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SARVATE-BEAM GROUP DIVISIBLE DESIGNS AND RESTRICTED SIMPLE 1-DESIGNS

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A t-Sarvate-Beam group divisible design and a restricted simple 1-design are defined. Some necessary conditions for existence and constructions of both designs are obtained.

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CONTENTS

CHAPTER I

INTRODUCTION

1.1 Motivation and preliminary

Group divisible designs and t-designs including simple 1-designs are well known combinatorial designs (see [10]). In 2007, Sarvate and Beam introduced a new type of block designs called *adesigns*, which were renamed by Stanton as the *Sarvate-Beam designs.* These designs have been studied by several mathematicians including Sarvate, Beam and Stanton etc. (see [3], [4], [5], [6], [7], [8], and [9]). These designs motivate us to study two new families of designs called *t-Sarvate-Beam group divisible designs* and *restricted simple 1-designs.* So in this thesis, we are using a very new concept to modify old designs with a new twist and proving existence and non-existence of certain families of these new designs we have developed. First, we recall definitions of group divisible designs, t-designs and t-Sarvate-Beam designs (see [1] and [4]).

Definition 1.1.1. Let m, n and k be positive integers such that $m \geq k \geq 2$. A **group divisible design GDD**(m, n, k), is a triple $(V, \mathscr{G}, \mathscr{B})$ satisfying the following properties :

1. V is a finite set of mn elements called **points**,

2. $\mathscr{G}$ is a partition of V into m nonempty subsets of size n called **groups**,

3. $\mathscr{B}$ is a collection of k-subsets of V called **blocks**,

4. any two points from the same group do not occur together in a block, and

5. each pair of points of V from distinct groups is contained in exactly one block.

Example 1.1.2. Let $V = \{1, 2, 3, 4, 5, 6\}$ be partitioned into three groups of size two, $\mathscr{G} = \{\{1, 2\}, \{3, 4\}, \{5, 6\}\}$, and a collection $\mathscr{B}$ of 4 blocks of size three, $\{\{1, 3, 5\}, \{1, 4, 6, \}, \{2, 3, 6\}, \{2, 4, 5\}\}$. Since each pair of points from distinct groups is contained in exactly one block, it follows that $(V, \mathscr{G}, \mathscr{B})$ is a GDD$(3, 2, 3)$.

Definition 1.1.3. Let v, k, r and t be positive integers such that $v > k \geq t$. A **t-(v, k, r)-design** is a pair $(V, \mathscr{B})$ satisfying the following properties :

1. V is a finite set of v elements called **points**,

2. $\mathscr{B}$ is a collection of k-subsets of V called **blocks**, and

3. every set of t distinct points is contained in exactly r blocks.

Note that we allow a t-(v, k, r)-design to contain repeated blocks. A t-(v, k, r)-design without repeated blocks is called a *simple t-(v, k, r)-design*. From now on, to save space, we denote $c\{x_1, x_2, \ldots, x_k\}$ the c copies of the block $\{x_1, x_2, \ldots, x_k\}$.

Example 1.1.4. Let $V = \{1, 2, 3, 4, 5, 6, 7\}$ and a collection $\mathscr{B}_1$ of 7 blocks of size three is $\{\{1, 2, 3\}, \{1, 4, 5\}, \{1, 6, 7\}, \{2, 4, 6\}, \{2, 5, 7\}, \{3, 4, 7\}, \{3, 5, 6\}\}$. Since each pair of points is contained in exactly one block, it follows that $(V, \mathscr{B}_1)$ is a simple 2-$(7, 3, 1)$-design. On the other hand, a collection $\mathscr{B}_2$ of 14 blocks of size three is $\{2\{1, 2, 3\}, \{1, 4, 5\}, \{1, 4, 7\}, \{1, 5, 6\}, \{1, 6, 7\}, \{2, 4, 5\}, \{2, 4, 6\}, \{2, 5, 7\}, \{2, 6, 7\}, \{3, 4, 6\}, \{3, 4, 7\}, \{3, 5, 6\}, \{3, 5, 7\}\}$. Since each pair of points is contained in two blocks, it follows that $(V, \mathscr{B}_2)$ is a 2-$(7, 3, 2)$-design.

Definition 1.1.5. Let v, k and t be positive integers. A **t-Sarvate-Beam design** t-SB(v, k)-design, is a pair $(V, \mathscr{B})$ such that the following properties are satisfied:

1. V is a finite set of v elements called **points**,

2. $\mathscr{B}$ is a collection of k-subsets of V called **blocks**, and

3. each t-subset of V occurs in distinct number of times in blocks.

In case $t = 1$, it is simply called a *Sarvate-Beam design*, $SB(v, k)$-design. A **strict t-SB(v, k)-design** is a t-SB(v, k)-design such that exactly one t-subset of points of V occurs i times for every positive integer $i = 1, 2, \ldots, \binom{v}{t}$.

Example 1.1.6. Let $V = \{1, 2, 3, 4, 5\}$ and a collection $\mathscr{B}$ of 14 blocks of size four is $\{\{1, 2, 3, 4\}, 2\{1, 2, 4, 5\}, 4\{1, 3, 4, 5\}, 7\{2, 3, 4, 5\}\}$. Since each 3-subset of V occurs in distinct number of times from 1 to 11, except 10 in $\mathscr{B}$, it follows that $(V, \mathscr{B})$ is a 3-SB$(5, 4)$.

t-Sarvate-Beam designs are completely new in the sense that earlier mathematicians were working on designs where the restriction was pairs should come same number of times. Sarvate and Beam were the first people who asked if all pairs come with different frequencies. The construction of such designs turned out to be very interesting from combinatorial point of view and raised interesting questions about counting. Staton who wrote five papers on these types of design called them *Sarvate-Beam designs* or *Sarvate-Beam type designs.* If $k = 3$, it is called *Sarvate-Beam triple system* and if $k = 4$, it is called *Sarvate-Beam quadruple system* etc.

This thesis is organized as follows. In chapter II, we give necessary conditions for the existence of *t-Sarvate-Beam group divisible designs*, study the first smallest example for $t = 2$ in detail and show the complete construction of strict 1-Sarvate-Beam group divisible designs. In chapter III, we give necessary conditions for the existence of *restricted simple 1-designs* and we present many methods to construct such designs.

CHAPTER II

t-SARVATE-BEAM GROUP DIVISIBLE DESIGNS

In this chapter we define a new design called a t-Sarvate-Beam group divisible design and investigate some of its properties. We give the construction of the first smallest case when $t = 2$ using Stanton's technique. We finish with the complete construction of 1-Sarvate-Beam group divisible designs.

2.1 Definitions and basic results

First, we introduce the definition of a t-Sarvate-Beam group divisible design. Next, we present a construction for a strict SBGDD$(3, 2, 3)$, the first smallest design when $t = 2$. In the rest of this section, we present a construction for an infinite SBGDD$(3, 3, 3)$.

Definition 2.1.1. Let m, n, k and t be positive integers such that $2 \leq k$ and $t \leq k \leq m$. A ***t*-Sarvate-Beam group divisible design** t-SBGDD(m, n, k), is a triple $(V, \mathscr{G}, \mathscr{B})$ such that the following properties are satisfied:

1. V is a finite set of mn elements called **points**,

2. $\mathscr{G}$ is a partition of V into m nonempty subsets of size n called **groups**,

3. $\mathscr{B}$ is a collection of k-subsets of V called **blocks**,

4. any two points from the same group do not occur together in a block, and

5. each t-subset of V from different groups occurs in distinct number of times in $\mathscr{B}$.

A **strict** t**-SBGDD**(m,n,k) is a t-SBGDD(m,n,k) such that exactly one t-subset of V from different groups occurs i times for every positive integer $i = 1,2,\ldots,\binom{m}{t}n^t$.

Denote by $\mathscr{D} = (V,\mathscr{G},\mathscr{B})$ the t-SBGDD(m,n,k) where parameters are not mentioned.

The general term t-SBGDD is used to indicate any t-SBGDD(m,n,k) and we allow a t-SBGDD to contain repeated blocks. We give a few examples of t-SBGDDs now. For convenience, a v-set V is assumed to be $\{1,2,\ldots,v\}$ unless V is specified as other set.

Example 2.1.2. Let $V = \{1,2,3,4,5,6,7,8\}$ be partitioned into four groups of size two, $\mathscr{G} = \{\{1,8\},\{2,7\},\{3,6\},\{4,5\}\}$ and a collection of 9 blocks of size four, $\mathscr{B} = \{\{1,2,3,4\},\{2,3,4,8\},\{3,4,7,8\},\{4,6,7,8\},5\{5,6,7,8\}\}$. Since single point occurs a different number of times from 1 to 8 in blocks, $(V,\mathscr{G},\mathscr{B})$ is a strict 1-SBGDD$(4,2,4)$.

Example 2.1.3. Let $V = \{1,2,3,4,5,6\}$ be partitioned into three groups of size two, $\mathscr{G} = \{\{1,2\},\{3,4\},\{5,6\}\}$. For the first collection of 28 blocks of size three, $\mathscr{B}_1 = \{\{1,3,5\},2\{1,4,5\},2\{1,4,6\},5\{2,3,5\},5\{2,3,6\},6\{2,4,5\},7\{2,4,6\}\}$, all twelve pairs of points from different groups come in distinct number of times from 1 to 13, except 7 in $\mathscr{B}_1$, this implies that $(V,\mathscr{G},\mathscr{B}_1)$ is a 2-SBGDD$(3,2,3)$, but the design is not strict. However, the second collection of 26 blocks of size three, $\mathscr{B}_2 = \{\{1,3,5\},8\{1,4,5\},4\{1,4,6\},2\{2,3,5\},5\{2,3,6\},6\{2,4,6\}\}$, among the twelve possible pairs of points from different groups, there is exactly one pair which occurs i times, for $i = 1,2,\ldots,12$ in $\mathscr{B}_2$, so $(V,\mathscr{G},\mathscr{B}_2)$ forms a strict 2-SBGDD$(3,2,3)$.

Example 2.1.4. Let $V = \{1,2,3,4,5,6,7,8,9\}$ be partitioned into three groups of size three, $\mathscr{G} = \{\{1,2,3\},\{4,5,6\},\{7,8,9\}\}$, and a collection of 340 blocks of size three, $\mathscr{B} = \{\{1,4,7\},\{1,4,8\},16\{1,4,9\},2\{1,5,7\},20\{1,6,7\},10\{1,6,9\},$ $39\{2,4,7\},10\{2,4,8\},5\{2,5,7\},40\{2,5,8\},6\{2,6,7\},50\{2,6,8\},60\{2,6,9\},$ $7\{3,4,7\},8\{3,5,7\},19\{3,5,8\},21\{3,5,9\},9\{3,6,7\},\{3,6,8\},15\{3,6,9\}\}$. Since all twenty-seven pairs of points from different groups come in distinct number of times in $\mathscr{B}$ which are $1,2,7,11,15,16,18,20,21,23,24,25,26,30,35,36,45,47,48,49,50,$ $51,59,60,85,100$ and 116, this implies that $(V,\mathscr{G},\mathscr{B})$ is a 2-SBGDD$(3,3,3)$.

For $t=2$, it is simply called a **Sarvate-Beam group divisible design**, denoted by **SBGDD**(m,n,k) instead of 2-SBGDD(m,n,k) and denote by b, d_t, r_i the number of the blocks, the number of distinct t-subsets of points and the replication number of the point i in the design, respectively.

If $(V,\mathscr{G},\mathscr{B})$ is a strict t-SBGDD(m,n,k), then we can easily obtain a t-SBGDD(m,n,k) such that each t-subset of V from different groups occurs in distinct number of time, for example, construct a new collection $\mathscr{B}'$ from $\mathscr{B}$ by making s copies (for some positive integer $s \geq 2$) of each blocks in $\mathscr{B}$. Then each t-subset of V from different groups still occurs in distinct number of times in $\mathscr{B}'$ from $s, 2s, \ldots, \binom{m}{t}n^t s$, i.e. $(V,\mathscr{G},\mathscr{B}')$ is a t-SBGDD(m,n,k). Therefore, we focus on constructions of a strict t-SBGDD(m,n,k).

Remark 2.1.5. [5] A strict t-SBGDD$(m,1,k)$ is a strict t-SB(m,k).

The proof of the following theorem follows immediately from the definition.

Theorem 2.1.6. *A strict k-SBGDD(m,n,k) exists for every positive integer m,n and k where $2 \leq k \leq m$.*

Proof. Suppose that an mn-set is partitioned into m groups of size n. Since the number of different blocks of size k for which points comes from different groups is $\binom{m}{k}n^k$ and equal to the number of k-subsets of points from different groups. For all positive integer i, $1 \leq i \leq \binom{m}{k}n^k$, we can take each k-subset of points from different groups, i copies as i blocks. Therefore, we have constructed a strict k-SBGDD(m,n,k). □

Example 2.1.7. Let $V = \{1,2,3,4,5,6,7,8\}$ be partitioned into four groups of size two, $\mathscr{G} = \{\{1,2\},\{3,4\},\{5,6\},\{7,8\}\}$ and a collection of 526 blocks of size three, $\mathscr{B} = \{\{1,3,5\}, 2\{1,3,6\}, 3\{1,3,7\}, 4\{1,3,8\}, 5\{1,4,5\}, 6\{1,4,6\}, 7\{1,4,7\}, 8\{1,4,8\}, 9\{1,5,7\}, 10\{1,5,8\}, 11\{1,6,7\}, 12\{1,6,8\}, 13\{2,3,5\}, 14\{2,3,6\}, 15\{2,3,7\}, 16\{2,3,8\}, 17\{2,4,5\}, 18\{2,4,6\}, 19\{2,4,7\}, 20\{2,4,8\}, 21\{2,5,7\}, 22\{2,5,8\}, 23\{2,6,7\}, 24\{2,6,8\}, 25\{3,5,7\}, 26\{3,5,8\}, 27\{3,6,7\}, 28\{3,6,8\}, 29\{4,5,7\}, 30\{4,5,8\}, 31\{4,6,7\}, 32\{4,6,8\}\}$. All triples of points from different groups occur in distinct number of times from 1 to 32 in blocks, this implies that $(V,\mathscr{G},\mathscr{B})$ forms a strict 3-SBGDD$(4,2,3)$.

2.2 Necessary conditions

In this section, we discuss necessary conditions for the existence of t-SBGDDs. Let m, n, k and t be positive integers and $k, m \geq 2$.

