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SUPERPOSITION OPERATORS
ON CLASSICAL SEQUENCE SPACES

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ปริภูมิลำดับ Φ, $\ell_p(1\leq p<\infty)$, cs, c_0, c, bs, ℓ_∞ และ ω เป็นปริภูมิลำดับคลาสสิก ในบรรดาปริภูมิลำดับเหล่านี้ Φ, $\ell_p(1\leq p<\infty)$, c_0, c, ℓ_∞ และ ω ยังเป็นปริภูมิลำดับที่มาตรฐานและเป็นที่รู้จักกันดีอีกด้วย

ในการวิจัยนี้ สำหรับ $g:\mathbb{N}\times\mathbb{R}\to\mathbb{R}$ ใด ๆ เราให้ลักษณะเฉพาะของตัวดำเนินการซ้อนทับ

(1) $P_g:X\to Y$ และ

(2) P_g ต่อเนื่องที่ทุกจุดของ X เมื่อ $P_g:X\to Y$

ในเทอมมาตรฐานการวิเคราะห์เชิงคณิตศาสตร์ของลำดับ $(g(k,\cdot))_{k=1}^{\infty}$ ของฟังก์ชันค่าจริง สำหรับทุก ๆ ปริภูมิลำดับ $X\in\{\Phi, \ell_q(1\leq q<\infty), cs, c_0, c, bs, \ell_\infty, \omega\}$ และ $Y\in\{\Phi, \ell_p(1\leq p<\infty), c_0, c, \ell_\infty, \omega\}$

ภาควิชา คณิตศาสตร์ ลายมือชื่อนิสิต สมกิจ เกตรานุรัตน์
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The sequence spaces Φ, $\ell_p(1 \leq p < \infty)$, cs, c_0, c, bs, ℓ_∞ and ω are classical. Among these sequence spaces, Φ, $\ell_p(1 \leq p < \infty)$, c_0, c, ℓ_∞ and ω are also standard and well-known.

In this research, for $g: \mathbb{N} \times \mathbb{R} \to \mathbb{R}$, the superposition operators

(1) $P_g: X \to Y$ and

(2) P_g is continuous at every point of X where $P_g: X \to Y$

are characterized in standard terms of mathematical analysis of the sequence $(g(k,\cdot))_{k=1}^{\infty}$ of real-valued functions for all $X \in \{\Phi, \ell_q(1 \leq q < \infty), cs, c_0, c, bs, \ell_\infty, \omega\}$ and $Y \in \{\Phi, \ell_p(1 \leq p < \infty), c_0, c, \ell_\infty, \omega\}$.

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CONTENTS

CHAPTER I

INTRODUCTION AND PRELIMINARIES

The theory of sequence spaces has long been studied. Linear operators, especially infinite matrix transformations, between sequence spaces have been widely studied in this area. However, some nonlinear operators between sequence spaces are well-known. Superposition operators between sequence spaces are nonlinear. The research on superposition operators has been continuously done. The definition of superposition operators is given as follows: For any $g: \mathbb{N}\times\mathbb{R}\rightarrow\mathbb{R}$, the *superposition operator* P_g is defined by $P_g((x_k)_{k=1}^{\infty}) = (g(k, x_k))_{k=1}^{\infty}$ for every real sequence $(x_k)_{k=1}^{\infty}$. In [1], J. Robert has characterized the superposition operator P_g between any two Orlicz sequence spaces which is continuous at the sequence (0) under the following additional conditions of g:

(*) $\quad g(k, 0) = 0$ for every $k \in \mathbb{N}$.

(**) $\quad g(k, \cdot)$ is continuous at 0 for every $k \in \mathbb{N}$.

I. V. Shragin [2] has introduced a class of sequence spaces which contains all Orlicz sequence spaces and he has characterized P_g between some types of sequence spaces in this class. Using the same idea of the proof given by I. V. Shragin [2], Chew Tuan Seng and Lee Peng Yee [3] have given characterizations in explicit forms of the superposition operators $P_g: \ell_p \rightarrow \ell_1$ $(1 \le p < \infty)$ and $P_g: c_0 \rightarrow \ell_1$ under the condition:

(***) $\quad g(k, \cdot)$ is continuous on $\mathbb{R}$ for every $k \in \mathbb{N}$.

Then they used their characterizations to give representations of orthogonally additive continuous functionals on ℓ_p $(1 \le p < \infty)$ and c_0. The superposition operator P_g with conditions (*) and (***) from w_0 into ℓ_1 has been characterized by Chew Tuan Seng in [4], where w_0 is the space of all real sequences $(x_k)_{k=1}^{\infty}$ such that $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n}|x_k| = 0$. The following condition of g is weaker than (***):

(****) For each $k \in \mathbb{N}$, $g(k, \cdot)$ is bounded on every bounded subset of $\mathbb{R}$.

By assuming only condition (****) of g, Ryszard Pluciennik [5] has proved the same result given by Chew Tuan Seng [4].

The sequence spaces Φ, ℓ_p $(1 \le p < \infty)$, cs, c_0, c, bs, ℓ_∞ and ω are classical. Among these sequence spaces, Φ, ℓ_p $(1 \le p < \infty)$, c_0, c, ℓ_∞ and ω are also standard and well-known. We consider the following P_g for any $X \in \{\Phi, \ell_q\ (1 \le q < \infty), cs, c_0, c, bs, \ell_\infty, \omega\}$ and $Y \in \{\Phi, \ell_p\ (1 \le p < \infty), c_0, c, \ell_\infty, \omega\}$:

(1) $P_g : X \to Y$.

(2) P_g is continuous at every point of X where $P_g : X \to Y$.

The purpose of this research is to give necessary and sufficient conditions of g for P_g in (1) and (2) above in terms of mathematical analysis of the sequence $((g(k,\cdot))_{k=1}^{\infty}$ of real-valued functions. These results are given in Chapter II and Chapter III, respectively.

Throughout this research, our scalar field is the field $\mathbb{R}$ of real numbers. By a sequence, we mean a sequence of real numbers. Let $\mathbb{N}$ denote the set of all natural numbers.

Let x be a sequence. For $k \in \mathbb{N}$, the k^{th} term of the sequence x is denoted by x_k. Then $x = (x_k)_{k=1}^{\infty}$. Let $|x|$ denote the sequence $(|x_k|)_{k=1}^{\infty}$. For $n \in \mathbb{N}$, let $e^{(n)}$ be the sequence such that

$$e_k^{(n)} = \begin{cases} 1 & \text{if } k = n, \\ 0 & \text{if } k \ne n. \end{cases}$$

For $t \in \mathbb{R}$, let (t) denote the constant sequence such that each coordinate is t.

For sequences x and y, we define $x \le y$ if $x_k \le y_k$ for all $k \in \mathbb{N}$. A sequence space X is said to be *solid* if for any sequence x, $|x| \le |y|$ for some $y \in X$ implies that $x \in X$.

A *K-space* is a topological sequence space in which each coordinate mapping is continuous.

For a normed linear space X, let $\|\cdot\|_X$ denote the norm of X.

The space of all sequences is denoted by ω and Φ denotes the space of all finite sequences, that is,

Φ = the space of all sequences x such that $x_k = 0$ for all but a finite number of k.

Hence for $x \in \omega$, $x \in \Phi$ if and only if $x = \sum_{k=1}^{N} x_k e^{(k)}$ for some $N \in \mathbb{N}$. The standard metric d_ω on ω is defined by

$$d_\omega(x,y) = \sum_{k=1}^{\infty} \frac{|x_k - y_k|}{2^k(1 + |x_k - y_k|)}.$$

We use the norm $\|\cdot\|_\Phi$ for Φ to be the sup-norm, that is,

$$\|x\|_\Phi = \sup_k |x_k| \quad (= \max_k |x_k|).$$

Other classical sequence spaces with their standard norms which are used in this research are as follows:

ℓ_∞ = the space of all bounded sequences,

$\|x\|_{\ell_\infty} = \sup_k |x_k|$,

c = the space of all convergent sequences,

$\|x\|_c = \sup_k |x_k|$,

c_0 = the space of all null sequences

= the space of all sequences x such that $\lim_{k\to\infty} x_k = 0$,

$\|x\|_{c_0} = \sup_k |x_k|$,

ℓ_p = the space of all sequences x such that $\sum_{k=1}^{\infty} |x_k|^p < \infty$

where $1 \le p < \infty$,

$$\|x\|_{\ell_p} = (\sum_{k=1}^{\infty} |x_k|^p)^{1/p},$$

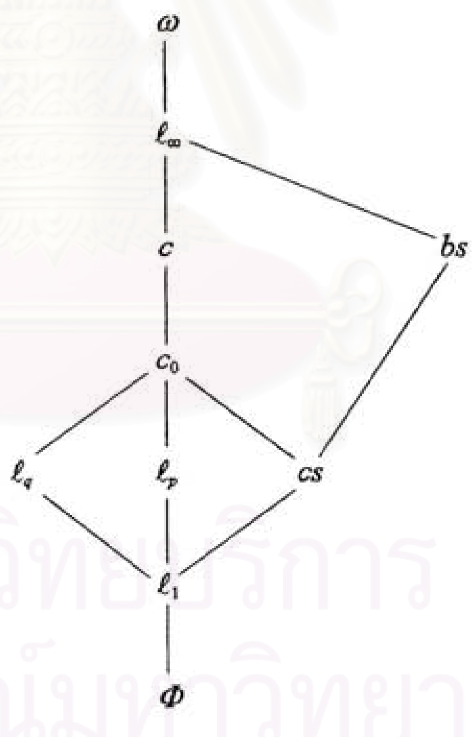
bs = the space of all sequences x such that $(\sum_{k=1}^{n} x_k)_{n=1}^{\infty}$ is bounded,

$$\|x\|_{bs} = \sup_n |\sum_{k=1}^{n} x_k|,$$

cs = the space of all sequences x such that $\sum_{k=1}^{\infty} x_k$ is a convergent series and

$$\|x\|_{cs} = \sup_n |\sum_{k=1}^{n} x_k|.$$

The following diagram shows the relationships under set inclusion among the sequence spaces mentioned above:



where $1 \le p, q < \infty$. The following statements are well-known:

(1) Φ, ℓ_p $(1 \le p < \infty)$, cs, c_0, c, bs, ℓ_∞ and ω are K-spaces.

