ค่าลำดับชั้นน้อยที่สุดของกราฟ

นายศราวุฒิ รัตนประยูร

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MINIMUM RANK OF GRAPHS

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ค่าลำดับชั้นน้อยที่สุดบนฟีลด์ F ของกราฟ G คือ ค่าลำดับชั้นน้อยที่สุดที่เป็นไปได้ในบรรดาเมทริกซ์สมมาตรบนฟีลด์ F ซึ่งสมาชิกแถวที่ i หลักที่ $j\ (i \neq j)$ ไม่เป็นศูนย์ ถ้า ij เป็นเส้นเชื่อมในกราฟ G และเป็นศูนย์ ถ้า ij ไม่เป็นเส้นเชื่อมในกราฟ G เมื่อ ศูนย์ คือ เอกลักษณ์การบวกบนฟีลด์ F เมทริกซ์เหมาะที่สุดเชิงเอกภพของกราฟ G คือ เมทริกซ์สมมาตร A ที่สมาชิกทุกตัวเป็นจำนวนเต็มแต่สมาชิกที่ไม่อยู่บนแนวทแยงมุมของเมทริกซ์ A คือ จำนวน 0, 1 หรือ -1 และสำหรับทุกฟีลด์ F ค่าลำดับชั้นของเมทริกซ์ A เท่ากับค่าลำดับชั้นน้อยที่สุดบนฟีลด์ F ของกราฟ G ซึ่งสมสัณฐานกับกราฟของเมทริกซ์ A เราแนะนำกราฟพัด กราฟหนังสือ กราฟดอกบัว และกราฟสะพานแขวน และแสดงค่าลำดับชั้นน้อยที่สุดของกราฟเหล่านี้บนทุกฟีลด์ เราใช้เมทริกซ์เหมาะที่สุดเชิงเอกภพเพื่อแสดงว่าค่าลำดับชั้นน้อยที่สุดของกราฟเหล่านี้ไม่ขึ้นอยู่กับฟีลด์ และให้ตัวอย่างกราฟที่มีค่าลำดับชั้นน้อยที่สุดขึ้นอยู่กับฟีลด์

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SARAWUT RATTANAPRAYOON : MINIMUM RANK OF GRAPHS.
THESIS ADVISOR : ASSOC. PROF. WANIDA HEMAKUL, Ph.D.
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The minimum rank over a field F of a graph G is the smallest possible rank among all symmetric matrices over F whose (i, j)th entry $(i \neq j)$ is nonzero whenever ij is an edge in G and is zero otherwise, where zero is the additive identity of F. A universally optimal matrix for a graph G is an integer symmetric matrix A such that every off-diagonal entry of A is 0, 1, or -1 and for all fields F, the rank of A is the minimum rank over F of G which is isomorphic to the graph of A. The fan graph, the book graph, the lotus graph and the hanging bridge graph are introduced and the minimum rank of these graphs over any field are presented. We use universally optimal matrices for these graphs to establish field independence of minimum rank. Examples verifying lack of field independence for some graphs are provided.

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CHAPTER I

INTRODUCTION

The *minimum rank problem* is, for a given graph and a field F, to determine the smallest possible rank among symmetric matrices over F whose off-diagonal pattern of zero-nonzero entries is described by the graph. Most work on minimum rank has been on the real minimum rank problem. S. Fallat and L. Hogben [5] provided a survey of known results and discussion of the motivation for the minimum rank problem. Catalogs of minimum rank and other parameters for families of graphs [7] and small graphs [8] were developed at the American Institute of Mathematics (AIM) workshop "Spectra of families of matrices described by graphs, digraphs, and sign patterns"[2] and are available on-line; these catalogs are updated routinely. The study of minimum rank over fields other than the real numbers was initiated in [3].

The minimum rank of a graph G is *field independent* if the minimum rank of G is the same for all fields. In [4], L.M. DeAlba et al. established the field independence or dependence of minimum rank for most of the families of graphs listed in the AIM on-line minimum rank graph catalog and established the minimum rank of several additional families. For almost every graph discussed that has field independent minimum rank, they exhibited a single integer matrix that over every field has the given graph and has rank in that field equal to the minimum rank over the field (what they call a universally optimal matrix described in chapter II).

Here is the outline of this thesis.

In chapter II, we recall definitions and review results of the relevant works.

In chapter III, we introduce the fan graph, the book graph, the lotus graph, and the hanging bridge graph and establish the field independence of minimum rank for these graphs by constructing universally optimal matrices.

In chapter IV, we provide examples verifying lack of field independence of minimum rank for some graphs, such as $P_4 \vee K_2$, $C_6 \vee K_4$, the clique path $KP(5, 4)$, and the clique-cycle path $KC(5; 4)$.

CHAPTER II

PRELIMINARIES

We recall definitions and review the known results that are needed in our work.

A *graph* G means a simple undirected graph (i.e., neither loops nor multiple edges allowed). Denote by $V(G)$ and $E(G)$ the set of vertices and edges of G, respectively. Also, $|G|$ denotes the number of vertices in G, and xy denotes the edge in $E(G)$ for some $x, y \in V(G)$.

The *adjacency matrix* of a graph G, denoted $\mathcal{A}(G) = [a_{ij}]$, is a $(0,1)$-matrix such that $a_{ij} = 1$ if and only if $ij \in E(G)$.

The *degree* of vertex v in a graph G, denoted $d(v)$, is the number of vertices adjacent to v. A *leaf* is a vertex of degree 1.

The *complement* of a graph G is the graph $\overline{G}$ such that vertex set is $V(G)$ and for each pair $u, v \in V(G)$, uv is an edge of $\overline{G}$ if and only if uv is not an edge of G.

A graph H is a *subgraph* of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. The subgraph $G[R]$ of G *induced* by $R \subseteq V(G)$ is the subgraph with vertex set R and edge set $\{ij \in E(G) : i, j \in R\}$. The subgraph induced by $\overline{R}$ is denoted by $G - R$, or in the case R is a single vertex v, by $G - v$.

An induced subgraph H of a graph G is a *clique* if H has an edge between every pair of vertices of H. A set of subgraphs of G, each of which is a clique and such that every edge of G is contained in at least one of these cliques, is called a *clique covering* of G.

Let u and v be vertices in a graph G, a *u,v-path* in G is a list $u = v_0, v_1, \ldots, v_n = v$ of vertices in $V(G)$ such that $v_{i-1}v_i \in E(G)$ and $v_0, v_1, \ldots,$ and v_n are all different.

A graph G is *connected* if it has a u,v-path in G whenever $u, v \in V(G)$; otherwise, G is *disconnected*.

A *path* is a graph P_n such that $V(P_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(P_n) = \{v_iv_{i+1} : i = 1, 2, \ldots, n-1\}$. A *cycle* is a graph C_n such that $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(C_n) = \{v_iv_{i+1} : i = 1, 2, \ldots, n-1\} \cup \{v_nv_1\}$. A *complete graph* is a graph K_n such that $V(K_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(K_n) = \{v_iv_j : 1 \leq i < j \leq n\}$.

A vertex v of a connected graph G is a *cut-vertex* if $G - v$ is disconnected.

A graph G is *isomorphic to* a graph H, denoted $G \cong H$, if there is a bijection $f : V(G) \to V(H)$ such that $uv \in E(G)$ if and only if $f(u)f(v) \in E(H)$.

The *union* of graphs $G_1, G_2, \ldots,$ and G_n, denoted $\bigcup_{i=1}^{n} G_i$, is the graph with vertex set $\bigcup_{i=1}^{n} V(G_i)$ and edge set $\bigcup_{i=1}^{n} E(G_i)$. When $V(G_i) \cap V(G_j) = \varnothing$ for all $i \neq j$, it is called the *disjoint union* of graphs $G_1, G_2, \ldots,$ and G_n, denoted $G_1 + G_2 + \cdots + G_n$. nG denotes the disjoint union of n copies of a graph G.

The *complete multipartite graph*, denoted $K_{n_1,n_2,\ldots,n_k}$, is the complement of $K_{n_1} + K_{n_2} + \cdots + K_{n_k}$. When $k = 2$, it is called a *complete bipartite graph*. A complete bipartite graph $K_{1,n-1}$ is called an *n-vertex star*.

The *join* of graphs G_1 and G_2 with disjoint vertex sets $V(G_1)$ and $V(G_2)$ and edge sets $E(G_1)$ and $E(G_2)$, denoted $G_1 \vee G_2$, is the union of G_1 and G_2 together with all the edges joining $V(G_1)$ and $V(G_2)$.

Next, we give the basic definitions and the association of matrices and graphs.

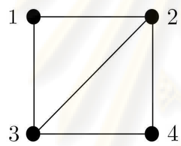
Let S_n^F denote the set of $n \times n$ symmetric matrices over a field F. For $A = [a_{ij}] \in S_n^F$, the *graph* of A, denoted $\mathcal{G}^F(A)$, is the graph with vertex set $\{1, 2, \ldots, n\}$ (or $\{v_1, v_2, \ldots, v_n\}$) and edge set $\{ij : a_{ij} \neq 0, 1 \leq i < j \leq n\}$. Note that the diagonal of A is ignored in determining $\mathcal{G}^F(A)$. The superscript F is used because the graph of an integer matrix may vary depending on the field in which the matrix is viewed.

Example 2.1. Let

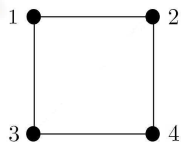
$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 & \sqrt{2} & 0 \\ 1 & 3.1 & -1.5 & 2 \\ \sqrt{2} & -1.5 & 1 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix}, \text{ and } C = \begin{bmatrix} 0 & 1 & 3 & 0 \\ 1 & -1 & 2 & 1 \\ 3 & 2 & 2 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix}.$$

These graphs $\mathcal{G}^{\mathbb{R}}(A)$, $\mathcal{G}^{\mathbb{R}}(B)$, and $\mathcal{G}^{\mathbb{R}}(C)$ are the graph G and graph $\mathcal{G}^{\mathbb{Z}_2}(C)$ is the graph H, as shown below. Note that $\mathcal{G}^{\mathbb{R}}(C)$ is not isomorphic to $\mathcal{G}^{\mathbb{Z}_2}(C)$.

G :



H :



The *minimum rank over a field F* of a graph G with n vertices is

$$\mathrm{mr}^F(G) = \min\{\mathrm{rank}(A) : A \in S_n^F, \mathcal{G}^F(A) \cong G\}.$$

In case $F = \mathbb{R}$, the superscript $\mathbb{R}$ may be omitted, so we write $\mathrm{mr}(G)$ for $\mathrm{mr}^{\mathbb{R}}(G)$ and $\mathcal{G}(A)$ for $\mathcal{G}^{\mathbb{R}}(A)$.

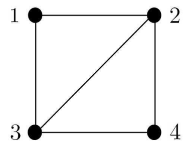
The minimum rank of a graph G is *field independent* if the minimum rank of G is the same for all fields.

Recall the result from basic linear algebra.

Proposition 2.2. [4] *Let S be a linearly dependent set of integer vectors over $\mathbb{Q}$. Then for every prime number p, S is linearly dependent over $\mathbb{Z}_p$. If A is a square integer matrix, then for every prime p,* $\mathrm{rank}^{\mathbb{Z}_p}(A) \leq \mathrm{rank}(A)$, *and if characteristic of a field F is 0, then* $\mathrm{rank}^F(A) = \mathrm{rank}(A)$.

Example 2.3. Let F be any field and G be the graph as shown below.

G :



with

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

and $\mathcal{G}^F(A) \cong G$. Note that $\text{rank}(A) = 2$. By Proposition 2.2, $\text{rank}^F(A) \leq 2$. Then $\text{mr}^F(G) \leq \text{rank}^F(A) \leq 2$. Next, show that $\text{mr}^F(G) \geq 2$. Let

$$B = \begin{bmatrix} d_1 & a & b & 0 \\ a & d_2 & c & d \\ b & c & d_3 & e \\ 0 & d & e & d_4 \end{bmatrix}$$

with $\mathcal{G}^F(B) \cong G$ where $a, b, c, d, e, d_1, d_2, d_3, d_4 \in F$ and $a, b, c, d,$ and e are nonzero in F. Since the third and the fourth rows of B are independent, $\text{rank}(B) \geq 2$. Then $\text{mr}^F(G) \geq 2$. Thus $\text{mr}^F(G) = 2$ for any field F. Therefore the minimum rank of G is field independent.

In [4], L.M. DeAlba et al. defined a universally optimal matrix to establish field independence of the minimum rank as follows. Recall that when A is an integer matrix and p is prime, A can be viewed as a matrix over $\mathbb{Z}_p$; the rank of A over $\mathbb{Z}_p$ will be denoted by $\text{rank}^{\mathbb{Z}_p}(A)$.

A *universally optimal matrix* for a graph G is an integer symmetric matrix A such that every off-diagonal entry of A is 0, 1, or -1 and $\mathcal{G}(A) \cong G$ and for all fields F, $\text{rank}^F(A) = \text{mr}^F(G)$.

Example 2.4. From example 2.3, the graph $G \cong \mathcal{G}(A)$ where

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

and $\text{rank}^F(A) = 2 = \text{mr}^F(G)$ for any field F. Therefore A is a universally optimal matrix for G.

In [4], L.M. DeAlba et al. showed the results about field independence of the minimum rank for families of graphs and these graphs have universally optimal matrices which is presented in Table 2.1. Definitions of graphs in this table can be found in the Appendix.

G	$\text{mr}^F(G)$	G	$\text{mr}^F(G)$
P_n (path)	$n-1$	$P_s \square P_s$	$s^2 - s$
C_n (cycle)	$n-2$	$C_s \square C_s$	$s^2 - (s + 2\left\lfloor \frac{s}{2} \right\rfloor)$
K_n (complete graph)	1	$K_s \square K_s$	$2s-2$
$K_{p,q}$ (complete bipartite graph)	2	claw-free block-clique (i.e., line graph of tree)	# of blocks
N_s (necklace)	$3s-2$	$K_t \circ K_s$	$t+1$
$P_{m,k}$ (pineapple), $m \geq 3, k \geq 2$	3	$C_t \circ K_1, t \geq 4$	$2t - \left\lfloor \frac{t}{2} \right\rfloor$
T (tree)		$C_t \circ K_s, s \geq 2$	$2t-2$
unicyclic		T_n (supertriangle)	$\frac{1}{2}n(n-1)$
polygonal path	$n-2$		

Table 2.1: Summary of field independence of the minimum rank over any field F for families of graphs

In [3], W. Barrett et al. showed that if

$$J = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \text{ and } A = \begin{bmatrix} \mathbf{0}_{3\times3} & J & J \\ J & \mathbf{0}_{3\times3} & J \\ J & J & \mathbf{0}_{3\times3} \end{bmatrix}$$

where $\mathbf{0}_{3\times3}$ is the 3×3 zero matrix, then the matrix A is a universally optimal matrix for the complete multipartite graph $K_{3,3,3}$ shown in Figure 2.1 because if characteristic of a field F is 2, $\text{rank}^F(A) = 2 = \text{mr}^F(K_{3,3,3})$; otherwise, $\text{rank}^F(A) = 3 = \text{mr}^F(K_{3,3,3})$. But $K_{3,3,3}$ does not have field independent minimum rank.

In [4], L.M. DeAlba et al. showed that if G is the disjoint union of $K_{3,3,3}$ and $\overline{P_3 \cup 2K_3}$ shown in Figure 2.1, then G has field independent minimum rank but G does not have a universally optimal matrix.

Figure 2.1: The complete multipartite graph $K_{3,3,3}$ and the graph $\overline{P_3 \cup 2K_3}$

Remark 2.5. [4] The existence of a universally optimal matrix for the graph G implies $\text{mr}^F(G) \leq \text{mr}(G)$ for all fields F, or equivalently, the existence of a field F such that $\text{mr}^F(G) > \text{mr}(G)$ implies that G does not have a universally optimal matrix.

In [4], L.M. DeAlba et al. showed the results about the minimum rank of graphs are dependent of the field and these graphs does not have a universally optimal matrix which is presented in Table 2.2. Definitions of graphs in this table can be found in the Appendix.