Theorem 2.2.1. *If a strict t-SBGDD(m,n,k) exists, then the number of blocks in the design is $\dfrac{d_t(d_t+1)}{2\binom{k}{t}}$. In particular, when $t=1$, $mn(mn+1) \equiv 0 \pmod{2k}$ and when $t=2$, $[m(m-1)n^2][m(m-1)n^2+2] \equiv 0 \pmod{4k(k-1)}$.*

Proof. First, since an mn-set is partitioned into m groups of size n, the number of distinct t-subsets from different m groups of size n in the design is $\binom{m}{t}n^t$. It follows that $d_t = \binom{m}{t}n^t$. Next, since the occurrence number of distinct t-subsets from different groups in the design must occur i times for any integer i, $1 \leq i \leq d_t$, the sum of occurrence number of distinct t-subsets from different groups is $1+2+\cdots+d_t = \dfrac{d_t(d_t+1)}{2}$. Also, looking at each block contains $\binom{k}{t}$

tuples, it follows that there must be a total of $\dfrac{d_t(d_t+1)}{2\binom{k}{t}}$ blocks. Since the number of blocks must be an integer, it follows that $d_t(d_t+1) \equiv 0 \pmod{2\binom{k}{t}}$. □

The following corollaries are immediate from Theorem 2.2.1.

Corollary 2.2.2. *If* $m \equiv 2 \pmod 3$ *and* $n \equiv 1, 2 \pmod 3$, *then a strict SBGDD* $(m, n, 3)$ *does not exist.*

Proof. Since $m \equiv 2 \pmod 3$ and $n \equiv 1, 2 \pmod 3$, $m(m-1) \equiv 2 \pmod 3$ and $n^2 \equiv 1 \pmod 3$. This implies that $[m(m-1)n^2][m(m-1)n^2+2] \equiv 2 \pmod 3$, and hence $[m(m-1)n^2][m(m-1)n^2+2] \not\equiv 0 \pmod{24}$. By Theorem 2.2.1, a strict SBGDD$(m, n, 3)$ does not exist. □

Corollary 2.2.3. *If a strict SBGDD*(k, k, k) *exists, then* $k \not\equiv 1 \pmod 4$.

Proof. Assume that $k \equiv 1 \pmod 4$. Then we have $k^2[k(k-1)k^2+2] \equiv 2 \pmod 4$. If $[k(k-1)k^2][k(k-1)k^2+2] \equiv 0 \pmod{4k(k-1)}$, then $[k(k-1)k^2][k(k-1)k^2+2] = 4k(k-1)q$ for some integer q. Since $k(k-1) \neq 0$, as $k \geq 2$, it follows that $k^2[k(k-1)k^2+2] = 4q$. Thus $k^2[k(k-1)k^2+2] \equiv 0 \pmod 4$ which contradicts the assumption. By Theorem 2.2.1, a strict SBGDD(k, k, k) does not exist. □

Corollary 2.2.4. *If a strict SBGDD*(k, n, k) *exists, then* $k \equiv 2, 3 \pmod 4$ *or* $n \equiv 0, 2 \pmod 4$.

Proof. Assume that $k \equiv 0, 1 \pmod 4$ and $n \equiv 1, 3 \pmod 4$. Since $k(k-1) \equiv 0 \pmod 4$ and $n^2 \equiv 1 \pmod 4$, $n^2[k(k-1)n^2+2] \equiv 2 \pmod 4$. If $[k(k-1)n^2][k(k-1)n^2+2] \equiv 0 \pmod{4k(k-1)}$, then $[k(k-1)n^2][k(k-1)n^2+2] = 4k(k-1)q$ for some integer q. Since $k(k-1) \neq 0$, as $k \geq 2$, it follows that $n^2[k(k-1)n^2+2] = 4q$. Thus $n^2[k(k-1)n^2+2] \equiv 0 \pmod 4$ which contradicts the assumption. By Theorem 2.2.1, a strict SBGDD(k, n, k) does not exist. □

Remark 2.2.5. A strict t-SBGDD(m, n, k) has the least number of blocks for any other t-SBGDD(m, n, k).

Theorem 2.2.6. *Let m, n, k and t be positive integers and $k, m \geq 2$. If a t-SBGDD(m, n, k) exists, then the sum of the replication numbers of points in group is less than or equal to the number of blocks in the design.*

Proof. Suppose that the sum of replication numbers of points in group is greater than the number of blocks in the design. It follows that there is a block in the design containing two points from the same group or there is a point occurs more than one time in a block, which contradicts the definition of such a design. Thus the sum of replication numbers of points in group is less than or equal to the number of blocks in the design. □

Remark 2.2.7. Theorem 2.2.6 will be useful for the construction of strict 1-SBGDDs in Section 2.4.

When $t = 2$, we present the first smallest example of a strict SBGDD$(3, 2, 3)$ by enumeration in the next section.

2.3 A strict SBGDD$(3, 2, 3)$

Stanton [5] suggested a method to construct a strict SB$(6, 3)$. We apply his idea to construct a strict SBGDD$(3, 2, 3)$ as follows.

Let $V = \{1, 2, 3, 4, 5, 6\}$ be partitioned into three groups of size two, $\mathscr{G} = \{\{1, 2\}, \{3, 4\}, \{5, 6\}\}$. There are 12 pairs of points from different groups and the sum of the number of times which pairs occur is $1 + 2 + 3 + \cdots + 12 = 78$, it follows that, we need 26 triples. We let the frequencies F of the various triples be as follows. Note that, for any $x, y, z \in V$, $F(xyz) = s$ means $s\{x, y, z\}$. Without loss of generality, suppose that $F(135) = 1$, $F(136) = 0$. Let $F(145) =$

a, $F(146) = b$, $F(235) = c$, $F(236) = d$, $F(245) = e$, $F(246) = f$. We can now write the pair frequencies:

$$F(13) = 1, \quad F(14) = a + b, \quad F(15) = 1 + a, \quad F(16) = b,$$

$$F(23) = c + d, \quad F(24) = e + f, \quad F(25) = c + e, \quad F(26) = d + f,$$

$$F(35) = 1 + c, \quad F(36) = d, \quad F(45) = a + e, \quad F(46) = b + f.$$

Let $A = F(13) + F(14) + F(15) + F(16) = 2(1 + a + b)$,

$B = F(23) + F(24) + F(25) + F(26) = 2(c + d + e + f)$,

$C = F(35) + F(36) + F(45) + F(46) = 1 + a + b + c + d + e + f$.

Thus $A + B = 2C$ and $C = 26$. Then $A + B = 52$. If we attempted to assign the pair frequencies exactly in order, then we would have $A = 10$ and $B = 42$. Since $A = 2(1 + a + b)$, we get $a + b = 4$. We first set $a = 2$, $b = 2$, we have that $F(13) = 1$, $F(14) = 4$, $F(15) = 3$, $F(16) = 2$. Since $B = 2(c + d + e + f)$, $c+d+e+f = 21$. Next, we assign $c = 4$, $d = 6$, $e = 5$, $f = 6$. This gives, $F(23) = 10$, $F(24) = 11$, $F(25) = 9$, $F(26) = 12$, and $F(35) = 5$, $F(36) = 6$, $F(45) = 7$, $F(46) = 8$. This implies that $(V, \mathscr{G}, \mathscr{B})$ form a strict SBGDD$(3, 2, 3)$ by taking $\mathscr{B} = \{\{1, 3, 5\}, 2\{1, 4, 5\}, 2\{1, 4, 6\}, 4\{2, 3, 5\}, 6\{2, 3, 6\}, 5\{2, 4, 5\}, 6\{2, 4, 6\}\}$ as the collection of twenty-six blocks of size three.

Obviously, we obtained other systems of different assignments of $a, b, \ldots, f$ by using a Fortran77 program. The following properties are useful for helping the Fortran77 program:

1. a, c, f are distinct positive integers and b, d, e are distinct integers, but $b, d \geq 2$ and $e = 0$ or $e \geq 2$ and $a, b, c, d, e, f \leq 11$.

2. The sum of the replication numbers of points from each group is equal to the number of triples because each block must have exactly one point from each group. In general, this property is satisfied if the number of groups is equal to the block size.

3. There is at most one group that its points have the same replication number. Suppose that $V = \{1,2,3,4,5,6\}$ is partitioned into three groups of size two, $\mathcal{G} = \{\{1,2\},\{3,4\},\{5,6\}\}$. If there are two groups such that its points have the same replication number, say $\{1,2\},\{3,4\}$. Then $F(1) = F(2) = F(3) = F(4) = 13$. This implies that, $F(1) = 1 + a + b = 13, a + b = 12$ or $F(14) = 12$, and $F(3) = 1 + c + d = 13, c + d = 12$ or $F(23) = 12$. Therefore $F(14) = F(23)$ which is not allowed for a strict design.

The 56 solutions of a strict SBGDD$(3,2,3)$ is obtained using a Fortran77 program as shown in Table 2.1.

This method can also be used to verify an infinite SBGDD$(3,3,3)$. Basically, we assign the copy number of various blocks as a variable and apply certain Fibonacci numbers. We illustrate this process now. Let $V = \{1,2,3,4,5,6,7,8,9\}$ be partitioned into three groups of size three, $\mathcal{G} = \{\{1,2,3\},\{4,5,6\},\{7,8,9\}\}$. There are 27 pairs of points of V from different groups. Let the frequencies F of the various triples be as follows:

$F(147) = x_1$, $F(148) = x_2$, $F(149) = x_3$, $F(157) = x_4$, $F(158) = x_5$,

$F(159) = x_6$, $F(167) = x_7$, $F(168) = x_8$, $F(169) = x_9$, $F(247) = x_{10}$,

$F(248) = x_{11}$, $F(249) = x_{12}$, $F(257) = x_{13}$, $F(258) = x_{14}$, $F(259) = x_{15}$,

$F(267) = x_{16}$, $F(268) = x_{17}$, $F(269) = x_{18}$, $F(347) = x_{19}$, $F(348) = x_{20}$,

$F(349) = x_{21}$, $F(357) = x_{22}$, $F(358) = x_{23}$, $F(359) = x_{24}$, $F(367) = x_{25}$,

$F(368) = x_{26}$, $F(369) = x_{27}$,

where x_1 is a positive integer, for $i = 3,\ldots,27$, define $x_i = x_{i-1} + x_{i-2}$ and $x_2 = x_1$. We can now write the pair frequencies:

$F(14) = 4x_1$, $F(15) = 16x_1$, $F(16) = 68x_1$, $F(17) = 17x_1$, $F(18) = 27x_1$, $F(19) = 44x_1$, $F(24) = 288x_1$, $F(25) = 1220x_1$, $F(26) = 5168x_1$, $F(27) = 1275x_1$, $F(28) =$

$2063x_1$, $F(29) = 3338x_1$, $F(34) = 21892x_1$, $F(35) = 92736x_1$, $F(36) = 392836x_1$, $F(37) = 96917x_1, F(38) = 156815x_1, F(39) = 253732x_1, F(47) = 4237x_1, F(48) = 6855x_1, F(49) = 11092x_1, F(57) = 17947x_1, F(58) = 29039x_1, F(59) = 46986x_1, F(67) = 76025x_1, F(68) = 123011x_1, F(69) = 199036x_1$. All 27 pairs of points from different groups come in distinct number of times, it follows that the design is a SBGDD(3, 3, 3) with $514228x_1$ blocks. Since x_1 is a variable, we set a variable as the copy number for the block. Therefore, we obtain the desired result.

2.4 A construction for strict 1-SBGDDs

In this section we consider a few special cases and introduce several constructions of the design. Now, we present a construction of a strict 1-SBGDD(m, n, k). Suppose that an mn-set is partitioned into m groups of size n. Recall if such a design exists, any two points in the same group do not occur in the same block and there are exactly $\frac{mn(mn+1)}{2k}$ blocks, it follows that for each group, the sum of the replication numbers of points in group is at most $\frac{mn(mn+1)}{2k}$. For each point i, a property of the replication number of the point i would be chosen one in $\{1, 2, \ldots, mn\}$.

From this observation, it follows that we can allocate a point in group such that the sum of points in group is not greater than number of blocks in the design. The following lemma is useful for the existence of a strict 1-SBGDD.

Lemma 2.4.1. *Let m and n be positive integers such that $2 \leq m, n$.*

(i) if n is even, then we can partition an mn-set into m groups of size n such that the sum of points in each group is $\frac{n(mn+1)}{2}$,

(ii) if n and m are odd, then we can partition an mn-set into m groups of size n such that the sum of points in each group is $\frac{n(mn+1)}{2}$, and

(iii) if n is odd and m is even, then we can partition an mn-set into m groups

of size n such that the sum of points in each group is $\frac{n(mn+1)+1}{2}$ *or* $\frac{n(mn+1)-1}{2}$.

Proof. Let $V = \{1, 2, \ldots, mn\}$.

(i) When n is even. Let $H_1 = \{1, 2, \ldots, \frac{n}{2}\}$, $H_2 = \{\frac{n}{2}+1, \frac{n}{2}+2, \ldots, n\}, \ldots,$ $H_{2m} = \{\frac{n(2m-1)}{2}+1, \frac{n(2m-1)}{2}+2, \ldots, mn\}$. In fact, for $i = 1, 2, \ldots, 2m$, $H_i = \{\frac{n}{2}(i-1)+j \mid j = 1, \ldots, \frac{n}{2}\}$. It is routine to check that $\{H_1, H_2, \ldots, H_{2m}\}$ is a partition on V into $2m$ groups of size $\frac{n}{2}$ and the sum of points in each group H_i is $\displaystyle\sum_{j=1}^{\frac{n}{2}}[\frac{n}{2}(i-1)+j] = \frac{2n^2i-n^2+2n}{8}$. Therefore, $\mathscr{G} = \{H_1 \cup H_{2m}, H_2 \cup H_{2m-1}, \ldots, H_m \cup H_{m+1}\}$ is the required partition on V and the sum of points in each group $H_i \cup H_{2m+1-i}$ is $\frac{2n^2i-n^2+2n}{8} + \frac{2n^2(2m+1-i)-n^2+2n}{8} = \frac{n(mn+1)}{2}$.

(ii) When n and m are odd. First, since $n-3$ is even, by (i) we can partition $\{1, 2, \ldots, m(n-3)\}$ into m groups of size $n-3$, say $X_1, X_2, \ldots, X_m$ such that the sum of points in X_i is $\frac{(n-3)(m(n-3)+1)}{2}$. Next, we partition $\{m(n-3)+1, m(n-3)+2, \ldots, m(n-3)+3m\}$ into m groups of size 3 as follows.

