. In view of Schauder theorem, it follows that for any $T \in [0, \infty)$ and for each $f \in \mathcal{P}_T$, there exists $u^\dagger \in \mathcal{U}_T$ such that

$$u^\dagger = \Phi(u^\dagger). \tag{3.16}$$

Let $e^\dagger \in \mathcal{E}_T$ denote the associated error of the system (3.8). Hence,

$$\begin{aligned} u^\dagger &= g_c * e^\dagger, \\ e^\dagger &= f - g_p * \left[Ku^\dagger + \phi(u^\dagger)\right]. \end{aligned} \tag{3.17}$$

Or equivalently,

$$\begin{aligned} u^\dagger &= g_c * e^\dagger, \\ e^\dagger &= f - g_p * \psi(u^\dagger). \end{aligned} \tag{3.18}$$

It readily follows from Assumption 3.1 that $e^\dagger$ and $u^\dagger$ are also the error and the controller output of the system (3.4) for any $T > 0$. As a result, conditions $e^\dagger \in \mathcal{E}_T$ and $u^\dagger \in \mathcal{U}_T$ imply that (3.2) is satisfied, and therefore the proof is completed. □

Obviously, $\hat{e}$, $\hat{e}'$, $\hat{u}$ and $\hat{u}$, depend on the design parameter $\mathbf{p}$. Theorem 3.2 states that if there exists a point $\mathbf{p}$ satisfying

$$\hat{e}' \le E_{\max} \quad \text{and} \quad \hat{u}' \le U_{\max},$$

then

$$\hat{e} \leq E_{\max} \quad \text{and} \quad \hat{u} \leq U_{\max}.$$

In other words, the design problem of the original nonlinear system can now be replaced by that of the auxiliary system (3.8). This is the key result of this section, providing an important step in deriving more tractable design inequalities.

It is important to note that the system (3.8) with two inputs f and $\phi(w)$ is linear. Now define

$$\mathcal{D}_w \triangleq \{d \in L_\infty \mid d = \phi(w),\ w \in \mathcal{U}\} \tag{3.19}$$

and consider the system in Figure 3.5 where $f \in \mathcal{P}$ and $d \in \mathcal{D}_w$.

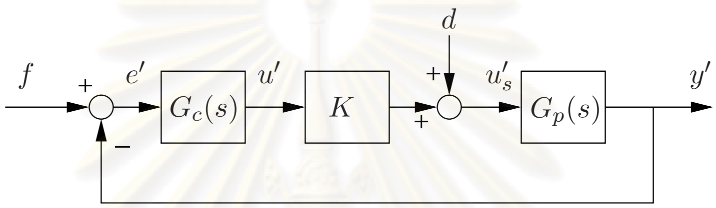


Figure 3.5: Nominal linear system of the system (3.4).

Obviously, this system is equivalent to the system (3.8). Consequently, the peak values $\hat{e}'$ and $\hat{u}'$ can be computed as follows

$$\begin{aligned} \hat{e}' &= \sup_{f \in \mathcal{P},\, d \in \mathcal{D}_w} \|e'\|_\infty \\ \hat{u}' &= \sup_{f \in \mathcal{P},\, d \in \mathcal{D}_w} \|u'\|_\infty \end{aligned} . \tag{3.20}$$

Since every $d \in \mathcal{D}_w$ depends on w, it follows that the set $\mathcal{D}_w$ cannot be readily employed in the design. Note, however, that d is always bounded for any $w \in \mathcal{U}$, or more specifically,

$$\sup_{w \in \mathcal{U}} \|d(w)\|_\infty \leq N, \quad \text{with} \quad N \triangleq \sup_{|x| \leq U_{\max}} |\psi(x) - Kx|. \tag{3.21}$$

Thus, by defining

$$\mathcal{D} \triangleq \{d \in L_\infty \mid \|d\|_\infty \leq N\}, \tag{3.22}$$

it is readily follows that $\mathcal{D}_w \subseteq \mathcal{D}$, and hence

$$\begin{aligned} \hat{e}' &\leq \tilde{e}, \quad \tilde{e} \triangleq \sup_{f \in \mathcal{P},\, d \in \mathcal{D}} \|e'\|_\infty \\ \hat{u}' &\leq \tilde{u}, \quad \tilde{u} \triangleq \sup_{f \in \mathcal{P},\, d \in \mathcal{D}} \|u'\|_\infty \end{aligned} . \tag{3.23}$$

As an immediate consequence, the following result is obvious.

Theorem 3.3. *Let Assumptions 3.1 and 3.2 be satisfied. The design criteria* (3.2) *are satisfied if*

$$\tilde{e} \leq E_{\max} \quad \textit{and} \quad \tilde{u} \leq U_{\max}. \tag{3.24}$$

Proof. The proof readily follows from the above discussion. □

Notice that in contrast to $\mathcal{D}_w$, any input in $\mathcal{D}$ does not depend on w. Therefore, $\tilde{e}$ and $\tilde{u}$ can be computed numerically by using available methods developed for linear systems (see Section 3.3). Hence, (3.24) become computationally tractable design inequalities.

3.3 Surrogate Design Criteria

Based on the results in Section 3.2, the aim of this section is to elaborate the computation of the performance measures $\tilde{e}$ and $\tilde{u}$, thereby providing design criteria in the form of inequalities that are suitable for solution by numerical methods [36].

By the linearity of the system in Figure 3.5, it follows immediately that

$$\begin{aligned} \tilde{e} &= \phi_{ef} + \phi_{ed} \\ \tilde{u} &= \phi_{uf} + \phi_{ud}, \end{aligned} \tag{3.25}$$

where

$$\begin{aligned} \phi_{ef} &\triangleq \sup_{f\in\mathcal{P},\, d=0} \|e'\|_\infty, & \phi_{ed} &\triangleq \sup_{f=0,\, d\in\mathcal{D}} \|e'\|_\infty, \\ \phi_{uf} &\triangleq \sup_{f\in\mathcal{P},\, d=0} \|u'\|_\infty, & \phi_{ud} &\triangleq \sup_{f=0,\, d\in\mathcal{D}} \|u'\|_\infty. \end{aligned} \tag{3.26}$$

From a well-known result in linear systems theory (see, for example, [28]), the numbers ϕ_{ed} and ϕ_{ud} are expressed as

$$\phi_{ed} = N\int_0^\infty |e'_d(\delta,t)|dt, \quad \phi_{ud} = N\int_0^\infty |u'_d(\delta,t)|dt, \tag{3.27}$$

where $e'_d(\delta,t)$ and $u'_d(\delta,t)$ are the values of e' and u' at time t, respectively, with f being zero and d being the Dirac delta function. From (3.27), it is clear that ϕ_{ed} and ϕ_{ud} can be obtained by standard numerical algorithms. Moreover, the numbers ϕ_{ef} and ϕ_{uf} can be computed by using known methods (see, for example, [13, 14, 40]). In this work, the approach developed in [14] is employed. Therefore, the values $\tilde{e}$ and $\tilde{u}$ can readily be obtained in practice.

From the above, define

$$\begin{aligned} \phi_1 &\triangleq \tilde{e} = \phi_{ef} + N\int_0^\infty |e'_d(\delta,t)|dt, \\ \phi_2 &\triangleq \tilde{u} = \phi_{uf} + N\int_0^\infty |u'_d(\delta,t)|dt. \end{aligned} \tag{3.28}$$

Evidently, the associated performance measures ϕ_{ef}, ϕ_{ed}, ϕ_{uf} and ϕ_{ud} depend on the gain K, and thus so do ϕ_1 and ϕ_2. As a result, to achieve a better design, K can as well be allowed to be an additional design parameter. To this end, the design problem is now to determine a controller transfer function $G_c(s,\mathbf{p})$ such that the following inequalities are satisfied

$$\begin{aligned} \phi_1(\tilde{\mathbf{p}}) &\leq E_{\max} \\ \phi_2(\tilde{\mathbf{p}}) &\leq U_{\max}, \end{aligned} \tag{3.29}$$

where

$$\tilde{\mathbf{p}} \triangleq [\mathbf{p}^T,\, K]^T. \tag{3.30}$$

From the above discussion, the main following providing a useful computational tool is stated.

Theorem 3.4. *Let Assumptions 3.1, and 3.2 be satisfied. If inequalities* (3.29) *hold, then the design criteria* (3.2) *are satisfied.*

Proof. By Theorems 3.3, the proof follows readily from the above. □

Notice that $\phi_1(\tilde{\mathbf{p}})$ and $\phi_2(\tilde{\mathbf{p}})$ are readily computable. Accordingly, inequalities (3.29) are practical sufficient conditions that are used for determining a controller $G_c(s, \mathbf{p})$ satisfying the design criteria (3.2) by numerical methods. In addition, (3.29) are appropriately called the *surrogate design criteria*.

3.4 Stability Condition

Following Zakian's framework [2, 3, 18, 36], it is readily appreciated that in searching for a design solution of (3.29) in $\mathbb{R}^n$, a numerical search algorithm needs to start from a stability point of the nominal system, that is, a point $\tilde{\mathbf{p}}$ for which

$$\phi_1(\tilde{\mathbf{p}}) < \infty \quad \text{and} \quad \phi_2(\tilde{\mathbf{p}}) < \infty. \tag{3.31}$$

It should be noted that for cases of linear systems, the problem of finding such a point has been investigated extensively (see, for example, [2, 11, 36, 41] and the references therein).

Moreover, it follows from Theorem 3.4 that the design solution of the nonlinear system can be found if the design criteria for the nominal (linear) system (3.29) are satisfied. Recall that the inequalities (3.29) are sufficient for (3.2). If (3.29) are not satisfied, then no conclusion can be drawn about either the existence of the solution of (3.2) or the stability of the original nonlinear systems. In this connection, either when inequalities (3.29) have no solution or when the search algorithm cannot locate a design solution, the stability of the nominal (linear) system ($\tilde{e} < \infty$ and $\tilde{u} < \infty$) is not enough to ensure the stability of the original nonlinear system. (It should be noted that finding a controller satisfying (3.29) is in general a non-convex problem. Consequently, the algorithm might be caught in a computational trap in the space of design parameters.) In such a case, it is desirable for designers to stabilize the original nonlinear system, in other words, to obtain a stability point of this system. Recall that a stability point of the nonlinear system is a point of design parameters $\mathbf{p}$ satisfying (1.8), that is,

$$\hat{e}(\mathbf{p}) < \infty \quad \text{and} \quad \hat{u}(\mathbf{p}) < \infty. \tag{3.32}$$

For the nonlinear system considered in the chapter, the problem can be solved by using the results developed in Chapter 2 where the nonlinearity is a sector-bounded function.

Note, in addition, that the points $\mathbf{p}$ satisfying that $\hat{e}(\mathbf{p}) = \infty$ or $\hat{u}(\mathbf{p}) = \infty$ form a connected region. Obviously, this region does not contain any solution of (3.2), and hence that of (3.29). As a result, this region should be excluded from the search-space of design parameters. From the computational point of view, this helps to narrow down the search-space, and thus facilitate the progress of the search algorithm. Therefore, from the above discussion, it follows that in finding a design solution of (3.2) by solving (3.29), a numerical algorithm needs to perform the search in the space of design parameters $\tilde{\mathbf{p}}$ satisfying both (3.31) and (3.32).

Let g denote the impulse response of the composite transfer function

$$G(s) \triangleq G_c(s, \mathbf{p})G_p(s). \tag{3.33}$$

Assumption 3.3. *The impulse responses g_p, g_c and g satisfies conditions that $g_p,\ g_c,\ \dot{g} \in \mathcal{A}$ and there exists $\alpha > 0$ such that*

$$\int_0^\infty \mathrm{e}^{2\alpha t} g^2(t)dt < \infty. \tag{3.34}$$

The boundedness of the responses e and u can be guaranteed by using the following theorem.

Theorem 3.5. *Consider the system in Figure 3.1 and let Assumption* (3.3) *hold. The responses e and u are bounded for any $f \in \mathcal{P}$ and for any $\psi \in \mathrm{sector}\,[0,\ k_0]$ if there exist $q \in \mathbb{R}$ and $\beta \in \mathbb{R}$ such that*

$$\mathrm{Re}\ [(1 + qj\omega)\, G\,(j\omega)] + \frac{1}{k_0} \geq \beta > 0, \quad \forall\omega \geq 0. \tag{3.35}$$

Proof. By noting that $\mathcal{P} \subset \mathcal{P}_\infty$, the proof immediately follows from Proposition 2.3 in Chapter 2. □

Also, from the stability results in Chapter 2, it should be noted that condition (3.35) is satisfied if the following holds:

$$\phi_0(\mathbf{p}) \leq -\gamma, \qquad \phi_0(\mathbf{p}) \triangleq k_0 - k_{\max}(\mathbf{p}), \tag{3.36}$$

where γ is a small positive number and $k_{\max}$ is the supremal value of the allowable sector bound obtained from the Popov test. Clearly, $\phi_0(\mathbf{p})$ can readily be computed in practice and (3.36) is in accordance with the method of inequalities [3]. As a consequence, condition (3.36) provides a useful inequality for obtaining stability points by numerical methods.

It should be noted further that the satisfaction of Popov criterion (3.35) implies that the Nyquist diagram of $G(s)$ does not encircle point $(-1/k_0,\ 0)$. See [30] for the detail on this. As a consequence, the Nyquist diagram of $G(s)$ also does not encircle any point $(-1/k,\ 0)$ with $k \in [0,\ k_{\max})$ since $k_{\max}$ is the supremal value of the allowable sector bound. Thus, from the theory of the Nyquist criterion [42–44] (see also [28]), it can be shown that the linear system shown in Figure 3.5 is BIBO stable for any $K \in [0,\ k_{\max})$. Accordingly, the following result is obvious.

Corollary 3.1. *If Assumption 3.3 and the Popov condition are satisfied by $G(s)$, then the system in Figure 3.5 is BIBO stable for any $K \in [0,\ k_{\max})$.*

Corollary 3.1 reveals that, during the search, if $0 \leq K < k_{\max}$, then a stability point $\mathbf{p}$ of the nonlinear system together with the value K form a stability point of the nominal linear system.

3.5 Controller Design for a Hydraulic System

In this section, a nonlinear hydraulic system [45] (see also [13]) is used to demonstrate the usefulness of the developed method where the plant is a hydraulic actuator equipped with a low-cost closed-center four-way proportional valve. The plant transfer function $G_p(s)$ is given by

$$G_p(s) = \frac{1.1411 \times 10^{10}}{(s + 0.0248)(s + 28.57)(s^2 + 35.14s + 25190)}. \tag{3.37}$$

The nonlinearity ψ is the dead-zone function shown in Figure 3.6, where $U_0 = 0.1$, $U_1 = 0.08$ and $z_0 = 0.02$.

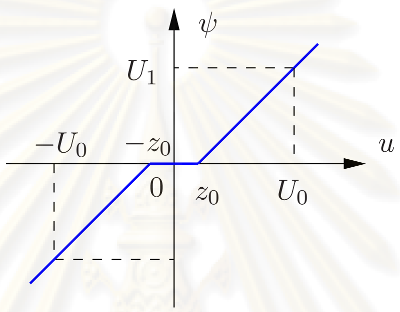


Figure 3.6: The dead-zone characteristic in the hydraulic actuator.

The design objective is to keep the tracking error and the control voltage within ± 65 N and ± 0.1 V, respectively, for all time and for any reference force f belonging to a possible set $\mathcal{P}$ where

$$M = 1000 \text{ N} \quad \text{and} \quad D = 1000 \text{ N/s}. \tag{3.38}$$

Consequently, the design criteria are expressed as

$$\hat{e} \le 65 \text{ N} \quad \text{and} \quad \hat{u} \le 0.1 \text{ V}. \tag{3.39}$$

It may be noted, in passing, that the error bound in (3.39) is smaller than that used in [13].

3.5.1 Linear System Design

Suppose that the dead-zone characteristic is neglected and replaced with a constant gain $K = 1$. As a result, the peak outputs $\hat{e}$ and $\hat{u}$ can be computed by using numerical algorithms developed for linear time-invariant systems.

The structure of the controller G_c is chosen as

$$G_c(s, \mathbf{p}) = \frac{p_4 s^2 + p_5 s + p_6}{(s + p_1)(s^2 + p_2 s + p_3)}, \tag{3.40}$$

where $\mathbf{p} = [p_1, p_2, p_3, p_4, p_5, p_6]^T \in \mathbb{R}^6$ denotes the vector of design parameters. It should be noted that, of all the possible controllers satisfying the design criteria, the one with simple structure is usually preferred. That is to say, designers should start searching a design solution from a simple controller first. After many attempts with controllers of different orders, the structure in (3.40) is arrived at.

Throughout this work, inequalities (3.39) are solved by using the MBP algorithm [3]. Alternatively, other algorithms for solving a set of inequalities may be used (see, for example, [2] and the references cited therein). After a number of iterations, a design solution $\mathbf{p}$ is located, where

$$\mathbf{p} = [4.0622 \times 10^{-4},\ 5.2633 \times 10^{2},\ 4.3564 \times 10^{3},\ 0.8252,\ 4.8476,\ 0.3817]^T \tag{3.41}$$

and the corresponding performance measures are

$$\hat{e}(\mathbf{p}) = 64.9975 \text{ N} \quad \text{and} \quad \hat{u}(\mathbf{p}) = 0.0961 \text{ V}. \tag{3.42}$$

To verify the performance of the obtained controller, a test input $\hat{f}$ is generated such that its magnitude and slope are bounded by 1000 N and 1000 N/s, respectively. The waveform of $\hat{f}$ and the responses of the nonlinear system are given in Figures 3.7 and 3.8, respectively.