G	$\mathrm{mr}(G)$	$\mathrm{mr}^{\mathbb{Z}_2}(G)$
W_6 (wheel)	3	4
M_5 (Möbius ladder)	6	8
$L(K_7)$	5	6
H_3 (half-graph)	3	4
$K_{2,2,2,2}$	2	4
complement of 2-tree H in Figure 2.2	4	5
complement of tree T in Figure 2.3	3	4
$\overline{3K_2 \cup K_1}$	2	4
$\overline{C_6} \cong K_3 \square K_2 \cong K_3 \square P_2 \cong C_3 \square P_2$	3	4
$C_5 \square K_3 \cong C_5 \square C_3$	9	10
$P_3 \boxtimes P_3$	4	6

Table 2.2: Summary of field dependence of the minimum rank for graphs

Figure 2.2: A 2-tree H and its complement $\overline{H}$

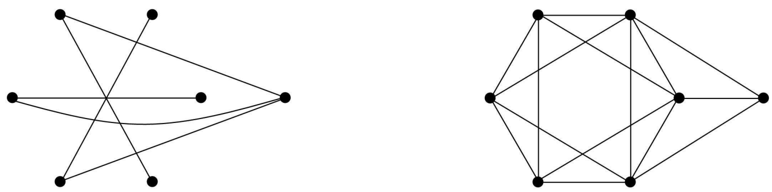


Figure 2.3: A tree T and its complement $\overline{T}$

In chapter IV, we present some graphs which do not have a universally optimal matrix by using Remark 2.5.

We introduce the following notation about specific matrices and a vector which will be used to determine universally optimal matrices.

1. I_n denotes the $n \times n$ identity matrix.

2. $\mathbf{0}_{m\times n}$ denotes the $m \times n$ zero matrix.

3. $\operatorname{diag}(a_1, a_2, \ldots, a_n)$ denotes the $n \times n$ matrix of the form

$$\begin{bmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & a_n \end{bmatrix}.$$

4. $\operatorname{diag}'(a_1, a_2, \ldots, a_{n-1})$ denotes the $n \times n$ matrix of the form

$$\begin{bmatrix} 0 & a_1 & 0 & 0 & 0 \\ a_1 & 0 & a_2 & 0 & 0 \\ 0 & a_2 & 0 & \ddots & 0 \\ 0 & 0 & \ddots & \ddots & a_{n-1} \\ 0 & 0 & 0 & a_{n-1} & 0 \end{bmatrix}.$$

5. $\operatorname{diag}''(a_1, a_2, \ldots, a_{n-2})$ denotes the $n \times n$ matrix of the form

$$\begin{bmatrix} 0 & 0 & a_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_2 & 0 & 0 \\ a_1 & 0 & 0 & 0 & \ddots & 0 \\ 0 & a_2 & 0 & 0 & \ddots & a_{n-2} \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & a_{n-2} & 0 & 0 \end{bmatrix}.$$

6. "repeat[ ]"means the sequence enclosed in parentheses appears as many times as needed (possibly zero times) to obtain a vector of the correct length.

For example, $(1,1,-1,0,0,-1,-1,0,0,-1,\ldots,-1,0,0,-1)^T = (1,1,\text{repeat}\,[-1,0,0,-1])^T$.

The result from the following proposition will be used to determine minimum ranks of graphs and universally optimal matrices.

Proposition 2.6. [4, 6]

1. *The path P_n has a universally optimal matrix of the form $\mathcal{A}(P_n) + D$ where*

$$D = \begin{cases} \text{diag}(\text{repeat}[0]) & \textit{if } n \textit{ is odd}, \\ \text{diag}(\text{repeat}[0], 1, 1) & \textit{if } n \textit{ is even}. \end{cases}$$

2. *The cycle C_n has a universally optimal matrix of the form $\mathcal{A}(C_n) + D$ where*

$$D = \begin{cases} \text{diag}(\text{repeat}[0]) & \textit{if } n \equiv 0 (\text{mod} 4), \\ \text{diag}(\underbrace{1,1,\ldots,1}_{9}, \text{repeat}[0]) & \textit{if } n \equiv 1 (\text{mod} 4) \textit{ and } n \neq 5, \\ \text{diag}(1,1,1,1,1,1,\text{repeat}[0]) & \textit{if } n \equiv 2 (\text{mod} 4), \\ \text{diag}(1,1,1,\text{repeat}[0]) & \textit{if } n \equiv 3 (\text{mod} 4), \\ \text{diag}(0,0,-1,-1,-1) & \textit{if } n = 5. \end{cases}$$

3. *The complete graph K_n has a universally optimal matrix of the form $\mathcal{A}(K_n) + I_n$.*

Example 2.7.

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \; A_2 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \text{ and } A_3 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

are universally optimal matrices for P_3, C_4, and K_5, respectively.

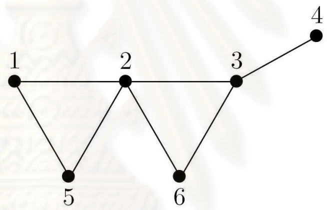
The next results are tools to determine lower bounds or upper bounds for the minimum rank of graphs.

Proposition 2.8. [3, 5]

1. *If H is an induced subgraph of a graph G, then $\mathrm{mr}^F(H) \leq \mathrm{mr}^F(G)$ for any field F.*
2. *If $G_1, G_2, \ldots,$ and G_n are graphs and $G = \bigcup_{i=1}^{n} G_i$, then $\mathrm{mr}(G) \leq \sum_{i=1}^{n} \mathrm{mr}(G_i)$.*

Example 2.9. We determine a lower bound and an upper bound for the minimum rank of a graph G.

G :



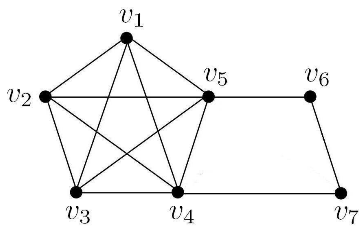
Let F be a field. Since the path P_4 is an induced subgraph of G and by Proposition 2.8 (1), $\mathrm{mr}^F(P_4) \leq \mathrm{mr}^F(G)$. By Table 2.1, $\mathrm{mr}^F(P_4) = 3$. Thus $3 \leq \mathrm{mr}^F(G)$ for any field F. We can view that G is the union of K_2 and 2 copies of K_3. By Table 2.1 and Proposition 2.8 (2), $\mathrm{mr}(G) \leq \mathrm{mr}(K_2)+ 2\mathrm{mr}(K_3) = 1 + 2 = 3$. Thus $\mathrm{mr}(G) \leq 3$.

In [1], F. Barioli et al. used the idea of covering the edges of a graph with subgraphs to determine the upper bound for the minimum rank of a graph G.

An (edge) *covering* of a graph G is a set of subgraphs $\mathcal{C} = \{G_i, i = 1, 2, \ldots, n\}$ such that G is the union $G = \bigcup_{i=1}^{n} G_i$. A graph has many possible coverings, but some, such as clique coverings, are more useful than others. For a given covering $\mathcal{C}$, $c_{\mathcal{C}}(e)$ denotes the number of subgraphs that have edge e as a member.

Example 2.10. Let G be the graph shown below.

G :



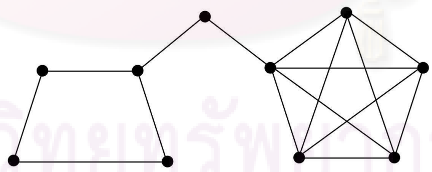
Since $\mathcal{C} = \{K_5, C_4\}$ is a covering of G and K_5 and C_4 have only one common edge v_4v_5, $c_{\mathcal{C}}(v_4v_5) = 2$ and $c_{\mathcal{C}}(e) = 1$ for every edge $e \in E(G) \setminus \{v_4v_5\}$.

Proposition 2.11. [4] *Let F be a field and let G be a graph. Suppose $\mathcal{C} = \{G_i, i = 1, 2, \dots, n\}$ is a covering of G such that for each G_i there is a universally optimal matrix of the form $\mathcal{A}(G_i) + D_i$, where D_i is a diagonal matrix. If* $\operatorname{char}F = 0$ *or if* $\operatorname{char}F = p$ *and $c_{\mathcal{C}}(e) \not\equiv 0 \pmod{p}$ where p is prime and for every edge $e \in E(G)$, then*

$$\operatorname{mr}^F(G) \le \sum_{i=1}^{n} \operatorname{mr}^F(G_i).$$

Example 2.12. Let G be the graph shown below.

G :



By Table 2.1, $\operatorname{mr}^F(P_3) = 2, \operatorname{mr}^F(C_4) = 2$ and $\operatorname{mr}^F(K_5) = 1$ for any field F. Since $\mathcal{C} = \{P_3, C_4, K_5\}$ is a covering of the graph G and $P_3, C_4,$ and K_5 have no common edges, $c_{\mathcal{C}}(e) = 1$ for every edge $e \in E(G)$. Then $c_{\mathcal{C}}(e) \not\equiv 0 \pmod{p}$ where p is prime. By Proposition 2.11, $\operatorname{mr}^F(G) \le \operatorname{mr}^F(P_3) + \operatorname{mr}^F(C_4) + \operatorname{mr}^F(K_5) = 2 + 2 + 1 = 5$ for any field F. Since the path P_6 is an induced subgraph of G and by Table 2.1 and Proposition 2.8 (1), $5 = \operatorname{mr}^F(P_6) \le \operatorname{mr}^F(G)$ for any field F. Then $\operatorname{mr}^F(G) = 5$ for any field F. By Example 2.7, $A_1, A_2,$ and A_3 are universally optimal matrices for P_3, C_4, and K_5, respectively. Consider

$$A = \begin{bmatrix} \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 4} \\ \mathbf{0}_{3\times 3} & A_1 & \mathbf{0}_{3\times 4} \\ \mathbf{0}_{4\times 3} & \mathbf{0}_{4\times 3} & \mathbf{0}_{4\times 4} \end{bmatrix} + \begin{bmatrix} A_2 & \mathbf{0}_{4\times 6} \\ \mathbf{0}_{6\times 4} & \mathbf{0}_{6\times 6} \end{bmatrix} + \begin{bmatrix} \mathbf{0}_{5\times 5} & \mathbf{0}_{5\times 5} \\ \mathbf{0}_{5\times 5} & A_3 \end{bmatrix},$$

which is

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Then $\operatorname{rank}(A) = 5$ and $\mathcal{G}(A) \cong G$. By Proposition 2.2, $\operatorname{rank}^F(A) \leq \operatorname{rank}(A) = 5$ for any field F. We have $5 = \operatorname{mr}^F(G) \leq \operatorname{rank}^F(A) \leq 5$ for any field F. Then $\operatorname{mr}^F(G) = \operatorname{rank}^F(A)$ for any field F. Thus A is a universally optimal matrix for G and G has field independent minimum rank.

In [1], F. Barioli et al. defined a zero forcing set as a tool to determine a lower bound for the minimum rank of a graph. First, they defined the color-change rule as follows: If G is a graph with each vertex colored either white or black, u is a black vertex of G, and exactly one neighbor v of u is white, then change the color of v to black. Given a coloring of G, the *derived coloring* is the result of applying the color-change rule until no more change are possible. A *zero forcing set* for a graph G is a subset Z of vertices such that if initially the vertices in Z are colored black and the remaining vertices are colored white, the derived coloring of G is all black. The *zero forcing number* for G, denoted $Z(G)$, is the minimum of $|Z|$ over all zero forcing sets $Z \subseteq V(G)$. The parameter $Z(G)$ is a tool to determine a lower bound for $\mathrm{mr}^F(G)$.

The next examples show zero forcing set and zero forcing numbers for some graph.

Example 2.13. The graph G, as shown below, has $\{v_3, v_4\}$ as a zero forcing set by applying the color-change rule shown in steps (a)-(d) as shown in Figure 2.4 and so $Z(G) \leq 2$. The derived coloring of G by the only one vertex is not all black since more than one white vertices are neighbors of a black vertex. Then any set of only one vertex of G cannot be a zero forcing set for G. Thus $Z(G) = 2$.

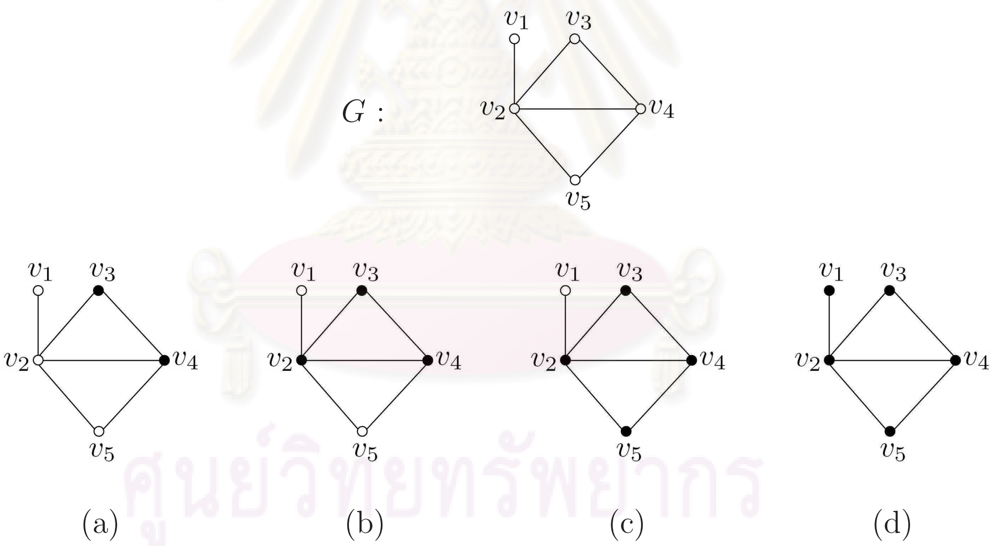


Figure 2.4: The graph G with $Z(G) = 2$

Example 2.14. Any set of $n-2$ leaves of the n-vertex star $K_{1,n-1}$ is a zero forcing set for $K_{1,n-1}$ and so $Z(K_{1,n-1}) \leq n-2$. The derived coloring of $K_{1,n-1}$ by any set of $n-3$ vertices is not all black since there are 2 or 3 vertices left which are colored white. Then any set of $n-3$ vertices of $K_{1,n-1}$ cannot be a zero forcing set for $K_{1,n-1}$. Thus $Z(K_{1,n-1}) = n-2$.

Proposition 2.15. [3] $Z(P_n) = 1$, $Z(C_n) = 2$ *and* $Z(K_n) = n - 1$.

Proposition 2.16. [1] *For any graph* G, $\mathrm{mr}^F(G) \geq |G| - Z(G)$ *for any field* F.

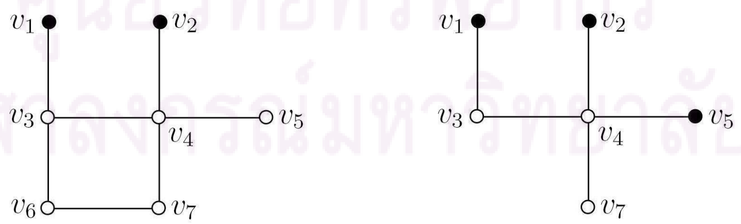
The next examples, we determine a lower bound for minimum rank over a field F of some graph G.

Example 2.17. Consider the graph G in Example 2.13. We have $Z(G) = 2$. By Proposition 2.16, $\mathrm{mr}^F(G) \geq |G| - Z(G) = 5 - 2 = 3$ for any field F. Thus $\mathrm{mr}^F(G) \geq 3$ for any field F.

Example 2.18. Consider the n-vertex star $K_{1,n-1}$. By Example 2.14, $Z(K_{1,n-1}) = n-2$. By Proposition 2.16, $\mathrm{mr}^F(K_{1,n-1}) \geq |K_{1,n-1}| - Z(K_{1,n-1}) = n-(n-2) = 2$ for any field F. Thus $\mathrm{mr}^F(K_{1,n-1}) \geq 2$ for any field F.

It is not true, if H is an induced subgraph of a graph G, then $Z(H) \geq Z(G)$ or $Z(H) \leq Z(G)$, as shown in the next examples.