(2) ℓ_p $(1 \le p < \infty)$, cs, c_0, c, bs, ℓ_∞ and ω are complete but Φ is not.

(3) Φ, ℓ_p $(1 \le p < \infty)$, c_0, ℓ_∞ and ω are solid but cs, c and bs are not.

(4) If x, $x^{(n)} \in \omega$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} x^{(n)} = x$ in ω if and only if $\lim_{n\to\infty} x_k^{(n)} = x_k$ in $\mathbb{R}$ for all $k \in \mathbb{N}$.

For $x \in \ell_p$ $(1 \le p < \infty)$, $n \in \mathbb{N}$, we have $|x_n| = (|x_n|^p)^{1/p} \le (\sum_{k=1}^{\infty} |x_k|^p)^{1/p} = \|x\|_{\ell_p}$. Also, for $x \in bs$, $n \in \mathbb{N}$, we have $|x_n| = |\sum_{k=1}^{n} x_k - \sum_{k=1}^{n-1} x_k| \le |\sum_{k=1}^{n} x_k| + |\sum_{k=1}^{n-1} x_k| \le 2\|x\|_{bs}$. These imply that $\|x\|_{\ell_\infty} \le \|x\|_{\ell_p}$ on ℓ_p $(1 \le p < \infty)$ and $\|x\|_{\ell_\infty} \le 2\|x\|_{bs}$ on bs.

Let $g: \mathbb{N} \times \mathbb{R} \to \mathbb{R}$. Then for each $k \in \mathbb{N}$, $g(k,\cdot): \mathbb{R} \to \mathbb{R}$ and so $(g(k,\cdot))_{k=1}^{\infty}$ is a sequence of real-valued functions on $\mathbb{R}$. Hence $(g(k,x_k))_{k=1}^{\infty} \in \omega$ for every $x \in \omega$. The map $P_g: \omega \to \omega$ defined by

$$P_g(x) = (g(k,x_k))_{k=1}^{\infty} \qquad (x \in \omega)$$

is called a *superposition operator*. Chew Tuan Seng and Lee Peng Yee [3] have characterized certain superposition operators as follows:

Theorem 1. ([3]) *Let $g: \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ be such that $g(k,\cdot)$ is continuous on $\mathbb{R}$ for all $k \in \mathbb{N}$. Then $P_g: c_0 \to \ell_1$ if and only if there exist $\alpha > 0$ and $(c_k)_{k=1}^{\infty} \in \ell_1$ such that for each $k \in \mathbb{N}$,*

$$|g(k,t)| \le c_k \quad \text{whenever } |t| \le \alpha.$$

An equivalent theorem of Theorem 1 is

Theorem 2. *Let $g: \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ be such that $g(k,\cdot)$ is continuous on $\mathbb{R}$ for all $k \in \mathbb{N}$. Then $P_g: c_0 \to \ell_1$ if and only if there exists $\alpha > 0$ such that*

$$\sum_{k=1}^{\infty} \sup_{t \in [-\alpha,\alpha]} |g(k,t)| < \infty.$$

Theorem 3. ([3]) *Let $g: \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ be such that $g(k,\cdot)$ is continuous on $\mathbb{R}$ for all $k \in \mathbb{N}$. Then for $1 \le p < \infty$, $P_g: \ell_p \to \ell_1$ if and only if there exist $\alpha > 0$, $\beta > 0$ and $(c_k)_{k=1}^{\infty} \in \ell_1$ such that for each $k \in \mathbb{N}$,*

$$|g(k,t)| \quad \leq \quad c_k + \beta|t|^p \quad \textit{whenever } |t| \leq \alpha.$$

The definition of continuous convergence of a sequence of real-valued functions is given in [6] as follows: Let $(f_n)_{n=1}^{\infty}$ be a sequence of real-valued functions on $\mathbb{R}$ and let $f:\mathbb{R}\to\mathbb{R}$. The sequence $(f_n)_{n=1}^{\infty}$ is said to be *continuously convergent* or *converge continuously at* $t\in\mathbb{R}$ *to* f if for any $\varepsilon>0$ there exist $\delta>0$ and $N\in\mathbb{N}$ such that for all $n\in\mathbb{N}$, $s\in\mathbb{R}$,

$$n\geq N \text{ and } |s-t|<\delta \quad \text{imply} \quad |f_n(s)-f(s)|<\varepsilon.$$

It is clear that if $(f_n)_{n=1}^{\infty}$ converges continuously at $t\in\mathbb{R}$ to f, then $\lim\limits_{n\to\infty} f_n(t) = f(t)$ and if $(f_n)_{n=1}^{\infty}$ converges uniformly to f on $\mathbb{R}$, then $(f_n)_{n=1}^{\infty}$ converges continuously to f at every point of $\mathbb{R}$. Moreover, it is routine to show that if $(f_n)_{n=1}^{\infty}$ converges continuously at t to f and f_n is continuous at t for every $n\in\mathbb{N}$, then f is continuous at t.

The following theorem will be used later.

Theorem 4. *Let $(f_n)_{n=1}^{\infty}$ be a sequence of real-valued functions on $\mathbb{R}$ and $f:\mathbb{R}\to\mathbb{R}$. Then:*

(*i*) *If $(f_n)_{n=1}^{\infty}$ converges continuously at $t\in\mathbb{R}$ to f, $\lim\limits_{s\to t} f(s)$ exists and $g:\mathbb{R}\to\mathbb{R}$ is such that $\lim\limits_{s\to t} g(s) = \lim\limits_{s\to t} f(s)$ and $g(t)=f(t)$, then $(f_n)_{n=1}^{\infty}$ converges continuously at t to g.*

(*ii*) *$(f_n)_{n=1}^{\infty}$ converges continuously to f at every point of $\mathbb{R}$ if and only if $(f_n)_{n=1}^{\infty}$ converges uniformly to f on every bounded subset of $\mathbb{R}$.*

(*iii*) *If f is continuous on $\mathbb{R}$, then $(f_n)_{n=1}^{\infty}$ converges continuously at t to $f(t)$ for all $t\in\mathbb{R}$ if and only if $(f_n)_{n=1}^{\infty}$ converges uniformly to f on every bounded subset of $\mathbb{R}$.*

Proof. (*i*) To prove $(f_n)_{n=1}^{\infty}$ converges continuously at t to g, let $\varepsilon>0$ be given. By assumption, $\lim\limits_{s\to t}[f(s)-g(s)] = 0 = f(t)-g(t)$. Then there exists $\delta>0$ such that for $s\in\mathbb{R}$,

$$|s-t| < \delta \quad \text{implies} \quad |f(s)-g(s)| < \frac{\varepsilon}{2}. \tag{1}$$

Since $(f_n)_{n=1}^{\infty}$ converges continuously at t to f, there exist $N\in\mathbb{N}$ and $\delta'\in\mathbb{R}$ with $0<\delta'\leq\delta$ such that for all $n\in\mathbb{N}$, $s\in\mathbb{R}$,

$$n\geq N \ \text{ and } \ |s-t|<\delta' \ \text{ imply } \ |f_n(s)-f(s)| < \frac{\varepsilon}{2}. \tag{2}$$

Then (1) and (2) yield that for all $n\in\mathbb{N}$, $s\in\mathbb{R}$,

$$n\geq N \text{ and } |s-t|<\delta' \text{ imply } |f_n(s)-g(s)|<\varepsilon.$$

Hence $(f_n)_{n=1}^{\infty}$ converges continuously at t to g.