$Y_1 = \{m(n-3)+1, m(n-3)+m+\frac{m+1}{2}, m(n-3)+3m\}$,

$Y_3 = \{m(n-3)+3, m(n-3)+m+\frac{m-1}{2}, m(n-3)+3m-1\}$,

$\vdots$

$Y_m = \{m(n-3)+m, m(n-3)+m+1, m(n-3)+2m+\frac{m+1}{2}\}$, and

$Y_2 = \{m(n-3)+2, m(n-3)+2m, m(n-3)+2m+\frac{m-1}{2}\}$

$Y_4 = \{m(n-3)+4, m(n-3)+2m-1, m(n-3)+2m+\frac{m-3}{2}\}$,

$\vdots$

$Y_{m-1} = \{m(n-3)+m-1, m(n-3)+m+\frac{m+3}{2}, m(n-3)+2m+1\}$.

In fact, for any integer i, $1 \leq i \leq m$,

if i is odd, $Y_i = \{m(n-3)+i, m(n-3)+m+\frac{(m-i+2)}{2}, m(n-3)+3m-\frac{(i-1)}{2}\}$,

if i is even, $Y_i = \{m(n-3)+i, m(n-3)+2m-\frac{(i-2)}{2}, m(n-3)+2m+\frac{m-i+1}{2}\}$.

Note that the sum of points in Y_i is $\frac{6mn-9m+3}{2}$. Therefore, $\mathscr{G} = \{X_1 \cup Y_1, X_2 \cup Y_2, \ldots, X_m \cup Y_m\}$ is the required partition on V and the sum of points in each group $X_i \cup Y_i$ is $\frac{(n-3)(m(n-3)+1)}{2} + \frac{6mn-9m+3}{2} = \frac{n(mn+1)}{2}$.

(iii) When n is odd and m is even. First, since $n-3$ is even, by (i) we can partition $\{1, 2, \ldots, m(n-3)\}$ into m groups of size $n-3$, say $X_1, X_2, \ldots, X_m$ such that the sum of points in X_i is $\frac{(n-3)(m(n-3)+1)}{2}$. Next, we partition $\{m(n-3)+1, m(n-3)+2, \ldots, m(n-3)+3m\}$ into m groups of size 3 as follows.

$Y_1 = \{m(n-3)+1, m(n-3)+m+\frac{m}{2}+1, m(n-3)+3m\}$,

$Y_3 = \{m(n-3)+3, m(n-3)+m+\frac{m}{2}, m(n-3)+3m-1\}$,

$\vdots$

$Y_{m-1} = \{m(n-3)+(m-1), m(n-3)+(m+2), m(n-3)+2m+\frac{m}{2}+1\}$,

and $Y_m = \{m(n-3)+m, m(n-3)+m+1, m(n-3)+2m+\frac{m}{2}\}$,

$Y_2 = \{m(n-3)+2, m(n-3)+2m, m(n-3)+2m+\frac{m}{2}-1\}$,

$Y_4 = \{m(n-3)+4, m(n-3)+2m-1, m(n-3)+2m+\frac{m}{2}-2\}$,

$\vdots$

$Y_{m-2} = \{m(n-3)+m-1, m(n-3)+m+\frac{m}{2}+2, m(n-3)+2m+1\}$.

In fact, for any integer i, $1 \leq i \leq m-1$,

if i is odd, $Y_i = \{m(n-3)+i, m(n-3)+m+\frac{(m-i+3)}{2}, m(n-3)+3m-\frac{(i-1)}{2}\}$,

if i is even, $Y_i = \{m(n-3)+i, m(n-3)+2m-\frac{(i-2)}{2}, m(n-3)+3m-\frac{(m+i)}{2}\}$,

and $Y_m = \{m(n-3)+m, m(n-3)+m+1, m(n-3)+2m+\frac{m}{2}\}$.

Note that when i is odd, the sum of points in Y_i is $\frac{6mn-9m+4}{2}$ and when i is even, the sum of points in Y_i is $\frac{6mn-9m+2}{2}$. Therefore, $\mathscr{G} = \{X_1 \cup Y_1, X_2 \cup Y_2, \ldots, X_m \cup Y_m\}$ is the required partition on V and the sum of points in each group $X_i \cup Y_i$ is $\frac{(n-3)(m(n-3)+1)}{2} + \frac{6mn-9m+4}{2} = \frac{n(mn+1)+1}{2}$, when i is odd and $\frac{(n-3)(m(n-3)+1)}{2} + \frac{6mn-9m+2}{2} = \frac{n(mn+1)-1}{2}$, when i is even. □

We illustrate the application of the above lemma in the following examples.

Example 2.4.2. Let $n = 4, m = 3$ and $V = \{1, 2, \ldots, 12\}$. Using Lemma 2.4.1 (i), we get $H_1 = \{1, 2\}, H_2 = \{3, 4\}, H_3 = \{5, 6\}, H_4 = \{7, 8\}, H_5 =$

$\{9,10\}, H_6 = \{11,12\}$. Therefore, $\mathscr{G} = \{H_1 \cup H_6, H_2 \cup H_5, H_3 \cup H_4\}$ is the required partition on V with the sum of points in group is 26.

Example 2.4.3. Let $n = 5, m = 3$ and $V = \{1, 2, \ldots, 15\}$. Using Lemma 2.4.1 (ii), we get $X_1 = \{1,6\}$, $X_2 = \{2,5\}$, $X_3 = \{3,4\}$ are three groups of size two and $Y_1 = \{7,15,11\}$, $Y_2 = \{8,12,13\}$, $Y_3 = \{9,14,10\}$ are three groups of size three. Therefore, $\mathscr{G} = \{X_1 \cup Y_1, X_2 \cup Y_2, X_3 \cup Y_3\}$ is the required partition on V with the sum of points in group is 40.

Example 2.4.4. Let $n = 5, m = 4$ and $V = \{1, 2, \ldots, 20\}$. Using Lemma 2.4.1 (iii), we get $X_1 = \{1,8\}$, $X_2 = \{2,7\}$, $X_3 = \{3,6\}$, $X_4 = \{4,5\}$ are four groups of size two and $Y_1 = \{9,20,15\}$, $Y_2 = \{10,16,17\}$, $Y_3 = \{11,19,14\}$, $Y_4 = \{12,13,18\}$ are four groups of size three. Therefore, $\mathscr{G} = \{X_1 \cup Y_1, X_2 \cup Y_2, X_3 \cup Y_3, X_4 \cup Y_4\}$ is the required partition on V with the sum of points in group are $53, 52, 53$ and 52, respectively.

Now, we prove the existence of a strict 1-SBGDD(m, n, k).

Theorem 2.4.5. *Let m, n and k be positive integers and $2 \le k \le m$. If $mn(mn+1) \equiv 0 \mod 2k$, then a strict 1-SBGDD$(m, n, k)$ exists.*

Proof. Let $V = \{1, 2, \ldots, mn\}$ be partitioned into m groups of size n, say $\mathscr{G} = \{G_1, G_2, \ldots, G_m\}$ by Lemma 2.4.1. Since $\frac{mn(mn+1)}{2k}$ is an integer, $b = \frac{mn(mn+1)}{2k}$. For convenience, call points in G_i, $a_{i1}, a_{i2}, \ldots, a_{in}$, for all $i = 1, 2, \ldots, m$. First, place point a_{11} of G_1 in a_{11} different blocks, say that $B_1, B_2, \ldots, B_{a_{11}}$. Then continue placing point a_{12} of G_1 in a_{12} different blocks, say that $B_{a_{11}+1}, B_{a_{11}+2}, \ldots, B_{a_{11}+a_{12}}$ where the subscripts are added modulo b. The same argument is applied to other points of G_1 and $G_2, G_3, \ldots, G_m$, respectively. Let $\mathscr{B} = \{B_1, B_2, \ldots, B_b\}$, claim that $(V, \mathscr{G}, \mathscr{B})$ is a strict 1-SBGDD(m, n, k).

To see this, each point a_{ij} occurs in a_{ij} blocks. Since $mn(mn+1) \equiv 0 \mod 2k$

and each point come in only one time in a block, it follows that the size of block is k and there are exactly $\frac{mn(mn+1)}{2k}$ blocks. Since $2 \leq k \leq m$ and the sum of the replication numbers of points in each group is equal to the sum of points in such group which is at most $\frac{mn(mn+1)}{2k}$, it follows that any two points in a block come from different groups. This completes the proof. □

Example 2.4.6. To illustrate the method in Theorem 2.4.5, let $m = 4$, $n = 2$ and $V = \{1,2,3,4,5,6,7,8\}$ be partitioned into four groups of size two, $\mathscr{G} = \{G_1, G_2, G_3, G_4\}$. Using Lemma 2.4.2 (i), let $G_1 = \{1,8\}, G_2 = \{2,7\}, G_3 = \{3,6\}$ and $G_4 = \{4,5\}$. Next, we are going to construct a collection $\mathscr{B}$ of blocks of size three for a strict 1-SBGDD$(4,2,3)$ as follows :

First, for the group $G_1 = \{1,8\}$, we place point 1 and continue placing point 8 in one block and eight different blocks, respectively, $\mathscr{B} = \{\{1, , \}, \{8, , \}, \{8, , \}, \{8, , \}, \{8, , \}, \{8, , \}, \{8, , \}, \{8 , \}, \{8, , \}, \{ , , \}, \{ , , \}, \{ , , \}\}$. Second, for the group $G_2 = \{2,7\}$, we continue placing points 2 and 7 in two and seven different blocks, respectively, $\mathscr{B} = \{\{1,7, \}, \{8,7, \}, \{8,7, \}, \{8,7, \}, \{8,7, \}, \{8,7, \}, \{8, , \}, \{8, , \}, \{8, , \}, \{2, , \}, \{2, , \}, \{7, , \}\}$. Third, for the group $G_3 = \{3,6\}$, we continue placing points 3 and 6 in three and six different blocks, respectively, $\mathscr{B} = \{\{1,7,6\}, \{8,7,6\}, \{8,7,6\}, \{8,7, \}, \{8,7, \}, \{8,7, \}, \{8,3, \}, \{8,3, \}, \{8,3, \}, \{2,6, \}, \{2,6, \}, \{7,6, \}\}$. Finally, for the group $G_4 = \{4,5\}$, we continue placing points 4 and 5 in four and five different blocks, respectively. Then the collection $\mathscr{B}$ of 12 blocks of size three as follows: $\{\{1,7,6\}, \{8,7,6\}, \{8,7,6\}, \{8,7,4\}, \{8,7,4\}, \{8,7,4\}, \{8,3,4\}, \{8,3,5\}, \{8,3,5\}, \{2,6,5\}, \{2,6,5\}, \{7,6,5\}\} = \{\{1,6,7\}, 2\{2,5,6\}, \{3,4,8\}, 2\{3,5,8\}, 3\{4,7,8\}, \{5,6,7\}, 2\{6,7,8\}\}$. Therefore, $(V, \mathscr{G}, \mathscr{B})$ is a strict 1-SBGDD$(4,2,3)$.

Table 2.1: The 56 solutions of a strict SBGDD(3, 2, 3).

a	b	c	d	e	f	13	14	15	16	23	24	25	26	35	36	45	46
1	3	4	7	5	5	1	4	2	3	11	10	9	12	5	7	6	8
1	3	4	8	6	3	1	4	2	3	12	9	10	11	5	8	7	6
1	3	5	5	7	4	1	4	2	3	10	11	12	9	6	5	8	7
1	3	6	6	4	5	1	4	2	3	12	9	10	11	7	6	5	8
1	4	2	6	9	3	1	5	2	4	8	12	11	9	3	6	10	7
1	4	5	3	6	6	1	5	2	4	8	12	11	9	6	3	7	10
1	5	6	4	2	7	1	6	2	5	10	9	8	11	7	4	3	12
1	7	2	9	3	3	1	8	2	7	11	6	5	12	3	9	4	10
1	7	3	3	9	2	1	8	2	7	6	11	12	5	4	3	10	9
1	8	3	7	2	4	1	9	2	8	10	6	5	11	4	7	3	12
1	10	3	3	6	2	1	11	2	10	6	8	9	5	4	3	7	12
2	2	4	6	5	6	1	4	3	2	10	11	9	12	5	6	7	8
2	2	5	7	6	3	1	4	3	2	12	9	11	10	6	7	8	5
2	2	6	5	6	4	1	4	3	2	11	10	12	9	7	5	8	6
2	2	7	5	4	5	1	4	3	2	12	9	11	10	8	5	6	7
2	4	6	2	3	8	1	6	3	4	8	11	9	10	7	2	5	12
2	4	9	2	3	5	1	6	3	4	11	8	12	7	10	2	5	9
2	5	8	4	0	6	1	7	3	5	12	6	8	10	9	4	2	11
2	6	1	4	9	3	1	8	3	6	5	12	10	7	2	4	11	9
2	6	3	2	7	5	1	8	3	6	5	12	10	7	4	2	9	11
2	8	6	5	0	4	1	10	3	8	11	4	6	9	7	5	2	12
2	9	1	7	3	3	1	11	3	9	8	6	4	10	2	7	5	12
3	2	2	6	7	5	1	5	4	2	8	12	9	11	3	6	10	7
3	2	5	3	4	8	1	5	4	2	8	12	9	11	6	3	7	10
3	3	1	7	9	2	1	6	4	3	8	11	10	9	2	7	12	5
3	3	1	10	6	2	1	6	4	3	11	8	7	12	2	10	9	5
3	7	1	8	2	4	1	10	4	7	9	6	3	12	2	8	5	11
3	7	4	2	8	1	1	10	4	7	6	9	12	3	5	2	11	8

Table 2.1: (Continued)The 56 solutions of a strict SBGDD(3, 2, 3).

a	b	c	d	e	f	13	14	15	16	23	24	25	26	35	36	45	46
3	8	5	7	0	2	1	11	4	8	12	2	5	9	6	7	3	10
4	2	3	7	8	1	1	6	5	2	10	9	11	8	4	7	12	3
4	2	7	3	0	9	1	6	5	2	10	9	7	12	8	3	4	11
4	3	10	2	0	6	1	7	5	3	12	6	10	8	11	2	4	9
4	6	2	2	5	6	1	10	5	6	4	11	7	8	3	2	9	12
4	7	1	3	5	5	1	11	5	7	4	10	6	8	2	3	9	12
4	8	1	3	6	3	1	12	5	8	4	9	7	6	2	3	10	11
5	3	1	4	6	6	1	8	6	3	5	12	7	10	2	4	11	9
5	3	3	2	4	8	1	8	6	3	5	12	7	10	4	2	9	11
5	5	1	3	7	4	1	10	6	5	4	11	8	7	2	3	12	9
5	7	2	2	6	3	1	12	6	7	4	9	8	5	3	2	11	10
5	7	3	8	0	2	1	12	6	7	11	2	3	10	4	8	5	9
6	2	2	4	3	8	1	8	7	2	6	11	5	12	3	4	9	10
6	2	8	3	4	2	1	8	7	2	11	6	12	5	9	3	10	4
6	4	2	5	2	7	1	10	7	4	6	9	3	12	2	5	8	11
6	4	7	2	5	1	1	10	7	4	9	6	12	3	8	2	11	5
6	5	2	2	6	4	1	11	7	5	4	10	8	6	3	2	12	9
6	5	2	8	0	4	1	11	7	5	10	4	2	12	3	8	6	9
6	6	1	3	4	5	1	12	7	6	4	9	5	8	2	3	10	11
7	2	6	4	5	1	1	9	8	2	10	6	11	5	7	4	12	3
7	3	4	2	0	9	1	10	8	3	6	9	4	11	5	2	7	12
7	4	9	3	0	2	1	11	8	4	12	2	9	5	10	3	7	6
7	5	2	2	4	5	1	12	8	5	4	9	6	7	3	2	11	10
8	3	6	2	4	2	1	11	9	3	8	6	10	4	7	2	12	5
8	4	2	5	0	6	1	12	9	4	7	6	2	11	3	5	8	10
9	2	2	4	3	5	1	11	10	2	6	8	5	9	3	4	12	7
9	3	7	4	0	2	1	12	10	3	11	2	7	6	8	4	9	5
10	2	4	3	0	6	1	12	11	2	7	6	4	9	5	3	10	8

CHAPTER III

RESTRICTED SIMPLE 1-DESIGNS

In this chapter, we begin with the definition of a restricted simple 1-design which generalize the notion of a simple 1-design and investigate some of its properties. Next, we present simple yet powerful tool to construct restricted simple 1-designs by many interesting methods.