Example 2.19. Consider the graph G shown below with H as an induced subgraph. We obtain $\{v_1, v_2\}$ and $\{v_1, v_2, v_5\}$ are zero forcing sets for G and H, respectively. Thus $Z(G) = 2 < 3 = Z(H)$.



G H

Example 2.20. Since the complete graph K_3 is an induced subgraph of the complete graph K_5, $Z(K_3) = 2 < 4 = Z(K_5)$.

CHAPTER III

FIELD INDEPENDENCE RESULTS

In this chapter, we introduce definitions of the book graph, the fan graph, the lotus graph, the hanging bridge graph, the path-cycle graph, and the path-clique graph and establish field independence of the minimum rank for the families of these graphs. We show that these graphs have field independent minimum rank and universally optimal matrices.

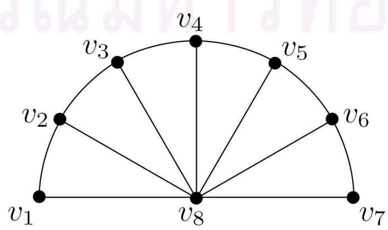
First, we present the definition of the fan graph and give results about this graph.

3.1 Fan Graphs

Let n be a positive integer greater than 3. The *fan graph* on n vertices, denoted F_n, is the graph for which $V(F_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(F_n) = \{v_iv_n : i = 1, 2, \ldots, n-1\} \cup \{v_iv_{i+1} : i = 1, 2, \ldots, n-2\}$.

Example 3.1. The fan graph F_8 on 8 vertices is shown below.

F_8 :



Proposition 3.2. *For $n \geq 4$, $Z(F_n) = 2$.*

Proof. We claim that $\{v_1, v_n\}$ is a zero forcing set for F_n, and so $Z(F_n) \leq 2$. Assign v_1 and v_n black and the other vertices white. For all k, $2 \leq k \leq n-1$, we can change the color of v_k to black since v_k is the only white vertex adjacent to v_{k-1}. Now, the derived coloring of F_n is all black. Then $\{v_1, v_n\}$ is a zero forcing set for F_n, as desired. Thus $Z(F_n) \leq 2$. We see that any one vertex in F_n cannot force the remaining vertices because its degree is greater than 1, $Z(F_n) \geq 2$. Thus $Z(F_n) = 2$. □

Next, we will show that for any field F, $\mathrm{mr}^F(F_n) = n-2$ by establishing a universally optimal matrix for F_n which yields an upper bound for $\mathrm{mr}^F(F_n)$.

Theorem 3.3. *For $n \geq 4$, there is a diagonal matrix D such that* $\mathrm{rank}(\mathcal{A}(F_n) + D) = n-2$. *Moreover, F_n has field independent minimum rank, and $\mathcal{A}(F_n) + D$ is a universally optimal matrix for F_n.*

Proof. Let D be an $n \times n$ diagonal matrix defined by

$$D = \begin{cases} \mathrm{diag}(0, \ldots, 0, \frac{n}{2}) & \text{if } n \equiv 0 (\mathrm{mod} 4), \\ \mathrm{diag}(1, 0, \ldots, 0, 1, \frac{n-1}{2}) & \text{if } n \equiv 1 (\mathrm{mod} 4), \\ \mathrm{diag}(1, 1, 0, \ldots, 0, 1, 1, \frac{n-2}{2}) & \text{if } n \equiv 2 (\mathrm{mod} 4), \\ \mathrm{diag}(1, 1, 0, \ldots, 0, \frac{n-1}{2}) & \text{if } n \equiv 3 (\mathrm{mod} 4). \end{cases}$$

Clearly, $\mathcal{G}(\mathcal{A}(F_n) + D) \cong F_n$. We exhibit two independent vectors $\vec{z}_1$ and $\vec{z}_2$ in the kernel of $\mathcal{A}(F_n) + D$ to show that $\mathrm{null}(\mathcal{A}(F_n) + D) \geq 2$. Consider the following 4 cases:

Case $n \equiv 0 \pmod 4$. Then $\vec{z}_1 = (1, 0, -1, \mathrm{repeat}[0, 1, 0, -1], 0)^T$ and $\vec{z}_2 = (0, -1, -1, \mathrm{repeat}[0, 0, -1, -1], 1)^T$.

Case $n \equiv 1 \pmod 4$. Then $\vec{z}_1 = (-1, 1, 1, \text{repeat}[-1, -1, 1, 1], -1, 0)^T$ and $\vec{z}_2 = (-1, 0, 0, \text{repeat}[-1, -1, 0, 0], -1, 1)^T$.

Case $n \equiv 2 \pmod 4$. Then $\vec{z}_1 = (0, -1, \text{repeat}[0, 0, -1, -1], 0, 0, -1, 1)^T$ and $\vec{z}_2 = (-1, 1, \text{repeat}[0, -1, 0, 1], 0, -1, 1, 0)^T$.

Case $n \equiv 3 \pmod 4$. Then $\vec{z}_1 = (1, -1, \text{repeat}[0, 1, 0, -1], 0)^T$ and $\vec{z}_2 = (-1, 0, \text{repeat}[0, -1, -1, 0], 1)^T$.

In any case, we obtain $\text{rank}(\mathcal{A}(F_n) + D) = n - \text{null}(\mathcal{A}(F_n) + D) \leq n - 2$. Let F be any field. By Proposition 2.16 and 3.2, $\text{mr}^F(F_n) \geq |F_n| - Z(F_n) = n - 2$. By Remark 2.2, $\text{rank}^F(\mathcal{A}(F_n) + D) \leq \text{rank}(\mathcal{A}(F_n) + D) \leq n - 2$. We have $n - 2 \leq \text{mr}^F(F_n) \leq \text{rank}^F(\mathcal{A}(F_n) + D) \leq n - 2$. Then $\text{mr}^F(F_n) = n - 2 = \text{rank}^F(\mathcal{A}(F_n)+D)$. Thus $\text{rank}(\mathcal{A}(F_n)+D) = n-2$. Hence F_n has field independent minimum rank, and $\mathcal{A}(F_n) + D$ is a universally optimal matrix for F_n. □

Example 3.4. For the fan graph F_8,

$$\mathcal{A}(F_8) + D = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 4 \end{bmatrix}$$

is a universally optimal matrix for F_8 where $D = \text{diag}(0, 0, 0, 0, 0, 0, 0, 4)$ and $\text{mr}^F(F_8) = 6$ for any field F.

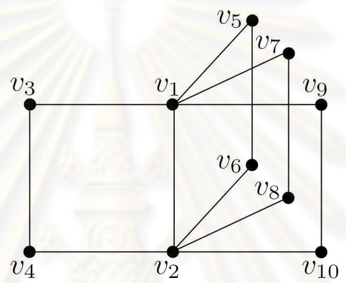
In the next section, we present the definition of the book graph and give results about this graph.

3.2 Book Graphs

Let n be a positive integer greater than 1. The *book graph* on $2(n+1)$ vertices, denoted B_n, is the graph for which $V(B_n) = \{v_1, v_2, \ldots, v_{2(n+1)}\}$ and $E(B_n) = \{v_1v_{2i+1} : i = 1, 2, \ldots, n\} \cup \{v_2v_{2(i+1)} : i = 1, 2, \ldots, n\} \cup \{v_iv_{i+1} : i = 1, 3, 5, \ldots, 2n+1\}$.

Example 3.5. The book graph B_4 on 10 vertices is shown below.

B_4 :



For any field F, the next result associates a lower bound for $\mathrm{mr}^F(B_n)$.

Proposition 3.6. *For $n \geq 2$, $Z(B_n) \leq n$.*

Proof. We claim that $\{v_1, v_5, v_7, v_9, \ldots, v_{2n+1}\}$ is a zero forcing set for B_n and so $Z(B_n) \leq n$. Assign $v_1, v_5, v_7, v_9, \ldots,$ and v_{2n+1} black and the other vertices white. For all k, $3 \leq k \leq n+1$, we can change the color of v_{2k} to black since v_{2k} is the only white vertex adjacent to v_{2k-1}. That is, $v_6, v_8, v_{10}, v_{12}, \ldots,$ and v_{2n+2} are black vertices. Then v_6 can force white vertex v_2 into black, also, v_1 and v_2 can force white vertices v_3 and v_4 into black, respectively. Now, the derived coloring of B_n is all black. Thus $\{v_1, v_5, v_7, v_9, \ldots, v_{2n+1}\}$ is a zero forcing set for B_n, as desired. Hence $Z(B_n) \leq n$. $\square$

Theorem 3.7. *For $n \geq 2$, there is a diagonal matrix D such that* $\operatorname{rank}(\mathcal{A}(B_n) + D) = n + 2$. *Moreover, B_n has field independent minimum rank, and $\mathcal{A}(B_n) + D$ is a universally optimal matrix for B_n.*

Proof. Let $D = \operatorname{diag}(0, n-2, 1, \ldots, 1)$. Clearly, $\mathcal{G}(\mathcal{A}(B_n) + D) \cong B_n$. We will exhibit n independent vectors $\vec{z}_1, \vec{z}_2, \ldots,$ and $\vec{z}_n$ in the kernel of $\mathcal{A}(B_n) + D$ to show that $\operatorname{null}(\mathcal{A}(B_n) + D) \geq n$. Then $\vec{z}_1 = (1, 1, -1, 0, \operatorname{repeat}[0, -1])^T$, $\vec{z}_2 = (0, 0, 1, -1, -1, 1, 0, \ldots, 0)^T$, $\vec{z}_3 = (0, 0, 0, 0, 1, -1, -1, 1, 0, \ldots, 0)^T, \ldots, \vec{z}_{n-1} = (0, \ldots, 0, 1, -1, -1, 1, 0, 0)^T$, and $\vec{z}_n = (0, \ldots, 0, 1, -1, -1, 1)^T$. We obtain $\operatorname{rank}(\mathcal{A}(B_n) + D) = 2n + 2 - \operatorname{null}(\mathcal{A}(B_n) + D) \leq 2n + 2 - n = n + 2$. Let F be any field. By Proposition 2.16 and 3.6, $\operatorname{mr}^F(B_n) \geq |B_n| - Z(B_n) \geq 2n+2-n = n+2$. By Remark 2.2, $\operatorname{rank}^F(\mathcal{A}(B_n) + D) \leq \operatorname{rank}(\mathcal{A}(B_n) + D) \leq n + 2$. We have $n + 2 \leq \operatorname{mr}^F(B_n) \leq \operatorname{rank}^F(\mathcal{A}(B_n) + D) \leq n + 2$. Then $\operatorname{mr}^F(B_n) = n + 2 = \operatorname{rank}^F(\mathcal{A}(B_n) + D)$. Thus $\operatorname{rank}(\mathcal{A}(B_n) + D) = n + 2$. Hence B_n has field independent minimum rank, and $\mathcal{A}(B_n) + D$ is a universally optimal matrix for B_n. □

Example 3.8. For the book graph B_4,

$$\mathcal{A}(B_4) + D = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 2 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

is a universally optimal matrix for B_4 where $D = \operatorname{diag}(0, 2, 1, 1, 1, 1, 1, 1, 1, 1)$ and $\operatorname{mr}^F(B_4) = 6$ for any field F.

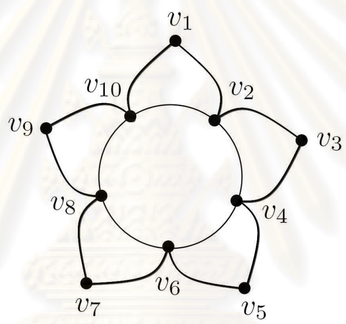
In the next section, we present the definition of the lotus graph and give results about this graph.

3.3 Lotus Graphs

Let n be a positive integer greater than 2. The *lotus graph* on $2n$ vertices, denoted Lt_n, is the graph for which $V(Lt_n) = \{v_1, v_2, \ldots, v_{2n}\}$ and $E(Lt_n) = \{v_iv_{i+1} : i = 1, 2, \ldots, 2n-1\} \cup \{v_1v_{2n}\} \cup \{v_{2i}v_{2(i+1)} : i = 1, 2, \ldots, n-1\} \cup \{v_2v_{2n}\}$.

Example 3.9. The lotus graph Lt_5 on 10 vertices is shown below.

Lt_5 :



For any field F, the next result associates lower bound for $\mathrm{mr}^F(Lt_n)$.

Proposition 3.10. *For $n \geq 3$, $Z(Lt_n) \leq n$.*

Proof. We claim that $\{v_1, v_2, v_4, v_6, \ldots, v_{2n-2}\}$ is a zero forcing set for Lt_n and so $Z(Lt_n) \leq n$. Assign $v_1, v_2, v_4, v_6, \ldots,$ and v_{2n-2} black and the other vertices white. We can change the color of v_{2n} to black since v_{2n} is the only white vertex adjacent to v_1. For $k = 1, 2, \ldots, n-1$, orderly, we can change the color of v_{2k+1} to black since v_{2k+1} is the only white vertex adjacent to v_{2k}, that is $v_3, v_5, v_7, \ldots,$ and v_{2n-1} are black vertices. Now, the derived coloring of Lt_n is all black. Then $\{v_1, v_2, v_4, v_6, \ldots, v_{2n-2}\}$ is a zero forcing set for Lt_n, as desired. Thus $Z(Lt_n) \leq n$. □

Theorem 3.11. *For $n \geq 3$, there is a matrix D such that* $\operatorname{rank}(\mathcal{A}(Lt_n) + D) = n$. *Moreover, Lt_n has field independent minimum rank, and $\mathcal{A}(Lt_n) + D$ is a universally optimal matrix for Lt_n.*

Proof. Let D be a $2n \times 2n$ matrix defined by

$$D = \begin{cases} \operatorname{diag}(\operatorname{repeat}[1,0,-1,0],1,2) + \operatorname{diag}'(\operatorname{repeat}[0,0,-2,-2],-2) & \\ + \operatorname{diag}''(\operatorname{repeat}[0,0,0,-2],0,0,0,0) & \text{if } n \text{ is odd,} \\ \operatorname{diag}(1,0,-1,-2,-1,\operatorname{repeat}[-2,-1,-2,-1],0,1,2) & \\ + \operatorname{diag}'(0,0,-2,0,\operatorname{repeat}[-2,0,-2,0],-2,-2,-2) & \text{if } n \text{ is even.} \end{cases}$$

Clearly, $\mathcal{G}(\mathcal{A}(Lt_n) + D) \cong Lt_n$. We exhibit n independent vectors $\vec{z}_1, \vec{z}_2, \ldots,$ and $\vec{z}_n$ in the kernel of $\mathcal{A}(Lt_n) + D$ to show that $\operatorname{null}(\mathcal{A}(Lt_n) + D) \geq n$. Then $\vec{z}_1 = (-1,1,1,0,\ldots,0)^T, \vec{z}_2 = (0,0,-1,1,1,0,\ldots,0)^T, \ldots, \vec{z}_{n-1} = (0,\ldots,0,-1,1,1,0)^T$, and $\vec{z}_n = (-1,0,\ldots,0,1,1)^T$. In any case, we obtain $\operatorname{rank}(\mathcal{A}(Lt_n) + D) = 2n - \operatorname{null}(\mathcal{A}(Lt_n)+D)) \leq 2n-n = n$. Let F be any field. By Proposition 2.16 and 3.10, $\operatorname{mr}^F(Lt_n) \geq |Lt_n| - Z(Lt_n) \geq 2n - n = n$. By Remark 2.2, $\operatorname{rank}^F(\mathcal{A}(Lt_n) + D) \leq \operatorname{rank}(\mathcal{A}(Lt_n) + D) = n$. We have $n \leq \operatorname{mr}^F(Lt_n) \leq \operatorname{rank}^F(\mathcal{A}(Lt_n) + D) \leq n$. Then $\operatorname{mr}^F(Lt_n) = n = \operatorname{rank}^F(\mathcal{A}(Lt_n) + D)$. Thus $\operatorname{rank}(\mathcal{A}(Lt_n) + D) = n$. Hence Lt_n has field independent minimum rank, and $\mathcal{A}(Lt_n) + D$ is a universally optimal matrix for Lt_n. □

Example 3.12. For the lotus graph Lt_5,

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 2 \end{bmatrix} = \mathcal{A}(Lt_5) + D$$

is a universally optimal matrix for Lt_5 where $D = \text{diag}(1, 0, -1, 0, 1, 0, -1, 0, 1, 2)+$ $\text{diag}^{'}(0, 0, -2, -2, 0, 0, -2, -2, -2) + \text{diag}^{''}(0, 0, 0, -2, 0, 0, 0, 0)$ and $\text{mr}^F(Lt_5) = 5$ for any field F.