(*ii*) Assume that $(f_n)_{n=1}^{\infty}$ converges continuously to f at every point of $\mathbb{R}$ and let S be a bounded subset of $\mathbb{R}$. To show that $(f_n)_{n=1}^{\infty}$ converges uniformly to f on S, let $\varepsilon>0$. Since S is bounded, $S\subseteq[-\alpha,\alpha]$ for some $\alpha>0$. Since $(f_n)_{n=1}^{\infty}$ converges continuously to f at every point of $\mathbb{R}$, for every $t\in[-\alpha,\alpha]$ there exist $\delta_t>0$ and $N_t\in\mathbb{N}$ such that for all $n\in\mathbb{N}$, $s\in\mathbb{R}$,

$$n\geq N_t \ \text{ and } \ |s-t|<\delta_t \ \text{ imply } \ |f_n(s)-f(s)|<\varepsilon. \tag{3}$$

Since $[-\alpha,\alpha]$ is compact in $\mathbb{R}$ and $[-\alpha,\alpha]\subseteq\bigcup\limits_{t\in[-\alpha,\alpha]}(t-\delta_t,t+\delta_t)$, $[-\alpha,\alpha]\subseteq\bigcup\limits_{i=1}^{m}(t_i-\delta_{t_i},t_i+\delta_{t_i})$ for some $m\in\mathbb{N}$ and $t_1,\dots,t_m\in[-\alpha,\alpha]$. If $s\in[-\alpha,\alpha]$, then $s\in(t_j-\delta_{t_j},t_j+\delta_{t_j})$ for some $j\in\{1,\dots,m\}$, so by (3), $|f_n(s)-f(s)|<\varepsilon$ for all $n\geq\max\{N_{t_1},\dots,N_{t_m}\}$. This implies that $(f_n)_{n=1}^{\infty}$ converges uniformly to f on $[-\alpha,\alpha]$ and hence it does on S.

Conversely, assume that $(f_n)_{n=1}^{\infty}$ converges uniformly to f on every bounded subset of $\mathbb{R}$. To prove that $(f_n)_{n=1}^{\infty}$ converges continuously to f at every point of $\mathbb{R}$, let $t\in\mathbb{R}$. Then by assumption, $(f_n)_{n=1}^{\infty}$ converges uniformly to f on

$(t-1, t+1)$. This implies that for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $s \in \mathbb{R}$,

$$n \geq N \text{ and } |s-t| < 1 \quad \text{imply} \quad |f_n(s) - f(s)| < \varepsilon.$$

Hence $(f_n)_{n=1}^{\infty}$ converges continuously at t to f.

(iii) follows directly from (i), (ii) and the fact that $\lim\limits_{s \to t} f(s) = f(t) = \lim\limits_{s \to t} f(t)$ for all $t \in \mathbb{R}$. □

CHAPTER II

CHARACTERIZATION OF SUPERPOSTION OPERATORS ON CLASSICAL SEQUENCE SPACES

The purpose of this chapter is to characterize the superposition operator $P_g: X \to Y$ where $g: \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ in terms of the sequence $(g(k,\cdot))_{k=1}^{\infty}$ of functions for all $X \in \{\Phi, \ell_p\,(1 \le p < \infty), cs, c_0, c, bs, \ell_\infty, \omega\}$ and $Y \in \{\Phi, \ell_p(1 \le p < \infty), c_0, c, \ell_\infty\}$.

For convenience, let p, q denote real numbers such that $1 < p, q < \infty$.

Throughout this research, let $g: \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ and P_g denote the superposition operator induced by g, that is, $P_g(x) = (g(k, x_k))_{k=1}^{\infty}$ for every $x \in \omega$. For $s \in \mathbb{R}$, the notation g_s denotes the function $g_s: \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ defined by $g_s(k,t) = g(k, t+s)$, so we have $P_{g_s}(x) = (g(k, x_k+s))_{k=1}^{\infty} = P_g(x+(s))$ for every $x \in \omega$ and the following statements:

(1) For sequence space X, Y, $P_g: (s)+X \to Y$ if and only if $P_{g_s}: X \to Y$.

(2) For $k \in \mathbb{N}$ and $\alpha > 0$, $\{g_s(k,t) \mid |t| < \alpha\} = \{g(k,t) \mid |t-s| < \alpha\}$.

Superposition Operators into ℓ_∞

First, we give a characterization of $P_g: \Phi \to Y$ for any sequence space Y containing Φ as follows:

Lemma 1. (*i*) *If X and Y are sequence spaces and $P_g: X \to Y$, then $(g(k,0))_{k=1}^{\infty} \in Y$.*

(ii) If Y is a sequence space containing Φ and $(g(k,0))_{k=1}^{\infty} \in Y$, then $P_g: \Phi \to Y$.

Proof. (i) follows from the fact that $(0) \in X$.

(ii) Let $x \in \Phi$. Then $x = \sum_{k=1}^{N} x_k e^{(k)}$ for some $N \in \mathbb{N}$ and hence

$$\begin{aligned} P_g(x) &= (g(k,x_k))_{k=1}^{\infty} \\ &= (g(1,x_1), g(2,x_2), \dots, g(N,x_N), g(N+1,0), g(N+2,0), \dots) \\ &= (g(1,x_1)-g(1,0), g(2,x_2)-g(2,0), \dots, g(N,x_N)-g(N,0), 0, 0, \dots) \\ &\quad + (g(1,0), g(2,0), g(3,0), \dots) \\ &= \sum_{k=1}^{N} [g(k,x_k)-g(k,0)]e^{(k)} + (g(k,0))_{k=1}^{\infty}. \qquad (*) \end{aligned}$$

If Y is a sequence space such that $\Phi \subseteq Y$ and $(g(k,0))_{k=1}^{\infty} \in Y$, then by $(*)$, $P_g(x) \in Y$. This proves (ii). □

By Lemma 1 and the definition of ℓ_∞, we have

Theorem 2. $P_g: \Phi \to \ell_\infty$ *if and only if* $(g(k,0))_{k=1}^{\infty}$ *is a bounded sequence.*

We know that the sequence spaces ℓ_p, cs and c_0 lie between ℓ_1 and c_0. Characterizations of P_g from these sequence spaces into ℓ_∞ are obtained from the following lemma.

Lemma 3. *Let X be a sequence space such that $\ell_1 \subseteq X \subseteq c_0$. Then the following statements are equivalent:*

(i) $P_g: c_0 \to \ell_\infty$.

(ii) $P_g: X \to \ell_\infty$.

(iii) $P_g: \ell_1 \to \ell_\infty$.

(*iv*) *There exists* $N\in\mathbb{N}$ *such that* $(g(k,\cdot))_{k=N}^{\infty}$ *is uniformly bounded on some neighborhood of* 0.

Proof. Since $\ell_1\subseteq X\subseteq c_0$, the implications $(i)\Rightarrow(ii)$ and $(ii)\Rightarrow(iii)$ are directly obtained.

To show $(iii)\Rightarrow(iv)$, suppose that (iv) is not true. Then for all $n\in\mathbb{N}$, $\alpha>0$, $(g(k,\cdot))_{k=n}^{\infty}$ is not uniformly bounded on $[-\alpha,\alpha]$. Hence for all $n\in\mathbb{N}$, there exist $k>n$ and $t\in[-2^{-n},2^{-n}]$ such that $|g(k,t)|>n$. This implies that there exist a subsequence $(n_k)_{k=1}^{\infty}$ of $(n)_{n=1}^{\infty}$ and a sequence $(x_{n_k})_{k=1}^{\infty}$ such that $x_{n_k}\in[-2^{-k},2^{-k}]$ and $|g(n_k,x_{n_k})|>k$ for all $k\in\mathbb{N}$. Then $(x_{n_k})_{k=1}^{\infty}\in\ell_1$ and $g(n_k,x_{n_k}))_{k=1}^{\infty}\notin\ell_\infty$. Let $(y_n)_{n=1}^{\infty}$ be a sequence defined by

$$y_n = \begin{cases} x_{n_k} & \text{if } n=n_k \text{ for some } k\in\mathbb{N},\\ 0 & \text{otherwise.}\end{cases}$$

Then $(y_n)_{n=1}^{\infty}\in\ell_1$ and $(g(n,y_n))_{n=1}^{\infty}\notin\ell_\infty$. Hence $P_g:\ell_1\not\to\ell_\infty$.

Finally, to show $(iv)\Rightarrow(i)$, assume that (iv) holds. Then there exist $N\in\mathbb{N}$, $\alpha>0$ and $M>0$ such that

$$|g(k,t)| \quad\le\quad M \quad \text{for all } k\ge N \text{ and } t\in[-\alpha,\alpha]. \tag{$*$}$$

Let $x\in c_0$. Then there exists $N'\ge N$ such that $|x_k|\le\alpha$ for all $k\ge N'$ which implies by $(*)$ that $|g(k,x_k)|\le M$ for all $k\ge N'$. Hence $(g(k,x_k))_{k=1}^{\infty}\in\ell_\infty$. This proves that $P_g: c_0\to\ell_\infty$. □

Theorem 4. *If X is one of the sequence spaces* ℓ_p, *cs and* c_0, *then* $P_g:X\to\ell_\infty$ *if and only if there exists* $N\in\mathbb{N}$ *such that* $(g(k,\cdot))_{k=N}^{\infty}$ *is uniformly bounded on some neighborhood of* 0.

The next theorem characterizes $P_g: c \to \ell_\infty$ and $P_g: \ell_\infty \to \ell_\infty$. It is shown in this theorem that these characterizations are the same.

Theorem 5. *The following statements are equivalent:*

(*i*) $P_g: \ell_\infty \to \ell_\infty$.

(*ii*) $P_g: c \to \ell_\infty$.

(*iii*) *For every bounded subset S of $\mathbb{R}$, there exists $N \in \mathbb{N}$ such that $(g(k,\cdot))_{k=N}^{\infty}$ is uniformly bounded on S.*

Proof. That $(i) \Rightarrow (ii)$ holds since $c \subseteq \ell_\infty$.