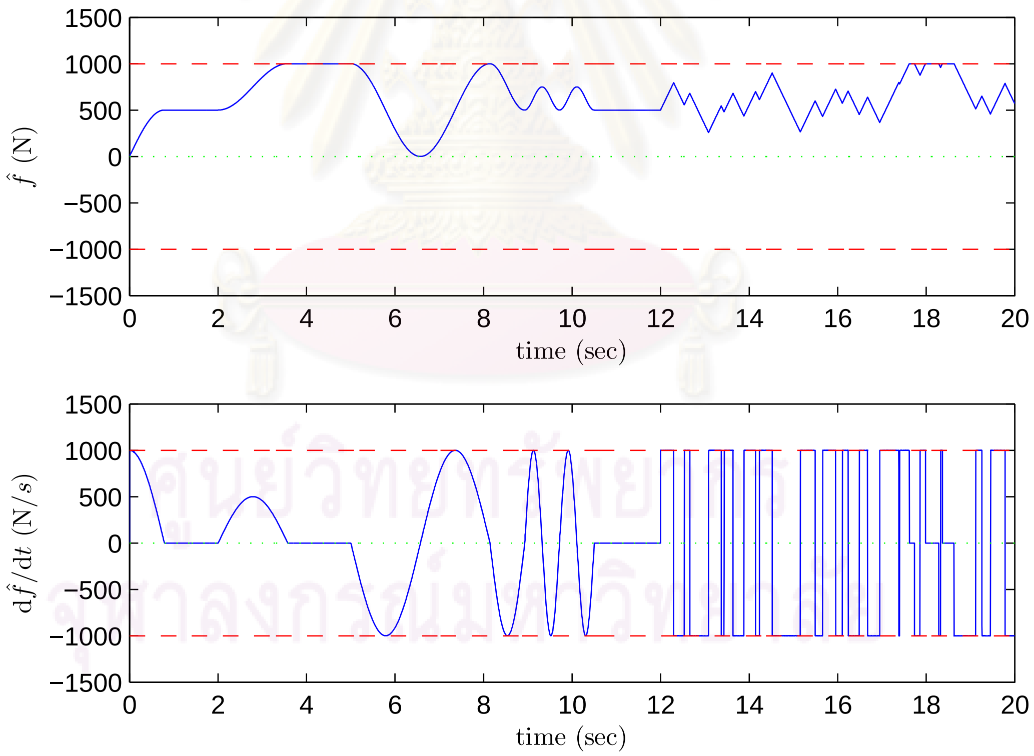


Figure 3.7: A test input $\hat{f} \in \mathcal{P}$ characterized by (3.38).

The simulation results show that the performance of the system using the controller obtained by neglecting the nonlinearity does not satisfy the design criteria (3.39). Specifically, in response to the input $\hat{f}$, the control signal u slightly violates the bound $U_{\max}$ and the maximal magnitude of e is 79.31 N, which exceeds the bound $E_{\max}$ by 22.02%.

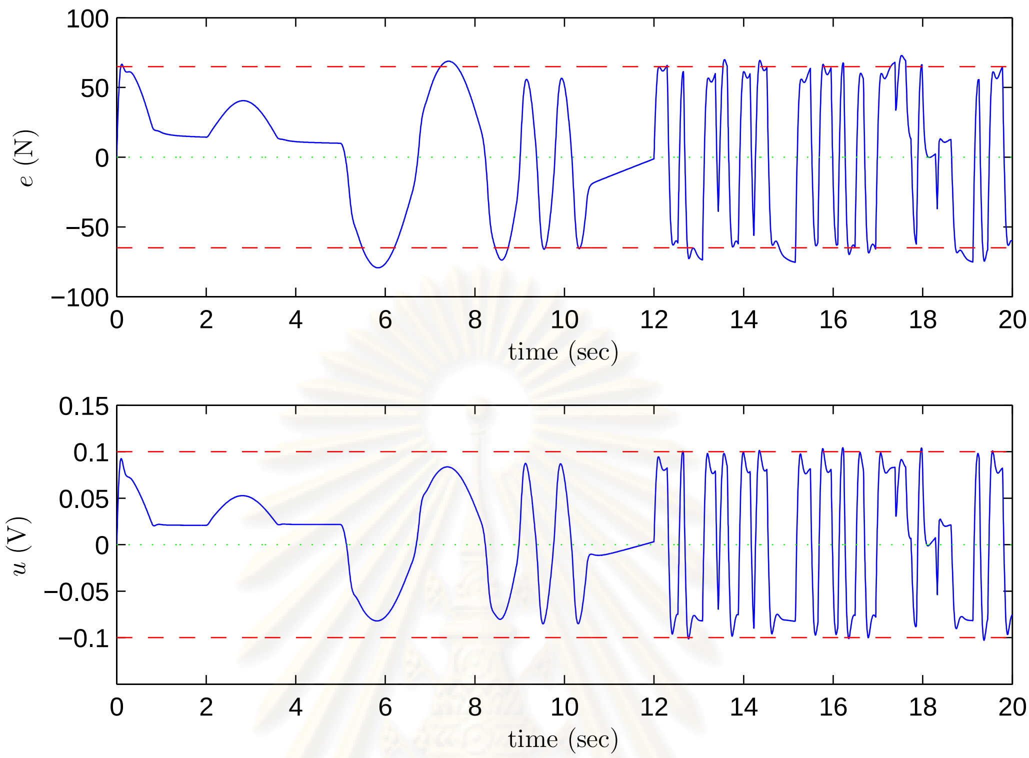


Figure 3.8: Responses of the nonlinear system to the input $\hat{f}$ using design solution (3.41).

3.5.2 Nonlinear System Design

Now the dead-zone element is taken into account in the design. A controller is obtained by solving the design criteria given by

$$\begin{aligned} \phi_0(\tilde{\mathbf{p}}) &\leq -10^{-6}, \\ \phi_1(\tilde{\mathbf{p}}) &\leq 65 \text{ N}, \\ \phi_2(\tilde{\mathbf{p}}) &\leq 0.1 \text{ V}. \end{aligned} \tag{3.43}$$

The nonlinearity $\psi \in$ sector $[0, 1]$ and is decomposed using (3.6). Accordingly, the relation between disturbance ϕ and control input u is given in Figure 3.9 and the bound

$$N = \max\{|Kz_0|, |KU_0 + z_0 - U_0|\}.$$

The structure of the controller $G_c(s)$ is chosen as in (3.40) where $\tilde{\mathbf{p}} = [\mathbf{p}^T, K]^T \in \mathbb{R}^7$. By using the MBP algorithm, a design solution

$$\tilde{\mathbf{p}} = [0.0110,\ 2.6954 \times 10^2,\ 1.7476 \times 10^4,\ 0.9097,\ 23.8560,\ 0.7325,\ 0.9680]^T \tag{3.44}$$

is found and the corresponding performance measures are

$$\begin{aligned} \phi_0(\tilde{\mathbf{p}}) &= -2.75, \\ \phi_1(\tilde{\mathbf{p}}) &= 62.67 \text{ N}, \\ \phi_2(\tilde{\mathbf{p}}) &= 0.099 \text{ V}. \end{aligned} \tag{3.45}$$

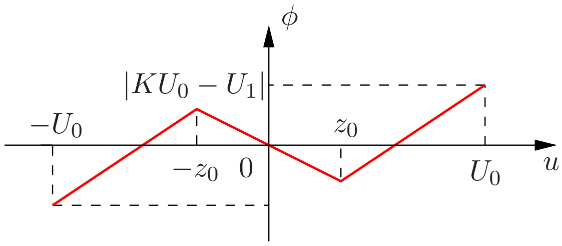


Figure 3.9: The relation between ϕ and u.

To verify the design, the simulation is performed with the test input $\hat{f}$. The responses of the closed-loop system are given in Figure 3.10. It is shown that the simulation results agree well with the results obtained by the design. Specifically, the maximal magnitudes of e and u in response to $\hat{f}$ are 61.82 N and 0.0846 V, respectively. Clearly, (3.43) and (3.39) are satisfied.

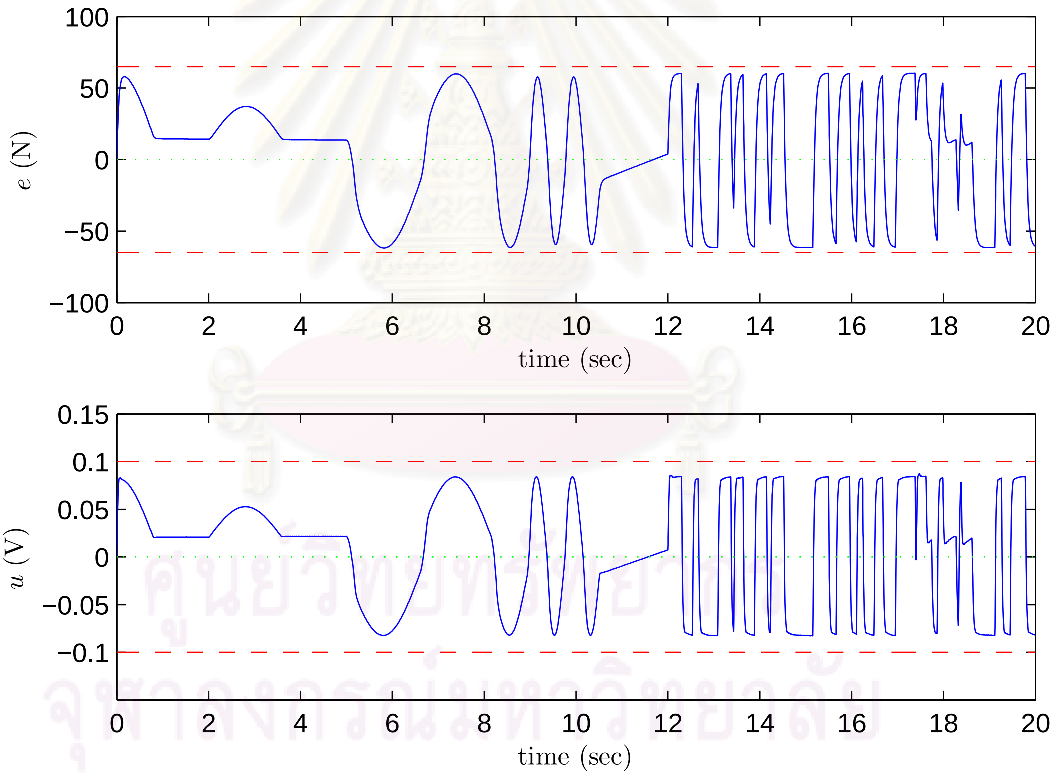


Figure 3.10: Responses of the nonlinear system to the input $\hat{f}$ using design solution (3.44).

From this example, it is seen that the design obtained by neglecting the nonlinearity ψ can fail to satisfy the design objective (3.2). This is critical in cases where any violation of (3.2) or (3.2) is unacceptable. Hence, the value of the method developed here is evident.

3.6 Conclusions and Discussion

This chapter has developed a practical method for designing a controller for the system (3.4) so as to ensure that the error e and the controller output u stay within the specified ranges $\pm E_{\max}$ and $\pm U_{\max}$ for all time and for all possible inputs $f \in \mathcal{P}$. In connection with Zakian's framework, the method can be seen as an adjunct to the principle of matching [1, 2].

Theorem 3.2 provides an essential basis for developing the surrogate design criteria (3.29), which are used to obtain a solution of the original design problem (3.2) by numerical methods. Because the system (3.4) uses the convolution representation, the developed method is applicable to both lumped- and distributed-parameter systems as long as Assumption 3.2 is satisfied.

The decomposition (3.6) is simple and useful. The nonlinearity is replaced with a fixed gain and an equivalent bounded disturbance. As a result, the nominal system used during the design becomes linear time-invariant so that the associated performance measures (which are ϕ_{ef}, ϕ_{ed}, ϕ_{uf} and ϕ_{ud}) are readily obtainable by known methods.

It is interesting to note further that the proposed method is also applicable to the case in which the set $\mathcal{P}$ is characterized by the bounds on the two-norms of inputs and their slope. See Chapter 6 for further details on this. Moreover, in connection with the results in [14], the method is also applicable to the case of the set $\mathcal{P}$ characterized by using more than two bounding conditions in order to eliminate *fictitious inputs* (that is, inputs that cannot happen in practice) and thus yields a better design.

CHAPTER IV

DESIGN OF FEEDBACK SYSTEMS WITH OUTPUT NONLINEARITY

This chapter deals with the design of feedback control systems where the plant is linear time-invariant and has a static memoryless nonlinearity in its output channel. The design problem is to determine a controller ensuring that the error function and the controller output stay within respective bounds for all time and for all inputs in a possible set $\mathcal{P}$. To this end, we extend the results developed in Chapter 3 in a straightforward manner, and demonstrate the usefulness of the method by a design example in which the plant is infinite-dimensional.

4.1 Introduction

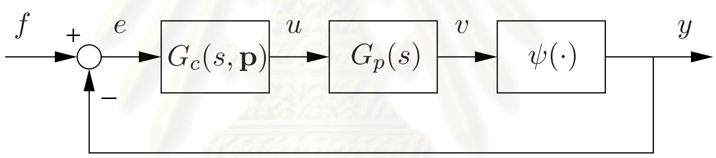


Figure 4.1: A feedback control system with an output nonlinearity.

Consider a feedback control system with an output nonlinearity as shown in Figure 4.1, where, as usual, $\psi : \mathbb{R} \to \mathbb{R}$ is a continuous, time-invariant and memoryless nonlinear function, $G_p(s)$ and $G_c(s, \mathbf{p})$ are the transfer functions of the plant and the controller with the design parameter $\mathbf{p} \in \mathbb{R}^n$, respectively.

The system is also assumed to be at rest for $t \leq 0$. Furthermore, as in Chapter 3, the input f is known only to the extent that it belongs to the possible set $\mathcal{P}$ described by (3.1), that is,

$$\mathcal{P} \triangleq \left\{ f \in L_\infty \;\middle|\; \|f\|_\infty \leq M,\ \|\dot{f}\|_\infty \leq D \right\}. \tag{4.1}$$

As mentioned previously, the design problem considered in the chapter is to determine a controller transfer function $G_c(s, \mathbf{p})$ such that the design objectives (1.7) are satisfied, that is

$$\begin{aligned} \hat{e} &\leq E_{\max} \\ \hat{u} &\leq U_{\max} \end{aligned} \tag{4.2}$$

where $\hat{e}$ and $\hat{u}$ are the peak values of e and u defined in (1.7), and the bounds $E_{\max}$ and $U_{\max}$ are given.

This chapter has two main objectives. First and foremost, by extending the results developed in previous chapters, we develop a practical method for designing the feedback system shown in Figure 4.1 so that the design criteria (4.2) are satisfied, where $G_p(s)$ can represent a lumped- or distributed-parameter system. To this end, we derive sufficient conditions of the form

$$\begin{aligned} \tilde{e} &\leq E_{\max} \\ \tilde{u} &\leq U_{\max} \end{aligned} \tag{4.3}$$

for ensuring the satisfaction of (4.2), where $\tilde{e}$ and $\tilde{u}$ are computable upper bounds of $\hat{e}$ and $\hat{u}$, respectively. Accordingly, the design criteria (4.3) are suitable for solution by numerical methods [36]. The key idea is to replace the nonlinearity with a constant gain and a bounded disturbance. Sufficient conditions for the satisfaction of the original design criteria are derived from a nominal linear system, and thus, provide surrogate design criteria that are more computationally tractable. Second, we demonstrate the usefulness of the method proposed by applying it to the design of a feedback system in which the plant is governed by a heat equation with an output nonlinearity.

The organization of this chapter is as follows. Section 4.2 derives sufficient conditions for ensuring the satisfaction of the original design criteria (4.2), where the main theoretical result is stated in Theorem 4.1. Practical sufficient conditions for (4.2) are then developed in the form of inequalities that are in keeping with the method of inequalities [3]. Section 4.3 presents a stability condition which essentially guarantees the boundedness of the nonlinear system outputs. The developed method is illustrated by a design example of a heat conduction process in Section 4.4. Finally, conclusions and discussion are given in Section 4.5.

4.2 Main Results

This section derives the main theoretical result of the chapter by making use of the technique due to [15, 24, 25], in which the nonlinearity is replaced with a constant gain and an equivalent bounded disturbance. The principal result is introduced in Theorem 4.1, providing sufficient conditions for the satisfaction of the design criteria (4.2). The conditions will be used subsequently to develop practical design inequalities that can be used for determining a controller $G_c(s)$ satisfying the original design criteria (4.2).

Assumption 4.1. *For every input $f \in \mathcal{P}$, there exist unique $e : \mathbb{R}_+ \to \mathbb{R}$, $u : \mathbb{R}_+ \to \mathbb{R}$ and $v : \mathbb{R}_+ \to \mathbb{R}$ that satisfy*

$$\begin{aligned} v &= g_p * u \\ u &= g_c * e \\ e &= f - \psi(v) \end{aligned} \tag{4.4}$$

where g_p and g_c are the impulse responses of G_p and G_c, respectively.

In the following, the decomposition technique used in Chapter 3 will be recapitulated. For a given value $K \in \mathbb{R}$, define a function $\phi : \mathbb{R} \to \mathbb{R}$ such that

$$\phi(x) \triangleq \psi(x) - Kx, \quad x \in \mathbb{R}. \tag{4.5}$$

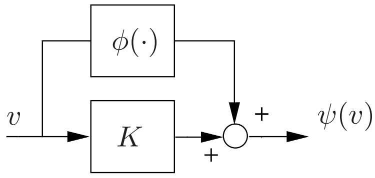


Figure 4.2: Decomposition of the nonlinearity ψ.

It readily follows that the nonlinearity ψ can be represented as in Figure 4.2.

Now consider another system described in Figure 4.3, where $f \in \mathcal{P}$ and $w \in \mathcal{V}$, defined by

$$\mathcal{V} \triangleq \{w \in L_\infty \mid \|w\|_\infty \leq V_{\max}\} \tag{4.6}$$

with an appropriate bound $V_{\max}$.

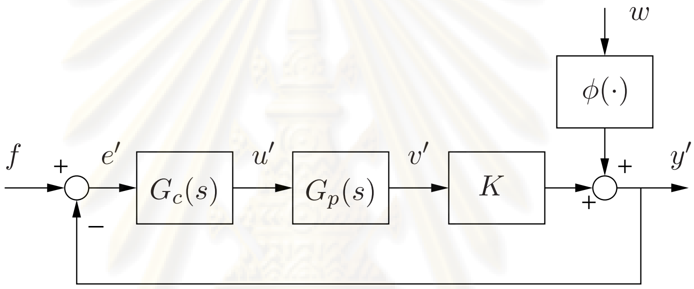


Figure 4.3: Auxiliary nonlinear system.

The system in Figure 4.3 is described by

$$\begin{aligned} v' &= g_p * u' \\ u' &= g_c * e' \\ e' &= f - Kv' - \phi(w) \end{aligned} \tag{4.7}$$

where $f \in \mathcal{P}$ and $w \in \mathcal{V}$.

In the following, we will show that the design objectives of the original nonlinear system can be ensured if the system in Figure 4.3 satisfies certain conditions. The result is stated and proved by using the technique in Chapter 3, which is essentially an application of the Schauder fixed point theorem (see, for example, [26, 27]).

Assumption 4.2. *The impulse response h of the transfer function*

$$H(s) \triangleq \frac{G_p(s)G_c(s)}{1 + KG_p(s)G_c(s)} \tag{4.8}$$

satisfies conditions that $h,\ \dot{h} \in \mathcal{A}$.