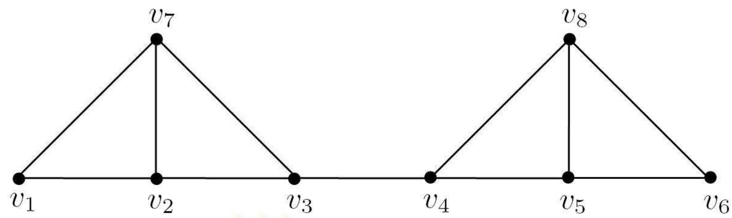
In the next section, we extend the definition of the path into the hanging bridge graph and we give results about this graph.

3.4 Hanging Bridge Graphs

Let n be a positive integer greater than 1. The *hanging bridge graph* on $4n$ vertices, denoted Hb_n, is the graph contructed from a path P_{3n} by appending n extra vertices, with each "extra"vertex adjacent to 3 sequential path vertices. Without loss of generality, let $v_1, v_2, \ldots,$ and v_{3n} be the vertices on path P_{3n} such that v_1 and v_{3n} have degree 2 and v_{3n+1}, $v_{3n+2}, \ldots,$ and v_{4n} be extra vertices in Hb_n.

Example 3.13. The hanging bridge graph Hb_2 on 8 vertices is shown below.

Hb_2 :



Proposition 3.14. *For $n \geq 2$, $Z(Hb_n) \leq n+1$.*

Proof. We claim that n extra vertices and one vertex of degree 2 form a zero forcing set for Hb_n. Let $V(Hb_n) = \{v_1, v_2, v_3, \ldots, v_{4n}\}$. Assign $v_1, v_{3n+1}, v_{3n+2}, \ldots,$ and v_{4n} black and the other vertices white. Claim that $\{v_1, v_{3n+1}, v_{3n+2}, \ldots, v_{4n}\}$ is a zero forcing set for Hb_n. For $k = 1, 2, \ldots, 3n$, orderly, we can change the color of v_{k+1} to black since v_{k+1} is the only white vertex adjacent to v_k, that is $v_2, v_3, v_4, \ldots,$ and v_{3n} are black vertices. Now, the derived coloring of Lt_n is all black. Thus $\{v_1, v_{3n+1}, v_{3n+2}, \ldots, v_{4n}\}$ is a zero forcing set for Hb_n, as desired. Hence $Z(Hb_n) \leq n+1$. □

Next we give result about the hanging bridge graph.

Lemma 3.15. *For $n \geq 2$, there exists a diagonal matrix D such that* $\text{rank}(\mathcal{A}(P_{3n}) + D) = 3n - 1$.

Proof. Let D be a $3n \times 3n$ diagonal matrix defined by

$$D = \begin{cases} \text{diag}(0, \text{repeat}[1], 0) & \text{if } n \text{ is odd,} \\ \text{diag}(0, 1, 0, \text{repeat}[0, 1, 1]) & \text{if } n \text{ is even.} \end{cases}$$

Clearly, $\mathcal{G}(\mathcal{A}(P_{3n}) + D) \cong P_{3n}$. We exhibit $\vec{z}$ in the kernel of $\mathcal{A}(P_{3n}) + D$ to show that $\text{null}(\mathcal{A}(P_{3n}) + D) \geq 1$. Consider the following 2 cases:

Case n is odd. Then $\vec{z} = (\text{repeat}[1, 0, -1])^T$.

Case n is even. Then $\vec{z} = (1, 0, -1, \text{repeat}[0, 1, -1])^T$.

In any case, we obtain $\text{rank}(\mathcal{A}(P_{3n}) + D) = 3n - \text{null}(\mathcal{A}(P_{3n}) + D) \leq 3n - 1$. By Table 2.1, $\text{mr}(P_{3n}) = 3n - 1$. We have $3n - 1 = \text{mr}(P_{3n}) \leq \text{rank}(\mathcal{A}(P_{3n}) + D) \leq 3n - 1$. Thus $\text{rank}(\mathcal{A}(P_{3n}) + D) = 3n - 1$. □

Theorem 3.16. *There exists a diagonal matrix D^* such that $\text{rank}(\mathcal{A}(Hb_n) + D^*) = 3n - 1$ for all $n \geq 2$. Moreover, Hb_n has field independent minimum rank, and $\mathcal{A}(Hb_n) + D^*$ is a universally optimal matrix for Hb_n.*

Proof. Let $D = \text{diag}(d_1, d_2, d_3, \ldots, d_{3n})$ be a diagonal matrix defined in the proof of Lemma 3.15 and $\text{rank}(\mathcal{A}(P_{3n}) + D) = 3n - 1$. Define $D^* = \text{diag}(d_1^*, d_2^*, \ldots, d_{4n}^*)$ where $d_i^* = d_i$ for all $i = 1, 2, 3, \ldots, 3n$ and $d_j^* = 1$ for all $j = 3n + 1, 3n + 2, 3n + 3, \ldots, 4n$. Clearly, $\mathcal{G}(\mathcal{A}(Hb_n) + D^*) \cong Hb_n$. The matrix $\mathcal{A}(Hb_n) + D^*$ has n duplicate rows and columns that can be deleted to leave $\mathcal{A}(P_{3n}) + D$ without changing the rank, that is $\text{rank}(\mathcal{A}(Hb_n) + D^*) = \text{rank}(\mathcal{A}(P_{3n}) + D) = 3n - 1$. Let F be any field. By Table 2.1, $\text{mr}^F(P_{3n}) = 3n - 1$. Since P_{3n} is an induced subgraph of Hb_n and by Proposition 2.8 (1), $\text{mr}^F(Hb_n) \geq \text{mr}^F(P_{3n}) = 3n - 1$. By Remark 2.2, $\text{rank}^F(\mathcal{A}(Hb_n) + D^*) \leq \text{rank}(\mathcal{A}(Hb_n) + D^*) = 3n - 1$. We have $3n - 1 \leq \text{mr}^F(Hb_n) \leq \text{rank}^F(\mathcal{A}(Hb_n) + D^*) \leq 3n - 1$. Then $\text{mr}^F(Hb_n) = 3n - 1 = \text{rank}^F(\mathcal{A}(Hb_n) + D^*)$. Hence Hb_n has field independent minimum rank, and $\mathcal{A}(Hb_n) + D^*$ is a universally optimal matrix for Hb_n. □

Example 3.17. For the hanging bridge graph Hb_2,

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix} = \mathcal{A}(Hb_n) + D$$

is a universally optimal matrix for Hb_2 where $D = \mathrm{diag}(0, 1, 0, 0, 1, 1, 1, 1)$ and $\mathrm{mr}^F(Hb_2) = 5$ for any field F.

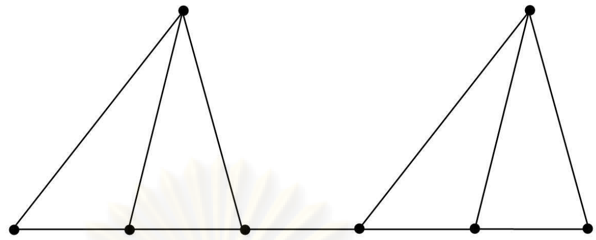
In the next example, we will construct the graph G by adding some "extra"vertex and appropriate edges to a hanging bridge graph Hb_n such that Hb_n is an induced subgraph of G and the minimum rank over a field F of G is equal to the minimum rank over F of Hb_n.

Example 3.18. Let

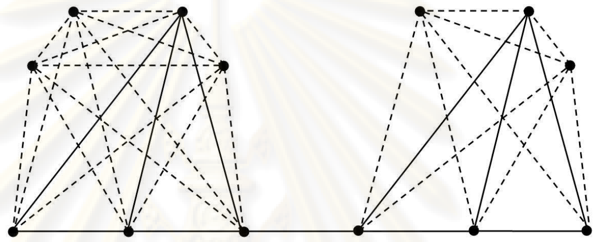
$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

with $\mathrm{rank}(A) = 5$ and G be the graph obtained from Hb_2 by adding 5 extra vertices and appropriate edges as shown below. We will show that $\mathrm{mr}^F(G) = \mathrm{mr}^F(Hb_2)$ for any field F.

Hb_2 :



G :

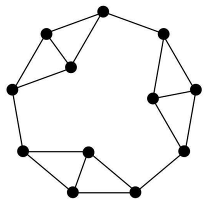


Let F be any field. By Theorem 3.16, $\mathrm{mr}^F(Hb_2) = 5$. Since Hb_2 is an induced subgraph of G and by Proposition 2.8 (1), $\mathrm{mr}^F(Hb_2) \leq \mathrm{mr}^F(G)$. By Remark 2.2, $\mathrm{rank}^F(A) \leq \mathrm{rank}(A) = 5$. We have $5 \leq \mathrm{mr}^F(G) \leq \mathrm{rank}^F(A) \leq 5$. Then $\mathrm{mr}^F(G) = 5 = \mathrm{rank}^F(A)$. Clearly, $\mathcal{G}(A) \cong G$. Thus A is a universally optimal matrix for G. Hence G has a universally optimal matrix, field independent minimum rank, and $\mathrm{mr}^F(G) = \mathrm{mr}^F(Hb_2)$ for any field F.

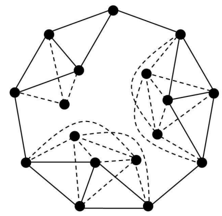
In [4], L.M. DeAlba et al. showed that a necklace with s diamonds N_s has a universally optimal matrix $\mathcal{A}(N_s) + I_{4s}$, has field independent minimum rank, and $\mathrm{mr}^F(N_s) = 3s - 2$ for any field F.

In the next example, we will construct the graph G by adding the "extra" vertex and appropriate edges to a necklace with s diamonds N_s such that N_s is an induced subgraph of G and the minimum rank over a field F of G is equal to the minimum rank over F of N_s.

Example 3.19.



N_3



G

Let

$$A = \begin{bmatrix}
1&1&0&0&0&0&0&0&1&1&1&0&0&0&0&0&0\\
1&1&1&0&0&0&0&0&0&0&0&1&1&1&0&0&0\\
0&1&1&1&0&0&0&0&0&0&0&1&1&1&0&0&0\\
0&0&1&1&1&0&0&0&0&0&0&1&1&1&0&0&0\\
0&0&0&1&1&1&0&0&0&0&0&0&0&0&1&1&1\\
0&0&0&0&1&1&1&0&0&0&0&0&0&0&1&1&1\\
0&0&0&0&0&1&1&1&0&0&0&0&0&0&1&1&1\\
0&0&0&0&0&0&1&1&1&1&1&0&0&0&0&0&0\\
1&0&0&0&0&0&0&1&1&1&1&0&0&0&0&0&0\\
1&0&0&0&0&0&0&1&1&1&1&0&0&0&0&0&0\\
1&0&0&0&0&0&0&1&1&1&1&0&0&0&0&0&0\\
0&1&1&1&0&0&0&0&0&0&0&1&1&1&0&0&0\\
0&1&1&1&0&0&0&0&0&0&0&1&1&1&0&0&0\\
0&1&1&1&0&0&0&0&0&0&0&1&1&1&0&0&0\\
0&0&0&0&1&1&1&0&0&0&0&0&0&0&1&1&1\\
0&0&0&0&1&1&1&0&0&0&0&0&0&0&1&1&1\\
0&0&0&0&1&1&1&0&0&0&0&0&0&0&1&1&1
\end{bmatrix}$$

with $\operatorname{rank}(A) = 7$ and G be the graph obtained from N_3 by adding 5 extra vertices and appropriate edges as shown above. We will show that $\operatorname{mr}^F(G) = \operatorname{mr}^F(N_3)$ for any field F. Let F be any field. We know that $\operatorname{mr}^F(N_3) = 7$. Since N_3 is an induced subgraph of G and by Proposition 2.8 (1), $\operatorname{mr}^F(N_3) \leq \operatorname{mr}^F(G)$. By Remark 2.2, $\operatorname{rank}^F(A) \leq \operatorname{rank}(A) = 7$. We have $7 \leq \operatorname{mr}^F(G) \leq \operatorname{rank}^F(A) \leq 7$. Then $\operatorname{mr}^F(G) = 7 = \operatorname{rank}^F(A)$. Clearly, $\mathcal{G}(A) \cong G$. Thus A is a universally optimal matrix for G. Hence G has a universally optimal matrix, field independent minimum rank, and $\text{mr}^F(G) = \text{mr}^F(N_3)$ for any field F.

In next section, we give the definition of the path-cycle graph and show that this graph has field independent minimum rank directly. Then we determine a universally optimal matrix for this graph as well.

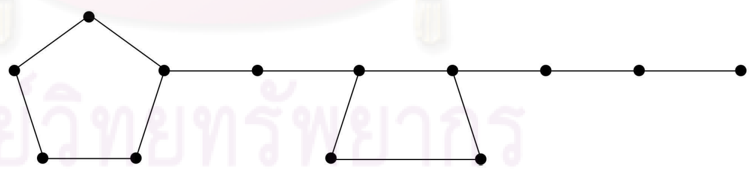
3.5 Path-cycle Graphs

Let k be a positive integer. A *path-cycle graph*, denoted $PC(m_1, m_2, \ldots, m_k; n_1, n_2, \ldots, n_{k-1})$, is obtained from paths $P_{m_1}, P_{m_2}, \ldots,$ and P_{m_k} and cycles $C_{n_1}, C_{n_2}, \ldots,$ and $C_{n_{k-1}}$ constructed so that for $i = 2, 3, \ldots, k$ and $j < i$, $V(P_{m_{i-1}}) \cap V(C_{n_{i-1}})$ and $V(P_{m_i}) \cap V(C_{n_{i-1}})$ have exactly one vertex and $V(P_{m_j}) \cap V(P_{m_i})$, $V(C_{n_{j-1}}) \cap V(C_{n_{i-1}})$, $V(C_{n_{j-1}}) \cap V(P_{m_i})$, and $V(P_{m_{j-1}}) \cap V(C_{n_{i-1}})$ have no vertices.

Clearly, $|PC(m_1, m_2, \ldots, m_k; n_1, n_2, \ldots, n_{k-1})| = \sum_{i=1}^{k} m_i + \sum_{i=1}^{k-1} n_i - 2(k-1)$.

Example 3.20. The path-cycle graph $PC(1, 3, 4; 5, 4)$ is shown below.