3.1 Definitions and basic results

From Chapter I, we first recall the definition of a simple 1-(v, k, r)-design and then develop a new design called a *restricted simple 1-(v, k, r)-design.* The general term a restricted simple 1-design is used to indicate any restricted simple 1-(v, k, r)-designs.

Definition 3.1.1. Let v, k and r be positive integers such that $2 \leq k \leq v$. A **simple 1-(v, k, r)-design** is a pair $(V, \mathscr{B})$ satisfying the following properties :

1. V is a finite set of v elements called **points**,
2. $\mathscr{B}$ is a collection of different k-subsets of V called **blocks**, and
3. each point of V is contained in exactly r blocks.

Denote by $\mathscr{D} = (V, \mathscr{B})$ the simple 1-(v, k, r)-design where parameters are not mentioned in the design.

Example 3.1.2. Let $V = \{1, 2, 3, 4, 5, 6\}$ and a collection of six blocks of size three $\mathscr{B} = \{\{1, 3, 5\}, \{1, 3, 6\}, \{1, 4, 5\}, \{2, 3, 6\}, \{2, 4, 5\}, \{2, 4, 6\}\}$, each point occur three times in blocks and no repeated block, this implies that $(V, \mathscr{B})$ is a simple 1-(6,3,3)-design.

We introduce the definition of a restricted simple 1-design as follows.

Definition 3.1.3. Let v, k and r be positive integers such that $2 \leq k \leq v$. A **restricted simple 1-(v, k, r)-design** is a triple $(V, \mathscr{P}, \mathscr{B})$ satisfying the following properties :

1. $(V, \mathscr{B})$ is a simple 1-(v, k, r)-design,
2. $\mathscr{P}$ is a partition of V into nonempty subsets called **parts**, and
3. any two points from the same part do not occur together in a block.

Denote by $\mathscr{D} = (V, \mathscr{P}, \mathscr{B})$ the restricted simple 1-(v, k, r)-design where parameters are not mentioned in the design.

Remark 3.1.4. A restricted simple 1-(v, k, r)-design in which all parts are size one is a simple 1-(v, k, r)-design.

Naturally for the same v-set, we can have many partitions and for certain partition a restricted simple 1-design may exist and for some other partition the design may not exist. The following example is instructive.

Example 3.1.5. Let $V = \{1, 2, 3, 4, 5, 6\}$ be given. We consider the existence or non-existence of a restricted simple 1-(6,3,3)-design in each of the possible cases:

(1) V can be partitioned into $\mathscr{P} = \{\{1, 2\}, \{3, 4\}, \{5, 6\}\}$ (each part of the same size 2) or $\{\{1, 2\}, \{3, 4\}, \{5\}, \{6\}\}$ or $\{\{1, 2\}, \{3\}, \{4\}, \{5\}, \{6\}\}$ or $\{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\}$. For the first partition $\mathscr{P} = \{\{1, 2\}, \{3, 4\}, \{5, 6\}\}$, and a collection of blocks of size three, $\mathscr{B} = \{\{1, 3, 5\}, \{1, 3, 6\}, \{1, 4, 5\}, \{2, 3, 6\}, \{2, 4, 5\}, \{2, 4, 6\}\}$, all six points occur three times in blocks, no repeated block and any

two points from the same part do not occur together in a block, this implies that $(V, \mathscr{P}, \mathscr{B})$ forms a restricted simple 1-(6,3,3)-design. Interestingly, for all other partitions given above the same blocks give a restricted simple 1-$(6, 3, 3)$-design.

(2) On the other hand, if V is partitioned into $\mathscr{P} = \{\{1, 2, 3\}, \{4, 5, 6\}\}$, a restricted simple 1-(6,3,3)-design does not exist, because each block must have three points that come from different parts but this partition has only two parts.

(3) Lastly suppose V is partitioned into $\mathscr{P} = \big\{\{1\}, \{2\}, \{3\}, \{4, 5, 6\}\big\}$, a restricted simple 1-(6,3,3)-design does not exist, because each element occurs three times in blocks and each block have size three, so the number of blocks is $\frac{6(3)}{3} = 6$, but the design needs at least nine distinct blocks to take care of the three points from $\{4, 5, 6\}$.

Denote by b the number of the blocks and r_i the replication number of the point i in the design. Summarizing the previous discussion, we obtain the following necessary conditions.

3.2 Necessary conditions

It is known that if a simple 1-(v, k, r)-design exists with b blocks, then $b \leq \binom{v}{k}$ and $vr = bk$ (see [2]).

Theorem 3.2.1. *In a restricted simple 1-(v, k, r)-design in which a v-set V is partitioned into m parts $\mathscr{P} = \big\{P_1, P_2, \ldots, P_m\big\}$ where $|P_i| = p_i$ for all $i = 1, 2, \ldots, m$ and $\mathscr{B}$ is a collection of b blocks of size k. Without loss of generality, suppose that $p_1 \geq p_2 \geq \cdots \geq p_m$. Then*

1. $vr = bk$,

2. $p_1 r \leq b$,

3. $k \leq min\{\frac{v}{p_1}, m\}$,

4. $b \leq \sum_{\{\alpha_1,\alpha_2,\ldots,\alpha_k\}\subseteq\{1,\ldots,m\}} p_{\alpha_1}p_{\alpha_2}\cdots p_{\alpha_k}$,

5. $r \leq \sum_{\{\alpha_1,\alpha_2,\ldots,\alpha_{k-1}\}\subseteq\{2,\ldots,m\}} p_{\alpha_1}p_{\alpha_2}\cdots p_{\alpha_{k-1}}$, *and*

6. *for all* $j = 1, \ldots, m$, $\sum_{\forall i\in P_j} r_i = p_j r$ *where* r_i *is the replication number of* i *for* $i \in V$.

Proof. 1. It follows from a property of a simple 1-(v, k, r)-design.

2. Since each point in the part P_1 must occur r times in blocks, the number of distinct blocks is at least $p_1 r$.

3. Since each block must have k points that come from different parts, $k \leq m$. From 1. and 2., it follows that $k \leq \frac{v}{p_1}$.

4. Since the number of distinct k-subsets such that each point comes from different parts is $\sum_{\{\alpha_1,\alpha_2,\ldots,\alpha_k\}\subseteq\{1,\ldots,m\}} p_{\alpha_1}p_{\alpha_2}\cdots p_{\alpha_k}$, it follows that the maximum number of blocks is $\sum_{\{\alpha_1,\alpha_2,\ldots,\alpha_k\}\subseteq\{1,\ldots,m\}} p_{\alpha_1}p_{\alpha_2}\cdots p_{\alpha_k}$.

5. Apply 2. and 4., the maximum number of r is obtained.

6. Since each point $i \in P_j$ is contained in exactly r blocks, the sum of replication number of $i \in P_j$ is $\underbrace{r + r + \cdots + r}_{p_j} = p_j r$. □

3.3 Restricted simple 1-designs with $k \geq 2$

There are many ways to construct such a restricted simple 1-design. We first present a construction for a restricted simple 1-$(v, k, 1)$-design as follows.

Theorem 3.3.1. *Let* v, k *and* m *be integers such that* $2 \leq k \leq m \leq v$ *and* k *divides* v. *Let* $V = \{1, 2, \ldots, v\}$ *be partitioned into* m *parts* $\mathscr{P} = \{P_1, P_2, \ldots, P_m\}$ *of size* $p_1, p_2, \ldots, p_m$, *respectively such that* $p_1 \geq p_2 \geq \cdots \geq p_m$ *and* $p_1 k \leq v$. *Then a restricted simple 1-*$(v, k, 1)$*-design exists.*

Proof. Without loss of generality, suppose that $P_1 = \{1, 2, \dots, p_1\}, P_2 = \{p_1 + 1, p_1 + 2, \dots, p_1 + p_2\}, \dots, P_m = \{p_1 + p_2 + \cdots + p_{m-1} + 1, \dots, p_1 + p_2 + \cdots + p_m\}$. For any integer i, $1 \le i \le \frac{v}{k}$, let $B_i = \{i, i + \frac{v}{k}, i + \frac{2v}{k}, \dots, i + \frac{(k-1)v}{k}\}$, note that $B_i \subseteq V$ and $B_i \cap B_j \neq \varnothing$ for $i \neq j$. Finally, define $\mathscr{B} = \{B_i \ : \ i = 1, 2, \dots, \frac{v}{k}\}$. It is not difficult to see that each point in V occurs in exactly one time in a block of $\mathscr{B}$, all blocks of $\mathscr{B}$ are different and every block of $\mathscr{B}$ contains exactly k points. Thus we need only to show that any two points in a block B_i comes from different parts of the partition $\mathscr{P}$. Let x and $y \in B_i$ in which $x > y$. This implies that there is a positive integer s such that $x - y = \frac{sv}{k}$. Note that for any integer i, $1 \le i \le m$, if a and $b \in P_i$ then $|a - b| \le p_i - 1$. Since $p_1 \le \frac{v}{k}$, it follows that $|x - y| = |\frac{sv}{k}| > p_1 - 1 \ge p_i - 1$, this forces that x and y comes from different parts. Therefore $(V, \mathscr{P}, \mathscr{B})$ is a restricted simple 1-$(v, k, 1)$-design. □

Example 3.3.2. An illustration of Theorem 3.3.1, let $v = 15, k = 3$ and $V = \{1, 2, \dots, 15\}$ be partitioned into four groups $\mathscr{P} = \{P_1, P_2, P_3, P_4\}$ where $P_1 = \{1, 2, 3, 4, 5\}$, $P_2 = \{6, 7, 8, 9\}$, $P_3 = \{10, 11, 12, 13\}$ and $P_4 = \{14, 15\}$. We obtain $B_1 = \{1, 6, 11\}$, $B_2 = \{2, 7, 12\}$, $B_3 = \{3, 8, 13\}$, $B_4 = \{4, 9, 14\}$ and $B_5 = \{5, 10, 15\}$. Set $\mathscr{B} = \{\{1, 6, 11\}, \{2, 7, 12\}, \{3, 8, 13\}, \{4, 9, 14\}, \{5, 10, 15\}\}$. Hence, $(V, \mathscr{P}, \mathscr{B})$ is a restricted simple 1-$(15, 3, 1)$-design.

This result provides another method to construct new restricted simple 1-designs from old.

Theorem 3.3.3. *Let v, k and m be positive integers such that $2 \le k \le v$. Suppose that there exists a restricted simple 1-$(v, k, 1)$-design. Then there also exists a restricted simple 1-$(v, k-1, k-1)$-design.*

Proof. Suppose that $(V, \mathscr{P}, \mathscr{B})$ is a restricted simple 1-$(v, k, 1)$-design. Let $\mathscr{B}'$ be a collection of all subsets of size $k - 1$ of each block in $\mathscr{B}'$. We claim that

$(V, \mathscr{P}, \mathscr{B}')$ is a restricted simple 1-$(v, k-1, k-1)$-design. Clearly, this design has v points, every block contains $k-1$ points from different parts because any two points x and y in a block $B' \in \mathscr{B}'$, there is only one block $B \in \mathscr{B}$ such that B contain points x and y, it follows that x and y comes form different parts of $\mathscr{P}$. Hence, we just need to show that every point occurs in exactly $k-1$ blocks, let $x \in V$. There is exactly one block $B \in \mathscr{B}$ of size k such that $x \in B$, it follows that we have $k-1$ blocks $B' \in \mathscr{B}'$ such that $x \in B' \subseteq B$. This implies that the replication number of any point of V is $k-1$ and the number of blocks is $|\mathscr{B}'| = v$. The proof is complete. □

Example 3.3.4. From Example 3.3.2, $v = 15, k = 3$ and we obtain a restricted simple 1-$(15, 3, 1)$-design $(V, \mathscr{P}, \mathscr{B})$ where $V = \{1, 2, \ldots, 15\}$, $\mathscr{P} = \{\{1, 2, 3, 4, 5\}, \{6, 7, 8, 9\}, \{10, 11, 12, 13\}, \{14, 15\}\}$ and a collection $\mathscr{B}$ of 5 blocks of size three $\{\{1, 6, 11\}, \{2, 7, 12\}, \{3, 8, 13\}, \{4, 9, 14\}, \{5, 10, 15\}\}$. Thus, let $\mathscr{B}' = \{\{1, 6\}, \{1, 11\}, \{6, 11\}, \{2, 7\}, \{2, 12\}, \{7, 12\}, \{3, 8\}, \{3, 13\}, \{8, 13\}, \{4, 9\}, \{4, 14\}, \{9, 14\}, \{5, 10\}, \{5, 15\}, \{10, 15\}\}$ be a collection of 12 blocks of size two. This implies that $(V, \mathscr{P}, \mathscr{B}')$ forms a restricted simple 1-$(15, 2, 2)$-design.