To show $(ii) \Rightarrow (iii)$, assume that $P_g: c \to \ell_\infty$. To prove (iii), it is equivalent to show the following statement:

For every $\alpha > 0$, there exists $N \in \mathbb{N}$ such that $(g(k,\cdot))_{k=N}^{\infty}$ is uniformly bounded on $[-\alpha, \alpha]$. (1)

Since for $\alpha > 0$, $[-\alpha, \alpha]$ is a compact set in $\mathbb{R}$, (1) follows from the statement:

For every $s \in \mathbb{R}$, there exist $r > 0$ and $N \in \mathbb{N}$ such that $(g(k,\cdot))_{k=N}^{\infty}$ is uniformly bounded on $(s-r, s+r)$. (2)

Next, we shall prove (2). Let $s \in \mathbb{R}$. Since $P_g: c \to \ell_\infty$ and $(s) + c_0 \subseteq c$, $P_g: (s) + c_0 \to \ell_\infty$. Then $P_{g_s}: c_0 \to \ell_\infty$. It follows from Theorem 4 that there exist $r > 0$ and $N \in \mathbb{N}$ such that $(g_s(k,\cdot))_{k=N}^{\infty}$ is uniformly bounded on $(-r, r)$. But $\{ g_s(k,t) \mid |t| < r \} = \{ g(k,t) \mid |t-s| < r \}$ for all $k \in \mathbb{N}$, so $(g(k,\cdot))_{k=N}^{\infty}$ is uniformly bounded on $(s-r, s+r)$.

To prove $(iii) \Rightarrow (i)$, assume that (iii) holds. Then (1) holds. To prove $P_g: \ell_\infty \to \ell_\infty$, let $x \in \ell_\infty$. Then there exists $\alpha > 0$ such that $|x_k| \le \alpha$ for all $k \in \mathbb{N}$. By (1), there exist $N \in \mathbb{N}$ and $M > 0$ such that $|g(k,t)| \le M$ for all $k \ge N$ and $t \in [-\alpha, \alpha]$. Since $|x_k| \le \alpha$ for all $k \in \mathbb{N}$, we have $|g(k, x_k)| \le M$ for all $k \ge N$. This implies that $(g(k, x_k))_{k=1}^{\infty} \in \ell_\infty$. □

To characterize $P_g: bs \to \ell_\infty$, the following lemma is required.

Lemma 6. *If Y is a solid sequence space and $P_g: bs \to Y$, then $P_g: \ell_\infty \to Y$.*

Proof. Let $x \in \ell_\infty$. Then there exists $\alpha > 0$ such that $|x_k| \le \alpha$ for all $k \in \mathbb{N}$. Define the sequences $(y_k)_{k=1}^{\infty}$ and $(z_k)_{k=1}^{\infty}$ as follows:

$$y_k = \begin{cases} x_k & \text{if } k \text{ is odd,} \\ -x_{k-1} & \text{if } k \text{ is even} \end{cases}$$

and

$$z_k = \begin{cases} -x_{k+1} & \text{if } k \text{ is odd,} \\ x_k & \text{if } k \text{ is even,} \end{cases}$$

that is, $(y_k)_{k=1}^{\infty} = (x_1, -x_1, x_3, -x_3, x_5, -x_5, \dots)$ and $(z_k)_{k=1}^{\infty} = (-x_2, x_2, -x_4, x_4, -x_6, x_6, \dots)$. Then for all $n \in \mathbb{N}$,

$$\sum_{k=1}^{n} y_k = \begin{cases} x_n & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

and

$$\sum_{k=1}^{n} z_k = \begin{cases} -x_{n+1} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

Since $x \in \ell_\infty$, we have $(\sum_{k=1}^{n} y_k)_{n=1}^{\infty} \in \ell_\infty$ and $(\sum_{k=1}^{n} z_k)_{n=1}^{\infty} \in \ell_\infty$. Then $(y_k)_{k=1}^{\infty}, (z_k)_{k=1}^{\infty} \in bs$. Since $P_g: bs \to Y$, $(g(k, y_k))_{k=1}^{\infty} \in Y$ and $(g(k, z_k))_{k=1}^{\infty} \in Y$. Define the sequences $(u_k)_{k=1}^{\infty}$ and $(v_k)_{k=1}^{\infty}$ by

$$u_k = \begin{cases} g(k, y_k) & \text{if } k \text{ is odd,} \\ 0 & \text{if } k \text{ is even} \end{cases}$$

and

$$v_k = \begin{cases} 0 & \text{if } k \text{ is odd,} \\ g(k, z_k) & \text{if } k \text{ is even.} \end{cases}$$

Then $|u_k| \le |g(k, y_k)|$ and $|v_k| \le |g(k, z_k)|$ for all $k \in \mathbb{N}$. Since Y is solid, we have $(u_k)_{k=1}^{\infty}, (v_k)_{k=1}^{\infty} \in Y$. But $g(k, y_k) = g(k, x_k)$ if k is odd and $g(k, z_k) = g(k, x_k)$ if k is even, so $(g(k, x_k))_{k=1}^{\infty} = (u_k)_{k=1}^{\infty} + (v_k)_{k=1}^{\infty} \in Y$. □

We know that ℓ_∞ is solid. The next theorem follows from this fact, $bs \subseteq \ell_\infty$, Lemma 6 and Theorem 5.

Theorem 7. *P_g: $bs \to \ell_\infty$ if and only if for every bounded subset S of $\mathbb{R}$, there exists $N \in \mathbb{N}$ such that $(g(k, \cdot))_{k=N}^{\infty}$ is uniformly bounded on S.*

The following corollary is obtained from Theorem 5 and Theorem 7.

Corollary 8. *If X is one of the sequence spaces c, bs and ℓ_∞, then P_g:$X \to \ell_\infty$ if and only if for every bounded subset S of $\mathbb{R}$, there exists $N \in \mathbb{N}$ such that $(g(k, \cdot))_{k=N}^{\infty}$ is uniformly bounded on S.*

The last theorem of this section gives a characterization of P_g: $\omega \to \ell_\infty$ as follows:

Theorem 9. *P_g: $\omega \to \ell_\infty$ if and only if there exists $N \in \mathbb{N}$ such that $(g(k, \cdot))_{k=N}^{\infty}$ is uniformly bounded on $\mathbb{R}$.*

Proof. Suppose that for every $n \in \mathbb{N}$, $(g(k, \cdot))_{k=n}^{\infty}$ is not uniformly bounded on $\mathbb{R}$. Then there exist a subsequence $(n_k)_{k=1}^{\infty}$ of $(n)_{n=1}^{\infty}$ and a sequence $(x_{n_k})_{k=1}^{\infty}$ such that $|g(n_k, x_{n_k})| > k$ for all $k \in \mathbb{N}$. Then $(g(n_k, x_{n_k}))_{k=1}^{\infty} \notin \ell_\infty$. Let $(y_n)_{n=1}^{\infty}$ be a sequence defined by

$$y_n = \begin{cases} x_{n_k} & \text{if } n = n_k \text{ for some } k \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Then $(y_n)_{n=1}^{\infty} \in \omega$ and $(g(n,y_n))_{n=1}^{\infty} \notin \ell_\infty$ since $(g(n_k,x_{n_k}))_{k=1}^{\infty}$ is a subsequence of $(g(n,y_n))_{n=1}^{\infty}$.

This proves that if $P_g: \omega \to \ell_\infty$, then there exists $N \in \mathbb{N}$ such that $(g(k,\cdot))_{k=N}^{\infty}$ is uniformly bounded on $\mathbb{R}$. The converse of this statement is obvious. $\square$

Superposition Operators into c

Since the sequence space c contains Φ, the following theorem follows directly from Lemma 1.

Theorem 10. *$P_g: \Phi \to c$ if and only if $\lim_{k\to\infty} g(k,0)$ exists.*

We obtain a necessary condition for P_g mapping a solid sequence space into c as follows:

Lemma 11. *For a solid sequence space X, if $P_g: X \to c$, then for any $x \in X$, $\lim_{k\to\infty} g(k,x_k) = \lim_{k\to\infty} g(k,0)$.*

Proof. Since $(0) \in X$, $P_g((0)) = (g(k,0))_{k=1}^{\infty} \in c$. Then $\lim_{k\to\infty} g(k,0)$ exists. Given $x \in X$, we have $P_g(x) = (g(k,x_k))_{k=1}^{\infty} \in c$, that is, $\lim_{k\to\infty} g(k,x_k)$ exists. Define the sequence $(y_k)_{k=1}^{\infty}$ by

$$y_k = \begin{cases} x_k & \text{if } k \text{ is odd,} \\ 0 & \text{if } k \text{ is even.} \end{cases}$$

Then $(y_{2k})_{k=1}^{\infty}=(0)$, $(y_{2k-1})_{k=1}^{\infty}=(x_{2k-1})_{k=1}^{\infty}$ and $|y_k|\leq|x_k|$ for all $k\in\mathbb{N}$. It follows that $(y_k)_{k=1}^{\infty}\in X$ since X is solid. Therefore, $\lim\limits_{k\to\infty} g(k,y_k)$ exists. Hence $\lim\limits_{k\to\infty} g(k,0)$ $=\lim\limits_{k\to\infty} g(2k,0)=\lim\limits_{k\to\infty} g(2k,y_{2k})=\lim\limits_{k\to\infty} g(k,y_k)=\lim\limits_{k\to\infty} g(2k-1,y_{2k-1})=\lim\limits_{k\to\infty} g(2k-1,x_{2k-1})$ $=\lim\limits_{k\to\infty} g(k,x_k)$. □

In order to characterize P_g from ℓ_p, cs and c_0 into c, the following lemma is required.