Theorem 4.1. *Let Assumptions 4.1 and 4.2 be satisfied. If*

$$\begin{array}{rclrl} \hat{e}' & \leq & E_{\max}, & \hat{e}' \triangleq & \sup\limits_{f\in\mathcal{P},\, w\in\mathcal{U}} \|e'\|_\infty \\ \hat{u}' & \leq & U_{\max}, & \hat{u}' \triangleq & \sup\limits_{f\in\mathcal{P},\, w\in\mathcal{U}} \|u'\|_\infty \\ \hat{v}' & \leq & V_{\max}, & \hat{v}' \triangleq & \sup\limits_{f\in\mathcal{P},\, w\in\mathcal{U}} \|v'\|_\infty \end{array} . \tag{4.9}$$

then the criteria (4.2) *are satisfied.*

Proof. The theorem can be proved by employing the method used in Chapter 3. The details of the proof are given, for the sake of completeness, in the following.

Consider the system in Figure 4.3 with $f \in \mathcal{P}$ and $w \in \mathcal{V}$. From (4.7), we have

$$v' = h * f - h * \phi(w), \tag{4.10}$$

where h is the impulse responses of $H(s)$ which satisfies Assumption 4.2.

Now, define

$$\begin{array}{rcl} \mathcal{E} & \triangleq & \{x \in L_\infty \mid \|x\|_\infty \leq E_{\max}\}, \\ \mathcal{U} & \triangleq & \{x \in L_\infty \mid \|x\|_\infty \leq U_{\max}\}. \end{array} \tag{4.11}$$

Let (4.9) hold. Consequently, it follows that $e' \in \mathcal{E}$, $u' \in \mathcal{U}$ and $v' \in \mathcal{V}$ for all $f \in \mathcal{P}$ and all $w \in \mathcal{V}$. Thus, for any $T \in [0, \infty)$ and for each $f \in \mathcal{P}_T$, equation (4.10) defines an operator $\Phi : \mathcal{V}_T \to \mathcal{V}_T$ such that

$$v'_T = \Phi(w_T). \tag{4.12}$$

Note that $\mathcal{V}_T$ is a bounded, closed and convex subset of the Banach space L_T for any $T \in [0, \infty)$.

Furthermore, it can be seen from (4.5) that ϕ is a continuous function on $\mathbb{R}$. Consequently, by virtue of Lemma D.1 (see Appendix D), it can be shown that if $h, \dot{h} \in \mathcal{A}$, then the operator Φ is compact over $\mathcal{V}_T$. In view of Schauder theorem, it follows that for any $T \in [0, \infty)$ and for each $f \in \mathcal{P}_T$, there exists $v^\dagger \in \mathcal{V}_T$ such that

$$v^\dagger = \Phi(v^\dagger). \tag{4.13}$$

Let $e^\dagger \in \mathcal{E}_T$ and $u^\dagger \in \mathcal{U}_T$ denote the associated error function and controller output of the system (4.7). Hence,

$$\begin{array}{rcl} v^\dagger & = & g_p * u^\dagger \\ u^\dagger & = & g_c * e^\dagger \\ e^\dagger & = & f - K v^\dagger - \phi(v^\dagger). \end{array} \tag{4.14}$$

Equivalently,

$$\begin{array}{rcl} v^\dagger & = & g_p * u^\dagger \\ u^\dagger & = & g_c * e^\dagger \\ e^\dagger & = & f - \psi(v^\dagger). \end{array} \tag{4.15}$$

It readily follows from Assumption 4.1 that $e^\dagger$, $u^\dagger$ and $v^\dagger$ are also the responses of system (4.4) for any $T > 0$. As a result, conditions $e^\dagger \in \mathcal{E}_T$, $u^\dagger \in \mathcal{U}_T$ and $v^\dagger \in \mathcal{V}_T$ imply that the criteria (4.2) are satisfied for any T, and therefore the proof is completed. □

Theorem 4.1 has a noteworthy consequence that the control problem of the nonlinear system (4.4) can be replaced by that of the system (4.7). Specifically, it is shown that, provided that the gain K and the bound $V_{\max}$ are chosen, the satisfaction of (4.9) implies that of original design criteria (4.2). This is the key result of this section, providing an important step in developing more tractable design inequalities. In the following, the derivation of such inequalities is presented.

It is important to note that the system (4.7) with two inputs f and $\phi(w)$ is linear. Now define

$$\mathcal{D}_w \triangleq \{d \in L_\infty \mid d = \phi(w),\ w \in \mathcal{V}\} \tag{4.16}$$

and consider the system in Figure 4.4 where $f \in \mathcal{P}$ and $d \in \mathcal{D}_w$.

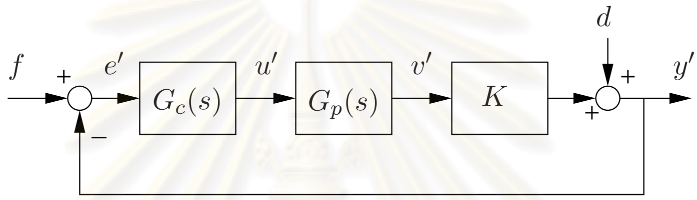


Figure 4.4: Nominal linear system of nonlinear system (4.1).

Evidently, it follows from the definition in (4.16) that the systems in Figures 4.3 and 4.4 are equivalent. Thus, the peak values $\hat{e}'$ and $\hat{u}'$ can be computed as follows

$$\begin{aligned} \hat{e}' &= \sup_{f\in\mathcal{P},\, d\in\mathcal{D}_w} \|e'\|_\infty \\ \hat{u}' &= \sup_{f\in\mathcal{P},\, d\in\mathcal{D}_w} \|u'\|_\infty \\ \hat{v}' &= \sup_{f\in\mathcal{P},\, d\in\mathcal{D}_w} \|v'\|_\infty \end{aligned} . \tag{4.17}$$

Apparently, the set $\mathcal{D}_w$ cannot be readily employed in the design since every d in $\mathcal{D}_w$ depends on w. However, it should be noted that d is always bounded for any $w \in \mathcal{V}$, or more specifically,

$$\sup_{w\in\mathcal{V}} \|\phi(w)\|_\infty \le N, \quad \text{with} \quad N \triangleq \sup_{|x|\le V_{\max}} |\psi(x) - Kx|. \tag{4.18}$$

Thus, by defining

$$\mathcal{D} \triangleq \{d \in L_\infty \mid \|d\|_\infty \le N\}, \tag{4.19}$$

it is readily follows that $\mathcal{D}_w \subseteq \mathcal{D}$, and hence

$$\begin{aligned} \hat{e}' &\le \tilde{e}, \quad \tilde{e} \triangleq \sup_{f\in\mathcal{P},\, d\in\mathcal{D}} \|e'\|_\infty \\ \hat{u}' &\le \tilde{u}, \quad \tilde{u} \triangleq \sup_{f\in\mathcal{P},\, d\in\mathcal{D}} \|u'\|_\infty \\ \hat{v}' &\le \tilde{v}, \quad \tilde{v} \triangleq \sup_{f\in\mathcal{P},\, d\in\mathcal{D}} \|v'\|_\infty \end{aligned} . \tag{4.20}$$

As an immediate consequence, we have the following result.

Theorem 4.2. *Let Assumptions 4.1 and 4.2 be satisfied. The design criteria* (4.2) *are satisfied if*

$$\begin{aligned} \tilde{e} &\leq E_{\max} \\ \tilde{u} &\leq U_{\max} \\ \tilde{v} &\leq V_{\max}. \end{aligned} \tag{4.21}$$

Proof. The proof readily follows from the above discussion. □

Notice that in contrast to $\mathcal{D}_w$, any input in $\mathcal{D}$ does not depend on w. Therefore, $\tilde{e}$, $\tilde{u}$ and $\tilde{v}$ can be computed numerically by using available methods developed for linear systems. In particular,

$$\begin{aligned} \tilde{e} &= \phi_1 \triangleq \sup_{f\in\mathcal{P},\, d=0} \|e'\|_\infty + N\int_0^\infty |e'_d(\delta,t)|dt, \\ \tilde{u} &= \phi_2 \triangleq \sup_{f\in\mathcal{P},\, d=0} \|u'\|_\infty + N\int_0^\infty |u'_d(\delta,t)|dt, \\ \tilde{v} &= \phi_3 \triangleq \sup_{f\in\mathcal{P},\, d=0} \|v'\|_\infty + N\int_0^\infty |v'_d(\delta,t)|dt, \end{aligned} \tag{4.22}$$

where $e'_d(\delta,t)$, $u'_d(\delta,t)$ and $v'_d(\delta,t)$ are the values of e', u' and v' at time t, respectively, with $f=0$ and $d=\delta(t)$. See Section 3.3 for more details on the computations of ϕ_1, ϕ_2 and ϕ_3.

Note further that the numbers ϕ_1, ϕ_2 and ϕ_3 depend on the gain K. Thus, in order to achieve a better design, K can be allowed to be an additional design parameter. To this end, define the augmented design parameter vector $\tilde{\mathbf{p}}$ as in (3.30), that is, $\tilde{\mathbf{p}} \triangleq [\mathbf{p}^T,\, K]^T$. As a consequence, (4.21) are equivalent to

$$\begin{aligned} \phi_1(\tilde{\mathbf{p}}) &\leq E_{\max}, \\ \phi_2(\tilde{\mathbf{p}}) &\leq U_{\max}, \\ \phi_3(\tilde{\mathbf{p}}) &\leq V_{\max}. \end{aligned} \tag{4.23}$$

Accordingly, (4.21), and hence (4.23) become computationally tractable design inequalities.

4.3 Stability Condition

The necessity of the stability requirement in this chapter is as that discussed in Section 3.4 of Chapter 3. In the following, only the stability condition is introduced.

Let g denote the impulse response of the composite transfer function

$$G(s) \triangleq G_c(s,\mathbf{p})G_p(s). \tag{4.24}$$

Assumption 4.3. *The impulse responses* g_0, g_c *and* g *satisfy conditions that* $g_0,\, g_c,\, \dot{g} \in \mathcal{A}$ *and there exists* $\alpha > 0$ *such that* $\int_0^\infty \mathrm{e}^{2\alpha t} g^2(t)dt < \infty$.

The boundedness of the responses e and u can be guaranteed by using the following theorem.

Theorem 4.3. *Consider system (4.4) and let Assumption 4.3 hold. The responses e, u and v are bounded for any $f \in \mathcal{P}$ and for any $\psi \in \text{sector}\,[0,\ k_0]$ if there exist $q \in \mathbb{R}$ and $\beta \in \mathbb{R}$ such that*

$$\text{Re}\,[(1 + qj\omega)\,G\,(j\omega)] + \frac{1}{k_0} \geq \beta > 0, \quad \forall\omega \geq 0. \tag{4.25}$$

Proof. By noting that $\mathcal{P} \subset \mathcal{P}_\infty$, the proof follows from Theorem 2.2. □

Following the results in Chapter 2, the Popov condition (4.25) can be replaced by a more tractable inequality

$$\phi_0(\mathbf{p}) \leq -\gamma, \quad \phi_0(\mathbf{p}) \triangleq k_0 - k_{\max}(\mathbf{p}), \tag{4.26}$$

where γ is a small positive number and $k_{\max}$ denotes the maximum value of the allowable sector bound obtained from the Popov test.

By making use of appropriate loop transformations, it is easy to see that Theorem 4.3 is also applicable to the case that $G(s)$ has one pole at the origin. For details on this, see Chapter 2.

4.4 Numerical Example

This section considers a design of the heat-conduction process that was previously introduced in Chapter 2. The transfer function of the plant given by

$$G_p(s) = \frac{20}{\sqrt{s}\,\sinh\sqrt{s}}. \tag{4.27}$$

Assume that the nonlinearity ψ is the output sensor and is described in Figure 4.5, where $z_0 = 0.2$, $k_1 = 0.5$ and $k_2 = 1$. Obviously, $\psi \in \text{sector}\,[k_1,\ k_2]$.

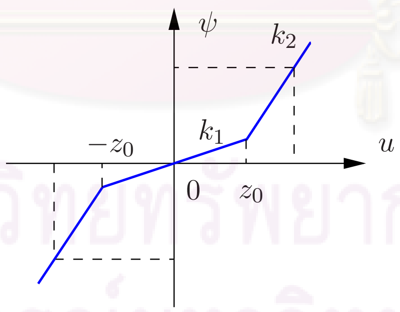


Figure 4.5: The output nonlinearity ψ of the heat-conduction system.

Suppose that the control objective is to keep the responses e and u stay within $\pm E_{\max} = \pm 0.3$ and $\pm U_{\max} = \pm 0.25$ for all time and for all inputs in the set $\mathcal{P}$ characterized by

$$M = 1 \quad \text{and} \quad D = 0.5. \tag{4.28}$$

Consequently, the criteria (4.2) become

$$\hat{e} \leq 0.3 \quad \text{and} \quad \hat{u} \leq 0.25. \tag{4.29}$$

In connection with the surrogate design criteria (4.23), let $V_{\max} = 1.2$.

The design problem is to find a controller satisfying inequalities (4.26) and (4.23). To this end, assume that the controller transfer function takes the form

$$G_c(s,\mathbf{p}) = \frac{p_1(s+p_2)}{s^2+p_3s+p_4}, \qquad \mathbf{p} = [p_1,p_2,p_3,p_4]^T \in \mathbb{R}^4. \tag{4.30}$$

Notice that, for G_p in (4.27), Assumption 4.2 is always satisfied since $H(s)$ is strictly proper whenever so is G_c and since the finiteness of ϕ_i in (4.23) implies that $h, \dot{h} \in \mathcal{A}$. (See [14] for the details on the connection between the stability issue and the finiteness of ϕ_i for linear problems). Moreover, it readily follows from Section 4.3 and from (2.34) that Assumption 4.3 is also fulfilled. That is to say, Theorem 4.3 is applicable.

For convenience, the nonrational transfer function G_p is approximated by a truncated eigenfunction expansion of order 20. When a higher order approximation is used, no significant difference in the computed results is found. Note, in passing, that one may avoid this approximation by employing the method developed in [11] for the design of retarded fractional delay differential systems.

By using the MBP algorithm to solve inequalities (4.26) and (4.23), a design solution

$$\tilde{\mathbf{p}} = [173.99,\ 8.59,\ 150.67,\ 5637.90,\ 0.95]^T \tag{4.31}$$

is located and the corresponding performance measures are

$$\phi_0(\mathbf{p}) = -72.79, \quad \phi_1(\tilde{\mathbf{p}}) = 0.28, \quad \phi_2(\tilde{\mathbf{p}}) = 0.20, \quad \phi_3(\tilde{\mathbf{p}}) = 1.15. \tag{4.32}$$

To verify the performance of the obtained controller, a test input $\hat{f} \in \mathcal{P}$ is generated such that its magnitude and slope are bounded by M and D, respectively. See Figure 4.6.

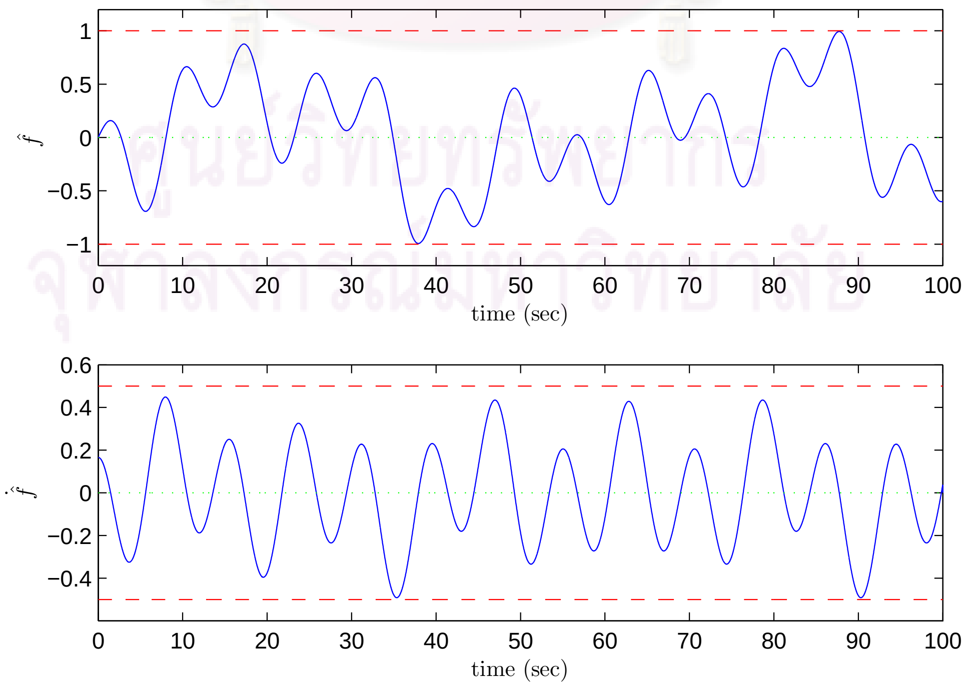


Figure 4.6: A test input $\hat{f} \in \mathcal{P}$ characterized by (4.28).

The responses of the system given in Figure 4.7. The simulation results show that the performances of the system by using the so-obtained controller satisfy the design criteria (4.29), thereby clearly illustrating the usefulness of the method.

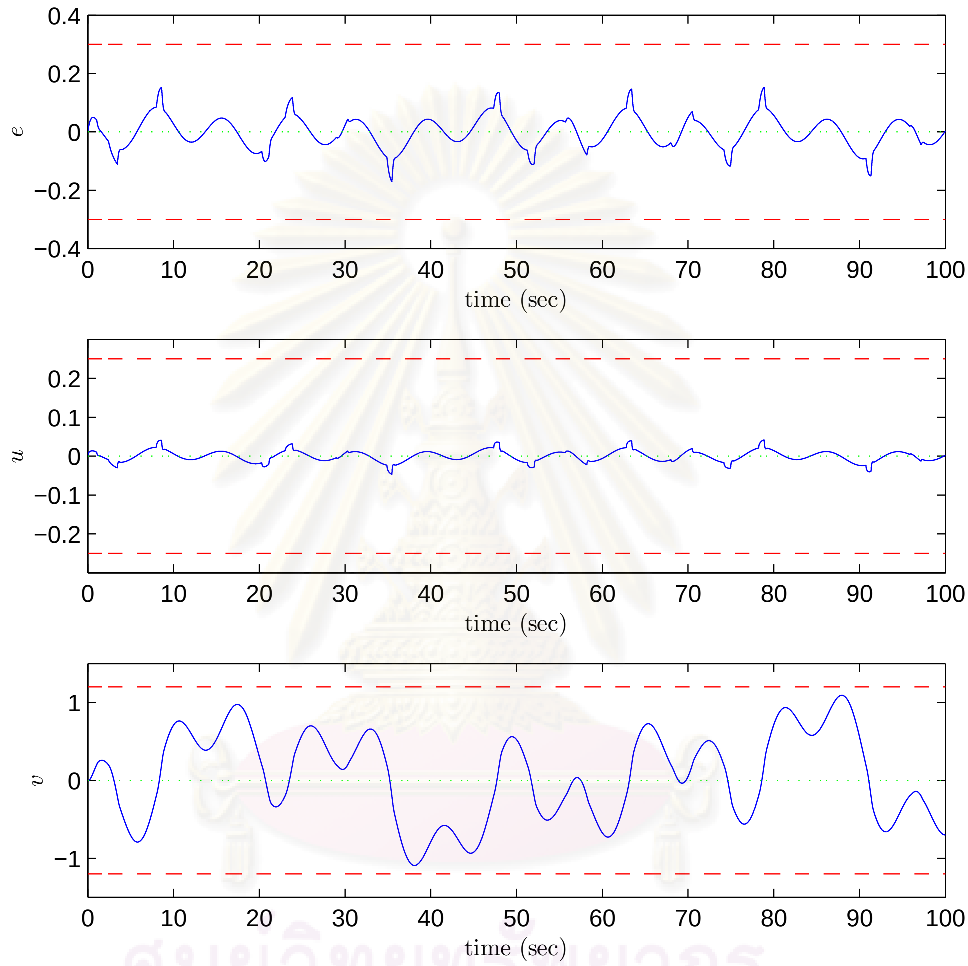


Figure 4.7: Responses of the nonlinear system to input $\hat{f}$ using design solution (4.31).