Next, we present a construction for a restricted simple 1-(v, k, r)-design when the partition of a v-set is of the same size. We begin with a rather nice result on a restricted simple 1-$(2n, 2, r)$-design.

Theorem 3.3.5. *Let n and r be positive integers. Suppose that an $2n$-set is partitioned into two parts of the same size n. Then there exists a restricted simple 1-$(2n, 2, r)$-design for all $r = 1, 2, \ldots, n$.*

Proof. Let $V = \{x_1, x_2, \ldots, x_n\} \cup \mathbb{Z}_n$ be partitioned into two parts of size n and $\mathscr{P} = \{\{x_1, x_2, \ldots, x_n\}, \mathbb{Z}_n\}$ where $\mathbb{Z}_n$ denoted $\{1, 2, \ldots, n\}$, the set of integers modulo n. Each positive integer r, $1 \le r \le n$, a collection $\mathscr{B}$ of blocks for a restricted simple 1-$(2n, 2, r)$-design constructed as follows. For each $i \in \{1, 2, \ldots, n\}$, define $B_i = (x_i, i)$ be n base blocks, note that (a, b) can be regarded as block $\{a, b\}$, the blocks are obtained by developing the second coordinates (modulo n) of base blocks B_i up to r times (keeping the first coordinates fixed), a collection $\mathscr{B}$ of nr blocks that contain every point of V exactly r times and no such block contain two points from the same part. The result $(V, \mathscr{P}, \mathscr{B})$ is a restricted simple 1-$(2n, 2, r)$-design. □

Example 3.3.6. Suppose that $V = \{x_1, x_2, x_3, x_4\} \cup \mathbb{Z}_4$ is partitioned into two parts of size four $\mathscr{P} = \big\{\{x_1, x_2, x_3, x_4\}, \{1, 2, 3, 4\}\big\}$. For the first, a restricted simple 1-$(8, 2, 1)$-design $(V, \mathscr{P}, \mathscr{B}_1)$ where a collection $\mathscr{B}_1$ of 4 blocks of size two is $\big\{\{x_1, 1\}, \{x_2, 2\}, \{x_3, 3\}, \{x_4, 4\}\big\}$. On the other hand, a restricted simple 1-$(8, 2, 2)$-design $(V, \mathscr{P}, \mathscr{B}_2)$ where a collection $\mathscr{B}_2$ of 8 blocks of size two is $\big\{\{x_1, 1\}, \{x_1, 2\}, \{x_2, 2\}, \{x_2, 3\}, \{x_3, 3\}, \{x_3, 4\}, \{x_4, 4\}, \{x_4, 1\}\big\}$. Next, a restricted simple 1-$(8, 2, 3)$-design $(V, \mathscr{P}, \mathscr{B}_3)$ where a collection $\mathscr{B}_3$ of 12 blocks of size two is $\big\{\{x_1, 1\}, \{x_1, 2\}, \{x_1, 3\}, \{x_2, 2\}, \{x_2, 3\}, \{x_2, 4\}, \{x_3, 3\}, \{x_3, 4\}, \{x_3, 1\}, \{x_4, 4\}, \{x_4, 1\}, \{x_4, 2\}\big\}$. Finally, $(V, \mathscr{P}, \mathscr{B}_4)$ is a restricted simple 1-$(8, 2, 4)$-design where a collection $\mathscr{B}_4$ of 16 blocks of size two is $\big\{\{x_1, 1\}, \{x_1, 2\}, \{x_1, 3\}, \{x_1, 4\}, \{x_2, 2\}, \{x_2, 3\}, \{x_2, 4\}, \{x_2, 1\}, \{x_3, 3\}, \{x_3, 4\}, \{x_3, 1\}, \{x_3, 2\}, \{x_4, 4\}, \{x_4, 1\}, \{x_4, 2\}, \{x_4, 3\}\big\}$.

We will show the existence of a restricted simple 1-(mn, m, r)-design for which an mn-set is partitioned into m parts of the same size n. There are two solutions, first solution, we construct new partitions on an mn-set from the original partition in which points in a new part come from different original parts together as follows.

Recall that, a *system of distinct representatives* (SDR) for a collection of finite nonempty sets $A_1, A_2, \ldots, A_m$ is a collection of distinct elements $x_1, x_2, \ldots, x_m$ such that $x_i \in A_i$ for each i. The sets $A_1, A_2, \ldots, A_m$ possess an SDR if and only if for each $k \leq m$, any k of the sets contain at least k elements in their union (*Hall's condition*). This condition is also sufficient to guarantee the existence of an SDR proved by Philip Hall in 1935.

Lemma 3.3.7. *Let m and n be positive integers such that $m, n \geq 2$. Suppose $\mathscr{P} = \{P_1, P_2, \ldots, P_m\}$ is a partition on an mn-set V, with each part of same size n. It can be constructed at least n^{m-1} new partitions $\mathscr{L}_1, \mathscr{L}_2, \ldots, \mathscr{L}_{n^{m-1}}$ on V where $\mathscr{L}_i=\{S_1^{(i)}, S_2^{(i)}, \ldots, S_n^{(i)}\}$ with each part of size m in which points in $S_j^{(i)}$ come from different parts of $\mathscr{P}$, for all $i = 1, 2, \ldots, n^{m-1}$ and $j = 1, 2, \ldots, n$.*

Proof. Since $P_i \cap P_j = \varnothing$ for $i \neq j$, it follows that for each $k \leq m$, the union of any k sets of P_i contains exactly kn points, the Hall's condition holds. Thus there exists an SDR for $\mathscr{P}$, call it $S_1^{(1)}$ and $S_1^{(1)}$ has size m. Let $\mathscr{P}' = \{P_1', P_2', \ldots, P_m'\}$ where $P_i' = P_i - {S_1}^{(1)}$, for all $i = 1, 2, \ldots, m$ and P_i' has size $n-1$. Since $P_i' \cap P_j' = \varnothing$ for $i \neq j$, it follows that for each $k \leq m$, the union of any k sets of P_i' contains exactly $k(n-1)$ points. Again $\mathscr{P}'$ satisfies the Hall's condition, there exists an SDR for $\mathscr{P}'$, call it $S_2^{(1)}$ and $S_2^{(1)}$ has size m. Repeat this process $n-1$ times, to get the last SDR for $\mathscr{P}$, call it $S_n^{(1)}$ and $S_n^{(1)}$ has size m. Thus $\mathscr{L}_1 = \{S_1^{(1)}, S_2^{(1)}, \ldots, S_n^{(1)}\}$ is clearly a new partition on V, with $|S_i^{(1)}| = m$, for all $i = 1, 2, \ldots, m$ and points in $S_i^{(1)}$ come from different parts of $\mathscr{P}$. Let $\mathscr{U} = P_1 \times P_2 \times \cdots \times P_m$. Each m-tuple in $\mathscr{U}$ can be regarded as an m-subset, it follows that $S_i^{(1)}$ belongs to $\mathscr{U}$ for all $i = 1, 2, \ldots, n$. For $k = 2, \ldots, n^{m-1}$, let $\mathscr{U}_{k-1} = \mathscr{U} - \cup_{j=1}^{k-1} \mathscr{L}_j$, by the same argument to obtain $\mathscr{L}_k=\{S_1^{(k)}, S_2^{(k)}, \ldots, S_n^{(k)}\}$ where $S_i^{(k)}$ belongs to $\mathscr{U}_{k-1}$, for all $i = 1, 2, \ldots, n$. Hence, there are at least n^{m-1} disjoint such partitions $\mathscr{L}_1, \mathscr{L}_2, \ldots, \mathscr{L}_{n^{m-1}}$ on V. □

Example 3.3.8. Suppose that $V = \{1, 2, \ldots, 12\}$ is partitioned into four parts of the same size three, $\mathscr{P} = \{\{1, 2, 3\}, \{4, 5, 6\}, \{7, 8, 9\}, \{10, 11, 12\}\}$. We obtain at least twenty-seven new partitions on V which have the required property as follows:

$$\mathscr{L}_1 = \{\{1,4,7,10\},\{2,5,8,11\},\{3,6,9,12\}\}$$

$$\mathscr{L}_2 = \{\{1,4,7,11\},\{2,5,8,12\},\{3,6,9,10\}\}$$

$$\mathscr{L}_3 = \{\{1,4,7,12\},\{2,5,8,10\},\{3,6,9,11\}\}$$

$$\mathscr{L}_4 = \{\{1,4,8,10\},\{2,5,9,11\},\{3,6,7,12\}\}$$

$$\mathscr{L}_5 = \{\{1,4,8,11\},\{2,5,9,12\},\{3,6,7,10\}\}$$

$$\mathscr{L}_6 = \{\{1,4,8,12\},\{2,5,9,10\},\{3,6,7,11\}\}$$

$$\mathscr{L}_7 = \{\{1,4,9,10\},\{2,5,7,11\},\{3,6,8,12\}\}$$

$$\mathscr{L}_8 = \{\{1,4,9,11\},\{2,5,7,12\},\{3,6,8,10\}\}$$

$$\mathscr{L}_9 = \{\{1,4,9,12\},\{2,5,7,10\},\{3,6,8,11\}\}$$

$$\mathscr{L}_{10} = \{\{1,5,7,10\},\{2,6,8,11\},\{3,4,9,12\}\}$$

$$\mathscr{L}_{11} = \{\{1,5,7,11\},\{2,6,8,12\},\{3,4,9,10\}\}$$

$$\mathscr{L}_{12} = \{\{1,5,7,12\},\{2,6,8,10\},\{3,4,9,11\}\}$$

$$\mathscr{L}_{13} = \{\{1,5,8,10\},\{2,6,9,11\},\{3,4,7,12\}\}$$

$$\mathscr{L}_{14} = \{\{1,5,8,11\},\{2,6,9,12\},\{3,4,7,10\}\}$$

$$\mathscr{L}_{15} = \{\{1,5,8,12\},\{2,6,9,10\},\{3,4,7,11\}\}$$

$$\mathscr{L}_{16} = \{\{1,5,9,10\},\{2,6,7,11\},\{3,4,8,12\}\}$$

$$\mathscr{L}_{17} = \{\{1,5,9,11\},\{2,6,7,12\},\{3,4,8,10\}\}$$

$$\mathscr{L}_{18} = \{\{1,5,9,12\},\{2,6,7,10\},\{3,4,8,11\}\}$$

$$\mathscr{L}_{19} = \{\{1,6,7,10\},\{2,4,8,11\},\{3,5,9,12\}\}$$

$$\mathscr{L}_{20} = \{\{1,6,7,11\},\{2,4,8,12\},\{3,5,9,10\}\}$$

$$\mathscr{L}_{21} = \{\{1,6,7,12\},\{2,4,8,10\},\{3,5,9,11\}\}$$

$$\mathscr{L}_{22} = \{\{1,6,8,10\},\{2,4,9,11\},\{3,5,7,12\}\}$$

$$\mathscr{L}_{23} = \{\{1,6,8,11\},\{2,4,9,12\},\{3,5,7,10\}\}$$

$$\mathscr{L}_{24} = \{\{1,6,8,12\},\{2,4,9,10\},\{3,5,7,11\}\}$$

$$\mathscr{L}_{25} = \{\{1,6,9,10\},\{2,4,7,11\},\{3,5,8,12\}\}$$
$$\mathscr{L}_{26} = \{\{1,6,9,11\},\{2,4,7,12\},\{3,5,8,10\}\}$$
$$\mathscr{L}_{27} = \{\{1,6,9,12\},\{2,4,7,10\},\{3,5,8,11\}\}.$$

Theorem 3.3.9. *Let m, n and r be positive integers such that $2 \le m$ and $r \le n^{m-1}$. Suppose that $\mathscr{P} = \{P_1, P_2, \ldots, P_m\}$ is a partition set of the same size n on an mn-set V. Then there exists a restricted simple 1-(mn, m, r)-design.*

Proof. By Lemma 3.3.7, there are at least n^{m-1} disjoint partitions $\mathscr{L}_1, \mathscr{L}_2, \ldots, \mathscr{L}_{n^{m-1}}$ on V. For each $r = 1, 2, \ldots, n^{m-1}$, let $J \subseteq \{1, 2, \ldots, n^{m-1}\}$ and $|J| = r$, define a collection $\mathscr{B}(J)$ of blocks to be the union of r partition sets $\mathscr{L}_i$, for all $i \in J$. Since all parts in each partition set $\mathscr{L}_i$ have size m, so every block must have size m. Since each point occurs only one time in each partition set $\mathscr{L}_i$, it follows that each point occurs r times in blocks and any two points in a block come from different parts of $\mathscr{P}$. Therefore, $(V, \mathscr{P}, \mathscr{B}(J))$ is a restricted simple 1-(mn, m, r)-design. □

Example 3.3.10. From Example 3.3.8, let $J_1 = \{2, 5, 6\}$ and $\mathscr{B}(J_1) = \mathscr{L}_2 \cup \mathscr{L}_5 \cup \mathscr{L}_6 = \{\{1,4,7,11\},\{2,5,8,12\},\{3,6,9,10\},\{1,4,8,11\},\{2,5,9,12\},\{3,6,7,10\},\{1,4,8,12\},\{2,5,9,10\},\{3,6,7,11\}\}$. Thus $(V, \mathscr{P}, \mathscr{B}(J_1))$ is a restricted simple 1-$(12, 4, 3)$-design. As another example, let $J_2 = \{6, 7, \ldots, 18\}$ and $\mathscr{B}(J_2) = \mathscr{L}_6 \cup \mathscr{L}_7 \cup \cdots \cup \mathscr{L}_{18} = \{\{1,4,8,12\},\{2,5,9,10\},\{3,6,7,11\},\{1,4,9,10\},\{2,5,7,11\},\{3,6,8,12\},\{1,4,9,11\},\{2,5,7,12\},\{3,6,8,10\},\{1,4,9,12\},\{2,5,7,10\},\{3,6,8,11\},\{1,5,7,10\},\{2,6,8,11\},\{3,4,9,12\},\{1,5,7,11\},\{2,6,8,12\},\{3,4,9,10\},\{1,5,7,12\},\{2,6,8,10\},\{3,4,9,11\},\{1,5,8,10\},\{2,6,9,11\},\{3,4,7,12\},\{1,5,8,11\},\{2,6,9,12\},\{3,4,7,10\},\{1,5,8,12\},\{2,6,9,10\},\{3,4,7,11\},\{1,5,9,10\},\{2,6,7,11\},\{3,4,8,12\},\{1,5,9,11\},\{2,6,7,12\},\{3,4,8,10\},\{1,5,9,12\},\{2,6,7,10\},\{3,4,8,11\}\}$. Therefore, $(V, \mathscr{P}, \mathscr{B}(J_2))$ is a restricted simple 1-$(12, 4, 13)$-design.