$PC(1, 3, 4; 5, 4)$:



Proposition 3.21. *For $k \geq 1$, $Z(PC(m_1, m_2, \ldots, m_k; n_1, n_2, \ldots, n_{k-1})) \leq k$.*

Proof. Let $v_2, v_3, \ldots,$ and v_k be any vertex of degree 2 in $C_{n_1}, C_{n_2}, \ldots,$ and $C_{n_{k-1}}$, respectively such that each is adjacent to the common vertex of P_{m_i} and C_{n_j}. If $m_1 = 1$, then let v_1 be the common vertex of P_{m_1} and C_{n_1}; otherwise, let v_1 be the end vertex of P_{m_1} but not the common vertex of P_{m_1} and C_{n_1}. Then

$\{v_1, v_2, \ldots, v_k\}$ is a zero forcing set of $PC(m_1, m_2, \ldots, m_k; n_1, n_2, \ldots, n_{k-1})$ because there is only one white vertex adjacent to a black vertex so the derived coloring of $PC(m_1, m_2, \ldots, m_k; n_1, n_2, \ldots, n_{k-1})$ is all black. Thus $Z(PC(m_1, m_2, \ldots, m_k; n_1, n_2, \ldots, n_{k-1})) \leq k$. □

Proposition 3.22. *For $k \geq 1$, $\mathrm{mr}^F(PC(m_1, m_2, \ldots, m_k; n_1, n_2, \ldots, n_{k-1})) = \sum_{i=1}^{k} m_i + \sum_{i=1}^{k-1} n_i - 2(k-1) - k$ for any field F. Thus $PC(m_1, m_2, \ldots, m_k; n_1, n_2, \ldots, n_{k-1})$ has field independent minimum rank.*

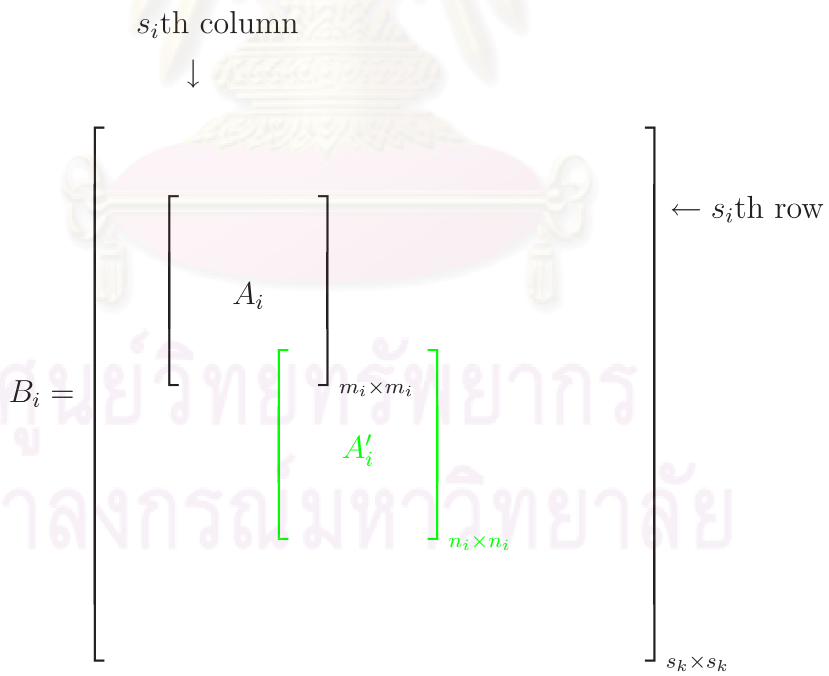
Proof. Let F be any field. By Proposition 2.16 and 3.21, $\sum_{i=1}^{k} m_i + \sum_{i=1}^{k-1} n_i - 2(k-1) - k \leq |PC(m_1, m_2, \ldots, m_k; n_1, n_2, \ldots, n_{k-1})| - Z(PC(m_1, m_2, \ldots, m_k; n_1, n_2, \ldots, n_{k-1})) \leq \mathrm{mr}^F(PC(m_1, m_2, \ldots, m_k; n_1, n_2, \ldots, n_{k-1}))$. Let $S = \{P_{m_1}, P_{m_2}, \ldots, P_{m_k}, C_{n_1}, C_{n_2}, \ldots, C_{n_{k-1}}\}$ and clearly S is a covering of $PC(m_1, m_2, \ldots, m_k; n_1, n_2, \ldots, n_{k-1})$. By Proposition 2.6, for any i, j, $1 \leq i \leq k$ and $1 \leq j \leq k-1$, P_{m_i} and C_{n_j} have universally optimal matrices of the from $\mathcal{A}(P_{m_i}) + D_i$ and $\mathcal{A}(C_{n_j}) + D_j^*$, respectively where D_i and D_j^* are diagonal matrices. Let e be any edge in $PC(m_1, m_2, \ldots, m_k; n_1, n_2, \ldots, n_{k-1})$. Since for $i = 2, 3, \ldots, k$ and $j < i$, $E(P_{m_j}) \cap E(P_{m_i})$, $E(C_{n_{j-1}}) \cap E(C_{n_{i-1}})$, $E(P_{m_i}) \cap E(C_{n_j})$ and $E(P_{m_1}) \cap E(C_{n_i})$ have no edges, $c_S(e) = 1$. By Proposition 2.11, $\mathrm{mr}^F(PC(m_1, m_2, \ldots, m_k; n_1, n_2, \ldots, n_{k-1})) \leq \sum_{i=1}^{k} \mathrm{mr}^F(P_{m_i}) + \sum_{i=1}^{k-1} \mathrm{mr}^F(C_{n_i})$. By Table 2.1, for any i, j, $1 \leq i \leq k$ and $1 \leq j \leq k-1$, $\mathrm{mr}^F(P_{m_i}) = m_i - 1$ and $\mathrm{mr}^F(C_{n_j}) = n_j - 2$. We have $\mathrm{mr}^F(PC(m_1, m_2, \ldots, m_k; n_1, n_2, \ldots, n_{k-1})) \leq \sum_{i=1}^{k} \mathrm{mr}^F(P_{m_i}) + \sum_{i=1}^{k-1} \mathrm{mr}^F(C_{n_i}) = (m_1 - 1) + (m_2 - 1) + \cdots + (m_k - 1) + (n_1 - 2) + (n_2 - 2) + \cdots + (n_{k-1} - 2) = m_1 + m_2 + \cdots + m_k - k + n_1 + n_2 + \cdots + n_{k-1} - 2(k-1) = \sum_{i=1}^{k} m_i + \sum_{i=1}^{k-1} n_i - 2(k-1) - k$.

Thus $\mathrm{mr}^F(PC(m_1, m_2, \ldots, m_k; n_1, n_2, \ldots, n_{k-1})) = \sum_{i=1}^{k} m_i + \sum_{i=1}^{k-1} n_i - 2(k-1) - k$ for any field F. □

We also establish a universally optimal matrix for $PC(m_1, m_2, \ldots, m_k; n_1, n_2, \ldots, n_{k-1})$.

Proposition 3.23. *For $k \geq 1$, $PC(m_1, m_2, \ldots, m_k; n_1, n_2, \ldots, n_{k-1})$ has a universally optimal matrix.*

Proof. Let $A_1, A_1', A_2, A_2', \ldots, A_{k-1}, A_{k-1}'$, and A_k be universally optimal matrices for $P_{m_1}, C_{n_1}, P_{m_2}, C_{n_2}, \ldots, P_{m_{k-1}}, C_{n_{k-1}}$, and P_{m_k}, respectively. Then $\operatorname{rank}(A_i) = \operatorname{mr}(P_{m_i}) = m_i - 1$ and $\operatorname{rank}(A_i') = \operatorname{mr}(C_{n_i}) = n_i - 2$ for all i. Let $s_j = m_1 + n_1 + m_2 + n_2 + \cdots + m_{j-1} + n_{j-1} - 2(j-1) + 1$ for all $j \in \{1, 2, \ldots, k\}$. For $i = 1, 2, \ldots, k$, we construct the matrix B_i by embedding A_i into the $s_k \times s_k$ zero matrix at the s_ith row and s_ith column as shown below



and for $i = 1, 2, \ldots, k-1$, we define the matrix B_i' by embedding A_i' into the $s_k \times s_k$ zero matrix at the $(s_i + m_i - 1)$th row and $(s_i + m_i - 1)$th column as shown below

$$
\begin{array}{c}
(s_i + m_i - 1)\text{th column} \\
\downarrow
\end{array}
$$

$$
B_i' = \begin{bmatrix} & & & \\ & \begin{bmatrix} & & \\ & A_i & \\ & & \end{bmatrix}_{m_i \times m_i} & & \\ & & \begin{bmatrix} & & \\ & A_i' & \\ & & \end{bmatrix}_{n_i \times n_i} & \\ & & & \end{bmatrix}_{s_k \times s_k} \leftarrow (s_i + m_i - 1)\text{th row}
$$

We see that $\text{rank}(A_i) = \text{rank}(B_i)$ and $\text{rank}(A_i') = \text{rank}(B_i')$ for all i. Let $A = \sum_{i=1}^{k-1}(B_i + B_i') + B_k$. Clearly, $\mathcal{G}(A) \cong PC(m_1, m_2, \ldots, m_k; n_1, n_2, \ldots, n_{k-1})$. We obtain $\text{rank}(A) \leq \sum_{i=1}^{k}\text{rank}(B_i) + \sum_{i=1}^{k-1}\text{rank}(B_i') = \sum_{i=1}^{k}\text{rank}(A_i) + \sum_{i=1}^{k-1}\text{rank}(A_i') = \sum_{i=1}^{k}(m_i - 1) + \sum_{i=1}^{k-1}(n_i - 2) = \sum_{i=1}^{k} m_i - k + \sum_{i=1}^{k-1} n_i - 2(k-1) = \sum_{i=1}^{k} m_i + \sum_{i=1}^{k-1} n_i - 2(k-1) - k$. Let F be any field. By Remark 2.2, $\text{rank}^F(A) \leq \text{rank}(A) \leq \sum_{i=1}^{k} m_i + \sum_{i=1}^{k-1} n_i - 2(k-1) - k$. By Proposition 3.22, $\sum_{i=1}^{k} m_i + \sum_{i=1}^{k-1} n_i - 2(k-1) - k = \text{mr}^F(PC(m_1, m_2, \ldots, m_k; n_1, n_2, \ldots, n_{k-1})) \leq \text{rank}^F(A) \leq \sum_{i=1}^{k} m_i + \sum_{i=1}^{k-1} n_i - 2(k-1) - k$. Then $\text{mr}^F(PC(m_1, m_2, \ldots, m_k; n_1, n_2, \ldots, n_{k-1})) = \text{rank}^F(A)$. Thus A is a universally optimal matrix for $PC(m_1, m_2, \ldots, m_k; n_1, n_2, \ldots, n_{k-1})$. $\square$

Example 3.24. By Proposition 3.23, $\mathrm{mr}^F(PC(1,3,4;5,4)) = 10$ for any field F with

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

is a universally optimal matrix and $PC(1,3,4;5,4)$ has field independent minimum rank.

The definition of the path-cycle graph can be extended by replacing some cycle in path-cycle graph with a polygonal path and show that the resulting graph has a universally optimal matrix and field independent minimum rank as shown in the next example.

Example 3.25. The graph G as shown in Figure 3.1 consists of paths $P_2^{(1)}$, $P_3^{(2)}, P_3^{(3)}$, and $P_3^{(4)}$, polygonal path G_1 consisted of $C_4^{(5)}$ and $C_6^{(6)}$, polygonal path G_2 consisted of $C_5^{(7)}, C_4^{(8)}$, and $C_6^{(9)}$, and polygonal path G_3 consisted of $C_5^{(10)}$ and $C_4^{(11)}$ which $V(P_2^{(1)}) \cap V(C_4^{(5)}), V(C_6^{(6)}) \cap V(P_3^{(2)}), V(P_3^{(2)}) \cap V(C_5^{(7)}), V(C_6^{(8)}) \cap V(P_3^{(3)}), V(P_3^{(3)}) \cap V(C_4^{(6)})$, and $V(C_4^{(7)}) \cap V(P_3^{(4)})$ have only one vertex. We show that G has a universally optimal matrix and field independent minimum rank.

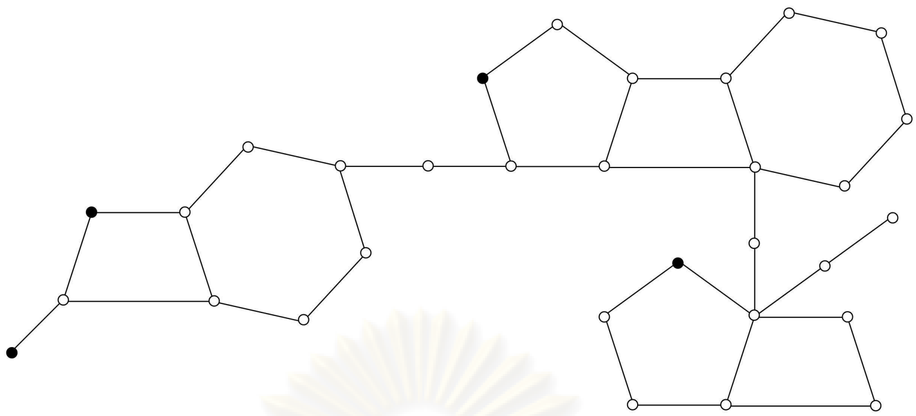


Figure 3.1: The graph G with $\mathrm{mr}^F(G) = 27$

We see that $Z(G) \leq 4$. Let F be any field. By Proposition 2.16, $\mathrm{mr}^F(G) \geq |G| - Z(G) \geq |G| - 4 = 27$. Let A_1, A_2, A_3, and A_4 be universally optimal matrices for P_2, P_3, P_3, and P_3, respectively. In Table 2.1, G_1, G_2, and G_3 have universally optimal matrices, say A_5, A_6, and A_7, respectively. For all $i = 1, 2, 3, \ldots, 7$, let B_i be constructed (similarly to the construction in Proposition 3.23) by embedding A_i in the appropriate place in a 27×27 matrix with $\mathrm{rank}(A_i) = \mathrm{rank}(B_i)$. Let $A = \sum_{i=1}^{7} B_i$. Then $\mathrm{rank}(A) \leq \sum_{i=1}^{7} \mathrm{rank}(B_i) = \sum_{i=1}^{7} \mathrm{rank}(A_i) = (2-1) + (3-1) + (3-1) + (3-1) + (8-2) + (11-2) + (7-2) = 27$. We obtain that $27 \leq \mathrm{mr}^F(G) \leq \mathrm{rank}^F(A) \leq \mathrm{rank}(A) \leq 27$. Thus $\mathrm{mr}^F(G) = 27 = \mathrm{rank}^F(A)$. Clearly, $\mathcal{G}(A) \cong G$. Hence G has a universally optimal matrix and field independent minimum rank.

In next section, we give the definition of the path-clique graph and show that this graph has field independent minimum rank directly. Also a universally optimal matrix for this graph is determined.

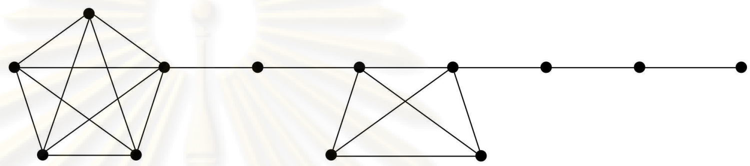
3.6 Path-clique Graphs

Let k be a positive integer. A *path-clique graph*, denoted $PK(m_1, m_2, \ldots, m_k; n_1, n_2, \ldots, n_{k-1})$, is obtained from paths $P_{m_1}, P_{m_2}, \ldots,$ and P_{m_k} and complete graphs $K_{n_1}, K_{n_2}, \ldots,$ and $K_{n_{k-1}}$ constructed so that for $i = 2, 3, \ldots, k$ and $j < i$, $V(P_{m_{i-1}}) \cap V(K_{n_{i-1}})$ and $V(P_{m_i}) \cap V(K_{n_{i-1}})$ have exactly one vertex and $V(P_{m_j}) \cap V(P_{m_i})$, $V(K_{n_{j-1}}) \cap V(K_{n_{i-1}})$, $V(K_{n_{j-1}}) \cap V(P_{m_i})$, and $V(P_{m_{j-1}}) \cap V(K_{n_{i-1}})$ have no vertices.

Clearly, $|PK(m_1, m_2, \ldots, m_k; n_1, n_2, \ldots, n_{k-1})| = \sum_{i=1}^{k} m_i + \sum_{i=1}^{k-1} n_i - 2(k-1)$.

Example 3.26. The path-clique graph $PK(1, 3, 4; 5, 4)$ is shown below.