4.5 Conclusions and Discussion

This chapter has developed a practical method for designing a controller for the system with an output nonlinearity, as shown in Figure 4.1, so as to ensure that e and u stay within the prescribed ranges $\pm E_{\max}$ and $\pm U_{\max}$ for all time and for all inputs $f \in \mathcal{P}$. Specifically, by using the decomposition (4.2), the nonlinearity is replaced with a constant gain and an bounded disturbance, and hence the original design problem becomes that of a linear time-invariant system subject to an additional disturbance. As a consequence, Theorem 4.1 provides an essential basis for developing the surrogate design criteria (4.23), which are used to obtain a solution of the original design problem by numerical methods. The simulation results of the heat conduction process have illustrated the advantage of the proposed method.

It should be noted that the method developed here is also applicable to the cases where $\mathcal{P} \triangleq \{f \in L_2 \mid \|f\|_2 \leq M_2, \|\dot{f}\|_2 \leq D_2\}$, with M_2 and D_2 being prescribed bounds or $\mathcal{P}$ is characterized by using more than two bounding conditions (see also Chapter 6).

CHAPTER V

DESIGN OF FEEDBACK CONTROL SYSTEMS WITH UNCERTAIN PLANT AND INPUT NONLINEARITY

This chapter is motivated by the fact that many real systems possess uncertainties. In this connection, the aim of this chapter is to develop a practical method for designing a class of uncertain nonlinear feedback control systems in such a way that the error function and the controller output are ensured to remain within respective bounds for all time and for all possible inputs in the presence of uncertainties. The key idea is to make use of the decomposition technique described in Chapters 3 and 4 together with Zakian's method for designing vague systems [1, 2]. Finally, a controller design for a heat-conduction process with uncertainties is carried out and the numerical results demonstrate the usefulness of the developed method.

5.1 Introduction

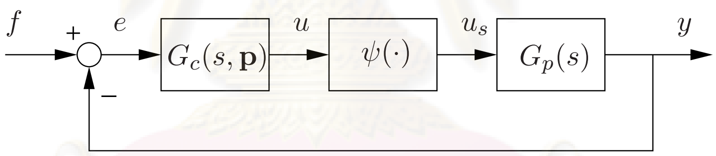


Figure 5.1: A feedback control system with uncertain plant and with input nonlinearity.

Consider a feedback control system shown in Figure 5.1, where $\psi(\cdot)$ is a continuous, time-invariant and memoryless nonlinear function such that $\psi(0) = 0$, $G_p(s)$ and $G_c(s, \mathbf{p})$ are the transfer functions of the plant and the controller with the design parameter $\mathbf{p} \in \mathbb{R}^n$, respectively. Suppose that $G_p(s)$ is uncertain and known only to the extent that it belongs to a set $\mathcal{G}_p$. Note, in passing, that if $\mathcal{G}_p$ consists of more than one distinct element, then the plant, and hence the system are said to be *vague* [1]. The input f is in the set $\mathcal{P}$ defined in (3.1). Moreover, it is also assumed that the system is at rest for $t \leq 0$.

Following Zakian's framework, the design problem is to determine a controller transfer function $G_c(s, \mathbf{p})$ such that the following design criteria are satisfied:

$$\left.\begin{aligned} |e(f,t)| &\leq E_{\max} \\ |u(f,t)| &\leq U_{\max} \end{aligned}\right\} \quad \forall f \in \mathcal{P} \quad \forall t \in \mathbb{R}_+ \quad \forall G_p \in \mathcal{G}_p, \tag{5.1}$$

where the bounds $E_{\max}$ and $U_{\max}$ are given. It is easy to see that the criteria (5.1) are equivalent to the following conditions

$$\begin{aligned}\sup_{G_p\in\mathcal{G}_p} \hat{e} &\leq E_{\max}\\ \sup_{G_p\in\mathcal{G}_p} \hat{u} &\leq U_{\max},\end{aligned} \tag{5.2}$$

where the peak values $\hat{e}$ and $\hat{u}$ are defined as in (1.7) for each $G_p \in \mathcal{G}_p$. Evidently, (5.2) become practical design criteria, provided that the terms on the left hand-side can be computed in practice. Nevertheless, the problems of computing $\sup_{G_p\in\mathcal{G}_p} \hat{e}$ and $\sup_{G_p\in\mathcal{G}_p} \hat{u}$ are obviously more difficult than those of $\hat{e}$ and $\hat{u}$, which are now still open. Therefore, the purpose of this chapter is to develop a method for designing a controller $G_c(s)$ satisfying the criteria (5.2). Since (5.2) are computationally intractable, we derive practical sufficient conditions of the form

$$\phi_1 \leq E_{\max} \quad \text{and} \quad \phi_2 \leq U_{\max} \tag{5.3}$$

for ensuring them, where ϕ_1 and ϕ_2 are readily computable. Consequently, (5.3) are more tractable and suitable for solution by numerical methods for a wide range of $G_p(s)$.

The key ideas are as follows. First, applying the decomposition technique presented in Chapters 3 and 4 to the system in Figure 5.1 results in an uncertain linear system subject to two inputs. Second, by extending Zakian's majorants [1, 22, 46], practical design inequalities (5.3) are obtained. Finally, in searching for a controller satisfying the criteria (5.2) by numerical methods, a useful condition for guaranteeing the robust stability of the nonlinear system is derived by using the results presented in Chapter 2. The condition also provides a readily computable inequality for determining a robust stabilizing controller.

The organization of this chapter is as follows. Section 5.2 presents an extension of Zakian's majorants. Section 5.3 casts the design problem of the uncertain nonlinear system by using the results developed in Section 5.2 and the design method presented in previous chapters. In Section 5.4, the stability condition of the uncertain nonlinear system is given. A design of a heat-conduction process with uncertainties is carried out in Section 5.5 to illustrate the usefulness of the method. Finally, conclusions and discussion are given in Section 5.6.

5.2 Extension of Zakian's Majorants

This section presents an extension of Zakian's majorants [1, 2, 22, 46] to the case of linear feedback systems with two inputs.

Consider the system in Figure 5.2 where $G_1(s)$ and $G_2(s)$ are the transfer functions of the controller and the plant, the input f belongs to a possible set $\mathcal{P} \subset L_\infty$ and the input d belongs to the set $\mathcal{D}$ defined by

$$\mathcal{D} \triangleq \{d \in L_\infty \mid \|d\|_\infty \leq N\}. \tag{5.4}$$

The system is described by

$$\begin{aligned} v_2 &= v_1 * g_1\\ v_1 &= f - g_2 * (d + v_2),\end{aligned} \tag{5.5}$$

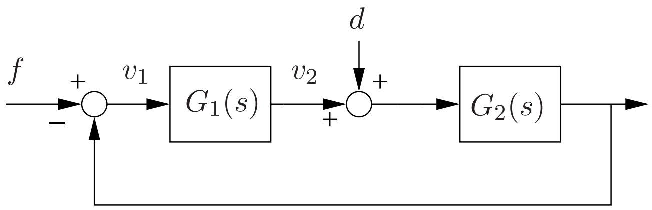


Figure 5.2: Uncertain linear system with two inputs.

where g_1 and g_2 are the impulse responses of G_1 and G_2, respectively.

Suppose that the design problem is to determine the transfer function $G_1(s)$ satisfying

$$\begin{aligned} \sup_{f\in\mathcal{P},\, d\in\mathcal{D}} \|v_1\|_\infty &\leq V_{1\,\max} \\ \sup_{f\in\mathcal{P},\, d\in\mathcal{D}} \|v_2\|_\infty &\leq V_{2\,\max}, \end{aligned} \tag{5.6}$$

where the bounds $V_{1\,\max}$ and $V_{2\,\max}$ are given. In case it is desirable to replace the transfer function $G_2(s)$ by $G_2^*(s)$ (which is more convenient to use than $G_2(s)$), the system in Figure 5.1 becomes the nominal one that is shown in Figure 5.3 and described by

$$\begin{aligned} v_2^* &= v_1^* * g_1 \\ v_1^* &= f - g_2^* * (d + v_2^*). \end{aligned} \tag{5.7}$$

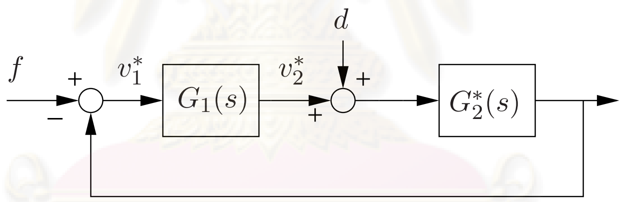


Figure 5.3: The nominal system connected with the one in Figure 5.2.

In the following, we will develop sufficient conditions expressed in terms of the nominal system for ensuring the satisfaction of the design criteria (5.6).

5.2.1 Zakian's Criterion of Approximation

First, consider the case in which the transfer function $G_2(s)$ is fixed. Accordingly, $G_2^*(s)$ can be seen as an approximant of $G_2(s)$. Define

$$\mu_1 \triangleq \int_0^\infty |w_1(\tau)|d\tau \quad \text{and} \quad \mu_2 \triangleq \int_0^\infty |w_2(\tau)|d\tau, \tag{5.8}$$

where w_1 and w_2 are the impulse responses of $W_1(s)$ and $W_2(s)$ given by

$$W_1(s) \triangleq \frac{G_2(s) - G_2^*(s)}{1 + G_1(s)G_2^*(s)}, \quad W_2(s) \triangleq G_1(s)\frac{G_2(s) - G_2^*(s)}{1 + G_1(s)G_2^*(s)}. \tag{5.9}$$

Let $\hat{v}_1^*$ and $\hat{v}_2^*$ denote the peak values of v_1^* and v_2^*, respectively, defined by

$$\hat{v}_1^* \triangleq \sup_{f\in\mathcal{P},d\in\mathcal{D}} \|v_1^*\|_\infty \quad \text{and} \quad \hat{v}_2^* \triangleq \sup_{f\in\mathcal{P},d\in\mathcal{D}} \|v_2^*\|_\infty. \tag{5.10}$$

Now the main result of this subsection is stated as follows.

Theorem 5.1. *Suppose that the system* (5.7) *is BIBO stable and that* $\mu_1 < \infty$ *and* $\mu_2 < \infty$*. The criteria* (5.6) *for the system* (5.5) *are satisfied if* $\mu_2 < 1$ *and if the following holds*

$$\begin{aligned} \phi_1 &\leq V_{1\max}, \quad \phi_1 \triangleq \frac{\hat{v}_1^* + \mu_1 N}{1-\mu_2} \\ \phi_2 &\leq V_{2\max}, \quad \phi_2 \triangleq \frac{\hat{v}_2^* + \mu_2 N}{1-\mu_2}. \end{aligned} \tag{5.11}$$

Proof. From (5.5), it is easy to verify that

$$\begin{aligned} V_1(s) &= F(s) - D(s)G_2(s) - G_1(s)G_2(s)V_1(s) \\ &= \frac{F(s) - D(s)G_2^*(s)}{1+G_1(s)G_2^*(s)} - \frac{G_2(s) - G_2^*(s)}{1+G_1(s)G_2^*(s)}D(s) - \frac{G_1(s)[G_2(s) - G_2^*(s)]}{1+G_1(s)G_2^*(s)}V_1(s). \end{aligned} \tag{5.12}$$

It follows from (5.7) and (5.9) that

$$V_1(s) = V_1^*(s) - W_1(s)D(s) - W_2(s)V_1(s). \tag{5.13}$$

Consequently,

$$v_1(t) = v_1^*(t) - \int_0^t w_1(t-\tau)d(\tau)d\tau - \int_0^t w_2(t-\tau)v_1(\tau)d\tau \tag{5.14}$$

and thus, by using (5.8), we have

$$\sup_{\tau\in[0,t]} |v_1(\tau)| \leq \sup_{\tau\in[0,t]} |v_1^*(\tau)| + \mu_1 \sup_{\tau\in[0,t]} |d(\tau)| + \mu_2 \sup_{\tau\in[0,t]} |v_1(\tau)|. \tag{5.15}$$

From (5.15), it is easy to verify that

$$(1-\mu_2) \sup_{\tau\in[0,t]} |v_1(\tau)| \leq \sup_{\tau\in[0,t]} |v_1^*(\tau)| + \mu_1 \sup_{\tau\in[0,t]} |d(\tau)|. \tag{5.16}$$

Since the system (5.7) is BIBO stable, the values $\hat{v}_1^*$ and $\hat{v}_2^*$ are always finite. As a result, provided that $\mu_1 < \infty$ and that $\mu_2 < 1$, it follows from (5.16) that letting $t \to \infty$ yields

$$\|v_1\|_\infty \leq \frac{\|v_1^*\|_\infty + \mu_1 \|d\|_\infty}{1-\mu_2}. \tag{5.17}$$

Furthermore, by multiplying both sides of (5.13) with $G_1(s)$ we arrive at

$$V_2(s) = V_2^*(s) - G_1(s)W_1(s)D(s) - W_2(s)V_2(s). \tag{5.18}$$

In the same way, by noting that $W_2(s) = G_1(s)W_1(s)$, it can be easily shown that

$$\|v_2\|_\infty \leq \frac{\|v_2^*\|_\infty + \mu_2 \|d\|_\infty}{1-\mu_2}. \tag{5.19}$$

Therefore, inequalities (5.11) are obtained as a consequence of (5.17) and (5.19). □

5.2.2 Zakian's Majorants for Vague Systems

Now consider the case in which the transfer function $G_2(s)$ belongs to a set $\mathcal{G}_2$ containing more than one distinct elements. As will be seen shortly, the result in the previous subsection will be used and developed further.

By noting that μ_1, μ_2, and hence ϕ_i depend on $G_2 \in \mathcal{G}_2$, the following result is obvious.

Proposition 5.1. *Suppose that the system* (5.7) *is BIBO stable and that* $\mu_1 < \infty$ *and* $\mu_2 < 1$ *for any* $G_2 \in \mathcal{G}_2$. *The criteria* (5.6) *for the system* (5.5) *are satisfied if the following holds.*

$$\frac{\hat{v}_1^* + N \sup\limits_{G_2 \in \mathcal{G}_2} \mu_1}{1 - \sup\limits_{G_2 \in \mathcal{G}_2} \mu_2} \quad \leq \quad V_{i\,\max}, \quad i = 1, 2. \tag{5.20}$$

Proof. By virtue of Theorem 5.1, the proof readily follows. □

Nevertheless, inequalities (5.20) cannot readily be used in practice for determining $G_1(s)$ by numerical methods, because, for each value of the design parameter $\mathbf{p}$, supremal operations over $\mathcal{G}_2$ are required in the evaluation of $\sup_{G_2 \in \mathcal{G}_2} \mu_i$, which is not computationally economical. In this connection, Zakian [1, 46] proposes to replace μ_i by its upper bounds $\tilde{\mu}_i$ and thus arrives at

$$\hat{\phi}_i \geq \phi_i, \quad i = 1, 2, \tag{5.21}$$

where $\hat{\phi}_i$ is a majorant of ϕ_i that is readily computable.

Next, the upper bound of μ_2 is derived. Note that

$$w_2(t) = z(0)v_2^*(t, \mathbf{1}) + \int_0^t \dot{z}(t-\tau)v_2^*(\tau, \mathbf{1})d\tau, \tag{5.22}$$

where $v_2(t, \mathbf{1})$ denotes the value of v_2^* at time t in response to the input $f = \mathbf{1}(t)$ and $d = 0$, and

$$z \triangleq g_2 - g_2^*. \tag{5.23}$$

Now assume that the nominal system is BIBO stable. As a consequence, the following limits exist

$$\sigma_1 \triangleq \lim_{t\to\infty} v_1^*(t, \mathbf{1}), \quad \sigma_2 \triangleq \lim_{t\to\infty} v_2^*(t, \mathbf{1}). \tag{5.24}$$

Hence, expression (5.22) is equivalent to

$$w_2(t) = z(0)\left[v_2^*(t, \mathbf{1}) - \sigma_2\right] + \int_0^t \dot{z}(t-\tau)\left[v_2^*(\tau, \mathbf{1}) - \sigma_2\right]d\tau + \sigma_2 z(t). \tag{5.25}$$

Using a known property of 1-norm of the convolution operator (see, for example, [28, p. 239]) yields

$$\|w_2\|_1 \leq A|\sigma_2| + B\|v_2^*(\mathbf{1}) - \sigma_2\|_1, \tag{5.26}$$

where

$$A \triangleq \sup\{\|z\|_1 : \ G_2 \in \mathcal{G}_2\}, \quad B \triangleq \sup\{|z(0)| + \|\dot{z}\|_1 : \ G_2 \in \mathcal{G}_2\}. \tag{5.27}$$

It follows from (5.26) that

$$\mu_2 \le \tilde{\mu}_2, \quad \tilde{\mu}_2 \triangleq A|\sigma_2| + B\|v_2^*(\mathbf{1}) - \sigma_2\|_1. \tag{5.28}$$

Similarly, an upper bound of μ_1 can be obtained as

$$\mu_1 \le \tilde{\mu}_1, \quad \tilde{\mu}_1 \triangleq A|\sigma_1| + B\|v_1^*(\mathbf{1}) - \sigma_1\|_1. \tag{5.29}$$

From the above, it should be noted that the values A and B need to be computed only once. Therefore, the upper bounds $\tilde{\mu}_1$ and $\tilde{\mu}_2$ can readily be used in practice.