Second solution, we define a function to construct a matrix such that the columns of the matrix form blocks in a restricted simple 1-(mn, m, r)-design as follows.

Theorem 3.3.11. *Let m, n and r be positive integers such that $m \geq 2$. Suppose that $\mathscr{P} = \{P_1, P_2, \ldots, P_m\}$ is a partition set of the same size n on an mn-set V and $r \leq n^{m-1}$. Then there exists a restricted simple 1-(mn, m, r)-design.*

Proof. For any integer k, $1 \leq k \leq n$, each part P_j regarded as a column matrix $(p_{1j}\ p_{2j} \cdots\ p_{nj})^t$ where $p_{1j}, \ldots, p_{nj} \in P_j$, defines a function r by $r(k, P_j) = r_k(P_j) = \left(p_{(k+1)j}\ p_{(k+2)j} \cdots p_{(k)j}\right)^t$ by developing the first coordinates modulo n, (keeping the second coordinates fixed). For each $i_1 \in \{1, 2, \ldots, n\}$, we construct a matrix $[P_1\ r_{i_1}(P_2)]$, all rows of $[P_1\ r_{i_1}(P_2)]$ form a partition on $P_1 \cup P_2$ and there are n disjoint partitions on $P_1 \cup P_2$. For each $i_1, i_2, \ldots, i_{m-2} \in \{1, 2, \ldots, n\}$, we construct a matrix $[P_1\ r_{i_1}(P_2)\ r_{i_2}(P_3) \cdots\ r_{i_{m-1}}(P_m)]$, for $i_{m-1} \in \{1, 2, \ldots, n\}$. All rows $\left[P_1\ r_{i_1}(P_2)\ r_{i_2}(P_3) \cdots r_{i_{m-1}}(P_m)\right]$ form a new partition on V. There are at least n^{m-1} new disjoint partitions on V.

Consider the rows of $\left[P_1\ r_{i_1}(P_2)\ r_{i_2}(P_3) \cdots r_{i_{m-1}}(P_m)\right]$ as blocks. Clearly, each block has size m. For each $r = 1, 2, \ldots, n^{m-1}$, we construct a restricted simple 1-(mn, m, r)-design $(V, \mathscr{P}, \mathscr{B})$ in which $\mathscr{B}$ is the union of r new partitions on V. □

We illustrate the application of the above theorem in the following example.

Example 3.3.12. Suppose that $V = \{1, 2, \ldots, 12\}$ is partitioned into four parts of size three $\mathscr{P} = \{P_1, P_2, P_3, P_4\}$, where $P_1 = \{1, 2, 3\}$, $P_2 = \{4, 5, 6\}$, $P_3 = \{7, 8, 9\}$ and $P_4 = \{10, 11, 12\}$.

Since $P_2 = (p_{12}\ p_{22}\ p_{32})^t = (4\ 5\ 6)^t$, we have $r_1(P_2) = (p_{22}\ p_{32}\ p_{12})^t = (5\ 6\ 4)^t$ and $[P_1\ r_1(P_2)] = [(1\ 2\ 3)^t\ (5\ 6\ 4)^t]$. Next, since $P_3 = (p_{13}\ p_{23}\ p_{33})^t =$

$(7\ 8\ 9)^t$, we have $r_1(P_3) = (p_{23}\ p_{33}\ p_{13})^t = (8\ 9\ 7)^t$ and $[P_1\ r_1(P_2)\ r_1(P_3)] = [(1\ 2\ 3)^t\ (5\ 6\ 4)^t\ (8\ 9\ 7)^t]$. Finally, since $P_4 = (p_{14}\ p_{24}\ p_{34})^t = (10\ 11\ 12)^t$, we have $r_1(P_4) = (p_{24}\ p_{34}\ p_{14})^t = (11\ 12\ 10)^t$ and $[P_1\ r_1(P_2)\ r_1(P_3)\ r_1(P_4)] = [(1\ 2\ 3)^t\ (5\ 6\ 4)^t\ (8\ 9\ 7)^t\ (11\ 12\ 10)^t] = \begin{bmatrix} 1 & 5 & 8 & 11 \\ 2 & 6 & 9 & 12 \\ 3 & 4 & 7 & 10 \end{bmatrix}$, thus $\big\{\{1,5,8,11\},\{2,6,9,12\},\{3,4,7,10\}\big\}$ is a new partition on V. Similarly, $[P_1\ r_2(P_2)\ r_3(P_3)\ r_3(P_4)] = \begin{bmatrix} 1 & 5 & 9 & 12 \\ 2 & 6 & 7 & 10 \\ 3 & 4 & 8 & 11 \end{bmatrix}$, thus $\big\{\{1,5,9,12\},\{2,6,7,10\},\{3,4,8,11\}\big\}$ is a new partition on V, $[P_1\ r_3(P_2)\ r_2(P_3)\ r_3(P_4)] = \begin{bmatrix} 1 & 6 & 8 & 12 \\ 2 & 4 & 9 & 10 \\ 3 & 5 & 7 & 11 \end{bmatrix}$, thus $\big\{\{1,6,8,12\},\{2,4,9,10\},\{3,5,7,11\}\big\}$ is a new partition on V and $[P_1\ r_3(P_2)\ r_3(P_3)\ r_1(P_4)] = \begin{bmatrix} 1 & 6 & 9 & 10 \\ 2 & 4 & 7 & 11 \\ 3 & 5 & 8 & 12 \end{bmatrix}$, thus $\big\{\{1,5,9,12\},\{2,6,7,10\},\{3,4,8,11\}\big\}$ is a new partition on V. Let $\mathscr{B}$ be the union of 4 new partitions mentioned above on V, that is $\mathscr{B} = \big\{\{1,5,8,11\},\{2,6,9,12\},\{3,4,7,10\},\{1,5,9,12\},\{2,6,7,10\},\{3,4,8,11\},\{1,6,8,12\},\{2,4,9,10\},\{3,5,7,11\},\{1,6,9,10\},\{2,4,7,11\},\{3,5,8,12\}\big\}$. Therefore, we obtain a restricted simple 1-(12, 4, 4)-design $(V, \mathscr{P}, \mathscr{B})$.

Finally, we present a construction for a restricted simple 1-(v,k,r)-design when the size of partition is arbitrary. Billington [2] gave an elegant proof for the existence of a simple 1-(v,k,r)-design. From the idea in the proof by Billington, we first introduce a new design as follows.

Definition 3.3.13. Let v, k and r be positive integers such that $2 \leq k \leq v$ and $r_1, r_2, \ldots, r_v$ be nonnegative integers. A **restricted simple** $(k; r_1, r_2, \ldots, r_v)$-

design is a triple $(V, \mathscr{P}, \mathscr{B})$ satisfying the following properties :

1. $V = \{1, 2, \ldots, v\}$ called **points**,

2. $\mathscr{P}$ is a partition of V into nonempty subsets called **parts**,

3. $\mathscr{B}$ is a collection of different k-subsets of V called **blocks**,

4. any two points from the same part do not occur together in a block, and

5. each point $i \in V$ is contained in exactly r_i blocks.

Denote by $\mathscr{D} = (V, \mathscr{P}, \mathscr{B})$ the restricted simple $(k; r_1, r_2, \ldots, r_v)$-design where parameters are not mentioned in the design and r_i may be called the *replication number of point i*.

Remark 3.3.14. A restricted simple $(k; r_1, r_2, \ldots, r_v)$-design is a restricted simple 1-(v, k, r)-design where $r_i = r$ for all $i \in V$.

Example 3.3.15. Let $V = \{1, 2, \ldots, 15\}$ be partitioned into four parts $\mathscr{P} = \{\{1,2,3,4,5\}, \{6,7,8,9\}, \{10,11,12,13\}, \{14,15\}\}$ and a collection of blocks of size three, $\mathscr{B} = \{\{1,6,10\}, \{2,6,10\}, \{3,6,10\}, \{4,6,10\}, \{5,6,10\}, \{1,7,10\}, \{1,8,10\}, \{1,9,10\}, \{1,6,11\}, \{1,6,12\}, \{1,6,13\}, \{2,7,10\}, \{1,6,14\}, \{2,6,14\}, \{3,6,14\}, \{4,6,14\}, \{1,10,14\}, \{2,10,14\}, \{3,10,14\}, \{4,10,14\}\}$. Thus, $(V, \mathscr{P}, \mathscr{B})$ form a restricted simple $(3; 9,4,3,3,1,12,2,1,1,13,1,1,1,8,0)$-design.

We first show the property of a restricted simple $(k; r_1, r_2, \ldots, r_v)$-design, then construct a restricted simple $(k; r_1, r_2, \ldots, r_v)$-design certain property. Finally, we construct a restricted simple 1-(v, k, r)-design.

Theorem 3.3.16. *Let v, m and k be positive integers such that $2 \leq k \leq m \leq v$ and $r_1, r_2, \ldots, r_v$ be nonnegative integers. Let $V = \{1, 2, \ldots, v\}$ be partitioned into* *m parts* $\mathscr{P} = \{P_1, P_2, \ldots, P_m\}$. *Suppose that a restricted simple* $(k; r_1, r_2, \ldots, r_v)$-*design exists. If* $x, y \in P_w$ *for some integer* w, $1 \leq w \leq m$ *with* $r_x > r_y$, *then there exists a restricted simple* $(k; r_1, r_2, \ldots, r_x - 1, \ldots, r_y + 1, \ldots, r_v)$-*design.*

Proof. Let $\mathscr{D} = (V, \mathscr{P}, \mathscr{B})$ be a restricted simple $(k; r_1, r_2, \ldots, r_v)$-design. Let $B_1, B_2, \ldots, B_i$ be all blocks of $\mathscr{B}$ which contain point x. Let $C_1, C_2, \ldots, C_j$ be all blocks of $\mathscr{B}$ which contain point y. First step, we set $\mathscr{B}^x = \{B_1{}^x, B_2{}^x, \ldots, B_i{}^x\}$ where $B_a{}^x = B_a - \{x\}$ and $\mathscr{C}^y = \{C_1{}^y, C_2{}^y, \ldots, C_j{}^y\}$ where $C_b{}^y = C_b - \{y\}$. Second step, we can choose a block $B \in \mathscr{B}^x - \mathscr{C}^y$ which has property that if $B^* = B \cup \{y\}$ then $B^* \notin \{C_1, C_2, \ldots, C_j\}$ (note that, $|\mathscr{B}^x - \mathscr{C}^y| \geq r_x - r_y$ or there are at least $r_x - r_y$ such B's). Final step, let $\mathscr{B}^* = [\mathscr{B} - (B \cup \{x\})] \cup B^*$, this implies that, $\mathscr{D}^* = (V, \mathscr{P}, \mathscr{B}^*)$ is a design with the replication number of x and y in $\mathscr{D}^*$ is changed to $r_x - 1$ and $r_y + 1$, respectively. For simple property, since $y \in B^*$ and $B^* \notin \{C_1, C_2, \ldots, C_j\}$, it follows that B^* is different from other blocks and for restriction property, since x and any points in B come from different parts and $x, y \in P_w$, it follows that y and any points in B^* must come from different parts of $\mathscr{P}$. □

Example 3.3.17. An illustration of Theorem 3.3.16, from Example 3.3.15, consider points $6, 7$ in the part $\{6, 7, 8, 9\}$ such that $r_6 = 12$ and $r_7 = 2$. Let $B_1 = \{1, 6, 10\}, B_2 = \{2, 6, 10\}, B_3 = \{3, 6, 10\}, B_4 = \{4, 6, 10\}, B_5 = \{5, 6, 10\}, B_6 = \{1, 6, 11\}, B_7 = \{1, 6, 12\}, B_8 = \{1, 6, 13\}, B_9 = \{1, 6, 14\}, B_{10} = \{2, 6, 14\}, B_{11} = \{3, 6, 14\}, B_{12} = \{4, 6, 14\}$ be all blocks in $\mathscr{B}$ which contain point 6. Let $C_1 = \{1, 7, 10\}, C_2 = \{2, 7, 10\}$ be all blocks in $\mathscr{B}$ which contain point 7. First step, we set $\mathscr{B}^6 = \{B_1{}^6, B_2{}^6, \ldots, B_{12}{}^6\}$ where $B_1{}^6 = B_1 - \{6\} = \{1, 10\}, B_2{}^6 = B_2 - \{6\} = \{2, 10\}, \ldots, B_{12}{}^6 = B_{12} - \{6\} = \{4, 14\}$ and $\mathscr{C}^7 = \{C_1{}^7, C_2{}^7\}$ where $C_1{}^7 = C_1 - \{7\} = \{1, 10\}, C_2{}^7 = C_2 - \{7\} = \{2, 10\}$. Second step, we choose $B = \{3, 10\} \in B^6 - B^7$ since $B^* = B \cup \{7\} = \{3, 7, 10\}$ and $\{3, 7, 10\} \notin \{C_1, C_2\}$. Final step, we delete $B = \{3, 6, 10\}$ from $\mathscr{B}$ and replace it by $B^* = \{3, 7, 10\}$. We obtain a restricted $(3; 9, 4, 3, 3, 1, 11, 3, 1, 1, 13, 1, 1, 1, 8, 0)$-design $(V, \mathscr{P}, \mathscr{B}^*)$ where a collection $\mathscr{B}^*$ of 20 blocks is $\{\{1, 6, 10\}, \{2, 6, 10\}, \{3, 7, 10\}, \{4, 6, 10\}, \{5, 6, 10\}, \{1, 7, 10\}, \{1, 8, 10\}, \{1, 9, 10\}, \{1, 6, 11\}, \{1, 6, 12\}, \{1, 6, 13\}, \{2, 7, 10\}, \{1, 6, 14\}, \{2, 6, 14\}, \{3, 6, 14\}, \{4, 6, 14\}, \{1, 10, 14\}, \{2, 10, 14\}, \{3, 10, 14\}, \{4, 10, 14\}\}$.