$PK(1, 3, 4; 5, 4)$:



Proposition 3.27. *For $k \geq 1$, $Z(PK(m_1, m_2, \ldots, m_k; n_1, n_2, \ldots, n_{k-1})) \leq \sum_{i=1}^{k-1} n_i - 2k - 3$.*

Proof. Let $v_1^{(j)}, v_2^{(j)}, \ldots,$ and $v_{n_j-2}^{(j)}$ be any vertex of degree $n_j - 1$ in K_{n_j} such that each is adjacent to the common vertex of P_{m_j} and K_{n_j}. If $m_1 = 1$, then let v_0 be the common vertex of P_{m_1} and K_{n_1}; otherwise, let v_0 be the end vertex of P_{m_1} but not the common vertex of P_{m_1} and K_{n_1}. Then $\{v_0, v_1^{(1)}, v_2^{(1)}, \ldots, v_{n_1-2}^{(1)}, v_1^{(2)}, v_2^{(2)}, \ldots, v_{n_2-2}^{(2)}, \ldots, v_1^{(k-1)}, v_2^{(k-1)}, \ldots, v_{n_{k-1}-2}^{(k-1)}\}$ is a zero forcing set of $PK(m_1, m_2, \ldots, m_k; n_1, n_2, \ldots, n_{k-1})$ because there is only one white vertex adjacent to a black vertex so the derived coloring of $PK(m_1, m_2, \ldots, m_k; n_1, n_2, \ldots, n_{k-1})$ is all black. Thus $Z(PK(m_1, m_2, \ldots, m_k; n_1, n_2, \ldots, n_{k-1})) \leq \sum_{i=1}^{k-1} n_i - 2k - 3$. □

Proposition 3.28. *For $k \geq 1$, $mr^F(PK(m_1, m_2, \ldots, m_k; n_1, n_2, \ldots, n_{k-1})) = \sum_{i=1}^{k} m_i - 1$ for any field F. Thus $PK(m_1, m_2, \ldots, m_k; n_1, n_2, \ldots, n_{k-1})$ has field independent minimum rank.*

Proof. Let F be any field. By Table 2.1, $\mathrm{mr}^F(P_{m_1+m_2+\cdots+m_k}) = \sum_{i=1}^{k} m_i - 1$. Since

$P_{m_1+m_2+\cdots+m_k}$ is an induced subgraph of $PK(m_1, m_2, \ldots, m_k;\ n_1, n_2, \ldots,\ n_{k-1})$ and by Proposition 2.8 (1), $\sum_{i=1}^{k} m_i - 1 = \text{mr}^F(P_{m_1+m_2+\cdots+m_k}) \leq \text{mr}^F(PK(m_1, m_2, \ldots, m_k; n_1, n_2, \ldots, n_{k-1}))$. Let $C = \{P_{m_1}, P_{m_2}, \ldots, P_{m_k}, K_{n_1}, K_{n_2}, \ldots, K_{n_{k-1}}\}$ and cleary C is a covering of $PK(m_1, m_2, \ldots, m_k; n_1, n_2, \ldots, n_{k-1})$. By Proposition 2.6, for any i, j, $1 \leq i \leq k$ and $1 \leq j \leq k-1$, P_{m_i} and K_{n_j} have universally optimal matrices of the from $\mathcal{A}(P_{m_i}) + D_i$ and $\mathcal{A}(K_{n_j}) + D_j^*$, respectively where D_i and D_j^* are diagonal matrices. Let e be any edge in $PK(m_1, m_2, \ldots, m_k; n_1, n_2, \ldots, n_{k-1})$. Since for $i = 2, 3, \ldots, k$ and $j < i$, $E(P_{m_j}) \cap E(P_{m_i})$, $E(K_{n_{j-1}}) \cap E(K_{n_{i-1}})$, $E(P_{m_i}) \cap E(K_{n_j})$, and $E(P_{m_1}) \cap E(K_{n_i})$ have no edges, $c_C(e) = 1$. By Proposition 2.11, $\text{mr}^F(PK(m_1, m_2, \ldots, m_k; n_1, n_2, \ldots, n_{k-1})) \leq \sum_{i=1}^{k} \text{mr}^F(P_{m_i}) + \sum_{i=1}^{k-1} \text{mr}^F(K_{n_i})$ $= \sum_{i=1}^{k} m_i - k + (k-1) = \sum_{i=1}^{k} m_i - 1$. Thus $\text{mr}^F(PK(m_1, m_2, \ldots, m_k; n_1, n_2, \ldots, n_{k-1})) = \sum_{i=1}^{k} m_i - 1$ for any field F. □

Proposition 3.29. *For $k \geq 1$, $PK(m_1, m_2, \ldots, m_k; n_1, n_2,\ \ldots, n_{k-1})$ has a universally optimal matrix.*

Proof. Let $A_1, A_1', A_2, A_2', \ldots, A_{k-1}, A_{k-1}'$, and A_k be universally optimal matrices for $P_{m_1}, K_{n_1}, P_{m_2}, K_{n_2}, \ldots, P_{m_{k-1}}, K_{n_{k-1}}$, and P_{m_k}, respectively. Then $\text{rank}(A_i) = \text{mr}(P_{m_i}) = m_i - 1$ and $\text{rank}(A_i') = \text{mr}(K_{n_i}) = 1$ for all i. Let $s_j = m_1 + n_1 + m_2 + n_2 + \cdots + m_{j-1} + n_{j-1} - 2(j-1) + 1$ for all $j \in \{1, 2, \ldots, k\}$. For $i = 1, 2, \ldots, k$, let B_i and B_i' be constructed (similarly to the construction in Proposition 3.23) by embedding A_i into the $s_k \times s_k$ zero matrix at the s_ith row and s_ith column with $\text{rank}(A_i) = \text{rank}(B_i)$ and $\text{rank}(A_i') = \text{rank}(B_i')$. Again, similar argument in Proposition 3.23 is applied. We obtain $\text{mr}^F(PC(m_1, m_2, \ldots, m_k; n_1, n_2, \ldots, n_{k-1})) = \sum_{i=1}^{k} m_i - 1 = \text{rank}^F(A)$. Thus A is a universally optimal matrix for $PK(m_1, m_2, \ldots, m_k;\ n_1, n_2, \ldots, n_{k-1})$. □

Example 3.30. By Proposition 3.29, $\mathrm{mr}^F(PK(1,3,4;5,4)) = 7$. for any field F with

$$\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1
\end{bmatrix}$$

is a universally optimal matrix and $PK(1,3,4;5,4)$ has field independent minimum rank.

CHAPTER IV

FIELD DEPENDENCE RESULTS

In our work, we also present some graphs which do not have field independence of minimum rank and these graphs do not have a universally optimal matrix.

4.1 The Join of Paths and Complete Graphs

Recall that for $t \geq 3, s \geq 2$, $P_t \vee K_s$ is the union of graphs P_t and K_s, with disjoint vertex sets $V(P_t)$ and $V(K_s)$, and all the edges joining $V(P_t)$ and $V(K_s)$.

First, we compute $\mathrm{mr}(P_t \vee K_s)$.

Proposition 4.1. *For $t \geq 3, s \geq 2$, $\mathrm{mr}(P_t \vee K_s) = t - 1$.*

Proof. By Table 2.1, $\mathrm{mr}(P_t) = t - 1$. Since P_t is an induced subgraph of $P_t \vee K_s$ and by Proposition 2.8 (1), $\mathrm{mr}(P_t) \leq \mathrm{mr}(P_t \vee K_s)$. We have $t - 1 \leq \mathrm{mr}(P_t \vee K_s)$. We will exhibit $s + 1$ independent vectors $\vec{z}_1, \vec{z}_2, \ldots$, and $\vec{z}_{s+1}$ in the kernel of a matrix A such that $\mathcal{G}(A) \cong P_t \vee K_s$. Let $V(P_t) = \{v_1, v_2, \ldots, v_t\}$ and $V(K_s) = \{v_{t+1}, v_{t+2}, \ldots, v_{t+s}\}$. Consider the following 4 cases:

Case $t = 3$. Let $A = \mathcal{A}(P_3 \vee K_s) +$ diag$(0, 0, 0, \underbrace{1, \ldots, 1}_{s})$. Then $\vec{z}_1 = (\underbrace{0, \ldots, 0}_{s+1}, 1, -1)^T$, $\vec{z}_2 = (\underbrace{0, \ldots, 0}_{s}, 1, -1, 0)^T$, $\vec{z}_3 = (\underbrace{0, \ldots, 0}_{s-1}, 1, -1, 0, 0)^T, \ldots, \vec{z}_{s-1} = (0, 0, 0, 1, -1, \underbrace{0, \ldots, 0}_{s-2})^T, \vec{z}_s = (1, 0, -1, \underbrace{0, \ldots, 0}_{s})^T$, and $\vec{z}_{s+1} = (0, 1, 0, -1, \underbrace{0, \ldots, 0}_{s-1})^T$.

Case $t \equiv 0(\text{mod}4)$. Let $A = \mathcal{A}(P_t \vee K_s) + D$ where

$$D = \text{diag}(1,0,0,\text{repeat}[0,0,0,0],1,\underbrace{1,\ldots,1}_{s}) + \begin{bmatrix} \mathbf{0}_{t\times t} & \mathbf{0}_{t\times s} \\ \mathbf{0}_{s\times t} & \begin{bmatrix} \frac{t-2}{2} & \cdots & \frac{t-2}{2} \\ \vdots & \ddots & \vdots \\ \frac{t-2}{2} & \cdots & \frac{t-2}{2} \end{bmatrix}_{s\times s} \end{bmatrix}.$$

Then $\vec{z}_1 = (\underbrace{0,\ldots,0}_{t+s-2},1,-1)^T, \vec{z}_2 = (\underbrace{0,\ldots,0}_{t+s-3},1,-1,0)^T, \vec{z}_3 = (\underbrace{0,\ldots,0}_{t+s-4},1,-1,0,0)^T$, $\ldots, \vec{z}_{s-1} = (\underbrace{0,\ldots,0}_{t},1,-1,\underbrace{0,\ldots,0}_{s-2})^T, \vec{z}_s = (\text{repeat}[1,-1,-1,1],\underbrace{0,\ldots,0}_{s})^T$, and $\vec{z}_{s+1} = (\text{repeat}[1,0,0,1],-1,\underbrace{0,\ldots,0}_{s-1})^T$.

Case $t \equiv 1(\text{mod}4)$. Let $A = \mathcal{A}(P_t \vee K_s) + D$ where

$$D = \text{diag}(1,1,0,1,\text{repeat}[0,0,0,0],1,\underbrace{1,\ldots,1}_{s}) + \begin{bmatrix} \mathbf{0}_{t\times t} & \mathbf{0}_{t\times s} \\ \mathbf{0}_{s\times t} & \begin{bmatrix} \frac{t-3}{2} & \cdots & \frac{t-3}{2} \\ \vdots & \ddots & \vdots \\ \frac{t-3}{2} & \cdots & \frac{t-3}{2} \end{bmatrix}_{s\times s} \end{bmatrix}.$$

Then $\vec{z}_1 = (\underbrace{0,\ldots,0}_{t+s-2},1,-1)^T, \vec{z}_2 = (\underbrace{0,\ldots,0}_{t+s-3},1,-1,0)^T$, $\vec{z}_3 = (\underbrace{0,\ldots,0}_{t+s-4},1,-1,0,0)^T$, $\ldots, \vec{z}_{s-1} = (\underbrace{0,\ldots,0}_{t},1,-1,\underbrace{0,\ldots,0}_{s-2})^T$, $\vec{z}_s = (1,-1,0,1,-1,\text{repeat}[-1,1,1,-1],$ $\underbrace{0,\ldots,0}_{s})^T$, and $\vec{z}_{s+1} = (1,0,0,1,0,\text{repeat}[0,1,1,0],-1,\underbrace{0,\ldots,0}_{s-1})^T$.

Case $t \equiv 2(\text{mod}4)$. Let $A = \mathcal{A}(P_t \vee K_s) + D$ where

$$D = \text{diag}(1,\text{repeat}[0,0,0,0],1,\underbrace{1,\ldots,1}_{s}) + \begin{bmatrix} \mathbf{0}_{t\times t} & \mathbf{0}_{t\times s} \\ \mathbf{0}_{s\times t} & \begin{bmatrix} \frac{t-2}{2} & \cdots & \frac{t-2}{2} \\ \vdots & \ddots & \vdots \\ \frac{t-2}{2} & \cdots & \frac{t-2}{2} \end{bmatrix}_{s\times s} \end{bmatrix}.$$

Then $\vec{z}_1 = (\underbrace{0,\ldots,0}_{t+s-2},1,-1)^T, \vec{z}_2 = (\underbrace{0,\ldots,0}_{t+s-3},1,-1,0)^T$, $\vec{z}_3 = (\underbrace{0,\ldots,0}_{t+s-4},1,-1,0,0)^T$, $\ldots, \vec{z}_{s-1} = (\underbrace{0,\ldots,0}_{t},1,-1,\underbrace{0,\ldots,0}_{s-2})^T$, $\vec{z}_s = (1,\text{repeat}[-1,-1,1,1],-1,\underbrace{0,\ldots,0}_{s})^T$, and $\vec{z}_{s+1} = (1,\text{repeat}[0,0,1,1],0,-1,\underbrace{0,\ldots,0}_{s-1})^T$.

Case $t \equiv 3(\text{mod}4)$ and $t \neq 3$. Let $A = \mathcal{A}(P_t \vee K_s) + D$ where

$$D = \text{diag}(1, 1, 0, 1, 0, 0, \text{repeat}[0, 0, 0, 0], 1, \underbrace{1, \ldots, 1}_{s}) + \begin{bmatrix} \mathbf{0}_{t\times t} & \mathbf{0}_{t\times s} \\ \mathbf{0}_{s\times t} & \begin{bmatrix} \frac{t-3}{2} & \cdots & \frac{t-3}{2} \\ \vdots & \ddots & \vdots \\ \frac{t-3}{2} & \cdots & \frac{t-3}{2} \end{bmatrix}_{s\times s} \end{bmatrix}.$$

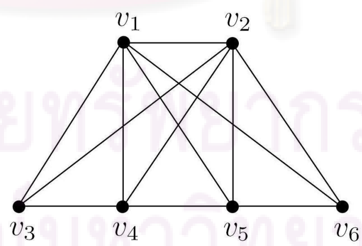
Then $\vec{z}_1 = (\underbrace{0, \ldots, 0}_{t+s-2}, 1, -1)^T, \vec{z}_2 = (\underbrace{0, \ldots, 0}_{t+s-3}, 1, -1, 0)^T$, $\vec{z}_3 = (\underbrace{0, \ldots, 0}_{t+s-4}, 1, -1, 0, 0)^T$, $\ldots, \vec{z}_{s-1} = (\underbrace{0, \ldots, 0}_{t}, 1, -1, \underbrace{0, \ldots, 0}_{s-2})^T, \vec{z}_s = (1, -1, 0, \text{repeat}[1, -1, -1, 1], \underbrace{0, \ldots, 0}_{s})^T$, and $\vec{z}_{s+1} = (1, 0, 0, \text{repeat}[1, 0, 0, 1], -1, \underbrace{0, \ldots, 0}_{s-1})^T$.

In any case, we obtain $s + 1 \leq \text{null}(A)$. Then $\text{rank}(A) = (t + s) - \text{null}(A) \leq (t + s) - (s + 1) = t - 1$. We have $t - 1 \leq \text{mr}(P_t \vee K_s) \leq \text{rank}(A) \leq t - 1$. Thus $\text{mr}(P_t \vee K_s) = t - 1$. □

The next example, it is shown that $P_4 \vee K_2$ does not have field independent minimum rank.

Example 4.2. $P_4 \vee K_2 (\cong P_4 \vee P_2)$ does not have field independent minimum rank.