Finally, let $\tilde{\mu_2} < 1$ and define

$$\hat{\phi}_i \triangleq \frac{\hat{v}_i^* + \tilde{\mu}_i N}{1 - \tilde{\mu_2}}, \quad i = 1, 2. \tag{5.30}$$

It is now ready to state the main result of this subsection.

Theorem 5.2. *Suppose that the nominal system* (5.7) *is BIBO stable and let* $\tilde{\mu}_1$ *and* $\tilde{\mu}_2$ *be finite. The criteria* (5.6) *are satisfied for all* $G_2(s) \in \mathcal{G}_2$ *if* $\tilde{\mu}_2 < 1$ *and if the following holds*

$$\begin{aligned} \hat{\phi}_1 &\le V_{1\max} \\ \hat{\phi}_2 &\le V_{2\max} \end{aligned} \tag{5.31}$$

Proof. The proof is completed by using Theorem 5.1 and the above discussion (see also [1, 46]). □

Since the majorants $\hat{\phi}_i$ can readily be computed in practice, conditions (5.31) provides useful design inequalities for determining $G_1(s)$ that satisfies the original criteria (5.6) by numerical methods.

5.3 Design of Uncertain Nonlinear Systems

This section casts the design problem of the uncertain nonlinear system shown in Figure 5.1. The key idea is, by replacing the nonlinearity with a constant gain and a bounded disturbance, to approximate the original uncertain nonlinear system by an uncertain linear one so that the results developed in Section 5.2 can be applied. The practical design inequalities are then developed based on the main results presented in Theorems 5.3 and 5.4.

Assumption 5.1. *For every input* $f \in \mathcal{P}$ *and every* $G_p \in \mathcal{G}_p$*, there are unique* $e : \mathbb{R}_+ \to \mathbb{R}$ *and* $u : \mathbb{R}_+ \to \mathbb{R}$ *that satisfy the following equations*

$$\begin{aligned} u &= g_c * e, \\ e &= f - u_s * g_p = f - \psi(u) * g_p. \end{aligned} \tag{5.32}$$

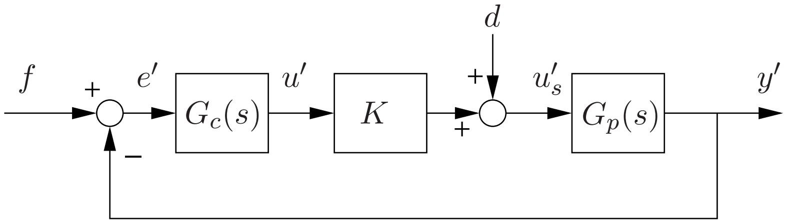


Figure 5.4: Auxiliary linear system.

By using the decomposition technique described in Chapters 3 and 4 (see also [15,16,24]), the auxiliary linear system as shown in Figure 5.4 is obtained, where d belongs to the set $\mathcal{D}$ given in (3.22) with the bound N defined in (3.21).

The system in Figure 5.4 is described by

$$\begin{aligned} u' &= g_c * e', \\ e' &= f - g_p * (Ku' + d)\,. \end{aligned} \tag{5.33}$$

where $G_p \in \mathcal{G}_p$, $f \in \mathcal{P}$ and $d \in \mathcal{D}$.

For each $G_p \in \mathcal{G}_p$, let $\hat{e}'$ and $\hat{u}'$ denote the peak values of e' and u', respectively, given by

$$\begin{aligned} \hat{e}' &\triangleq \sup_{f\in\mathcal{P},\, d\in\mathcal{D}} \|e'\|_\infty, \\ \hat{u}' &\triangleq \sup_{f\in\mathcal{P},\, d\in\mathcal{D}} \|u'\|_\infty. \end{aligned} \tag{5.34}$$

Also, let h be the impulse response of the transfer function

$$H(s) \triangleq \frac{G_p(s)G_c(s)}{1 + KG_p(s)G_c(s)}. \tag{5.35}$$

Assumption 5.2. *The function h satisfies conditions that $h \in \mathcal{A}$ and $\dot{h} \in \mathcal{A}$ for any $G_p \in \mathcal{G}_p$.*

It should be noted that by virtue of the convolution representation, the plant transfer function $G_p(s)$ in (5.32) can be lumped or distributed-parameters systems as long as h satisfies Assumption 5.2. For example, the plant can be a system with time-delays or a heat conduction process.

In the following, the relation between the design problem of the original system (5.32) and the auxiliary system (5.33) is stated. (One may notice that this result differs from Theorem 3.3 only to the extent that the system uncertainties are now taken into account).

Theorem 5.3. *Let Assumptions 5.1 and 5.2 be satisfied. The criteria* (5.2) *for the system in Figure 5.1 are satisfied if the following conditions for the system in Figure 5.4 hold*

$$\begin{aligned} \sup_{G_p\in\mathcal{G}_p} \hat{e}' &\leq E_{\max}, \\ \sup_{G_p\in\mathcal{G}_p} \hat{u}' &\leq U_{\max}. \end{aligned} \tag{5.36}$$

Proof. The proof can be completed by the same technique used to prove Theorem 3.2 and Theorem 3.3 in Chapter 3. □

According to Theorem 5.3, the design problem of the nonlinear system can be replaced by that of the auxiliary linear system subject to an additional disturbance d. However, computing the performances $\hat{e}'$ and $\hat{u}'$ given by (5.36), in general, are very difficult due to the plant uncertainty. Therefore, it is desirable to replace (5.36) with sufficient conditions by using the result developed in Section 5.2.

Consider the nominal system as shown in Figure 5.5 where $f \in \mathcal{P}$, $d \in \mathcal{D}$ and $G_p^*(s)$ denote the nominal transfer function of $G_p(s) \in \mathcal{G}_p$.

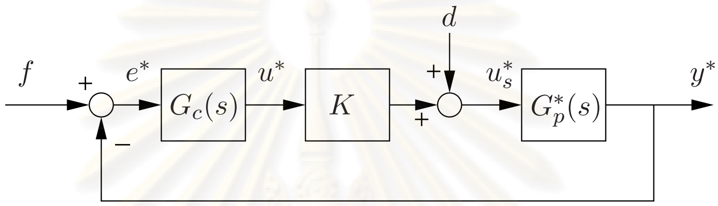


Figure 5.5: The nominal linear system.

Assume that the nominal system is BIBO stable. As a result, the following limits exist

$$\sigma_1 \triangleq \lim_{t\to\infty} e^*(t,\mathbf{1}), \quad \text{and} \quad \sigma_2 \triangleq \lim_{t\to\infty} u^*(t,\mathbf{1}). \tag{5.37}$$

where $e^*(t,\mathbf{1})$ and $u^*(t,\mathbf{1})$ are the values of e^* and u^* at time t in response to the inputs $f(t) = \mathbf{1}(t)$ and $d(t) = 0$. Next, define

$$\begin{aligned} \tilde{\mu}_1 &\triangleq A|\sigma_1| + B\|e^*(\mathbf{1}) - \sigma_1\|_1, \\ \tilde{\mu}_2 &\triangleq A|\sigma_2| + B\|u^*(\mathbf{1}) - \sigma_2\|_1, \end{aligned} \tag{5.38}$$

where

$$\begin{aligned} z &\triangleq g_p - g_p^* \\ A &\triangleq \sup\{\|z\|_1 : \ G_p \in \mathcal{G}_p\} \\ B &\triangleq \sup\{|z(0)| + \|\dot{z}\|_1 : \ G_p \in \mathcal{G}_p\}. \end{aligned} \tag{5.39}$$

Let $\hat{e}^*$ and $\hat{u}^*$ denote the peak values of e^* and u^* given by

$$\begin{aligned} \hat{e}^* &\triangleq \sup_{f\in\mathcal{P},\, d\in\mathcal{D}} \|e^*\|_\infty, \\ \hat{u}^* &\triangleq \sup_{f\in\mathcal{P},\, d\in\mathcal{D}} \|u^*\|_\infty. \end{aligned} \tag{5.40}$$

It is now ready to state the sufficient conditions to ensure the satisfaction of inequalities (5.36).

Theorem 5.4. *Suppose that the nominal system in Figure 5.5 is BIBO stable and that $\tilde{\mu}_1$ and $\tilde{\mu}_2$ defined in* (5.38) *are finite. The criteria* (5.36) *for the system in Figure 5.4 are satisfied if $\tilde{\mu}_2 < 1$ and the following inequalities hold*

$$\begin{aligned} \frac{\hat{e}^* + \tilde{\mu}_1 N}{1 - \tilde{\mu}_2} &\leq E_{\max}, \\ \frac{K\hat{u}^* + \tilde{\mu}_2 N}{K(1 - \tilde{\mu}_2)} &\leq U_{\max}. \end{aligned} \tag{5.41}$$

Proof. By virtue of Theorem 5.2, the proof immediately follows. □

It should be noted that $\hat{e}^*$ and $\hat{u}^*$ are the peak outputs of the nominal linear system without any uncertainties, and therefore they can be computed numerically by using available methods (see Section 3.3). As a consequence, conditions (5.41) become useful design inequalities, which are more tractable than those in (5.36).

Define

$$\begin{aligned} \phi_1 &\triangleq \frac{N}{1-\tilde{\mu_2}}\left(\|e_d^*(\delta)\|_1 + \frac{\phi_{ef}}{N} + \tilde{\mu_1}\right) \\ \phi_2 &\triangleq \frac{N}{1-\tilde{\mu_2}}\left(\|u_d^*(\delta)\|_1 + \frac{\phi_{uf}}{N} + \frac{\tilde{\mu_2}}{K}\right) \end{aligned} \tag{5.42}$$

where $e_d^*(\delta)$ and $u_d^*(\delta)$ are the values of e^* and u^* at time t, respectively, with $f = 0$ and $d = \delta(t)$. Evidently, ϕ_1 and ϕ_2 depend on the value of the gain K. Let $\tilde{\mathbf{p}} \triangleq [\mathbf{p}^T, K]^T$. Accordingly, the design problem is now to determine a controller transfer function $G_c(s, \mathbf{p})$ such that the following surrogate design criteria are satisfied

$$\begin{aligned} \phi_1(\tilde{\mathbf{p}}) &\leq E_{\max} \\ \phi_2(\tilde{\mathbf{p}}) &\leq U_{\max}. \end{aligned} \tag{5.43}$$

Notice that $\phi_1(\tilde{\mathbf{p}})$ and $\phi_2(\tilde{\mathbf{p}})$ are readily computable. For further details, see Section 3.3.

5.4 Stability Conditions

The usefulness of the stability condition for ensuring the boundedness of the outputs of the nonlinear system with respect to the possible set $\mathcal{P}$ is as explained in Section 3.4. In the following, only stability conditions are introduced.

For each $G_c(s, \mathbf{p})$, define

$$\mathcal{G} \triangleq \{G : \mathbb{C} \to \mathbb{C} \mid G(s) = G_c(s, \mathbf{p})G_p(s),\ \forall G_p(s) \in \mathcal{G}_p\}\,. \tag{5.44}$$

Let g denote the impulse response of the composite transfer function $G(s) \in \mathcal{G}$.

Assumption 5.3. *The impulse responses g_0, g_c and g satisfies conditions that $g_0, g_c, \dot{g} \in \mathcal{A}$ for all $G_p(s) \in \mathcal{G}_p$ and there exists $\alpha > 0$ satisfying*

$$\int_0^\infty \mathrm{e}^{2\alpha t} g^2(t)dt < \infty, \quad \forall\, G(s) \in \mathcal{G}. \tag{5.45}$$

The boundedness of e and u can be guaranteed by using the following theorem.

Theorem 5.5. *Let Assumption 5.3 be satisfied. The responses e and u are bounded for any $f \in \mathcal{P}$ and for any $\psi \in \mathrm{sector}\,[0,\ k_0]$ if there exist $q \in \mathbb{R}$ and $\beta \in \mathbb{R}$ such that the following condition is satisfied*

$$\mathrm{Re}\,\left[(1 + qj\omega)\,G\,(j\omega)\right] + \frac{1}{k_0} \geq \beta > 0, \quad \forall\, G(s) \in \mathcal{G}, \quad \forall\, \omega \geq 0. \tag{5.46}$$

Proof. The proof is completed by the direct application of Theorem 2.2 in Section 2.2 (see also [47]). □

Following the results in Section 2.3, one can see that condition (5.46) can be checked graphically by using the Popov plots of all $G(s) \in \mathcal{G}$. Now define $\mathbf{\Omega}$ as the convex hull of all the plots, (that is, the minimal convex set containing all the plots). Then, the following result is obtained.

Proposition 5.2. *The Popov plots of all $G(s) \in \mathcal{G}$ lie to the right of the Popov line if and only if so does $\mathbf{\Omega}$.*

Proof. Note that if $g, \dot{g} \in \mathcal{A}$, then $\omega \mapsto \mathrm{Re}\ [G(j\omega)]$ and $\omega \mapsto \omega\mathrm{Im}\ [G(j\omega)]$ are continuous and bounded on $\mathbb{R}_+$ (see, for example, [28]). Therefore, all the Popov plots, and hence $\mathbf{\Omega}$ lie in the finite plane. The rest of the proof follows Proposition B.1 in Appendix B. □

By virtue of this proposition, we can proceed with the Popov test developed in Chapter 2. Consequently, condition (5.46) is satisfied if the following holds:

$$\phi_0(\mathbf{p}) \leq -\gamma, \qquad \phi_0(\mathbf{p}) \triangleq k_0 - k_{\max}(\mathbf{p}), \tag{5.47}$$

where γ is a small positive number and $k_{\max}$ is the supremal value of the allowable sector bound obtained from the modified Popov test.

Clearly, once $\mathbf{\Omega}$ can be obtained in practice, condition (5.47) provides a useful inequality for determining stability points of the system by numerical methods. Note that the number of elements in $\mathcal{G}$ may be infinite. In such a case, it is desirable to approximate $\mathbf{\Omega}$ by the convex hull of the Popov plots for a sufficiently large number of $G(s) \in \mathcal{G}$. Thus, designers should use the number γ as a marginal tolerance for the error caused by this approximation.

Furthermore, by a straightforward extension of Corollary 3.1 in Chapter 3, the following result is obtained.

Corollary 5.1. *If Assumption 5.3 and the Popov condition* (5.46) *are satisfied by $G(s) \in \mathcal{G}$, then the nominal system in Figure 5.5 is BIBO stable for any $K \in [0,\ k_{\max})$.*

The corollary implies that if $\mathbf{p}$ is a stability point of the nonlinear system and if $0 \leq K < k_{\max}$, then $\tilde{\mathbf{p}}$ is a stability point of the nominal linear system.

5.5 Numerical Example

Consider the heat-conduction process considered in Chapter 2. Suppose now that the plant possesses uncertainties and thus its transfer function is described by

$$G_p(s) = \frac{a}{\sqrt{\lambda s}\,\sinh\sqrt{\lambda s}}, \quad a \in [18,\ 21], \quad \lambda \in [0.9,\ 1.1]. \tag{5.48}$$

Note that $G_p(s)$ has one pole at the origin and the others on the negative real axis (see, for example, [35]). Moreover, the impulse response g_p is given by

$$g_p(t) = a + \frac{2a}{\lambda}\sum_{n=1}^{\infty}(-1)^n \mathrm{e}^{-\frac{n^2\pi^2 t}{\lambda}}, \quad t > 0. \tag{5.49}$$

Now assume that the nonlinearity ψ is described in Figure 5.6, where $z_0 = 0.2$, $k_1 = 0.2$ and $k_2 = 1$. Obviously, $\psi \in \text{sector}\,[k_1,\ k_2]$.

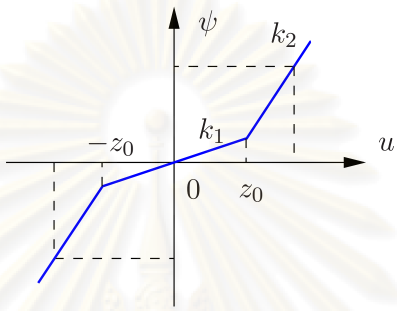


Figure 5.6: The input nonlinearity of the heat-conduction process.

Assume that the control objective is to keep the error e and the control input u staying within $\pm E_{\max}$ and $\pm U_{\max}$, respectively, for all time and for all inputs belonging to a possible set $\mathcal{P}$ given by (3.1) where

$$E_{\max} = 6, \quad U_{\max} = 5, \quad M = 50 \quad \text{and} \quad D = 25. \tag{5.50}$$

Accordingly, the design problem is to determine the controller transfer function $G_c(s)$ so that the following criteria are fulfilled:

$$\begin{aligned} \phi_0(\mathbf{p}) &\le -0.1, \\ \phi_1(\tilde{\mathbf{p}}) &\le 6, \\ \phi_2(\tilde{\mathbf{p}}) &\le 5. \end{aligned} \tag{5.51}$$

To this end, assume that the controller transfer function takes the form

$$G_c(s, \mathbf{p}) = \frac{p_1(s^2 + p_2 s + p_3)}{s^2 + p_4 s + p_5}, \tag{5.52}$$

where $\mathbf{p} = [p_1, p_2, p_3, p_4, p_5]^T \in \mathbb{R}^5$. The nominal model $G_p^*(s)$ of the plant is chosen with

$$a = 20, \quad \lambda = 1. \tag{5.53}$$

Since $G_c(0) \neq 0$ and $G_p^*(0) = \infty$, it follows that

$$\sigma_1 = 0, \quad \sigma_2 = 0. \tag{5.54}$$

Thus,

$$\tilde{\mu}_1 = B\|e^*(\mathbf{1})\|_1, \quad \tilde{\mu}_2 = B\|u^*(\mathbf{1})\|_1. \tag{5.55}$$

Note that the computation of B involves an extensive numerical search in $\mathbb{R}^2$. However, B is required to be computed only once. The search reveals that

$$B = 4.1032 \quad \text{at } a = 21 \text{ and } \lambda = 0.9. \tag{5.56}$$

In this example, inequalities (5.51) are solved by using the MBP algorithm. A design solution

$$\tilde{\mathbf{p}} = [7.2442,\ 28.7856,\ 200.4369,\ 142.0602,\ 3785.7,\ 0.9730]^T \tag{5.57}$$

is located and the corresponding performance measures are

$$\begin{aligned} \phi_0(\mathbf{p}) &= -2.03, \\ \phi_1(\tilde{\mathbf{p}}) &= 5.86, \\ \phi_2(\tilde{\mathbf{p}}) &= 4.74. \end{aligned} \tag{5.58}$$

Figure 5.7 shows the Popov plots of the systems with various pairs of a and λ.