Theorem 3.3.18. *Let v, m, k and r be positive integers such that $2 \le k \le m \le v$. Let $V = \{1, 2, \ldots, v\}$ be partitioned into m parts $\mathscr{P} = \{P_1, P_2, \ldots, P_m\}$ of size $p_1, p_2, \ldots, p_m$, respectively such that $p_1 \ge p_2 \ge \cdots \ge p_m$. Suppose that $vr \equiv 0 \pmod{k}$, $p_1 k \le v$ and $r \le p_{\alpha_1} p_{\alpha_2} \cdots p_{\alpha_{k-1}}$ for all $\{\alpha_1, \alpha_2, \ldots, \alpha_{k-1}\} \subseteq \{2, 3, \ldots, m\}$. Then there are nonnegative integers $r_1, r_2, \ldots, r_v$ such that $\sum_{i \in P_j} r_i = p_j r$ for all $j = 1, 2, \ldots, m$ and a restricted simple $(k; r_1, r_2, \ldots, r_v)$-design exists with $\frac{vr}{k}$ blocks where r_i is the replication number of i for $i \in V$.*

Proof. Without loss of generality, suppose that $P_1 = \{1, 2, \ldots, p_1\}, P_2 = \{p_1 + 1, p_1 + 2, \ldots, p_1 + p_2\}, \ldots, P_m = \{p_1 + p_2 + \cdots + p_{m-1} + 1, \ldots, p_1 + p_2 + \cdots + p_m\}$. For convenience in our construction, we make use of two subscripts to describe points in each part of $\mathscr{P}$ by the formula : for $i = 1, \ldots, m$, x_{ij} corresponds to $\sum_{k=1}^{i-1} p_k + j$ where $j = 1, \ldots, p_i$. Thus for $i = 1, \ldots, m$, $P_i = \{x_{ij} \in V \mid j = 1, \ldots, p_i\}$. First, construct a restricted $(k; p_1 r, \underbrace{0, \ldots, 0}_{(p_1 - 1)\text{ terms}}, p_2 r, \underbrace{0, \ldots, 0}_{(p_2 - 1)\text{ terms}}, \ldots, p_m r, \underbrace{0, \ldots, 0}_{(p_m - 1)\text{ terms}})$-design, say $\mathscr{D}_1$ as follows.

Let call all blocks of size k as $B_1, B_2, \ldots, B_{\frac{vr}{k}}$. Place point x_{11} in $p_1 r$ blocks $B_1, \ldots, B_{p_1 r}$. Then continue placing point x_{21} in $p_2 r$ blocks $B_{p_1 r + 1}, \ldots, B_{p_1 r + p_2 r}$ where the subscripts are added modulo $\frac{vr}{k}$. The same argument is applied to the points $x_{31}, \ldots, x_{m1}$. By this method, each point $x_{11}, x_{21}, \ldots, x_{m1}$ occurs $p_1 r, p_2 r, \ldots, p_m r$ times, respectively and distributes evenly in each block. Since the total of occurrence of $x_{11}, x_{21}, \ldots, x_{m1}$ is $p_1 r + p_2 r + \cdots + p_m r = (p_1 + p_2 + \cdots + p_m) r = vr$ and $vr \equiv 0 \pmod{k}$, it follows that $\mathscr{D}_1$ is clearly a restricted

$(k; p_1r, \underbrace{0,\ldots,0}_{(p_1-1)\text{ terms}}, p_2r, \underbrace{0,\ldots,0}_{(p_2-1)\text{ terms}}, \ldots, p_mr, \underbrace{0,\ldots,0}_{(p_m-1)\text{ terms}})$-design, with $\frac{vr}{k}$ blocks, but $\mathscr{D}_1$ might not be simple. Note that for all $j = 1, 2, \ldots, m$, $\sum_{i \in P_j} r_i = p_jr + \underbrace{0 + 0 + \cdots + 0}_{(p_j-1)\text{ terms}} = p_jr$.

Suppose $\mathscr{D}_1$ is not simple, let $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_l$ for some a positive integer l be different collections of $\mu_1, \mu_2, \ldots, \mu_l$ repeated blocks, respectively. Note that $\mu_1+\mu_2+\cdots+\mu_l = \frac{vr}{k}$. Without loss of generality, suppose that $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_l$. Next, construct a restricted simple $(k; s_{11}, \ldots, s_{1p_1}, s_{21}, \ldots, s_{2p_2}, \ldots, s_{m1}, \ldots, s_{mp_m})$-design, where $s_{i1} + \cdots + s_{ip_i} = p_ir$, for $i = 1, \ldots, m$ as follows.

Let $\{a_{\alpha_1 1}, a_{\alpha_2 1}, \ldots, a_{\alpha_k 1}\}$ be μ_1 repeated blocks in $\mathscr{A}_1$, where $a_{\alpha_i 1} \in P_{\alpha_i}$, for $i = 1, \ldots, k$ and $\{\alpha_1, \alpha_2, \ldots, \alpha_k\} \subseteq \{1, \ldots, m\}$. Without loss of generality, suppose that $p_{\alpha_1} \geq p_{\alpha_2} \geq \cdots \geq p_{\alpha_k}$.

Since there are at most rp_{α_k} repeated blocks in $\mathscr{D}_1$, this implies that $\mu_1 \leq rp_{\alpha_k}$. Since there are $p_{\alpha_1}p_{\alpha_2}\cdots p_{\alpha_k}$ k-subsets for which each point comes from $P_{\alpha_1}, P_{\alpha_2}, \ldots, P_{\alpha_k}$ and $r \leq p_{\alpha_1}p_{\alpha_2}\cdots p_{\alpha_{k-1}}$ for all $\{\alpha_1, \alpha_2, \ldots, \alpha_{k-1}\} \subseteq \{2, 3, \ldots, m\}$ it follows that $\mu_1 \leq p_{\alpha_1}p_{\alpha_2}\cdots p_{\alpha_k}$. Replace μ_1 repeated blocks by any different k-subsets from the $P_{\alpha_1}, P_{\alpha_2} \ldots, P_{\alpha_k}$. The replication number of each point in each part $P_{\alpha_1}, P_{\alpha_2}, \ldots, P_{\alpha_k}$ is changed from $p_ir, 0, \ldots, 0$ to ${s^{(1)}}_{i1}, \ldots, {s^{(1)}}_{ip_i}$ such that ${s^{(1)}}_{i1} + \cdots + {s^{(1)}}_{ip_i} = p_ir$, for all $i = 1, \ldots, k$. An apply the same process to $\mathscr{A}_2, \mathscr{A}_3$, until $\mathscr{A}_l$. Since $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_l$ are different collections of repeated blocks, it follows that each time the repeat blocks are replaced by different new k-subsets and the last design will be a required design with $r_1 = s^{(l)}_{11}, r_2 = s^{(l)}_{12}, \ldots, r_v = s^{(l)}_{mp_m}$, as desired. □

Example 3.3.19. Let $v = 15, k = 3$ and $r = 4$. Let $V = \{1, 2, \ldots, 15\}$ be partitioned into 4 parts $\mathscr{P} = \{P_1, P_2, P_3, P_4\}$ of size $p_1 = 5, p_2 = 4, p_3 = 4, p_4 = 2$, respectively. Note that $vr \equiv 0 \pmod{k}$, $p_1k \leq v$ and $r \leq p_{\alpha_1}p_{\alpha_2}$

for all $\{\alpha_1, \alpha_2\} \subseteq \{2, 3, 4\}$. Without loss of generality, suppose that $P_1 = \{1, 2, 3, 4, 5\} = \{x_{11}, x_{12}, x_{13}, x_{14}, x_{15}\}$, $P_2 = \{6, 7, 8, 9\} = \{x_{21}, x_{22}, x_{23}, x_{24}\}$, $P_3 = \{10, 11, 12, 13\} = \{x_{31}, x_{32}, x_{33}, x_{34}\}$ and $P_4 = \{14, 15\} = \{x_{41}, x_{42}\}$. First, we construct a restricted $(3; 20, 0, 0, 0, 0, 16, 0, 0, 0, 16, 0, 0, 0, 8, 0)$-design, say $\mathscr{D}_1$ with blocks are :

$\{x_{11}, x_{21}, x_{31}\}, \{x_{11}, x_{21}, x_{31}\}, \{x_{11}, x_{21}, x_{31}\}, \{x_{11}, x_{21}, x_{31}\}, \{x_{11}, x_{21}, x_{31}\},$

$\{x_{11}, x_{21}, x_{31}\}, \{x_{11}, x_{21}, x_{31}\}, \{x_{11}, x_{21}, x_{31}\}, \{x_{11}, x_{21}, x_{31}\}, \{x_{11}, x_{21}, x_{31}\},$

$\{x_{11}, x_{21}, x_{31}\}, \{x_{11}, x_{21}, x_{31}\}, \{x_{11}, x_{21}, x_{41}\}, \{x_{11}, x_{21}, x_{41}\}, \{x_{11}, x_{21}, x_{41}\},$

$\{x_{11}, x_{21}, x_{41}\}, \{x_{11}, x_{31}, x_{41}\}, \{x_{11}, x_{31}, x_{41}\}, \{x_{11}, x_{31}, x_{41}\}, \{x_{11}, x_{31}, x_{41}\}.$

Note that $r_1 + r_2 + r_3 + r_4 + r_5 = 20 + 0 + 0 + 0 + 0 = 5 \times 4$, $r_6 + r_7 + r_8 + r_9 = 16 + 0 + 0 + 0 = 4 \times 4$, $r_{10} + r_{11} + r_{12} + r_{13} = 16 + 0 + 0 + 0 = 4 \times 4$ and $r_{14} + r_{15} = 8 + 0 = 2 \times 4$. Since $\mathscr{D}_1$ is not simple, let $\mathscr{A}_1 = 12\{x_{11}, x_{21}, x_{31}\}$ be a collection of $\mu_1 = 12$ repeated blocks. There are eighty 3-subsets for which each point comes from P_1, P_2 and P_3, then replace all repeated blocks with any different 3-subsets from the eighty 3-subsets.

We obtain a restricted $(3; 15, 2, 1, 1, 1, 12, 2, 1, 1, 13, 1, 1, 1, 8, 0)$-design with blocks are :

$\{x_{11}, x_{21}, x_{31}\}, \{x_{12}, x_{21}, x_{31}\}, \{x_{13}, x_{21}, x_{31}\}, \{x_{14}, x_{21}, x_{31}\}, \{x_{15}, x_{21}, x_{31}\},$

$\{x_{11}, x_{22}, x_{31}\}, \{x_{11}, x_{23}, x_{31}\}, \{x_{11}, x_{24}, x_{31}\}, \{x_{11}, x_{21}, x_{32}\}, \{x_{11}, x_{21}, x_{33}\},$

$\{x_{11}, x_{21}, x_{34}\}, \{x_{12}, x_{22}, x_{31}\}, \{x_{11}, x_{21}, x_{41}\}, \{x_{11}, x_{21}, x_{41}\}, \{x_{11}, x_{21}, x_{41}\},$

$\{x_{11}, x_{21}, x_{41}\}, \{x_{11}, x_{31}, x_{41}\}, \{x_{11}, x_{31}, x_{41}\}, \{x_{11}, x_{31}, x_{41}\}, \{x_{11}, x_{31}, x_{41}\}.$

Note that $r_1 + r_2 + r_3 + r_4 + r_5 = 15 + 2 + 1 + 1 + 1 = 5 \times 4$, $r_6 + r_7 + r_8 + r_9 = 12 + 2 + 1 + 1 = 4 \times 4$, $r_{10} + r_{11} + r_{12} + r_{13} = 13 + 1 + 1 + 1 = 4 \times 4$ and $r_{14} + r_{15} = 8 + 0 = 2 \times 4$. Again the design is not simple, let $\mathscr{A}_2 = 4\{x_{11}, x_{21}, x_{41}\}$ be a collection of $\mu_2 = 4$ repeated blocks. There are forty 3-subsets for which each point comes from P_1, P_2 and P_4, then replace all repeated blocks with any different 3-subsets from the forty 3-subsets.

We obtain a restricted $(3; 12, 3, 2, 2, 1, 12, 2, 1, 1, 13, 1, 1, 1, 8, 0)$-design with blocks are :

$\{x_{11}, x_{21}, x_{31}\}, \{x_{12}, x_{21}, x_{31}\}, \{x_{13}, x_{21}, x_{31}\}, \{x_{14}, x_{21}, x_{31}\}, \{x_{15}, x_{21}, x_{31}\},$
$\{x_{11}, x_{22}, x_{31}\}, \{x_{11}, x_{23}, x_{31}\}, \{x_{11}, x_{24}, x_{31}\}, \{x_{11}, x_{21}, x_{32}\}, \{x_{11}, x_{21}, x_{33}\},$
$\{x_{11}, x_{21}, x_{34}\}, \{x_{12}, x_{22}, x_{31}\}, \{x_{11}, x_{21}, x_{41}\}, \{x_{12}, x_{21}, x_{41}\}, \{x_{13}, x_{21}, x_{41}\},$
$\{x_{14}, x_{21}, x_{41}\}, \{x_{11}, x_{31}, x_{41}\}, \{x_{11}, x_{31}, x_{41}\}, \{x_{11}, x_{31}, x_{41}\}, \{x_{11}, x_{31}, x_{41}\}.$

Note that $r_1 + r_2 + r_3 + r_4 + r_5 = 12 + 3 + 2 + 2 + 1 = 5 \times 4$, $r_6 + r_7 + r_8 + r_9 = 12 + 2 + 1 + 1 = 4 \times 4$, $r_{10} + r_{11} + r_{12} + r_{13} = 13 + 1 + 1 + 1 = 4 \times 4$ and $r_{14} + r_{15} = 8 + 0 = 2 \times 4$. Again the design is not simple, let $\mathscr{A}_3 = 4\{x_{11}, x_{31}, x_{41}\}$ be a collection of $\mu_3 = 4$ repeated blocks. There are forty 3-subsets for which each point comes from P_1, P_3 and P_4, then replace all repeated blocks with any different 3-subsets from the forty 3-subsets.