Lemma 12. *Let X be a sequence space such that $\ell_1\subseteq X\subseteq c_0$. Then the following statements are equivalent:*

(*i*) $P_g: c_0\to c$.

(*ii*) $P_g: X\to c$.

(*iii*) $P_g: \ell_1\to c$.

(*iv*) $\lim\limits_{k\to\infty} g(k,0)$ *exists and* $(g(k,\cdot))_{k=1}^{\infty}$ *converges continuously at* 0 *to* $\lim\limits_{k\to\infty} g(k,0)$.

Proof. Since $\ell_1\subseteq X\subseteq c_0$, $(i)\Rightarrow(ii)$ and $(ii)\Rightarrow(iii)$ hold.

To prove $(iii)\Rightarrow(iv)$, suppose that (iv) does not hold. If $\lim\limits_{k\to\infty} g(k,0)$ does not exist, then $P_g: \ell_1\not\to c$ since $(0)\in\ell_1$. Assume that $\lim\limits_{k\to\infty} g(k,0)$ exists but $(g(k,\cdot))_{k=1}^{\infty}$ does not converge continuously at 0 to $\lim\limits_{k\to\infty} g(k,0)$. Let $L=\lim\limits_{k\to\infty} g(k,0)$. Then there exists $\varepsilon>0$ such that for all $\delta>0$, $n\in\mathbb{N}$, there exist $k\in\mathbb{N}$, $t\in\mathbb{R}$ such that $k>n$, $|t|<\delta$ and $|g(k,t)-L|\geq\varepsilon$. This implies that there exist a subsequence $(n_k)_{k=1}^{\infty}$ of $(n)_{n=1}^{\infty}$ and a sequence $(t_{n_k})_{k=1}^{\infty}$ such that $|t_{n_k}|<2^{-k}$ and $|g(n_k,t_{n_k})-L|\geq\varepsilon$ for all $k\in\mathbb{N}$. Then $(t_{n_k})_{k=1}^{\infty}\in\ell_1$. Define the sequence $(x_n)_{n=1}^{\infty}$ by

$$x_n = \begin{cases} t_{n_k} & \text{if } n=n_k \text{ for some } k\in\mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Then $(x_n)_{n=1}^{\infty} \in \ell_1$ and $|g(n_k, x_{n_k}) - L| \geq \varepsilon$ for all $k \in \mathbb{N}$. If $\lim_{n\to\infty} g(n, x_n)$ does not exist, then $P_g: \ell_1 \not\to c$. Assume that $\lim_{n\to\infty} g(n, x_n)$ exists. Then $\lim_{n\to\infty} g(n, x_n) = \lim_{k\to\infty} g(n_k, x_{n_k})$ $\neq L = \lim_{k\to\infty} g(k, 0)$. Since ℓ_1 is solid, it follows by Lemma 11 that $P_g: \ell_1 \not\to c$.

Finally, to show that $(iv) \Rightarrow (i)$, assume that (iv) holds. Let $L = \lim_{k\to\infty} g(k, 0)$ and let $\varepsilon > 0$ be given. Since $(g(k,\cdot))_{k=1}^{\infty}$ converges continuously at 0 to L, there exist $\delta > 0$ and $N \in \mathbb{N}$ such that for all $k \in \mathbb{N}$, $t \in \mathbb{R}$,

$$k \geq N \text{ and } |t| < \delta \text{ imply } |g(k, t) - L| < \varepsilon. \qquad (*)$$

If $x \in c_0$, then there exists $N' \geq N$ such that $|x_k| < \delta$ for all $k \geq N'$, which implies by $(*)$ that $|g(k, x_k) - L| \leq \varepsilon$ for all $k \geq N'$. Hence $\lim_{k\to\infty} g(k, x_k) = L$ for all $x \in c_0$. This shows that $P_g: c_0 \to c$. □

Theorem 13. *If X is one of the sequence spaces ℓ_p, cs* and *c_0, then $P_g: X \to c$ if and only if $\lim_{k\to\infty} g(k, 0)$ exists and $(g(k,\cdot))_{k=1}^{\infty}$ converges continuously at 0 to $\lim_{k\to\infty} g(k, 0)$.*

The next theorem gives a characterization of $P_g: c \to c$. Theorem I.4 is required to prove this theorem.

Theorem 14. $P_g: c \to c$ *if and only if*

(*i*) $\lim_{k\to\infty} g(k, t)$ *exists for all* $t \in \mathbb{R}$,

(*ii*) $h \equiv \lim_{k\to\infty} g(k, \cdot)$ *is continuous on* $\mathbb{R}$ *and*

(*iii*) $(g(k,\cdot))_{k=1}^{\infty}$ *converges uniformly to h on every bounded subset of* $\mathbb{R}$.

Proof. Assume that $P_g: c \to c$. Since $(t) \in c$ for all $t \in \mathbb{R}$, we have $\lim_{k\to\infty} g(k, t)$ exists for all $t \in \mathbb{R}$. Then $h(t) = \lim_{k\to\infty} g(k, t) = \lim_{k\to\infty} g_t(k, 0)$ for all $t \in \mathbb{R}$. To prove (iii), it suffices by Theorem I.4 (iii) to show that (ii) holds and $(g(k,\cdot))_{k=1}^{\infty}$ converges continuously at t to $h(t)$ for all $t \in \mathbb{R}$.

Let $t \in \mathbb{R}$. Since $(t)+c_0 \subseteq c$, $P_g:(t)+c_0 \to c$ which implies that $P_{g_t}: c_0 \to c$. By Theorem 13, $(g_t(k,\cdot))_{k=1}^{\infty}$ converges continuously at 0 to $h(t)$. Since $\{ g_t(k,s) \mid |s|<\alpha \} = \{ g(k,s) \mid |s-t|<\alpha \}$ for $k \in \mathbb{N}$ and $\alpha>0$, it follows that $(g(k,\cdot))_{k=1}^{\infty}$ converges continuously at t to $h(t)$. It remains to show that h is continuous at t, let $\varepsilon>0$ be given. Since $(g(k,\cdot))_{k=1}^{\infty}$ converges continuously at t to $h(t)$, there exist $\delta>0$ and $N \in \mathbb{N}$ such that for all $k \in \mathbb{N}$, $s \in \mathbb{R}$,

$$k \geq N \text{ and } |s-t|<\delta \text{ imply } |g(k,s)-h(t)| < \frac{\varepsilon}{2}. \tag{1}$$

Let $s \in \mathbb{R}$ be such that $|s-t|<\delta$. Since $\lim\limits_{k\to\infty} g(k,t) = h(t)$, there exists $N' \geq N$ such that $|g(N',t)-h(t)| < \frac{\varepsilon}{2}$. Then by (1), $|h(s)-h(t)| \leq |h(s)-g(N',s)| + |g(N',s)-h(t)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

Conversely, assume that (*i*), (*ii*) and (*iii*) hold. To prove that $P_g: c \to c$, let $x \in c$ and let $t = \lim\limits_{k\to\infty} x_k$. Claim that $\lim\limits_{k\to\infty} g(k,x_k) = h(t)$. Let $\varepsilon>0$ be given. Since h is continuous at t, there exists $\delta>0$ such that for all $s \in \mathbb{R}$,

$$|s-t|<\delta \text{ implies } |h(s)-h(t)| < \frac{\varepsilon}{2}. \tag{2}$$

Since $t = \lim\limits_{k\to\infty} x_k$ and $(g(k,\cdot))_{k=1}^{\infty}$ converges uniformly to h on $(t-\delta, t+\delta)$, there exists $N \in \mathbb{N}$ such that for each $k \in \mathbb{N}$, $s \in \mathbb{R}$,

$$k \geq N \text{ implies } |x_k - t| < \delta. \tag{3}$$

and

$$k \geq N \text{ and } |s-t|<\delta \text{ imply } |g(k,s)-h(s)| < \frac{\varepsilon}{2} \tag{4}$$

By (2), (3) and (4), we have that $|h(x_k)-h(t)| < \frac{\varepsilon}{2}$ and $|g(k,x_k)-h(x_k)| < \frac{\varepsilon}{2}$ for all $k \geq N$. These imply that $|g(k,x_k)-h(t)| < \varepsilon$ for all $k \geq N$. Then $\lim\limits_{k\to\infty} g(k,x_k) = h(t)$ and hence $P_g(x) = (g(k,x_k))_{k=1}^{\infty} \in c$. □

We know that the space ℓ_∞ is solid. Using Lemma 11, we obtain a characterization of $P_g: \ell_\infty \to c$ as follows:

Theorem 15. *$P_g: \ell_\infty \to c$ if and only if $\lim\limits_{k\to\infty} g(k,0)$ exists and $(g(k,\cdot))_{k=1}^{\infty}$ converges uniformly to $\lim\limits_{k\to\infty} g(k,0)$ on every bounded subset of $\mathbb{R}$.*