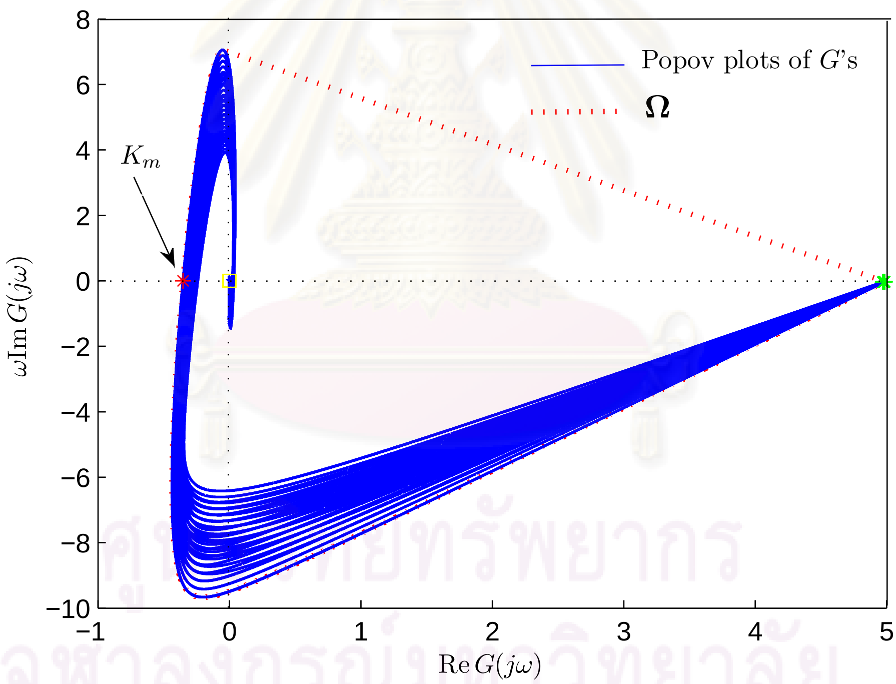


Figure 5.7: Popov plots of $G(s)$ with $a = 18 : 0.5 : 21$ and $\lambda = 0.9 : 0.05 : 1.1$.

To verify the design, the simulation is carried out with the nonlinear system subject to a test input $\hat{f}$, which is generated such that its magnitude and slope are bounded by M and D, respectively. See Figure 5.8. The responses of the system are given in Figure 5.9, clearly illustrating the usefulness of the method.

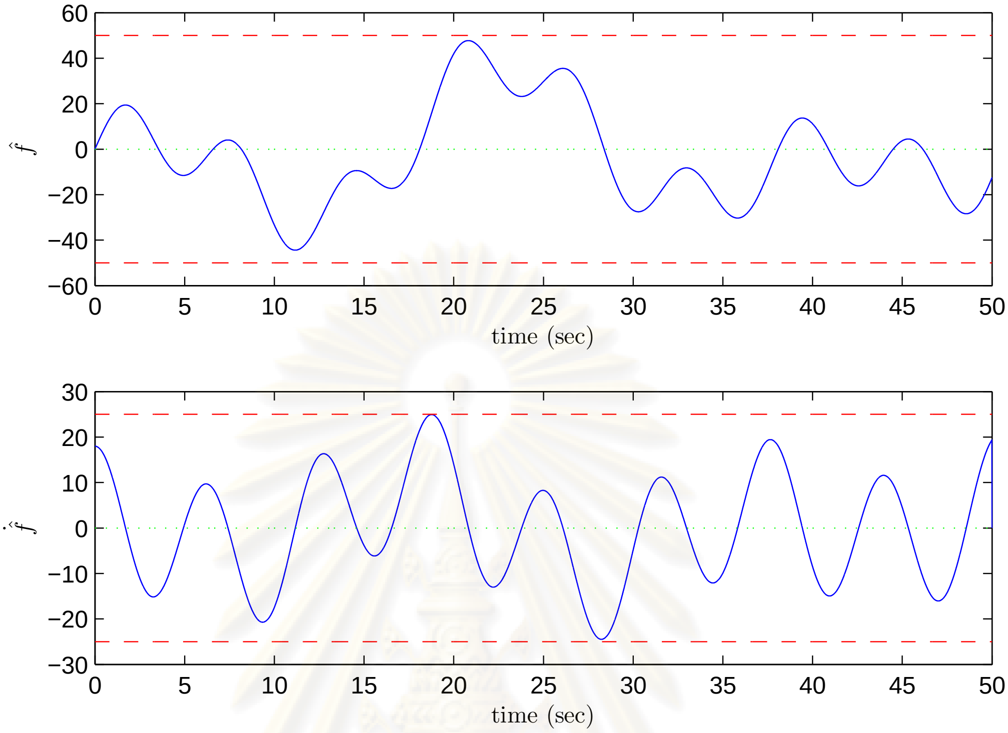


Figure 5.8: A test input $\hat{f} \in \mathcal{P}$ characterized by (5.50).

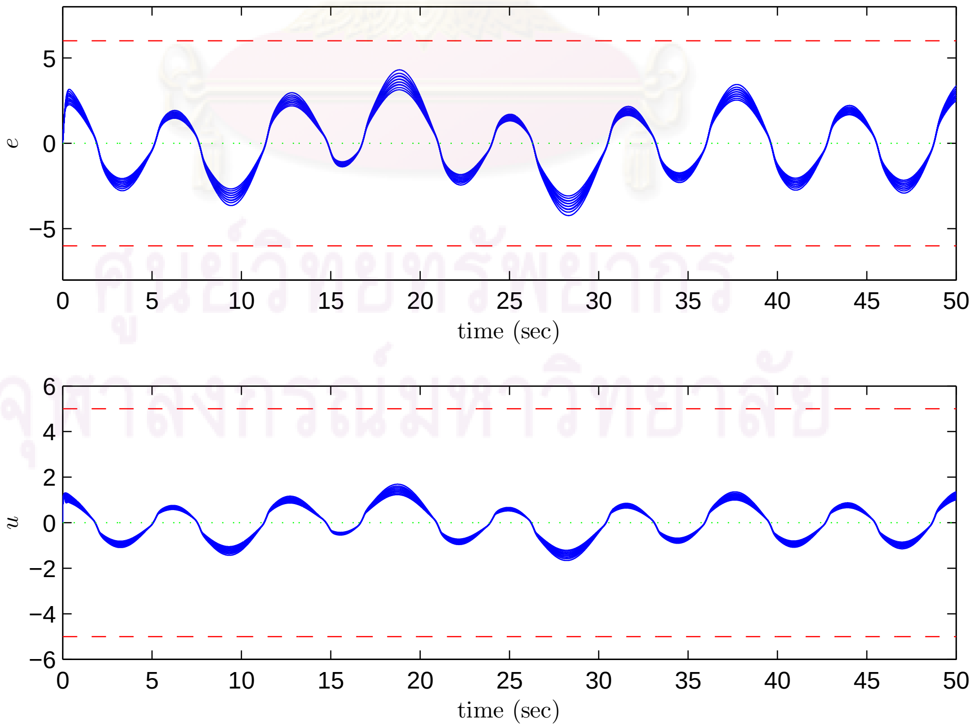


Figure 5.9: Responses e and u of the nonlinear system to input $\hat{f}$ with $a = 18, 19, 20, 21$ and $\lambda = 0.9, 1.0, 1.1$.

5.6 Conclusions and discussion

In this chapter, we have developed a practical method for designing the feedback control system shown in Figure 5.1. Following Zakian's principle of matching, the control objective is to ensure that e and u stay within the specified ranges $\pm E_{\max}$ and $\pm U_{\max}$, respectively, for all time and for all inputs $f \in \mathcal{P}$ in the presence of uncertainties. The results of this chapter are derived from those in Chapter 3 and from the extension of Zakian's majorants, which provides a useful tool to deal with vague systems. The effectiveness of the developed method is illustrated by the design example of an uncertain heat-conduction process.

It should be noted that the design method can also be applied to the case of uncertain systems with an output nonlinearity. This can be seen as an adjunct to the results presented in Chapter 4.

CHAPTER VI

CONCLUSIONS

In this chapter, we summarize our contributions and discuss some directions for future research.

6.1 Contributions

The contributions of this thesis are as follows.

First, we develop a practical approach for stabilizing Lur'e systems with inputs in the possible set $\mathcal{P}_2$ or $\mathcal{P}_\infty$. Specifically, input-output stability properties of Lur'e systems, in which the linear subsystem is allowed to be a nonrational transfer function belonging to a subclass of $\mathcal{A}$, are revisited by extending some results in [30] and [32]. It is shown that if the magnitudes of inputs and their slopes are bounded in the sense of two norms or infinity norms, then the well-known Popov criterion can be used to ensure the boundedness of the system outputs for any nonlinearity lying in a given sector bound. Based on the obtained results, we develop the Popov test and devise a practical condition for obtaining stability points of the system that is readily soluble by numerical methods.

Second, in connection with Zakian's framework, this thesis also develops a practical method for designing nonlinear feedback systems where the plant is possibly uncertain and consists of a linear time-invariant subsystem and a nonlinearity in its input or output channels, so as to ensure that the error function and the controller output stay within respective bounds for all time, for all possible inputs and in the presence of plant uncertainties. This is considered as the most significant contribution of the work in this thesis. A unified and systematic methodology has been introduced. In particular, the design procedure is as follows.

1. Replacing the nonlinearity ψ with constant gain K and equivalent disturbance ϕ yields the equivalent linear systems as shown in Figures 3.4 and 4.3.

2. Using the Schauder fixed point theorem to prove that the design problem of the nonliear system can be repalced by that of the equivalent linear system (see Theorems 3.2 and 4.1).

3. Replacing the set $\mathcal{D}_w$ in (3.19) and (4.16) by a tractable set $\mathcal{D}$ defined in (3.22) and (4.19) results in the nominal linear system used during the design.

4. Using the linearity of the nominal system, sufficient conditions for the satisfaction of the design criteria of the original nonlinear system can be obtained. If the plant transfer function $G_p(s)$ is known, these conditions are also surrogate design criteria (see (3.29) and (4.23)).

5. If the plant is vague, surrogate design criteria (5.43) is obtained by using the extension of Zakian's majorants (see Section 5.2).

6. Surrogate design criteria (3.29), (4.23), and (5.43) are in accordance with the method of inequalities and suitable for solutions by numerical methods.

It is shown clearly from the stability results and the design procedure that the developed method can be used for the design of a large class of nonlinear, possibly uncertain and possibly infinite-dimensional systems. This can be seen as a considerable advantage of the research work presented in this thesis.

The effectiveness of the method has been demonstrated through some numerical design examples. In addition, when the system is critical, the value of the method becomes evident.

6.2 Future Works

Possible extensions of this thesis are as follows.

First, the design method can also be applied to the case of systems with inputs in the set $\mathcal{P}$ given by

$$\mathcal{P} \triangleq \{f \in L_2 \mid \|f\|_2 \leq M_2, \|\dot{f}\|_2 \leq D_2\}, \tag{6.1}$$

where M_2 and D_2 are the given bounds. This is because the method used to compute the peak outputs of linear systems [14] is applicable to many possible sets of which $\mathcal{P}$ described in (3.1) or in (6.1) is only a special case. In this connection, the possible set $\mathcal{P}$ can be characterized with many (two or more than two) bounding conditions on the two- and/or infinity-norms of the inputs and their slopes. See [14] for further details. Moreover, it can be seen obviously from the results in Chapter 2 that the stability conditions with respect to the set $\mathcal{P}_\infty$ are stricter than those with respect to $\mathcal{P}_2$. Therefore, for the set $\mathcal{P}$ given in (6.1), the design method can be used in a straightforward manner.

Second, from the decomposition technique (see Chapters 3, 4) and stability conditions in Chapter 2, it follows that the nonlinearity ψ in the system is allowed to be uncertain as long as it lies in a given sector bound and the bound N on the magnitude of the equivalent disturbance can be obtained. Consider, for example, the nonlinearity $\psi : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$ is a continuous function of its first argument and is parameterized by $\theta \in \mathbf{\Theta}$ with

$$\mathbf{\Theta} \triangleq \{\theta \in \mathbb{R}^m \mid \theta_i \in [\theta_{i\,\min}, \theta_{i\,\max}], \ \forall i = 1, \ldots, m\}, \tag{6.2}$$

where the values $\theta_{i\,\min}, \theta_{i\,\max}$ are given. In this case, the decomposition of the nonlinearity is as in Figure 6.1.

Assume that the condition that $\|u\|_\infty \leq U_{\max}$ is part of the design. Then,

$$\phi(u, \theta) \leq N, \tag{6.3}$$

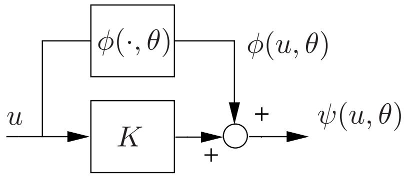


Figure 6.1: Decomposition of the nonlinearity $\psi(\cdot,\theta)$.

where

$$N \triangleq \sup_{\theta\in\boldsymbol{\Theta}} \sup_{|x|\le U_{\max}} |\psi(x,\theta) - Kx| . \tag{6.4}$$

Note that, in general, this optimization problem is not easy to solve. However, if the number of parameters m is small and ψ has a simple structure, then the bound N is obtainable. Furthermore, N needs to be computed only once before the design process, provided that K is fixed.

REFERENCES

[1] V. Zakian. Perspectives of the principle of matching and the method of inequalities. *International Journal of Control* 65 (1996): 147–175.

[2] V. Zakian, ed.. *Control Systems Design: A New Framework*. London: Springer-Verlag, 2005.

[3] V. Zakian and U. Al-Naib. Design of dynamical and control systems by the method of inequalities. *Proceedings of the Institution of Electrical Engineers* 120 (1973): 1421–1427.

[4] V. Zakian. Well matched systems. *IMA Journal of Mathematical Control and Information* 8 (1991): 29–38.

[5] O. Taiwo. The design of robust control systems for plants with recycle. *International Journal of Control* 43 (1986): 671–678.

[6] P. G. Lane. *Design of Control Systems with Inputs and Outputs Satisfying Certain Bounding Conditions*. Doctoral dissertation, University of Manchester Institute of Science and Technology, Manchester, 1992.

[7] P. G. Lane. The principle of matching: a necessary and sufficient condition for inputs restricted in magnitude and rate of change. *International Journal of Control* 62 (1995): 893–915.

[8] J. F. Whidborne. EMS control system design for a maglev vehicle a critical system. *Automatica* 29 (1993): 1345–1349.

[9] W. Khaisongkram, D. Banjerdpongchai, and S. Arunsawatwong. Controller design for a binary distillation column under disturbances with bounds on magnitudes and derivatives using Zakian's framework. In *Proceedings of the 5th Asian Control Conference*, pp. 1676–1684, Melbourne, Australia, 2004.

[10] S. Arunsawatwong. Critical control of building under seismic disturbance. In V. Zakian (ed.), *Control Systems Design: A New Framework*, pp. 339–353. London: Springer-Verlag, 2005.

[11] S. Arunsawatwong and V. Q. Nguyen. Design of retarded delay differential systems using the method of inequalities. *International Journal of Automation and Computing* 6 (2009): 22–28.

[12] T. Ono. Design of a hard disk drive. In V. Zakian (ed.), *Control Systems Design: A New Framework*, pp. 355–367. London: Springer-Verlag, 2005.

[13] T. Satoh. Matching to environment generating persistent disturbances. In V. Zakian (ed.), *Control Systems Design: A New Framework*, pp. 339–353. London: Springer-Verlag, 2005.

[14] W. Silpsrikul and S. Arunsawatwong. Computation of peak output for inputs satisfying many bounding conditions on magnitude and slope. *International Journal of Control* 83 (2010): 49–65.

[15] V. S. Mai, S. Arunsawatwong, and E. H. Abed. Design of nonlinear feedback systems with inputs and outputs satisfying bounding conditions. In *Proceedings of IEEE Multi-Conference on Systems and Control*, pp. 2017–2022, Pacifico Yokohoma, Japan, 2010.

[16] V. S. Mai, S. Arunsawatwong, and E. H. Abed. Design of feedback systems with output nonlinearity and with inputs and outputs satisfying bounding conditions. In *Proceedings of SICE Annual Conference*, pp. 1586–1591, Taipei, Taiwan, 2010.

[17] B. J. Birch and R. Jackson. The behaviour of linear systems with inputs satisfying certain bounding conditions. *Journal of Electronics and Control* 6 (1959): 366–375.

[18] V. Zakian. New formulation for the method of inequalities. *Proceedings of the Institution of Electrical Engineers* 126 (1979): 579–584. (reprinted in *Systems and Control Encyclopedia* 5 (1987): 3206–3215).

[19] V. Zakian. Critical systems and tolerable inputs. *International Journal of Control* 49 (1989): 1285–1289.

[20] J. F. Whidborne and G. P. Liu. *Critical Control Systems: Theory, Design and Applications*. Taunton: Research Studies Press, 1993.

[21] H. K. Khalil. *Nonlinear Systems*, 3rd Edition. Upper Saddle River, New Jersey: Pearson Prentice-Hall, 2000.

[22] V. Zakian. A criterion of approximation for the method of inequalities. *International Journal of Control* 37 (1983): 1103–1111.

[23] S. Arunsawatwong. Private communication, 2009.

[24] S. Oldak, C. Baril, and P. O. Gutman. Quantitative design of a class of nonlinear systems with parameter uncertainty. *International Journal of Robust Nonlinear Control* 4 (1994): 101–117.

[25] S. M. Shahruz. Performance enhancement of a class of nonlinear systems by disturbance observers. *IEEE/ASME Transactions on Mechatronics* 5 (2000): 319–323.

[26] E. Zeidler. *Nonlinear Functional Analysis and Its Applications, I - Fixed-Point Theorems*. New York: Springer Verlag, 1986.

[27] A. Baños and I. M. Horowitz. Nonlinear quantitative stability. *International Journal of Robust Nonlinear Control* 65 (2004): 289–306.

[28] C. A. Desoer and M. Vidyasagar. *Feedback Systems: Input-Output Properties*. New York: Academic Press, 1975.

[29] M. Vidyasagar. *Nonlinear Systems Analysis*. 2nd Edition. Englewood Cliffs, New Jersey: Prentice-Hall, 1993.

[30] C. A. Desoer. A generalization of the Popov criterion. *IEEE Transactions on Automatic Control* 10 (1965): 182–185.

[31] M. A. Aizerman and F. R. Gantmacher. *Absolute Stability of Regulator System*. San Francisco: Holden-Day, 1964.