$P_4 \vee K_2$:



Let $A \in S_6^{\mathbb{Z}_2}$ be such that $\mathcal{G}^{\mathbb{Z}_2}(A) \cong P_4 \vee K_2$. We can write

$$A = \begin{bmatrix} d_1 & 1 & 1 & 1 & 1 & 1 \\ 1 & d_2 & 1 & 1 & 1 & 1 \\ 1 & 1 & d_3 & 1 & 0 & 0 \\ 1 & 1 & 1 & d_4 & 1 & 0 \\ 1 & 1 & 0 & 1 & d_5 & 1 \\ 1 & 1 & 0 & 0 & 1 & d_6 \end{bmatrix}$$

where $d_1, d_2, \ldots, d_6 \in \mathbb{Z}_2$. It is easily to show that vectors $(1,1,d_3,1,0,0), (1,1,1,d_4,1,0)$, and $(1,1,0,1,d_5,1)$ are linearly independent. Then $\text{rank}(A) \geq 3$. Suppose that $\text{rank}(A) = 3$. Then $\{(1,1,d_3,1,0,0), (1,1,1,d_4,1,0), (1,1,0,1,d_5,1)\}$ is maximal independent subset of the row vector space of A. Thus $(1,1,0,0,1,d_6) = a \cdot (1,1,d_3,1,0,0) + b \cdot (1,1,1,d_4,1,0) + c \cdot (1,1,0,1,d_5,1)$ for some $a,b,c \in \mathbb{Z}_2$. We obtain $a = 1, b = 1, c = 1, d_3 = 1, d_4 = 0, d_5 = 0$, and $d_6 = 1$. Then $(1,d_2,1,1,1,1)$ cannot be written as a linear combination of $(1,1,1,1,0,0), (1,1,1,0,1,0)$, and $(1,1,0,1,1,1)$, a contradiction. Thus $\text{rank}(A) \geq 4$. Since A is arbitrary, $\text{mr}^{\mathbb{Z}_2}(P_4 \vee K_2) \geq 4$. Let $B \in S_6^{\mathbb{Z}_2}$ be such that

$$B = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}$$

with $\text{rank}(B) = 4$. Clearly, $\mathcal{G}^{\mathbb{Z}_2}(B) \cong P_4 \vee K_2$. Then $\text{mr}^{\mathbb{Z}_2}(P_4 \vee K_2) = 4$. By Proposition 4.1, $\text{mr}(P_t \vee K_s) = 3$. Thus $\text{mr}(P_4 \vee K_2) = 3 < 4 = \text{mr}^{\mathbb{Z}_2}(P_4 \vee K_2)$, i.e., $P_4 \vee K_2$ does not have field independent minimum rank. By Remark 2.5, $P_4 \vee K_2$ does not have a universally optimal matrix.

4.2 The Join of Cycles and Complete Graphs

Recall that for $t \geq 3, s \geq 2$, $C_t \vee K_s$ is the union of graphs C_t and K_s, with disjoint vertex sets $V(C_t)$ and $V(K_s)$, and all the edges joining $V(C_t)$ and $V(K_s)$.

First, we compute $\text{mr}(C_t \vee K_s)$.

Proposition 4.3. *For* $t \geq 3, s \geq 2$, $\text{mr}(C_t \vee K_s) = t - 2$.

Proof. By Table 2.1, $\text{mr}(P_{t-1}) = t - 2$. Since P_{t-1} is an induced subgraph of $C_t \vee K_s$ and by Proposition 2.8, $t - 2 = \text{mr}(P_{t-1}) \leq \text{mr}(C_t \vee K_s)$. We will exhibit

$s+2$ independent vectors $\vec{z}_1, \vec{z}_2, \ldots,$ and $\vec{z}_{s+2}$ in the kernel of a matrix A such that $\mathcal{G}(A) \cong C_t \vee K_s$. Let $V(C_t) = \{v_1, v_2, \ldots, v_t\}$ and $V(K_s) = \{v_{t+1}, v_{t+2}, \ldots, v_{t+s}\}$. Consider the following 4 cases:

Case $t = 3$. Let $A = \mathcal{A}(C_3 \vee K_s) + I_{s+3}$. Then $\vec{z}_1 = (1, \underbrace{0, \ldots, 0}_{s+1}, -1)^T, \vec{z}_2 = (1, \underbrace{0, \ldots, 0}_{s}, -1, 0)^T, \vec{z}_3 = (1, \underbrace{0, \ldots, 0}_{s-1}, -1, 0, 0)^T, \ldots,$ and $\vec{z}_{s+2} = (1, -1, \underbrace{0, \ldots, 0}_{s+1})^T$.

Case $t = 5$. Let

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 2 & 2 & \cdots & 2 \\ 1 & -1 & 1 & 0 & 0 & 1 & 1 & \cdots & 1 \\ 0 & 1 & -1 & 1 & 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & -1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 0 & 1 & 0 & 2 & 2 & \cdots & 2 \\ 2 & 1 & 1 & 1 & 2 & 7 & 7 & \cdots & 7 \\ 2 & 1 & 1 & 1 & 2 & 7 & 7 & \cdots & 7 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 2 & 1 & 1 & 1 & 2 & 7 & 7 & \cdots & 7 \end{bmatrix}_{(s+5)\times(s+5)}.$$

Then $\vec{z}_1 = (1, 1, 1, 1, 1, \underbrace{0, \ldots, 0}_{s-1}, -1)^T, \vec{z}_2 = (1, 1, 1, 1, 1, \underbrace{0, \ldots, 0}_{s-2}, -1, 0)^T, \vec{z}_3 = (1, 1, 1, 1, 1, \underbrace{0, \ldots, 0}_{s-3}, -1, 0, 0)^T, \ldots, \vec{z}_s = (1, 1, 1, 1, 1, -1, \underbrace{0, \ldots, 0}_{s-1})^T, \vec{z}_{s+1} = (0, 1, 1, 0, -1, \underbrace{0, \ldots, 0}_{s})^T,$ and $\vec{z}_{s+2} = (-1, 0, 1, 1, 0, \underbrace{0, \ldots, 0}_{s})^T$.

Case $t \equiv 0 (\mathrm{mod} 4)$. Let $A = \mathcal{A}(C_t \vee K_s) + D$ where

$$D = \mathrm{diag}(\underbrace{0, \ldots, 0}_{t}, \underbrace{1, \ldots, 1}_{s}) + \begin{bmatrix} \mathbf{0}_{t\times t} & \mathbf{0}_{t\times s} \\ \mathbf{0}_{s\times t} & \begin{bmatrix} \frac{t-2}{2} & \cdots & \frac{t-2}{2} \\ \vdots & \ddots & \vdots \\ \frac{t-2}{2} & \cdots & \frac{t-2}{2} \end{bmatrix}_{s\times s} \end{bmatrix}.$$

Then $\vec{z}_1 = (\mathrm{repeat}[1, 1, 0, 0], \underbrace{0, \ldots, 0}_{s-1}, -1)^T, \vec{z}_2 = (\mathrm{repeat}[1, 1, 0, 0], \underbrace{0, \ldots, 0}_{s-2}, -1, 0)^T,$ $\vec{z}_3 = (\mathrm{repeat}[1, 1, 0, 0], \underbrace{0, \ldots, 0}_{s-3}, -1, 0, 0)^T, \ldots, \vec{z}_s = (\mathrm{repeat}[1, 1, 0, 0], -1, \underbrace{0, \ldots, 0}_{s-1})^T,$ $\vec{z}_{s+1} = (\mathrm{repeat}[1, 0, -1, 0], \underbrace{0, \ldots, 0}_{s})^T,$ and $\vec{z}_{s+2} = (\mathrm{repeat}[0, 1, 0, -1], \underbrace{0, \ldots, 0}_{s})^T$.

Case $t \equiv 1(\mathrm{mod}4)$ and $t \neq 5$. Let $A = \mathcal{A}(C_t \vee K_s) + D$ where

$$D = \mathrm{diag}(1,2,1,1,0,\mathrm{repeat}[0,0,0,0],1,1,1,1,\underbrace{1,\ldots,1}_{s}) + \begin{bmatrix} \mathbf{0}_{t\times t} & \mathbf{0}_{t\times s} \\ \mathbf{0}_{s\times t} & \begin{bmatrix} \frac{t-5}{2} & \cdots & \frac{t-5}{2} \\ \vdots & \ddots & \vdots \\ \frac{t-5}{2} & \cdots & \frac{t-5}{2} \end{bmatrix}_{s\times s} \end{bmatrix}.$$

Then $\vec{z}_1 = (1,0,0,1,0,\mathrm{repeat}[0,1,1,0],0,1,0,0,\underbrace{0,\ldots,0}_{s-1},-1)^T, \vec{z}_2 = (1,0,0,1,0,$ $\mathrm{repeat}[0,1,1,0],0,1,0,0,\underbrace{0,\ldots,0}_{s-2},-1,0)^T$, $\vec{z}_3 = (1,0,0,1,0,\mathrm{repeat}[0,1,1,0],0,1,0,$ $0,\underbrace{0,\ldots,0}_{s-3},-1,0,0)^T,\ldots,\vec{z}_s = (1,0,0,1,0,\mathrm{repeat}[0,1,1,0],0,1,0,0,-1,\underbrace{0,\ldots,0}_{s-1})^T$, $\vec{z}_{s+1} = (-1,0,1,-1,0,\mathrm{repeat}[1,0,-1,0],1,-1,0,1,\underbrace{0,\ldots,0}_{s})^T$, and $\vec{z}_{s+2} = (1,-1,$ $1,0,-1,\mathrm{repeat}[0,1,0,-1],0,1,-1,\underbrace{0,\ldots,0}_{s+1})^T$.

Case $t \equiv 2(\mathrm{mod}4)$. Let $A = \mathcal{A}(C_t \vee K_s) + D$ where

$$D = \mathrm{diag}(1,1,1,1,1,1,\mathrm{repeat}[0,0,0,0],\underbrace{1,\ldots,1}_{s}) + \begin{bmatrix} \mathbf{0}_{t\times t} & \mathbf{0}_{t\times s} \\ \mathbf{0}_{s\times t} & \begin{bmatrix} \frac{t-4}{2} & \cdots & \frac{t-4}{2} \\ \vdots & \ddots & \vdots \\ \frac{t-4}{2} & \cdots & \frac{t-4}{2} \end{bmatrix}_{s\times s} \end{bmatrix}.$$

Then $\vec{z}_1 = (0,1,0,0,1,0,\mathrm{repeat}[0,1,1,0],\underbrace{0,\ldots,0}_{s-1},-1)^T, \vec{z}_2 = (0,1,0,0,1,0,\mathrm{repeat}$ $[0,1,1,0],\underbrace{0,\ldots,0}_{s-2},-1,0)^T$, $\vec{z}_3 = (0,1,0,0,1,0,\mathrm{repeat}[0,1,1,0],\underbrace{0,\ldots,0}_{s-3},-1,0,0)^T$, $\ldots$, $\vec{z}_s = (0,1,0,0,1,0,\mathrm{repeat}[0,1,1,0],-1,\underbrace{0,\ldots,0}_{s-1})^T$, $\vec{z}_{s+1} = (1,-1,0,1,-1,$ 0, $\mathrm{repeat}[1,0,-1,0],\underbrace{0,\ldots,0}_{s})^T$, and $\vec{z}_{s+2} = (0,1,-1,0,1,-1$, $\mathrm{repeat}[0,1,0,-1]$, $\underbrace{0,\ldots,0}_{s})^T$.

Case $t \equiv 3(\mathrm{mod}4)$ and $t \neq 3$. Let $A = \mathcal{A}(C_t \vee K_s) + D$ where

$$D = \mathrm{diag}(1,1,1,\mathrm{repeat}[0,0,0,0],\underbrace{1,\ldots,1}_{s}) + \begin{bmatrix} \mathbf{0}_{t\times t} & \mathbf{0}_{t\times s} \\ \mathbf{0}_{s\times t} & \begin{bmatrix} \frac{t-3}{2} & \cdots & \frac{t-3}{2} \\ \vdots & \ddots & \vdots \\ \frac{t-3}{2} & \cdots & \frac{t-3}{2} \end{bmatrix}_{s\times s} \end{bmatrix}.$$

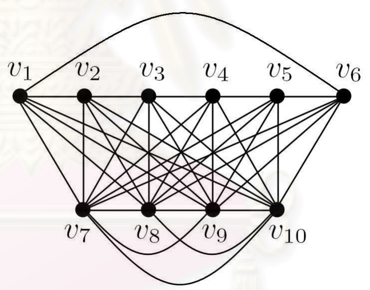
Then $\vec{z}_1 = (0,1,0,\text{repeat}[0,1,1,0],\underbrace{0,\ldots,0}_{s-1},-1)^T$, $\vec{z}_2 = (0,1,0,\text{repeat}[0,1,1,0],\underbrace{0,\ldots,0}_{s-2},-1,0)^T$, $\vec{z}_3 = (0,1,0,\text{repeat}[0,1,1,0],\underbrace{0,\ldots,0}_{s-3},-1,0,0)^T,\ldots,\vec{z}_s = (0,1,0,\text{repeat}[0,1,1,0],-1,\underbrace{0,\ldots,0}_{s-1})^T$, $\vec{z}_{s+1} = (1,-1,0,\text{repeat}[1,0,-1,0],\underbrace{0,\ldots,0}_{s})^T$, and $\vec{z}_{s+2} = (0,1,-1,\text{repeat}[0,1,0,-1],\underbrace{0,\ldots,0}_{s})^T$.

In any case, we obtain $s+2 \leq \text{null}(A)$. Then $\text{rank}(A) = (t+s) - \text{null}(A) \leq (t+s)-(s+2) = t-2$. We have $t-2 \leq \text{mr}(C_t \vee K_s) \leq \text{rank}(A) \leq t-2$. Thus $\text{mr}(C_t \vee K_s) = t-2$. □

The next example, it is shown that $C_6 \vee K_4$ does not have field independent minimum rank.

Example 4.4. $C_6 \vee K_4$ does not have field independent minimum rank.

$C_6 \vee K_4$:



Let $A \in S_{10}^{\mathbb{Z}_2}$ be such that $\mathcal{G}^{\mathbb{Z}_2}(A) \cong C_6 \vee K_4$. We can write

$$A = \begin{bmatrix} d_1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & d_2 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & d_3 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & d_4 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & d_5 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & d_6 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & d_7 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & d_8 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & d_9 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & d_{10} \end{bmatrix}$$

where $d_1, d_2, \ldots, d_{10} \in \mathbb{Z}_2$. Claim that vectors $(1, d_2, 1, 0, 0, 0, 1, 1, 1, 1), (0, 1, d_3, 1, 0, 0, 1, 1, 1, 1), (0, 0, 1, d_4, 1, 0, 1, 1, 1, 1), (0, 0, 0, 1, d_5, 1, 1, 1, 1, 1),$ and $(1, 1, 1, 1, 1, 1, d_7, 1, 1, 1)$ are linearly independent. Let $\alpha_1, \alpha_2, \ldots, \alpha_5 \in \mathbb{Z}_2$ be such that $\alpha_1(1, d_2, 1, 0, 0, 0, 1, 1, 1, 1) + \alpha_2(0, 1, d_3, 1, 0, 0, 1, 1, 1, 1) + \alpha_3(0, 0, 1, d_4, 1, 0, 1, 1, 1, 1) + \alpha_4(0, 0, 0, 1, d_5, 1, 1, 1, 1, 1) + \alpha_5(1, 1, 1, 1, 1, 1, d_7, 1, 1, 1) = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$ Then $\alpha_1 + \alpha_5 = 0, \alpha_1 d_2 + \alpha_2 + \alpha_5 = 0, \alpha_1 + \alpha_2 d_3 + \alpha_3 + \alpha_5 = 0, \alpha_2 + \alpha_3 d_4 + \alpha_4 + \alpha_5 = 0, \alpha_3 + \alpha_4 d_5 + \alpha_5 = 0, \alpha_4 + \alpha_5 = 0, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 d_7 = 0, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 = 0$. Suppose that $\alpha_5 = 1$. Then $\alpha_1 = 1, \alpha_4 = 1$, and $\alpha_2 + \alpha_3 = 1$. If $\alpha_2 = 0$, then $\alpha_3 = 1$ which is impossible. If $\alpha_2 = 1$, then $\alpha_3 = 0$ which is impossible. Thus $\alpha_5 = 0$ which implies $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$, as desired. Then $\text{rank}(A) \geq 5$. Since A is arbitrary, $\text{mr}^{\mathbb{Z}_2}(C_6 \vee K_4) \geq 5$. Let $B \in S_{10}^{\mathbb{Z}_2}$ be such that

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

with $\text{rank}(B) = 5$. Clearly, $\mathcal{G}^{\mathbb{Z}_2}(B) \cong C_6 \vee K_4$. Then $\text{mr}^{\mathbb{Z}_2}(C_6 \vee K_4) = 5$. By Proposition 4.3, $\text{mr}(C_6 \vee K_4) = 4$. Thus $\text{mr}(C_6 \vee K_4) = 4 < 5 = \text{mr}^{\mathbb{Z}_2}(C_6 \vee K_4)$, i.e., $C_6 \vee K_4$ does not have field independent minimum rank. By Remark 2.5, $C_6 \vee K_4$ does not have a universally optimal matrix.