We obtain a restricted simple $(3; 9, 4, 3, 3, 1, 12, 2, 1, 1, 13, 1, 1, 1, 8, 0)$-design with blocks are :

$\{x_{11}, x_{21}, x_{31}\}, \{x_{12}, x_{21}, x_{31}\}, \{x_{13}, x_{21}, x_{31}\}, \{x_{14}, x_{21}, x_{31}\}, \{x_{15}, x_{21}, x_{31}\},$
$\{x_{11}, x_{22}, x_{31}\}, \{x_{11}, x_{23}, x_{31}\}, \{x_{11}, x_{24}, x_{31}\}, \{x_{11}, x_{21}, x_{32}\}, \{x_{11}, x_{21}, x_{33}\},$
$\{x_{11}, x_{21}, x_{34}\}, \{x_{12}, x_{22}, x_{31}\}, \{x_{11}, x_{21}, x_{41}\}, \{x_{12}, x_{21}, x_{41}\}, \{x_{13}, x_{21}, x_{41}\},$
$\{x_{14}, x_{21}, x_{41}\}, \{x_{11}, x_{31}, x_{41}\}, \{x_{12}, x_{31}, x_{41}\}, \{x_{13}, x_{31}, x_{41}\}, \{x_{14}, x_{31}, x_{41}\},$
which correspond to $\{1, 6, 10\}, \{2, 6, 10\}, \{3, 6, 10\}, \{4, 6, 10\}, \{5, 6, 10\}, \{1, 7, 10\}, \{1, 8, 10\}, \{1, 9, 10\}, \{1, 6, 11\}, \{1, 6, 12\}, \{1, 6, 13\}, \{2, 7, 10\}, \{1, 6, 14\}, \{2, 6, 14\}, \{3, 6, 14\}, \{4, 6, 14\}, \{1, 10, 14\}, \{2, 10, 14\}, \{3, 10, 14\}, \{4, 10, 14\}$.

Note that $r_1 + r_2 + r_3 + r_4 + r_5 = 9 + 4 + 3 + 3 + 1 = 5 \times 4$, $r_6 + r_7 + r_8 + r_9 = 12 + 2 + 1 + 1 = 4 \times 4$, $r_{10} + r_{11} + r_{12} + r_{13} = 13 + 1 + 1 + 1 = 4 \times 4$ and $r_{14} + r_{15} = 8 + 0 = 2 \times 4$. Therefore, we obtain a restricted simple $(3; 9, 4, 3, 3, 1, 12, 2, 1, 1, 13, 1, 1, 1, 8, 0)$-design.

Now, we are ready to show the existence of a restricted simple 1-(v, k, r)-design.

Theorem 3.3.20. *Let v, m, k and r be positive integers such that $2 \leq k \leq m \leq v$. Let $V = \{1, 2, \ldots, v\}$ be partitioned into m parts $\mathscr{P} = \{P_1, P_2, \ldots, P_m\}$ of size $p_1, p_2, \ldots, p_m$, respectively such that $p_1 \geq p_2 \geq \cdots \geq p_m$. Suppose that $vr \equiv 0 \pmod{k}$, $p_1 k \leq v$ and there exists a restricted simple $(k; r_1, r_2, \ldots, r_v)$-design and $\frac{1}{p_j}\sum\limits_{i \in P_j} r_i$ is a constant r, for all $j = 1, \ldots, m$ where r_i is the replication number of i for $i \in V$. Then there exists a restricted simple 1-(v, k, r)-design.*

Proof. Let $(V, \mathscr{P}, \mathscr{B})$ be a restricted simple $(k; r_1, r_2, \ldots, r_v)$-design and for each $j = 1, \ldots, m$, $\sum\limits_{i \in P_j} r_i = rp_j$ and $bk = rp_1 + rp_2 + \cdots + rp_m = (p_1 + p_2 + \cdots + p_m)r = vr$. In order to change the replication number r_i to r for all $i \in V$, any two points are considered at a time, by Theorem 3.3.16 and this theorem may be applied $\frac{1}{2}\sum\limits_{i=1}^{v}|r - r_i|$ times. Since $vr = r_1 + r_2 + \cdots + r_v$, it follows that $\mathscr{B}$ is transformed into a restricted simple $(k; r, r, \ldots, r)$-design, say $\mathscr{B}'$. Hence, $(V, \mathscr{P}, \mathscr{B}')$ forms a restricted simple 1-(v, k, r)-design and so the proof is complete. □

Example 3.3.21. From Example 3.3.19, we obtain a restricted simple $(3; 9, 4, 3, 3, 1, 12, 2, 1, 1, 13, 1, 1, 1, 8, 0)$-design, say $(V, \mathscr{P}, \mathscr{B})$ where a collection $\mathscr{B}$ of twenty blocks of size three is $\{\{1, 6, 10\}, \{2, 6, 10\}, \{3, 6, 10\}, \{4, 6, 10\}, \{5, 6, 10\}, \{1, 7, 10\}, \{1, 8, 10\}, \{1, 9, 10\}, \{1, 6, 11\}, \{1, 6, 12\}, \{1, 6, 13\}, \{2, 7, 10\}, \{1, 6, 14\}, \{2, 6, 14\}, \{3, 6, 14\}, \{4, 6, 14\}, \{1, 10, 14\}, \{2, 10, 14\}, \{3, 10, 14\}, \{4, 10, 14\}\}$. Since for all $j = 1, 2, 3, 4, 5$, $\sum\limits_{i \in P_j} r_i = 4p_j$, Theorem 3.3.16 may be applied $\frac{1}{2}\sum\limits_{i=1}^{15}|4 - r_i|$=26 times to change the replication number r_i to 4 for all $i \in V$. For the last transformation, we obtain a restricted simple $(3; 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4)$-design, say $(V, \mathscr{P}, \mathscr{B}')$ where a collection $\mathscr{B}'$ of twenty blocks of size three is $\{\{1, 8, 11\}, \{2, 8, 11\}, \{3, 8, 11\}, \{4, 7, 12\}, \{5, 7, 12\}, \{3, 7, 12\}, \{4, 8, 13\}, \{5, 9, 13\}, \{5, 9, 11\}, \{5, 9, 12\}, \{1, 9, 13\}, \{2, 7, 13\}, \{1, 6, 15\}, \{2, 6, 15\}, \{3, 6, 15\}, \{4, 6, 15\}, \{1, 10, 14\}, \{2,$

$10, 14\}, \{3, 10, 14\}, \{4, 10, 14\}\}$. Therefore, $(V, \mathscr{P}, \mathscr{B}')$ is a restricted simple 1-$(15, 3, 4)$-design.

3.4 New restricted simple 1-designs from old

We give two simple methods of constructing new restricted simple 1-designs from the existing one. The first construction may be called a *sum construction* and the second construction may be called a *refinement construction.*

Theorem 3.4.1. *Suppose that there are a restricted simple 1-(v_1, k, r)-design, $(V_1, \mathscr{P}_1, \mathscr{B}_1)$ and a restricted simple 1-(v_2, k, r)-design $(V_2, \mathscr{P}_2, \mathscr{B}_2)$ where $V_1 \cap V_2 = \varnothing$. Then there exists a restricted simple 1-$(v_1 + v_2, k, r)$-design.*

Proof. $(V_1 \cup V_2, \mathscr{P}_1 \cup \mathscr{P}_2, \mathscr{B}_1 \cup \mathscr{B}_2)$ is a restricted simple 1-$(v_1 + v_2, k, r)$-design. □

Example 3.4.2. Let $V_1 = \{1, 2, \ldots, 15\}$ be partitioned into four parts $\mathscr{P}_1 = \{\{1, 2, 3, 4, 5\}, \{6, 7, 8, 9\}, \{10, 11, 12, 13\}, \{14, 15\}\}$ and a collection $\mathscr{B}_1$ of twenty blocks of size three be $\{\{1, 8, 11\}, \{2, 8, 11\}, \{3, 8, 11\}, \{4, 7, 12\}, \{5, 7, 12\}, \{3, 7, 12\}, \{4, 8, 13\}, \{5, 9, 13\}, \{5, 9, 11\}, \{5, 9, 12\}, \{1, 9, 13\}, \{2, 7, 13\}, \{1, 6, 15\}, \{2, 6, 15\}, \{3, 6, 15\}, \{4, 6, 15\}, \{1, 10, 14\}, \{2, 10, 14\}, \{3, 10, 14\}, \{4, 10, 14\}\}$. It follows that $(V_1, \mathscr{P}_1, \mathscr{B}_1)$ forms a restricted simple 1-$(15, 3, 4)$-design. Let $V_2 = \{16, 17, 18, 19, 20, 21\}$ be partitioned into three parts $\mathscr{P}_2 = \{\{16, 17\}, \{18, 19\}, \{20, 21\}\}$ and a collection $\mathscr{B}_2$ of eight blocks of size three be $\{\{16, 18, 20\}, \{16, 18, 21\}, \{16, 19, 20\}, \{16, 19, 21\}, \{17, 19, 21\}, \{17, 19, 20\}, \{17, 18, 21\}, \{17, 18, 20\}\}$. It follows that $(V_2, \mathscr{P}_2, \mathscr{B}_2)$ forms a restricted simple 1-$(6, 3, 4)$-design. Therefore, $(V_1 \cup V_2, \mathscr{P}_1 \cup \mathscr{P}_2, \mathscr{B}_1 \cup \mathscr{B}_2)$ is a restricted simple 1-$(21, 3, 4)$-design.

Recall that, any partition set $\mathscr{A}$ on a set X is a *refinement* of a partition set $\mathscr{P}$ on X, if every element of $\mathscr{A}$ is a subset of some element of $\mathscr{P}$.

Theorem 3.4.3. *Suppose that there exists a restricted simple 1-(v,k,r)-design, $(V,\mathscr{P},\mathscr{B})$. Then there also exists a restricted simple 1-(v,k,r)-design with any refinement of $\mathscr{P}$.*

Proof. Since each point in a block comes from different parts in $\mathscr{P}$, also it comes from different parts in the refinement of $\mathscr{P}$. □

Example 3.4.4. From Example 3.1.5, $(V,\mathscr{P},\mathscr{B})$ is a restricted simple 1-(6,3,3)-design where $V = \{1,2,3,4,5,6\}$ is partitioned into three parts $\mathscr{P} = \{\{1,2\},\{3,4\},\{5,6\}\}$ and a collection $\mathscr{B}$ of six blocks of size three is $\{\{1,3,5\},\{1,3,6\},\{1,4,5\},\{2,3,6\},\{2,4,5\},\{2,4,6\}\}$. Therefore, $(V,\mathscr{P},\mathscr{B})$ forms a restricted simple 1-$(6,3,3)$-design with each refinement of $\mathscr{P}$, i.e., $\mathscr{X}_1 = \{\{1,2\},\{3,4\},\{5\},\{6\}\}$, $\mathscr{X}_2 = \{\{1,2\},\{3\},\{4\},\{5\},\{6\}\}$ and $\mathscr{X}_3 = \{\{1\},\{2\},\{3\},\{4\},\{5\},\{6\}\}$.

CHAPTER IV

CONCLUSIONS AND OPEN PROBLEMS

4.1 Conclusions

From our work, the results can be concluded as follows.

1. The 56 solutions of a strict SBGDD$(3,2,3)$ and a construction for an infinite SBGDD$(3,3,3)$.

2. If $mn(mn+1) \equiv 0 \bmod 2k$, then a strict 1-SBGDD$(m,n,k)$ exists for every positive integer m, n and k where $2 \leq k \leq m$.

3. A strict k-SBGDD(m,n,k) exists for every positive integer m, n and k where $2 \leq k \leq m$.

4. Let v, k and m be integers such that $2 \leq k \leq m \leq v$ and k divides v. Let $V = \{1, 2, \ldots, v\}$ be partitioned into m parts $\mathscr{P} = \{P_1, P_2, \ldots, P_m\}$ of size $p_1, p_2, \ldots, p_m$, respectively such that $p_1 \geq p_2 \geq \cdots \geq p_m$ and $p_1 k \leq v$. Then there exist a restricted simple 1-$(v,k,1)$-design and a restricted simple 1-$(v,k-1,k-1)$-design.

5. Let n and r be positive integers. Suppose that an $2n$-set is partitioned into two parts of the same size n. Then there exists a restricted simple 1-$(2n,2,r)$-design for all $r = 1, 2, \ldots, n$.

6. Let m, n and r be positive integers such that $2 \leq m$ and $r \leq n^{m-1}$. Suppose that $\mathscr{P} = \{P_1, P_2, \ldots, P_m\}$ is a partition set of the same size n on an mn-set V. Then there exists a restricted simple 1-(mn,m,r)-design.

7. Let v, m, k and r be positive integers such that $2 \leq k \leq m \leq v$.

Let $V = \{1, 2, \ldots, v\}$ be partitioned into m parts $\mathscr{P} = \{P_1, P_2, \ldots, P_m\}$ of size $p_1, p_2, \ldots, p_m$, respectively such that $p_1 \geq p_2 \geq \cdots \geq p_m$. Suppose that $vr \equiv 0 \pmod{k}$, $p_1 k \leq v$ and $r \leq p_{\alpha_1} p_{\alpha_2} \cdots p_{\alpha_{k-1}}$ for all $\{\alpha_1, \alpha_2, \ldots, \alpha_{k-1}\} \subseteq \{2, 3, \ldots, m\}$. Then there exists a restricted simple 1-(v, k, r)-design.

8. There are 2 simple methods of constructing new restricted simple 1-designs from old as follows.

8.1 (sum construction). Suppose that there are a restricted simple 1-(v_1, k, r)-design, $(V_1, \mathscr{P}_1, \mathscr{B}_1)$ and a restricted simple 1-(v_2, k, r)-design $(V_2, \mathscr{P}_2, \mathscr{B}_2)$ where $V_1 \cap V_2 = \varnothing$. Then $(V_1 \cup V_2, \mathscr{P}_1 \cup \mathscr{P}_2, \mathscr{B}_1 \cup \mathscr{B}_2)$ is a restricted simple 1-(v_1+v_2, k, r)-design.

8.2 (refinement construction). Suppose that there exists a restricted simple 1-(v, k, r)-design, $(V, \mathscr{P}, \mathscr{B})$. Then there also exists a restricted simple 1-(v, k, r)-design with any refinement of $\mathscr{P}$.

4.2 Open problems

There are open problems that can be further discussed as follows.

1. To investigate some necessary conditions for existence and constructions of a t-SBGDD(m, n, k) for some t, $2 \leq t \leq k-1$.

2. To investigate some necessary conditions for existence and constructions of a restricted simple 1-(v, k, r)-design when the size of partition is arbitrary and $r \geq p_{\alpha_1} p_{\alpha_2} \cdots p_{\alpha_{k-1}} + 1$ for some $\{\alpha_1, \alpha_2, \ldots, \alpha_{k-1}\} \subseteq \{2, 3, \ldots, m\}$.

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