[32] A. R. Bergen, R. P. Iwens, and A. J. Rault. On input-output stability of nonlinear feedback systems. *IEEE Transactions on Automatic Control* 11 (1966): 742–744.

[33] C. B. Barber, D. P. Dobkin, and H. T. Huhdanpaa. The quickhull algorithm for convex hulls. *ACM Transactions on Mathematical Software* 22 (1996): 469–483.

[34] R. J. Schwarz and B. Friedland. *Linear Systems*. New York: McGraw-Hill, 1965.

[35] R. F. Curtain and H. J. Zwart. *An Introduction to Infinite-Dimensional Linear Systems Theory*. New York: Springer-Verlag, 1995.

[36] V. Zakian. Design formulations. *International Journal of Control* 46 (1987): 403–408.

[37] O. Yaniv. *Quantitative Feedback Design of Linear and Nonlinear Control Systems*. London: Kluwer Academic Publishers, 1999.

[38] E. J. McShane. *Integration*. Princeton, N. J.: Princeton University Press, 1944.

[39] E. Kreyszig. *Introductory Functional Analysis with Applications*. New York: John Wiley and Son, 1978.

[40] W. Reinelt. Maximum output amplitude of linear systems for certain input constraints. In *Proceedings of the 39th IEEE Conference on Decision and Control*, pp. 1075–1080, Sydney, Australia, 2000.

[41] V. Q. Nguyen and S. Arunsawatwong. Stability and stabilization of retarded fractional delay differential systems. In *Proceedings of IFAC World Congress*, pp. 3928–3933, Seoul, Korea, 2008.

[42] C. A. Desoer. A general formulation of the Nyquist criterion. *IEEE Transactions on Circuit Theory* 12 (1965): 230–234.

[43] C. A. Desoer and M. Y. Wu. Stability of linear time-invariant systems. *IEEE Transactions on Automatic Control* 15 (1968): 245–250.

[44] F. M. Callier and C. A. Desoer. A graphical test for checking the stability of a linear time-invariant feedback system. *IEEE Transactions on Automatic Control* 17 (1972): 773–780.

[45] N. Niksefat and N. Sepehri. Designing robust force control of hydraulic actuators despite system and environmental uncertainties. *IEEE Control Systems Magazine* 21 (2001): 66–77.

[46] V. Zakian. A framework for design: theory of majorants. *Control Systems Centre Report* 604, UMIST, Manchester, England, 1984.

[47] V. S. Mai, S. Arunsawatwong, and E. H. Abed. Input-output stability of Lur'e systems with inputs satisfying bounding conditions on magnitude and slope. In *Proceedings of the 7th ECTI Conference*, pp. 198–202, Chiang Mai, Thailand, 2010.

APPENDICES

APPENDIX A

Lemma A.1. *In the proof of Theorem 2.1, q can be assumed to be nonnegative.*

Proof. Assume that Theorem 2.1 has been proved for $q \geq 0$. Now let condition (2.5) be satisfied by the given $G(s)$, k, for some $\beta > 0$ and $q < 0$. Note that the given system is equivalent to the one shown in Figure A.1.

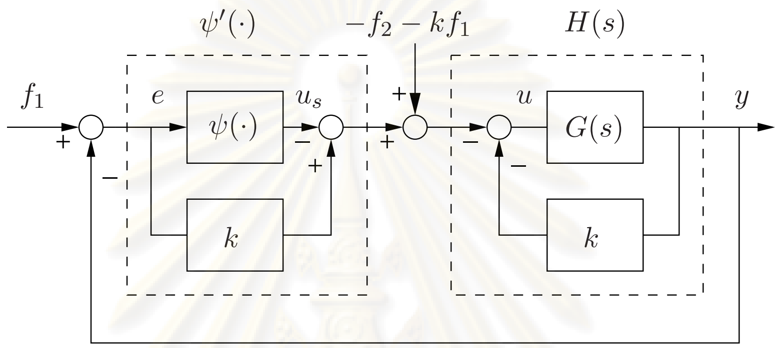


Figure A.1: Equivalent Lur'e system.

Define

$$\begin{aligned} \psi'(e) &\triangleq ke - \psi(e) \\ H(s) &\triangleq -\frac{G(s)}{1+kG(s)} \end{aligned} \quad . \tag{A.1}$$

First, it is easy to see that $f_2' \triangleq -f_2 - kf_1 \in \mathcal{P}_2$ and that $\psi \in \operatorname{sector}[0,\, k]$ implies $\psi' \in \operatorname{sector}[0,\, k]$.

Second, we show that the impulse response h of $H(s)$ satisfies the same conditions imposed on g. Note that the Popov condition (2.5) implies that the Nyquist plot of $G(s)$ does not encircle or go through the point $(-1/k,\, 0)$. Therefore, it follows from the results in [42], (also in [28, p. 85] and [29, p. 311]) that h belongs to $\mathcal{A}$. Moreover,

$$\begin{aligned} h &= -g - k(h * g) \\ \dot{h} &= -\dot{g} - kg(0)h - k(h * \dot{g}). \end{aligned} \tag{A.2}$$

Since each term of the right-hand side of the last equation is in $\mathcal{A}$, hence so is $\dot{h}$.

Finally, for all ω, we have

$$\begin{aligned}
\mathrm{Re}\left[(1-qj\omega)H(j\omega)+\frac{1}{k}\right] &= \mathrm{Re}\left[(-1+qj\omega)\frac{G(j\omega)}{1+kG(j\omega)}+\frac{1}{k}\right] = \mathrm{Re}\,\frac{qj\omega G(j\omega)+\frac{1}{k}}{1+kG(j\omega)} \\
&= \frac{1}{|1+kG(j\omega)|^2}\mathrm{Re}\left[\left(qj\omega G(j\omega)+\frac{1}{k}\right)\left(1+k\bar{G}(j\omega)\right)\right] \\
&= \frac{1}{|1+kG(j\omega)|^2}\mathrm{Re}\left[\bar{G}(j\omega)+qj\omega|G(j\omega)|^2+qj\omega G(j\omega)+\frac{1}{k}\right] \\
&= \frac{1}{|1+kG(j\omega)|^2}\mathrm{Re}\left[(1+qj\omega)G(j\omega)+\frac{1}{k}\right] \\
&\geq \frac{\beta}{|1+kG(j\omega)|^2} \triangleq \beta_1 > 0.
\end{aligned} \tag{A.3}$$

From the above and note that $-q > 0$, it readily follows that Theorem 2.1 is applicable to the equivalent system in Figure A.1. That is to say, the system responses are in L_∞, implying that the responses e, u_s, u and y in the system in Figure 2.1 are also in L_∞. This completes the proof of the lemma. □

Lemma A.2. *In the proof of Theorem 2.2, q can be assumed to be nonnegative.*

Proof. By using the same method in Lemma A.1, the proof is completed by showing that there exists a number $\gamma > 0$ such that

$$\int_0^\infty \mathrm{e}^{2\gamma t}h^2(t)dt < \infty. \tag{A.4}$$

First we show that if g, $\dot{g} \in \mathcal{A}$ and if there exists a $\alpha > 0$ such that (2.19) is satisfied, then g decays to zero exponentially. Since $g \in \mathcal{A}$ and $\dot{g} \in \mathcal{A}$, it follows that g does not contain any impulse, $\lim_{t\to\infty} g(t) = 0$ and $\int_0^\infty |\dot{g}(t)|\, dt < \infty$. Thus,

$$|g(t)-g(0)| = \left|\int_0^t \dot{g}(t)dt\right| \leq \int_0^t |\dot{g}(t)|\, dt < \infty. \tag{A.5}$$

It is easy to see that $|g(0)| < \infty$. Then (A.5) implies that $|g(t)| < \infty,\ \forall t > 0$. Furthermore, for any $\gamma \in (0,\, \alpha)$, by using Cauchy–Schwarz inequality, we have

$$\begin{aligned}
\int_0^\infty |\mathrm{e}^{\gamma t}g(t)|dt &= \int_0^\infty |\mathrm{e}^{\alpha t}g(t)|\, |\mathrm{e}^{(\gamma-\alpha)t}|dt \\
&\leq \left\{\int_0^\infty |\mathrm{e}^{\alpha t}g(t)|^2dt \int_0^\infty \mathrm{e}^{-2(\alpha-\gamma)t}dt\right\}^{1/2} \\
&= \left\{\frac{1}{2(\alpha-\gamma)}\int_0^\infty \mathrm{e}^{2\alpha t}g^2(t)dt\right\}^{1/2}.
\end{aligned} \tag{A.6}$$

It follows from (2.19) and (A.6) that

$$\int_0^\infty |\mathrm{e}^{\gamma t}g(t)|dt < \infty. \tag{A.7}$$

As a result, $\mathrm{e}^{\gamma t}g(t) \to 0$ as $t \to \infty$. Since g is bounded and $\mathrm{e}^{\gamma t}$ is a continuous function, it follows that $|\mathrm{e}^{\gamma t}g(t)|$ is finite for any $t \in [0, \infty]$. That is to say, there exists a number $C > 0$ such that

$$|\mathrm{e}^{\gamma t}g(t)| \leq C, \quad \forall t \geq 0 \tag{A.8}$$

$$\Rightarrow |g(t)| \leq C\mathrm{e}^{-\gamma t}, \quad \forall t \geq 0. \tag{A.9}$$

Second, we prove that $|\mathrm{e}^{\gamma t}h(t)|$ is also finite for any $t \in [0, \infty]$. From (A.1) we have

$$-h = g + kg * h. \tag{A.10}$$

By defining

$$\begin{aligned} h_\gamma(t) &\triangleq \mathrm{e}^{\gamma t}h(t) \\ g_\gamma(t) &\triangleq \mathrm{e}^{\gamma t}g(t) \end{aligned}, \tag{A.11}$$

we arrive at

$$-h_\gamma = g_\gamma + kg_\gamma * h_\gamma. \tag{A.12}$$

Thus,

$$\begin{aligned} |h_\gamma(t)| &\leq |g_\gamma(t)| + k\left|\int_0^t g_\gamma(t-\tau)h_\gamma(\tau)d\tau\right| \\ &\leq C + k\int_0^t |g_\gamma(t-\tau)|\,|h_\gamma(\tau)|\,d\tau, \quad \forall t > 0. \end{aligned} \tag{A.13}$$

By applying Bellman-Gronwall lemma (see, for example, [28]) for the above inequality and using (A.7), we have

$$|h_\gamma(t)| \leq C\,\mathrm{e}^{k\int_0^t |g_\gamma(t-\tau)|d\tau} < C\,\mathrm{e}^{k\,\|g_\gamma\|_1} < \infty. \tag{A.14}$$

That is to say, $|\mathrm{e}^{\gamma t}h(t)|$ is finite for all $t \geq 0$. Hence, it is easy to see that h decays to zero exponentially and satisfies condition (A.4). □

Lemma A.3. [28] *If $f : \mathbb{R}_+ \to \mathbb{R}$ satisfies the condition that $f, \dot{f} \in L_2$, then $f \in L_\infty$.*

Proof. Let $v \triangleq f\dot{f}$. Using the Hölder's inequality gives

$$\int_0^\infty |v(t)|\,dt = \int_0^\infty \left|f(t)\dot{f}(t)\right| dt \leq \|f\|_2\,\|\dot{f}\|_2. \tag{A.15}$$

Since $f, \dot{f} \in L_2$, $\|f\|_2$ and $\|\dot{f}\|_2$ are finite. Thus, $v \in L_1$.
Moreover, $f \in L_2$ implies that $f^2 \to 0$ as $t \to \infty$. Accordingly,

$$f_\infty \triangleq \lim_{t\to\infty} f(t) = 0, \tag{A.16}$$

$$\frac{1}{2}\|v\|_1 \geq \frac{1}{2}\left|\int_0^\infty f(t)\dot{f}(t)dt\right| = \left|f_\infty^2 - f^2(0)\right|. \tag{A.17}$$

Hence, $f^2(0) < \infty$. As a result, for any $t > 0$

$$\frac{1}{2}\int_0^t \left|f(\tau)\dot{f}(\tau)\right| d\tau \geq \frac{1}{2}\left|\int_0^t f(\tau)\dot{f}(\tau)d\tau\right| = \left|f^2(t) - f^2(0)\right|. \tag{A.18}$$

Therefore,

$$f^2(t) < \infty \qquad \forall t \geq 0. \tag{A.19}$$

That is to say, $f \in L_\infty$. Moreover, $f(t) \to 0$ as $t \to \infty$. □

Lemma A.4. [31] *If the three real functions $f_1(t), f_2(t)$ and $f_3(t)$ belong to L_2 and if their Fourier transforms are related by the equation*

$$F_1(j\omega) = F_2(j\omega) + H(j\omega)F_3(j\omega), \tag{A.20}$$

where $\operatorname{Re} H(j\omega) \geq \delta > 0,\ \forall\omega \geq 0$*, then*

$$\int_0^\infty f_1(t)f_3(t)dt + \frac{1}{4\delta}\int_0^\infty f_2^2(t)dt \geq 0. \tag{A.21}$$

Proof. Let $I = \displaystyle\int_0^\infty f_1(t)f_3(t)dt$. Using Parseval formula yields

$$\begin{aligned}
I &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \bar{F}_1(j\omega)F_3(j\omega)d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left[\bar{F}_2(j\omega) + \bar{H}(j\omega)\bar{F}_3(j\omega)\right]F_3(j\omega)d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left[\text{Re}H(j\omega)\left|F_3(j\omega)\right|^2 + \frac{1}{2}\bar{F}_2(j\omega)F_3(j\omega) + \frac{1}{2}F_2(j\omega)\bar{F}_3(j\omega)\right]d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left|[\text{Re}H(j\omega)]^{1/2}F_3(j\omega) + \frac{F_2(j\omega)}{2\left[\text{Re}H(j\omega)\right]^{1/2}}\right|^2 d\omega - \frac{1}{8\pi}\int_{-\infty}^{+\infty}\frac{|F_2(j\omega)|^2}{\text{Re}H(j\omega)}d\omega \\
&\geq -\frac{1}{8\pi}\int_{-\infty}^{+\infty}\frac{|F_2(j\omega)|^2}{\text{Re}H(j\omega)}d\omega \\
&\geq -\frac{1}{4\delta}\int_0^\infty f_2^2(t)dt.
\end{aligned} \tag{A.22}$$

This completes the proof of Lemma A.4. □

Lemma A.5. [32] *Consider the system from the input f to the output y_s in Figure 2.3. Let Assumptions 2.1 and 2.2 be satisfied. If $f \in \mathcal{P}_\infty$, g, $\dot{g} \in \mathcal{A}$ and if there exist $q \in \mathbb{R}$, $\beta \in \mathbb{R}$ and sufficiently small $\alpha > 0$ such that* (2.5) *and* (2.19) *are satisfied, then the following inequality holds*

$$\int_0^t \mathrm{e}^{2\alpha\tau}u_s^2(\tau)d\tau \leq \int_0^t \frac{\mathrm{e}^{2\alpha\tau}}{\beta^2}\left[f(\tau) + q\dot{f}(\tau)\right]^2 d\tau + \frac{2q}{\beta}\int_0^{e(0)}\psi(e)de, \quad \forall t \geq 0. \tag{A.23}$$

Proof. From equation (2.20), we have

$$\dot{e}(t) = \dot{f}(t) - \int_0^t \dot{g}(t-\tau)u_s(\tau)d\tau - g(0)u_s(t). \tag{A.24}$$

Let $f_T, \dot{f}_T$ and u_{sT} be the truncations at T of $f, \dot{f}$ and u_s respectively. Then

$$e_T(t) = f_T(t) - \int_0^t g(t-\tau)u_{sT}(\tau)d\tau \tag{A.25}$$

$$\dot{e}_T(t) = \dot{f}_T(t) - \int_0^t \dot{g}(t-\tau)u_{sT}(\tau)d\tau - g(0)u_{sT}(t). \tag{A.26}$$

It turns out that

$$\begin{aligned} -e_T(t) - q\dot{e}_T(t) &= -\left[f_T(t) + q\dot{f}_T(t)\right] + g(0)u_{sT}(t) \\ &\quad + \int_0^t \left[g(t-\tau) + q\dot{g}(t-\tau)\right] u_{sT}(\tau)d\tau. \end{aligned} \tag{A.27}$$

By adding $(1/k - \sigma)u_{sT}(t)$ with $\sigma \in (0, \beta)$, to both sides and multiplying by $\mathrm{e}^{\alpha t}$ with sufficiently small $\alpha > 0$, we obtain

$$\begin{aligned} f_1(t) &= f_2(t) + \left(\frac{1}{k} - \sigma\right)\mathrm{e}^{\alpha t}u_{sT}(t) + qg(0)\mathrm{e}^{\alpha t}u_{sT}(t) \\ &\quad + \int_0^t \mathrm{e}^{\alpha(t-\tau)}\left[g(t-\tau) + q\dot{g}(t-\tau)\right]\mathrm{e}^{\alpha\tau}u_{sT}(\tau)d\tau \end{aligned} \tag{A.28}$$

where

$$\begin{aligned} f_1(t) &\triangleq \left[-e_T(t) - q\dot{e}_T(t) + \left(\frac{1}{k} - \sigma\right)u_{sT}(t)\right]\mathrm{e}^{\alpha t} \\ f_2(t) &\triangleq -\left[f_T(t) + q\dot{f}_T(t)\right]\mathrm{e}^{\alpha t} \end{aligned}. \tag{A.29}$$

Note that all terms in (A.28) belong to L_2 due to the truncation at T. Thus, Fourier-transforming this equation yields

$$F_1(j\omega) = F_2(j\omega) + \left\{\left[1 + q(j\omega - \alpha)\right]G(j\omega - \alpha) + \frac{1}{k} - \sigma\right\}U_{sT}(j\omega - \alpha). \tag{A.30}$$

From (2.5), there always exists a sufficiently small $\alpha > 0$ such that

$$\mathrm{Re}\left\{\left[1 + q(j\omega - \alpha)\right]G(j\omega - \alpha)\right\} + \frac{1}{k} - \sigma \geq \beta - \sigma > 0. \tag{A.31}$$

Hence, in view of Lemma A.4 with $\delta = \beta - \sigma$, one obtains

$$-\int_0^\infty f_1(t)u_{sT}(t)\mathrm{e}^{\alpha t}dt \leq \frac{1}{4\delta}\int_0^\infty f_2^2(t)dt. \tag{A.32}$$