Proof. Assume that $P_g: \ell_\infty \to c$. Then $\lim\limits_{k\to\infty} g(k,0)$ exists. Let $L = \lim\limits_{k\to\infty} g(k,0)$. By Lemma 11,

$$\lim_{k\to\infty} g(k, x_k) \;=\; L \quad \text{for all } x \in \ell_\infty. \tag{1}$$

To show that $(g(k,\cdot))_{k=1}^{\infty}$ converges uniformly to L on every bounded subset of $\mathbb{R}$, suppose not. Then there exist $\alpha > 0$ and $\varepsilon > 0$ such that for each $n \in \mathbb{N}$, there exist $k > n$ and $t \in [-\alpha, \alpha]$ such that $|g(k,t) - L| \ge \varepsilon$. It follows that there exist a subsequence $(n_k)_{k=1}^{\infty}$ of $(n)_{n=1}^{\infty}$ and a sequence $(t_{n_k})_{k=1}^{\infty}$ such that for each $k \in \mathbb{N}$,

$$|t_{n_k}| \le \alpha \quad \text{and} \quad |g(n_k, t_{n_k}) - L| \ge \varepsilon. \tag{2}$$

Define the sequence $(y_n)_{n=1}^{\infty}$ by

$$y_n \;=\; \begin{cases} t_{n_k} & \text{if } n = n_k \text{ for some } k \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Then $|y_n| \le \alpha$ for all $n \in \mathbb{N}$, so $(y_n)_{n=1}^{\infty} \in \ell_\infty$. By (1), $\lim\limits_{k\to\infty} g(k, y_k) = L$. Thus $\lim\limits_{k\to\infty} g(n_k, t_{n_k}) = \lim\limits_{k\to\infty} g(n_k, y_{n_k}) = L$ which contradicts (2).

Conversely, assume that $\lim\limits_{k\to\infty} g(k,0)$ exists and $(g(k,\cdot))_{k=1}^{\infty}$ converges uniformly to L on every bounded subset of $\mathbb{R}$. Let $L = \lim\limits_{k\to\infty} g(k,0)$. To show that $P_g: \ell_\infty \to c$, let $x \in \ell_\infty$. Then there exists $\alpha > 0$ such that $|x_k| \le \alpha$ for all $k \in \mathbb{N}$. By assumption, $(g(k,\cdot))_{k=1}^{\infty}$ converges uniformly to L on $[-\alpha, \alpha]$. This implies directly that $\lim\limits_{k\to\infty} g(k, x_k) = L$ since $x_k \in [-\alpha, \alpha]$ for all $k \in \mathbb{N}$. Then $P_g(x) = (g(k, x_k))_{k=1}^{\infty} \in c$. □

To characterize $P_g: bs \to c$, the following two lemmas are required.

Lemma 16. *If $P_g: bs \to c$, then $\lim\limits_{k\to\infty} g(k, x_k) = \lim\limits_{k\to\infty} g(k,0)$ for all $x \in bs$.*

Proof. By assumption, $\lim_{k\to\infty} g(k,0)$ exists. Let $x \in bs$. Then $(\sum_{k=1}^{n} x_k)_{n=1}^{\infty} \in \ell_\infty$ and $\lim_{k\to\infty} g(k,x_k)$ exists. Define the sequence $(y_k)_{k=1}^{\infty}$ by

$$y_k = \begin{cases} x_k & \text{if } k \equiv 0 \pmod 3, \\ x_k + x_{k+1} & \text{if } k \equiv 1 \pmod 3, \\ 0 & \text{if } k \equiv 2 \pmod 3, \end{cases}$$

that is, $(y_k)_{k=1}^{\infty} = (x_1+x_2, 0, x_3, x_4+x_5, 0, x_6, \dots)$. Then for $n \in \mathbb{N}$,

$$\sum_{k=1}^{n} y_k = \begin{cases} \sum_{k=1}^{n+1} x_k & \text{if } n \equiv 1 \pmod 3, \\ \sum_{k=1}^{n} x_k & \text{otherwise.} \end{cases}$$

Since $(\sum_{k=1}^{n} x_k)_{n=1}^{\infty} \in \ell_\infty$, $(\sum_{k=1}^{n} y_k)_{n=1}^{\infty} \in \ell_\infty$. Then $(y_k)_{k=1}^{\infty} \in bs$ and $\lim_{k\to\infty} g(k,y_k)$ exists. Hence $\lim_{k\to\infty} g(k,x_k) = \lim_{k\to\infty} g(3k,x_{3k}) = \lim_{k\to\infty} g(3k,y_{3k}) = \lim_{k\to\infty} g(k,y_k) = \lim_{k\to\infty} g(3k-1,y_{3k-1})$ $= \lim_{k\to\infty} g(3k-1,0) = \lim_{k\to\infty} g(k,0)$. □

Lemma 17. *$P_g : bs \to c$ if and only if $P_g : \ell_\infty \to c$.*

Proof. Assume that $P_g : bs \to c$. To show that $P_g : \ell_\infty \to c$, let $x \in \ell_\infty$. Define the sequences $(y_k)_{k=1}^{\infty}$ and $(z_k)_{k=1}^{\infty}$ by

$$y_k = \begin{cases} x_k & \text{if } k \text{ is odd,} \\ -x_{k-1} & \text{if } k \text{ is even} \end{cases}$$

and

$$z_k = \begin{cases} -x_{k+1} & \text{if } k \text{ is odd,} \\ x_k & \text{if } k \text{ is even,} \end{cases}$$

that is, $(y_k)_{k=1}^{\infty} = (x_1, -x_1, x_3, -x_3, x_5, -x_5, \dots)$ and $(z_k)_{k=1}^{\infty} = (-x_2, x_2, -x_4, x_4, -x_6, x_6, \dots)$. Then for $n \in \mathbb{N}$,

$$\sum_{k=1}^{n} y_k = \begin{cases} x_n & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

and

$$\sum_{k=1}^{n} z_k = \begin{cases} -x_{n+1} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

Since $x \in \ell_\infty$, $(\sum_{k=1}^{n} y_k)_{n=1}^{\infty} \in \ell_\infty$ and $(\sum_{k=1}^{n} z_k)_{n=1}^{\infty} \in \ell_\infty$. Then $(y_k)_{k=1}^{\infty}, (z_k)_{k=1}^{\infty} \in bs$. By assumption, $\lim_{k\to\infty} g(k, y_k)$ and $\lim_{k\to\infty} g(k, z_k)$ exist. By Lemma 16, we have that $\lim_{k\to\infty} g(k, y_k) = \lim_{k\to\infty} g(k, z_k)$. It follows that $\lim_{k\to\infty} g(2k-1, x_{2k-1}) = \lim_{k\to\infty} g(2k-1, y_{2k-1})$ $= \lim_{k\to\infty} g(k, y_k) = \lim_{k\to\infty} g(k, z_k) = \lim_{k\to\infty} g(2k, z_{2k}) = \lim_{k\to\infty} g(2k, x_{2k})$. This implies that $\lim_{k\to\infty} g(k, x_k)$ exists. Thus $P_g(x) = (g(k, x_k))_{k=1}^{\infty} \in c$.

The converse follows from the fact that $bs \subseteq \ell_\infty$. □

Theorem 18. *$P_g: bs \to c$ if and only if $\lim_{k\to\infty} g(k, 0)$ exists and $(g(k, \cdot))_{k=1}^{\infty}$ converges uniformly to $\lim_{k\to\infty} g(k, 0)$ on every bounded subset of $\mathbb{R}$.*

Proof. It follows directly from Lemma 17 and Theorem 15. □

The last theorem of this section gives a characterization of $P_g: \omega \to c$. We recall that ω is solid. Lemma 11 is required to prove this theorem.

Theorem 19. *$P_g: \omega \to c$ if and only if $\lim_{k\to\infty} g(k, 0)$ exists and $(g(k, \cdot))_{k=1}^{\infty}$ converges uniformly to $\lim_{k\to\infty} g(k, 0)$ on $\mathbb{R}$.*

Proof. Assume that $P_g: \omega \to c$. Then $\lim_{k\to\infty} g(k, 0)$ exists. Let $L = \lim_{k\to\infty} g(k, 0)$. To show that $(g(k, \cdot))_{k=1}^{\infty}$ converges uniformly to L on $\mathbb{R}$, suppose not. Then there exist $\varepsilon > 0$, a subsequence $(n_k)_{k=1}^{\infty}$ of $(n)_{n=1}^{\infty}$ and a sequence $(t_{n_k})_{k=1}^{\infty}$ such that for each $k \in \mathbb{N}$,

$$|g(n_k, t_{n_k}) - L| \geq \varepsilon. \tag{*}$$

Define the sequence $(x_n)_{n=1}^{\infty}$ by

$$x_n = \begin{cases} t_{n_k} & \text{if } n = n_k \text{ for some } k \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Then $(x_n)_{n=1}^{\infty} \in \omega$ and by Lemma 11, $\lim_{k\to\infty} g(k, x_k) = L$. This contradicts (*) since $(t_{n_k})_{k=1}^{\infty}$ is a subsequence of $(x_n)_{n=1}^{\infty}$.