4.3 Clique Paths

For $i = 1, 2, \ldots, k, m_i \geq 3$. A *clique path*, denoted $KP(m_1, m_2, \ldots, m_k)$, is the "path" of complete graphs built from complete graphs $K_{m_1}, K_{m_2}, \ldots,$ and K_{m_k} constructed so that for $i = 2, 3, \ldots, k$ and $j < i - 1$, $E(K_{m_{i-1}}) \cap E(K_{m_i})$ has exactly one edge and $V(K_{m_j}) \cap V(K_{m_i})$ has no vertices.

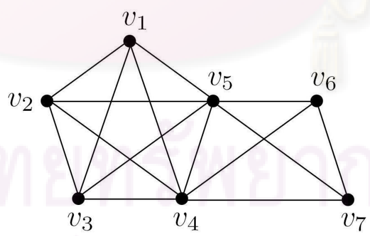
Remark 4.5. $|KP(m_1, m_2, \ldots, m_k)| = \sum_{i=1}^{k} m_i - 2(k-1)$.

Proposition 4.6. $\text{mr}(KP(m_1, m_2, \ldots, m_k)) = k$.

Proof. Clearly, P_{k+1} is an induced subgraph of $KP(m_1, m_2, \ldots, m_k)$. By Table 2.1 and Proposition 2.8 (1), $k = \text{mr}(P_{k+1}) \leq \text{mr}(KP(m_1, m_2, \ldots, m_k))$. We can view that $KP(m_1, m_2, \ldots, m_k)$ is the union of the complete graphs $K_{m_1}, K_{m_2}, \ldots,$ and K_{m_k}. By Table 2.1 and Proposition 2.8 (2), $\text{mr}(KP(m_1,\ m_2, \ldots, m_k)) \leq \text{mr}(K_{m_1}) + \text{mr}(K_{m_2}) + \cdots + \text{mr}(K_{m_k}) = k$. Thus $\text{mr}(KP(m_1, m_2, \ldots, m_k)) = k$. □

Example 4.7. $KP(5, 4)$ does not have field independent minimum rank.

$KP(5,4)$:



Let $A \in S_7^{\mathbb{Z}_2}$ be such that $\mathcal{G}(A) \cong KP(5, 4)$. We can write

$$A = \begin{bmatrix} d_1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & d_2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & d_3 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & d_4 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & d_5 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & d_6 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & d_7 \end{bmatrix}$$

where $d_1, d_2, \ldots, d_7 \in \mathbb{Z}_2$. It is easily to show that vectors $(1,1,d_3,1,1,0,0)$, $(1,1,1,d_4,1,1,1)$, and $(0,0,0,1,1,d_6,1)$ are linearly independent. Then $\text{rank}(A) \geq 3$. Since A is arbitrary, $\text{mr}^{\mathbb{Z}_2}(KP(5,4)) \geq 3$. Let $B \in S_7^{\mathbb{Z}_2}$ be such that

$$B = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

with $\text{rank}(B) = 3$. Clearly, $\mathcal{G}^{\mathbb{Z}_2}(B) = KP(5,4)$. Then $\text{mr}^{\mathbb{Z}_2}(KP(5,4)) = 3$. By Proposition 4.6, $\text{mr}(KP(5,4)) = 2$. Thus $\text{mr}(KP(5,4)) = 2 < 3 = \text{mr}^{\mathbb{Z}_2}(KP(5,4))$, i.e., $KP(5,4)$ does not have field independent minimum rank. By Remark 2.5, $KP(5,4)$ does not have a universally optimal matrix.

4.4 Clique-cycle Paths

For $i = 1,2,\ldots,k, m_i \geq 3$. A *clique-cycle path*, denoted $KC(m_1, m_2, \ldots, m_k; n_1, n_2, \ldots, n_k)$, is obtained from complete graphs $K_{m_1}, K_{m_2}, \ldots,$ and K_{m_k} and cycles $C_{n_1}, C_{n_2}, \ldots,$ and C_{n_k} constructed so that for $i = 2,3,\ldots,k$ and $j < i$, $E(K_{m_1}) \cap E(C_{n_1})$, $E(K_{m_i}) \cap E(C_{n_i})$, and $E(K_{m_i}) \cap E(C_{n_{i-1}})$ have exactly one edge and $V(K_{m_j}) \cap V(K_{m_i})$, $V(C_{n_j}) \cap V(C_{n_i})$, $V(K_{m_j}) \cap V(C_{n_i})$, and $V(C_{n_{j-1}}) \cap V(K_{m_i})$ have no vertices.

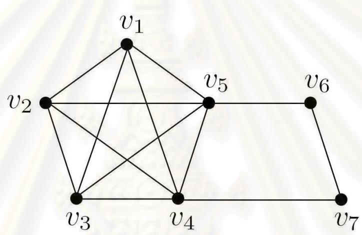
Remark 4.8. $|KC(m_1, m_2, \ldots, m_k; n_1, n_2, \ldots, n_k)| = \sum_{i=1}^{k} m_i + \sum_{i=1}^{k} n_i - 4k + 2.$

Proposition 4.9. $\mathrm{mr}(KC(m_1, m_2, \ldots, m_k; n_1, n_2, \ldots, n_k)) \leq \sum_{i=1}^{k} n_i - k.$

Proof. We can view that $KC(m_1, m_2, \ldots, m_k; n_1, n_2, \ldots, n_k)$ is the union of complete graphs K_{m_1}, $K_{m_2}, \ldots,$ and K_{m_k} and cycles $C_{n_1}, C_{n_2}, \ldots,$ and C_{n_k}. By Table 2.1, $\mathrm{mr}(K_{m_i}) = 1$ and $\mathrm{mr}(C_{n_i}) = n_i - 2$ for all $i = 1, 2, \ldots, k$. By Proposition 2.8 (2), $\mathrm{mr}(KC(m_1, m_2, \ldots, m_k; n_1, n_2, \ldots, n_k)) \leq \sum_{i=1}^{k} \mathrm{mr}(K_{m_i}) + \sum_{i=1}^{k} \mathrm{mr}(C_{n_i}) = k + \sum_{i=1}^{k} n_i - 2k = \sum_{i=1}^{k} n_i - k.$ □

Example 4.10. $KC(5; 4)$ does not have field independent minimum rank.

$KC(5; 4)$:



Let $A \in S_7^{\mathbb{Z}_2}$ be such that $\mathcal{G}(A) \cong KC(5; 4)$. We can write

$$A = \begin{bmatrix} d_1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & d_2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & d_3 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & d_4 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & d_5 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & d_6 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & d_7 \end{bmatrix}$$

where $d_1, d_2, \ldots, d_7 \in \mathbb{Z}_2$. It is easily to show that vectors $(1, 1, d_3, 1, 1, 0, 0), (1, 1, 1, 1, d_5, 1, 0), (0, 0, 0, 0, 1, d_6, 1),$ and $(0, 0, 0, 1, 0, 1, d_7)$ are linearly independent. Then $\mathrm{rank}(A) \geq 4$. Since A is arbitrary, $\mathrm{mr}^{\mathbb{Z}_2}(KC\ (5; 4)) \geq 4$. Let $B \in S_7^{\mathbb{Z}_2}$ be such that

$$B = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

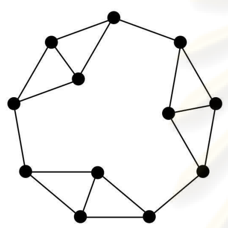
with $\text{rank}(B) = 4$. Clearly, $\mathcal{G}^{\mathbb{Z}_2}(B) = KC(5;4)$. Then $\text{mr}^{\mathbb{Z}_2}(KC(5;4)) = 4$. By Proposition 4.9, $\text{mr}(KC(5;4)) \leq 3$. Clearly, P_4 is an induced subgraph of $KC(5;4)$. By Table 2.1 and Proposition 2.8, $3 = \text{mr}(P_4) \leq \text{mr}(KC(5;4))$. Then $\text{mr}(KC(5;4)) = 3$. Thus $\text{mr}(KC(5;4)) = 3 < 4 = \text{mr}^{\mathbb{Z}_2}(KC(5;4))$, i.e., $KC(5;4)$ does not have field independent minimum rank. By Remark 2.5, $KC(5;4)$ does not have a universally optimal matrix.

Question. Which values of t and k that the family $P_t \vee K_s$, $C_t \vee K_s$, $KP(m_1, m_2, \ldots, m_k)$, and $KC(m_1, m_2, \ldots, m_k; n_1, n_2, \ldots, n_k)$ have field independent minimum rank?

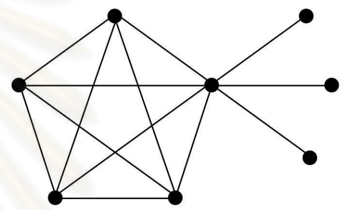
APPENDIX

The *necklace* with s diamonds, denoted N_s, is a graph that can be constructed from a cycle C_{3s} by appending s extra vertices, with each "extra" vertex adjacent to 3 sequential cycle vertices.

The m, k-*pineapple* (with $m \geq 3, k \geq 2$), denoted $P_{m,k}$, is the graph $K_m \cup K_{1,k}$ such that a vertex in $V(K_m) \cap V(K_{1,k})$ is the vertex of $K_{1,k}$ of degree k.



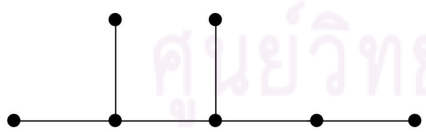
N_3



$P_{5,3}$

A *tree* is a connected graph with n vertices and $n-1$ edges.

A *unicyclic* is a connected graph containing exactly one cycle.



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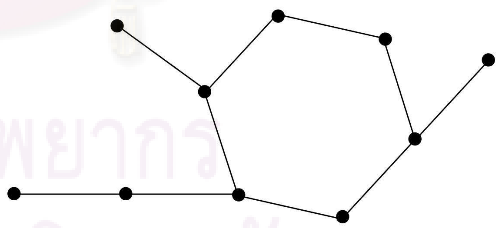
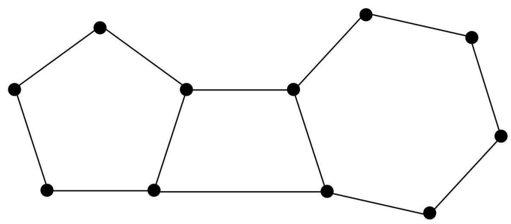


unicyclic

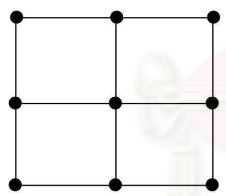
A *polygonal path* is a "path" of cycles built from cycles $C_{m_1}, C_{m_2}, \ldots,$ and C_{m_k} constructed so that for $i = 2, 3, \ldots, k$ and $j < i-1$, $E(C_{m_{i-1}}) \cap E(C_{m_i})$ has exactly one edge and $E(C_{m_j}) \cap E(C_{m_i})$ has no edges.



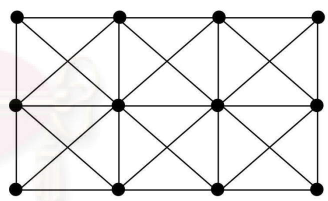
polygonal path built from C_5, C_4 and C_6

The *Cartesian product* of two graphs G and H, denoted $G \Box H$, is the graph with vertex set $V(G) \times V(H)$ such that (u, v) is adjacent to (u', v') if and only if (1) $u = u'$ and $vv' \in E(H)$, or (2) $v = v'$ and $uu' \in E(G)$.

The *strong product* of two graphs G and H, denoted $G \boxtimes H$, is the graph with vertex set $V(G) \times V(H)$ such that (u, v) is adjacent to (u', v') if and only if (1) $uu' \in E(G)$ and $vv' \in E(H)$, or (2) $u = u'$ and $vv' \in E(H)$, or (3) $v = v'$ and $uu' \in E(G)$.



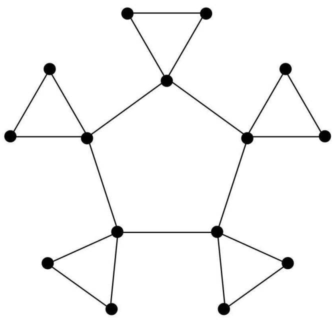
$P_3 \Box P_3$



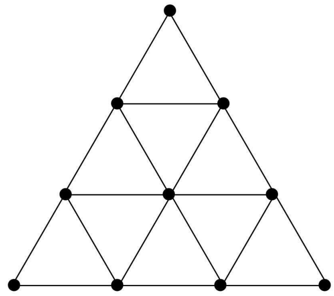
$P_3 \boxtimes P_4$

The *corona* of a graph G with a graph H, denoted $G \circ H$, is the graph on $|G||H| + |G|$ vertices obtained by taking one copy of G and $|G|$ copies of H, and joining all the vertices in the ith copy of H to the ith vertex of G.

The nth *supertriangle*, denoted T_n, is a graph G with vertex set $V(G) = \{(i, j) : i = 1, 2, \ldots, n \text{ and } j = 1, 2, \ldots, i\}$ such that (i, j) is adjacent to (i', j') if and only if (1) $|i - i'| = 1$ and $|j - j'| = 0$, or (2) $|i - i'| = 0$ and $|j - j'| = 1$, or (3) $|i - i'| = 1$ and $|j - j'| = 1$. Clearly, $|T_n| = \frac{1}{2}n(n + 1)$.



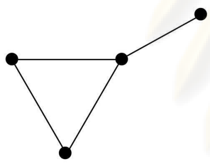
$C_5 \circ K_2$



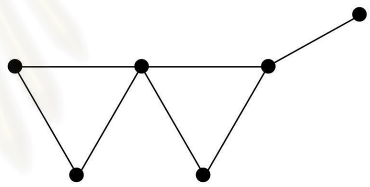
T_4

A *block* of a graph G is a maximal connected subgraph of G that has no cut-vertex. A *block-clique graph* is a graph in which every block is a clique.

A graph is *claw-free* if it dose not contain an induced $K_{1,3}$.



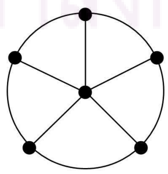
block-clique graph G



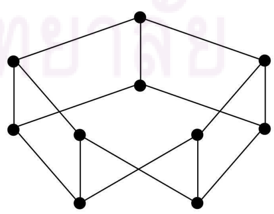
claw-free block-clique H

The nth wheel, denoted W_n, is the graph $K_1 \vee C_{n-1}$.

The sth *Möbius ladder*, denoted M_s, is obtained from $C_s \square P_2$ by replacing one pair of parallel cycle edges with a crossed pair.

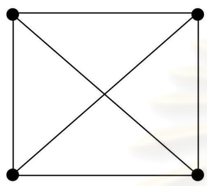


W_6

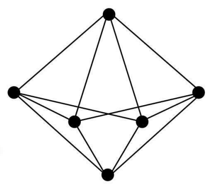


M_5

The *line graph* of a graph G, denoted $L(G)$, is the graph having vertex set $E(G)$, with two vertices in $L(G)$ adjacent if and only if the corresponding edges share an endpoint in G. Since we require a graph to have a nonempty set of vertices, the line graph $L(G)$ is defined only for a graph G that has at least one edge.

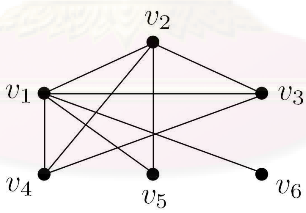


K_4



$L(K_4)$

The sth *half-graph*, denoted H_s, is the graph is constructed from (disjoint) graphs K_s and $\overline{K_s}$, having vertices $u_1, u_2, \ldots,$ and u_s and $v_{s+1}, v_{s+2}, \ldots,$ and v_{2s}, respectively, by adding all edges $u_i u_j$ such that $i + j \leq 2s + 1$.



H_3

A *2-tree* is a graph built from K_3 by adding to it one vertex at a time adjacent to exactly a pair of existing adjacent vertices.

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