Let J denote the left-hand side of (A.32). It follows from (A.29) that

$$\begin{aligned} J &= \int_0^T \left(e - \frac{u_s}{k}\right)u_s\mathrm{e}^{2\alpha t}dt + q\int_0^T \dot{e}u_s\mathrm{e}^{2\alpha t}dt + \sigma\int_0^T u_s^2\mathrm{e}^{2\alpha t}dt \\ &= \int_0^T \left[e - \frac{\psi(e)}{k}\right]\psi(e)\mathrm{e}^{2\alpha t}dt + \sigma\int_0^T u_s^2(t)\mathrm{e}^{2\alpha t}dt \\ &\quad -2q\alpha\int_0^T \mathrm{e}^{2\alpha t}\left[\int_0^{e(t)}\psi(e)de\right]dt + q\mathrm{e}^{2\alpha T}\int_0^{e(T)}\psi(e)de - q\int_0^{e(0)}\psi(e)de. \end{aligned} \tag{A.33}$$

Since $\psi(e) \in \text{sector}\,[\epsilon,\ k-\epsilon]$, $\epsilon > 0$ arbitrarily small, we have

$$\begin{aligned} \int_0^{e(t)} \psi(e)de &\le \frac{k}{2}e^2(t) \\ \left[e - \frac{\psi(e)}{k}\right]\psi(e) &\ge \frac{\epsilon^2}{k}e^2(t). \end{aligned} \tag{A.34}$$

Accordingly,

$$\begin{aligned} \frac{J}{\sigma} \quad &\ge \frac{1}{\sigma}\int_0^T \left(\frac{\epsilon^2}{k} - kq\alpha\right) e^2 \mathrm{e}^{2\alpha t}dt \\ &+ \int_0^T u_s^2(t)\mathrm{e}^{2\alpha t}dt - \frac{q}{\sigma}\int_0^{e(0)} \psi(e)de. \end{aligned} \tag{A.35}$$

For any $\epsilon > 0$, $q < \infty$, $k < \infty$, there always exists a sufficiently small $\alpha > 0$ such that $\epsilon^2 - k^2 q\alpha \ge 0$. Hence

$$\frac{J}{\sigma} \ge \int_0^T u_s^2(t)\mathrm{e}^{2\alpha t}dt - \frac{q}{\sigma}\int_0^{e(0)} \psi(e)de, \quad \forall\, T \ge 0. \tag{A.36}$$

On the other hand,

$$\frac{J}{\sigma} \le \frac{1}{4\sigma(\beta - \sigma)}\int_0^T \left(f + q\dot{f}\right)^2 \mathrm{e}^{2\alpha t}dt, \quad \forall\, T \ge 0. \tag{A.37}$$

Since $\sigma \in (0,\, \beta)$ is arbitrary, $\sigma = \beta/2$ minimizes the right-hand side of (A.37). Substituting this value of σ and comparing the right-hand side terms of (A.36) and (A.37) will complete the proof. $\square$

APPENDIX B

Proposition B.1. *Let $P \subset \mathbb{R}^2$ be a bounded set. Define Ω as the convex hull of P. Let L be a straight line in $\mathbb{R}^2$. Then, it follows that P lies to the right of L if and only if so does Ω.*

Proof. ($\Leftarrow$) This can be seen easily by noting that $P \subset \Omega$.

($\Rightarrow$) We prove by contradiction. Assume that the line L lies to the left of P but not Ω. It is easy to see that L intersects Ω. Let L be represented by

$$L \triangleq \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 \mid ax_1 + bx_2 = c\},$$

where a, b and c are real numbers. Accordingly, Ω can be decomposed as

$$\Omega = \Omega_1 \cup \Omega_2,$$

where

$$\Omega_1 \triangleq \{\mathbf{x}(x_1, x_2) \in \Omega \mid ax_1 + bx_2 \leq c\}$$
$$\Omega_2 \triangleq \{\mathbf{x}(x_1, x_2) \in \Omega \mid ax_1 + bx_2 > c\}$$

Obviously, Ω_1 and Ω_2 are non-empty and convex. Furthermore, $\Omega_1 \cap \Omega_2 = \{\}$, where $\{\}$ denotes the empty set. Since P lies to the right of L, it readily follows that $P \in \Omega_2$. That is to say, Ω is not the minimal convex set containing P. This contradicts the definition of Ω, and therefore, the above assumption must be false. This completes the proof. $\square$

APPENDIX C

Definition C.1. *Let $\hat{\mathcal{A}}$ denote the set of all functions $G : \mathbb{C}+ \to \mathbb{C}$ that are Laplace transforms of elements of $\mathcal{A}$.*

Lemma C.1 ([28, 29])**.** *Suppose $G(s) \in \hat{\mathcal{A}}$; then $G^{-1}(s) \in \hat{\mathcal{A}}$ if and only if*

$$\inf_{\operatorname{Re} s \geq 0} |G(s)| > 0. \tag{C.1}$$

Proposition C.1. *Consider the system shown in Figure C.1, where the impulse responses k, g of the transfer functions $K(s), G(s)$ satisfy the following conditions:* (i) $g(t) = c + g_1(t), \forall t \geq 0$ *with $g_1, \dot{g}_1 \in \mathcal{A}$,* (ii) $k * g(t) = r + h_1(t), \forall t \geq 0$ *with $0 < r < \infty$ and $k, h_1 \in \mathcal{A}$.*

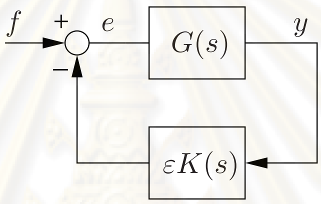


Figure C.1: Linear feedback system

Then, the impulse response h of $H(s) = G(s)[1 + \varepsilon K(s)G(s)]^{-1}$ satisfies conditions that $h \in \mathcal{A}$ and $\dot{h} \in \mathcal{A}$ for a sufficiently small $\varepsilon > 0$.

Proof. We have

$$\begin{aligned} \frac{1}{1 + \varepsilon K(s)G(s)} &= \frac{1}{1 + \varepsilon \left[\frac{r}{s} + H_1(s)\right]} \\ &= \frac{s}{s + \varepsilon r + \varepsilon s H_1(s)} \\ &= \frac{s}{s + \varepsilon r}\left[1 + \frac{\varepsilon s}{s + \varepsilon r} H_1(s)\right]^{-1}. \end{aligned} \tag{C.2}$$

First, by defining

$$N(s) \triangleq 1 + \frac{\varepsilon s}{s + \varepsilon r} H_1(s), \tag{C.3}$$

we prove that $N^{-1}(s) \in \hat{\mathcal{A}}$ for a sufficiently small $\varepsilon > 0$. By using the triangular inequality, we have

$$|N(s)| \geq 1 - \varepsilon \left|\frac{s}{s + \varepsilon r}\right| |H_1(s)|. \tag{C.4}$$

It can be shown (see, for example, [28]) that if $H_1(s) \in \hat{\mathcal{A}}$ then there exists $M > 0$ such that

$$\sup_{\operatorname{Re} s \geq 0} |H_1(s)| \leq M. \tag{C.5}$$

In addition, it is easy to see that

$$\sup_{\mathrm{Re}\, s \geq 0} \left| \frac{s}{s + \varepsilon r} \right| \leq 2. \tag{C.6}$$

Hence

$$\inf_{\mathrm{Re}\, s \geq 0} |N(s)| \geq 1 - 2\varepsilon M. \tag{C.7}$$

Thus, for a sufficiently small $\varepsilon > 0$,

$$\inf_{\mathrm{Re}\, s \geq 0} |N(s)| > 0. \tag{C.8}$$

Since $N \in \hat{\mathcal{A}}$, it follows from Lemma C.1 that (C.8) is equivalent to $N^{-1}(s) \in \hat{\mathcal{A}}$.

Next, we have

$$\begin{aligned} H(s) &= \frac{s}{s + \varepsilon r} G(s) N^{-1}(s) \\ &= \frac{s}{s + \varepsilon r} \left[\frac{c}{s} + G_1(s) \right] N^{-1}(s) \\ &= \left[\frac{c}{s + \varepsilon r} + \frac{s}{s + \varepsilon r} G_1(s) \right] N^{-1}(s). \end{aligned} \tag{C.9}$$

Since $G_1(s), N(s), \frac{c}{s+\varepsilon r}$ and $\frac{s}{s+\varepsilon r} \in \hat{\mathcal{A}}$, it follows immediately that $H(s) \in \hat{\mathcal{A}}$. That is to say, $h \in \mathcal{A}$.

Moreover, by differentiating both sides of the following equation

$$h = g - \varepsilon h * k * g, \tag{C.10}$$

we arrive at

$$\dot{h} = \dot{g} - \varepsilon h * k * \dot{g}. \tag{C.11}$$

Since $k, h \in \mathcal{A}$ and $\dot{g} = c\delta(t) + \dot{g}_1 \in \mathcal{A}$, then it follows that $\dot{h} \in \mathcal{A}$. □

Loop Transformations for Systems Having One Integrator

G_p Contains One Integrator

Consider the system in Figure 2.5. Assume that $\psi \in$ sector $[\epsilon,\ k + \epsilon]$ for a sufficiently small $\epsilon > 0$, that $g_p(t) = c + g_{p1},\ \forall t \geq 0$ with $g_{p1},\ \dot{g}_{p1} \in \mathcal{A}$, and that $g_c * g_p(t) = r + h_1(t)$, $\forall t \geq 0$ with $0 < r < \infty$ and $g_c, h_1 \in \mathcal{A}$. Then, by using the following loop transformation,

$$\begin{aligned} \psi'(u) &\triangleq \psi(u) - \epsilon u \\ G'(s) &\triangleq \frac{G_p(s)G_c(s)}{1 + \epsilon G_p(s)(s)G_c(s)} \end{aligned} \tag{C.12}$$

the given system is equivalent to the one in Figure C.2.

Clearly,

$$\psi' \in \text{sector}\, [0, k]. \tag{C.13}$$

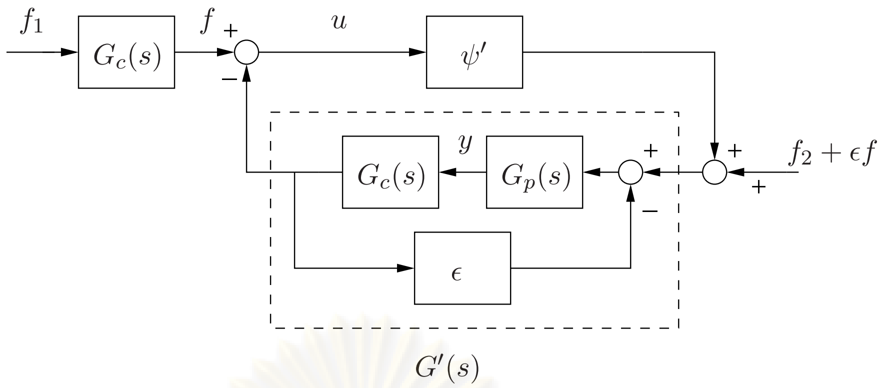


Figure C.2: The loop transformation when G_p contains one integrator.

and, provided that $f_1, f_2 \in \mathcal{P}_n$ $(n = 2, \infty)$,

$$f = f_1 * g_c \in \mathcal{P}_n, \quad f_2 + \epsilon f \in \mathcal{P}_n. \tag{C.14}$$

Let g' denote the impulse response of $G'(s)$. Then, it follows from Proposition C.1 that $g', \dot{g}' \in \mathcal{A}$ for a sufficiently small $\epsilon > 0$. Therefore, Theorems 2.1 and 2.2 can be applied to the system in Figure C.2.

G_c Contains One Integrator

Consider the system in Figure 2.5. Assume that $\psi \in \text{sector}\,[\epsilon,\ k + \epsilon]$ for a sufficiently small $\epsilon > 0$, that $g_c(t) = c + g_{c1},\ \forall t \geq 0$ with $g_{c1},\ \dot{g}_{c1} \in \mathcal{A}$, and that $g_c * g_p(t) = r + h_1(t), \forall t \geq 0$ with $0 < r < \infty$ and $g_p, h_1 \in \mathcal{A}$. Then, by using the following loop transformation,

$$\begin{aligned} \psi'(u) &\triangleq \psi(u) - \epsilon u \\ G'(s) &\triangleq \frac{G_c(s)}{1 + \epsilon G_p(s)(s)G_c(s)} \end{aligned} \tag{C.15}$$

the given system is equivalent to the one in Figure C.3.

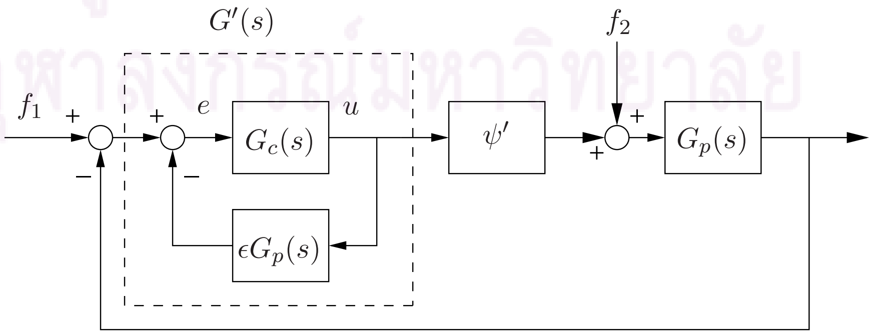


Figure C.3: The loop transformation when G_c contains one integrator.

Clearly,

$$\psi' \in \text{sector}\,[0, k]. \tag{C.16}$$

Let g' denote the impulse response of $G'(s)$. Then, it follows from Proposition C.1 that $g', \dot{g}' \in \mathcal{A}$ for a sufficiently small $\epsilon > 0$. Therefore, Proposition 2.3 can be applied to the system in Figure C.3.

APPENDIX D

Definition D.1 ([38, p. 336])**.** Let $\mathcal{F}$ be a set of functions defined and finite-valued on a set E. The functions of $\mathcal{F}$ are called equicontinuous if, for every $\epsilon > 0$, there is a $\delta > 0$ such that

$$\|x_1 - x_2\| < \delta, \quad \forall x_1, x_2 \in E \implies |f(x_1) - f(x_2)| < \epsilon, \quad \forall f \in \mathcal{F}. \tag{D.1}$$

The functions of $\mathcal{F}$ are said to be uniformly bounded if there exists a finite M such that

$$|f(x)| \le M, \quad \forall f \in \mathcal{F}, \forall x \in E. \tag{D.2}$$

Theorem D.1 (**Ascoli's Theorem** [38, p. 336])**.** *If $\mathcal{F}$ is a set of functions defined, equicontinuous and uniformly bounded on a bounded closed set, then from every sequence $\{f_n\} \in \mathcal{F}$ it is possible to select a uniformly convergent subsequence.*

Theorem D.2 (**Compactness Criterion** [39, p. 407])**.** *Let X and Y be normed spaces and $H : X \to Y$ be a linear operator. Then H is compact if and only if it maps every bounded sequence $\{x_n\}$ in X onto a sequence $\{Hx_n\}$ in Y which has a convergent subsequence.*

Lemma D.1. *Define $X \triangleq \{x \in L_\infty \mid \|x\|_\infty \le C\}$ where C is a finite number. For a fixed $T > 0$, let H denote the convolution operator over X_T, given by*

$$Hx(t) = \int_0^T h(t-\tau)x(\tau)d\tau. \tag{D.3}$$

If $h \in \mathcal{A}$ and if $\dot{h} \in \mathcal{A}$, then H is compact.

Proof. Assume that $h, \dot{h} \in \mathcal{A}$. Let $\{x_n\}$ be any sequence in X_T and let $\{y_n\}$ be defined by

$$y_n(t) = Hx_n(t). \tag{D.4}$$

First we show that $\{y_n\}$ is uniformly bounded. For every $x \in X_T$, we have

$$\begin{aligned} \left| \int_0^T h(t-\tau)x(\tau)d\tau \right| &\le \int_0^T |h(t-\tau)|\, |x(\tau)|\, d\tau \\ &\le C \int_0^\infty |h(\tau)|\, d\tau. \end{aligned} \tag{D.5}$$

Since $h \in \mathcal{A}$, it follows that there exists $M_0 < \infty$ such that $\int_0^\infty |h(\tau)|\, d\tau < M_0$. Thus,

$$\|y\|_\infty = \sup_{t \ge 0} \left| \int_0^T h(t-\tau)x(\tau)d\tau \right| \le M_0 C. \tag{D.6}$$

Therefore, $\{y_n\}$ is uniformly bounded on $[0, T]$ for any $T > 0$.

Next, we show that $\{y_n\}$ is equicontinuous. For any $t_1, t_2 \in [0, T]$ and any $k > 0$, we have

$$\begin{aligned}
|y_k(t_1) - y_k(t_2)| &= \left| \int_0^T \{h(t_1 - \tau) - h(t_2 - \tau)\} x_k(\tau) d\tau \right| \\
&\leq \int_0^T |h(t_1 - \tau) - h(t_2 - \tau)| \, |x_k(\tau)| \, d\tau \qquad \text{(D.7)} \\
&\leq C \, |\Delta t| \int_0^T \left| \frac{\Delta h}{\Delta t} \right| d\tau,
\end{aligned}$$

where $\Delta t = t_1 - t_2$ and $\Delta h = h(t_1 - \tau) - h(t_2 - \tau)$. Since $\dot{h} \in \mathcal{A}$ by assumption, it follows that there exists $M_1 < \infty$ such that $\int_0^\infty |\dot{h}(\tau)| d\tau < M_1$. Thus,

$$\lim_{\Delta t \to 0} \int_0^T \left| \frac{\Delta h}{\Delta t} \right| d\tau \ \leq \ \int_0^\infty \left| \dot{h}(\tau) \right| d\tau \ < \ M_1. \qquad \text{(D.8)}$$

From (D.7) and (D.8), it follows that $|y_k(t_1) - y_k(t_2)| \to 0$ as $\Delta t \to 0$ for any $k > 0$. Therefore, y_n is equicontinuous by definition.

Note that $[0, T]$ is a bounded and closed set. Thus, in view of Theorem D.1, $\{y_n\}$ has a convergent subsequence. Since $\{x_n\}$ is an arbitrary sequence in X_T and $y_n = Hx_n$, the compactness of H follows immediately from Theorem D.2. □

BIOGRAPHY

Van Sy Mai was born in Thanh Hoa, Vietnam, in 1985. He received his Bachelor's degree in electrical engineering from Hanoi University of Technology, Vietnam, in 2008. He has been granted a scholarship by the AUN/SEED-Net (www.seed-net.org) to pursue his Master's degree in electrical engineering at Chulalongkorn University, Thailand, since 2008. He conducted his graduate study with the Control Systems Research Laboratory, Department of Electrical Engineering, Faculty of Engineering, Chulalongkorn University. His research interests include nonlinear systems and control systems design by the Method of Inequalities and the Principle of Matching.