Conversely, assume that $\lim_{k\to\infty} g(k, 0)$ exists and $(g(k,\cdot))_{k=1}^{\infty}$ converges uniformly to $\lim_{k\to\infty} g(k, 0)$ on $\mathbb{R}$. Let $L = \lim_{k\to\infty} g(k, 0)$. Then for $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|g(k, t) - L| < \varepsilon$ for all $k \geq N$ and $t \in \mathbb{R}$ and hence $|g(k, x_k) - L| < \varepsilon$ for all $k \geq N$ and $x \in \omega$. This proves that $\lim_{k\to\infty} g(k, x_k) = L$ for all $x \in \omega$. Hence $P_g: \omega \to c$. □

Superposition Operators into c_0

Since 0 is the limit of x for all $x \in c_0$, by Lemma 1, we have

Theorem 20. *$P_g: \Phi \to c_0$ if and only if $\lim_{k\to\infty} g(k, 0) = 0$.*

The next lemma is similar to Lemma 12. It is used to characterize $P_g: X \to c_0$ where X is one of the sequence spaces ℓ_p, cs and c_0.

Lemma 21. *Let X be a sequence space such that $\ell_1 \subseteq X \subseteq c_0$. Then the following statements are equivalent:*

(*i*) $P_g: c_0 \to c_0$.

(*ii*) $P_g: X \to c_0$.

(*iii*) $P_g: \ell_1 \to c_0$.

(*iv*) $(g(k,\cdot))_{k=1}^{\infty}$ *converges continuously at* 0 *to* 0.

Proof. The implications $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (iii)$ follow directly from the assumption that $\ell_1 \subseteq X \subseteq c_0$.

Assume that (*iii*) holds. Then $\lim_{k\to\infty} g(k,0)=0$ and $P_g:\ell_1\to c$, so by Lemma 12, $(g(k,\cdot))_{k=1}^{\infty}$ converges continuously at 0 to 0. Hence (*iv*) holds.

Next, assume that (*iv*) holds. Then $\lim_{k\to\infty} g(k,0)=0$ (see Chapter I, page 6). Now we have that $(g(k,\cdot))_{k=1}^{\infty}$ converges continuously at 0 to $\lim_{k\to\infty} g(k,0)$. By Lemma 12, $P_g: c_0\to c$. Since c_0 is solid, by Lemma 11, $\lim_{k\to\infty} g(k,x_k)=\lim_{k\to\infty} g(k,0)$ $=0$ for all $x\in c_0$. This proves $P_g: c_0\to c_0$. Hence (*i*) holds. □

Theorem 22. *If X is one of the sequence spaces ℓ_p, cs and c_0, then $P_g:X\to c_0$ if and only if $(g(k,\cdot))_{k=1}^{\infty}$ converges continuously at* 0 *to* 0.

The following theorem characterizes $P_g:X\to c_0$ where X is one of the sequence spaces c, bs and ℓ_∞.

Theorem 23. *If X is one of the sequence spaces c, bs and ℓ_∞, then $P_g:X\to c$ if and only if $(g(k,\cdot))_{k=1}^{\infty}$ converges uniformly to* 0 *on every bounded subset of* $\mathbb{R}$.

Proof. To prove the theorem, we shall prove that the following statements are equivalent:

(*i*) $P_g: c\to c_0$.

(*ii*) $(g(k,\cdot))_{k=1}^{\infty}$ converges uniformly to 0 on every bounded subset of $\mathbb{R}$.

(*iii*) $P_g:\ell_\infty\to c_0$.

(*iv*) $P_g: bs\to c_0$.

If (*i*) holds, then $P_g: c\to c$ and $\lim_{k\to\infty} g(k,t)=0$ for all $t\in\mathbb{R}$ which implies by Theorem 14 that (*ii*) holds.

Next, assume that (*ii*) holds. Then $\lim_{k\to\infty} g(k,0)=0$ and $(g(k,\cdot))_{k=1}^{\infty}$ converges uniformly to $\lim_{k\to\infty} g(k,0)$ on every bounded subset of $\mathbb{R}$. By Theorem

15, $P_g: \ell_\infty \to c$. Since ℓ_∞ is solid, by Lemma 11, $\lim_{k\to\infty} g(k, x_k) = \lim_{k\to\infty} g(k, 0) = 0$ for all $x \in \ell_\infty$. Hence $P_g: \ell_\infty \to c_0$.

The implications $(iii) \Rightarrow (i)$ and $(iii) \Rightarrow (iv)$ hold since $c \subseteq \ell_\infty$ and $bs \subseteq \ell_\infty$, respectively. Since c_0 is solid, $(iv) \Rightarrow (iii)$ follows from Lemma 6. □

The last theorem of this section gives a characterization of $P_g: \omega \to c_0$.

Theorem 24. $P_g: \omega \to c_0$ *if and only if* $(g(k, \cdot))_{k=1}^{\infty}$ *converges uniformly to* 0 *on* $\mathbb{R}$.

Proof. Assume that $P_g: \omega \to c_0$. Then $P_g: \omega \to c$ and $\lim_{k\to\infty} g(k, 0) = 0$, so by Theorem 19, $(g(k, \cdot))_{k=1}^{\infty}$ converges uniformly to 0 on $\mathbb{R}$.

Conversely, assume that $(g(k, \cdot))_{k=1}^{\infty}$ converges uniformly to 0 on $\mathbb{R}$. Then $\lim_{k\to\infty} g(k, 0) = 0$. By Theorem 19, $P_g: \omega \to c$. Since ω is solid, by Lemma 11, $\lim_{k\to\infty} g(k, x_k) = \lim_{k\to\infty} g(k, 0) = 0$ for all $x \in \omega$. Hence $P_g: \omega \to c_0$. □

Superposition Operators into ℓ_p

The first theorem of this section follows directly from Lemma 1 and the definition of ℓ_p.

Theorem 25. $P_g: \Phi \to \ell_p$ *if and only if* $\sum_{k=1}^{\infty} |g(k, 0)|^p < \infty$.

A characterization of $P_g: \ell_q \to \ell_1$ has been given by Chew Tuan Seng and Lee Peng Yee in [3] under the condition that $g(k, \cdot)$ is continuous on $\mathbb{R}$ for all $k \in \mathbb{N}$ (see Theorem I.3). Our generalization of this fact is given by the following theorem. However, the idea of our proof is taken from the one given by Chew Tuan Seng and Lee Peng Yee.

Theorem 26. *$P_g: \ell_q \to \ell_p$ if and only if there exist a neighborhood V_0 of 0 in $\mathbb{R}$, $\alpha>0$, $N\in\mathbb{N}$ and $(c_k)_{k=1}^{\infty}\in \ell_1$ such that*

$$|g(k,t)|^p \quad \le \quad c_k + \alpha\,|t|^q \tag{*}$$

for all $t\in V_0$ and $k\ge N$

Proof. Assume that $P_g: \ell_q \to \ell_p$. Then $P_g: \ell_q \to \ell_\infty$ since $\ell_p \subseteq \ell_\infty$. By Theorem 4, there exist $\beta_0>0$ and $N\in\mathbb{N}$ such that $(g(k,\cdot))_{k=N}^{\infty}$ is uniformly bounded on $[-\beta_0,\beta_0]$. Then for $k\ge N$, $\sup\limits_{t\in[-\beta_0,\beta_0]}|g(k,t)|^p<\infty$. For all $k\ge N$, $\alpha>0$ and $\beta>0$, define $A(k,\alpha,\beta)\subseteq[-\beta_0,\beta_0]$ by

$$A(k,\alpha,\beta) \quad = \quad \{t\in[-\beta_0,\beta_0] \mid |t|^q\le \min\{\beta, \alpha^{-1}|g(k,t)|^p\}\} \tag{1}$$

and

$$B(k,\alpha,\beta) \quad = \quad \sup_{t\in A(k,\alpha,\beta)}|g(k,t)|^p \tag{2}$$

Observe that $0\in A(k,\alpha,\beta)$ for all $k\ge N$, $\alpha>0$ and $\beta>0$. To show that (*) holds, we consider the following statements:

(*i*) For all $\alpha>0$ and $\beta>0$, $\sum\limits_{k=N}^{\infty}B(k,\alpha,\beta)$ diverges.

(*ii*) There exists $x\in\ell_q$ such that $P_g(x)\notin\ell_p$, that is, $P_g: \ell_q \not\to \ell_p$.

We claim that (*i*) implies (*ii*). First, we assume that this claim is true. Since $P_g: \ell_q\to\ell_p$, (*i*) is false. Then there exist $\alpha>0$ and $\beta>0$ such that $\sum\limits_{k=N}^{\infty}B(k,\alpha,\beta)<\infty$. Let $\gamma=\min\{\beta_0,\beta^{1/q}\}$, $V_0=[-\gamma,\gamma]$ and

$$c_k \quad = \quad \begin{cases} 0 & \text{if } 1\le k<N,\\ B(k,\alpha,\beta) & \text{if } k\ge N.\end{cases}$$

This implies that $V_0\subseteq[-\beta_0,\beta_0]$, $|t|^q\le\beta$ for all $t\in V_0$ and $(c_k)_{k=1}^{\infty}\in\ell_1$. Let $k\ge N$ and $t\in V_0$. If $|t|^q\le\alpha^{-1}|g(k,t)|^p$, then by (1), $t\in A(k,\alpha,\beta)$ and hence by (2), we have

$|g(k,t)|^p \le B(k,\alpha,\beta) = c_k \le c_k + \alpha|t|^q$. If $|t|^q > \alpha^{-1}|g(k,t)|^p$, then $|g(k,t)|^p < \alpha|t|^q \le c_k + \alpha|t|^q$. Hence (*) holds.

It remains to show that (*i*) implies (*ii*). Assume that (*i*) holds. Then $\sum_{k=N}^{\infty} B(k, 2^j, 2^{-j})$ diverges for all $j \in \mathbb{N}\cup\{0\}$. This implies that for all $j \in \mathbb{N}\cup\{0\}$, $n \ge N$, there exists $m > n$ such that

$$\sum_{k=n+1}^{m} B(k, 2^j, 2^{-j}) \; > \; 1. \tag{3}$$

Then there exists $m_1 > N$ such that $\sum_{k=N+1}^{m_1} B(k, 2^0, 2^{-0}) > 1$. Let $n_1 = \min\{\, m \in \mathbb{N} \mid m > N \text{ and } \sum_{k=N+1}^{m} B(k, 2^0, 2^{-0}) > 1 \,\}$. By (3), there exists $m_2 > n_1$ such that $\sum_{k=n_1+1}^{m_2} B(k, 2^1, 2^{-1}) > 1$. Let $n_2 = \min\{\, m \in \mathbb{N} \mid m > n_1 \text{ and } \sum_{k=n_1+1}^{m} B(k, 2^1, 2^{-1}) > 1 \,\}$. By induction process, there exists a subsequence $(n_k)_{k=1}^{\infty}$ of $(n)_{n=1}^{\infty}$ such that $n_1 > N$ and for all $j \in \mathbb{N}$,

$$n_{j+1} \;=\; \min\{m \in \mathbb{N} \mid m > n_j \text{ and } \sum_{k=n_j+1}^{m} B(k, 2^j, 2^{-j}) > 1\}. \tag{4}$$

Hence for all $j \in \mathbb{N}$,

$$\sum_{k=n_j+1}^{n_{j+1}-1} B(k, 2^j, 2^{-j}) \; \le \; 1. \tag{5}$$

If $k \in \{1, \ldots, n_1\}$, let $x_k = 0$. If $k > n_1$, then there exists unique $j \in \mathbb{N}$ such that $n_j < k \le n_{j+1}$ and hence by (2), there exists x_k such that

$$x_k \in A(k, 2^j, 2^{-j}) \;\text{ and }\; 0 \le B(k, 2^j, 2^{-j}) < |g(k, x_k)|^p + 2^{-k}, \tag{6}$$

and by (1),

$$|x_k|^q \;\le \min\{2^{-j}, 2^{-j}|g(k, x_k)|^p\} \tag{7}$$

Then for all $m \in \mathbb{N}$,

$$\begin{aligned} m \quad &< \quad \sum_{j=1}^{m} \sum_{k=n_j+1}^{n_{j+1}} B(k, 2^j, 2^{-j}) \quad \text{(by (4))} \\ &< \quad \sum_{k=n_1+1}^{n_{m+1}} \{\, |g(k, x_k)|^p + 2^{-k} \} \quad \text{(by (6))} \\ &< \quad \sum_{k=1}^{n_{m+1}} |g(k, x_k)|^p + \sum_{k=1}^{\infty} 2^{-k}, \end{aligned}$$

which implies that $\left(\sum_{k=1}^{n}|g(k,x_k)|^p\right)_{n=1}^{\infty}$ is not bounded. Thus $P_g((x_k)_{k=1}^{\infty}) = (g(k,x_k))_{k=1}^{\infty} \notin \ell_p$. Next, to show that $(x_k)_{k=1}^{\infty} \in \ell_q$, it is equivalent to show that $\left(\sum_{k=1}^{n}|x_k|^q\right)_{n=1}^{\infty}$ is bounded. Let $s = \sum_{k=1}^{n_1}|x_k|^q$ and let $n \in \mathbb{N}$. Then there exists $m \in \mathbb{N}$ such that $n \leq n_{m+1}$, and hence

$$
\begin{aligned}
\sum_{k=1}^{n}|x_k|^q &\leq \sum_{k=1}^{n_{m+1}}|x_k|^q \\
&= s + \sum_{k=n_1+1}^{n_{m+1}}|x_k|^q \\
&= s + \sum_{j=1}^{m}\sum_{k=n_j+1}^{n_{j+1}}|x_k|^q \\
&= s + \sum_{j=1}^{m}\Big\{\sum_{k=n_j+1}^{n_{j+1}-1}|x_k|^q + |x_{n_{j+1}}|^q\Big\} \\
&\leq s + \sum_{j=1}^{m}\Big\{2^{-j}\sum_{k=n_j+1}^{n_{j+1}-1}|g(k,x_k)|^p + 2^{-j}\Big\} \quad \text{(by (7))} \\
&\leq s + \sum_{j=1}^{m}\Big\{2^{-j}\sum_{k=n_j+1}^{n_{j+1}-1}B(k,2^j,2^{-j}) + 2^{-j}\Big\} \quad \text{(by (6) and (2))} \\
&\leq s + \sum_{j=1}^{m}\{2^{-j}\cdot 1 + 2^{-j}\} \quad \text{(by (5))} \\
&< s + 2\sum_{j=1}^{\infty}2^{-j}.
\end{aligned}
$$

This shows that $\left(\sum_{k=1}^{n}|x_k|^q\right)_{n=1}^{\infty}$ is bounded. Hence (ii) holds.

Conversely, assume that there exist a neighborhood V_0 of 0 in $\mathbb{R}$, $\alpha>0$, $N \in \mathbb{N}$ and $(c_k)_{k=1}^{\infty} \in \ell_1$ such that $|g(k,t)|^p \leq c_k + \alpha|t|^q$ for all $t \in V_0$ and $k \in \mathbb{N}$. Since V_0 is a neighborhood of 0, $[-\beta,\beta] \subseteq V_0$ for some $\beta>0$. To show that $P_g: \ell_q \to \ell_p$, let $x \in \ell_q$. Then $\sum_{k=1}^{\infty}|x_k|^q < \infty$ and $\lim_{k\to\infty}x_k = 0$, so there exists $N' \geq N$ such that $|x_k| < \beta$ for all $k \geq N'$. Thus, by assumption, $|g(k,x_k)|^p \leq c_k + \alpha|x_k|^q$ for all $k \geq N'$. This implies that $\sum_{k=1}^{\infty}|g(k,x_k)|^p < \infty$ since $\sum_{k=1}^{\infty}[c_k + \alpha|x_k|^q] = \sum_{k=1}^{\infty}c_k + \alpha\sum_{k=1}^{\infty}|x_k|^q < \infty$. Hence $P_g(x) = (g(k,x_k))_{k=1}^{\infty} \in \ell_p$. □

Chew Tuan Seng and Lee Peng Yee have given a characterization of $P_g: c_0 \to \ell_1$ under the condition that $g(k,\cdot)$ is continuous on $\mathbb{R}$ for all $k \in \mathbb{N}$ (see Theorem I.1). Using the idea of their proof, we generalize this result by characterizing $P_g: c_0 \to \ell_p$. The continuity of each $g(k, \cdot)$ is not assumed in our generalization.

Theorem 27. *$P_g: c_0 \to \ell_p$ if and only if there exist a neighborhood V_0 of 0 and $N \in \mathbb{N}$ such that $\sum_{k=N}^{\infty} \sup_{t \in V_0} |g(k,t)|^p < \infty$.*

Proof. Assume that $P_g: c_0 \to \ell_p$. Then $P_g: c_0 \to \ell_\infty$, so by Theorem 4, there exist $\alpha > 0$ and $N \in \mathbb{N}$ such that $(g(k, \cdot))_{k=N}^{\infty}$ is uniformly bounded on $[-\alpha, \alpha]$. Then for each $k \geq N$, $\sup_{t \in [-\alpha, \alpha]} |g(k,t)|^p < \infty$. Set

$$B(k, \beta) \quad = \quad \sup_{t \in [-\beta, \beta]} |g(k,t)|^p \tag{1}$$

for all $\beta \in \mathbb{R}$, $0 < \beta \leq \alpha$. Claim that $\sum_{k=N}^{\infty} B(k, \beta) < \infty$ for some $\beta \in \mathbb{R}$, $0 < \beta \leq \alpha$. If the claim holds, let $V_0 = [-\beta, \beta]$, so we have $\sum_{k=N}^{\infty} \sup_{t \in V_0} |g(k, x_k)|^p < \infty$, as required. To prove that the claim is true, suppose not. Then $\sum_{k=N}^{\infty} B(k, \frac{\alpha}{j})$ diverges for all $j \in \mathbb{N}$. This implies that for all $j \in \mathbb{N}$, $n \geq N$, there exists $m > n$ such that $\sum_{k=n+1}^{m} B(k, \frac{\alpha}{j}) > 1$. It follows that there exists a subsequence $(n_k)_{k=1}^{\infty}$ of $(n)_{n=1}^{\infty}$ such that $n_1 > N$ and for all $j \in \mathbb{N}$,

$$\sum_{k=n_j+1}^{n_{j+1}} B(k, \tfrac{\alpha}{j}) \;>\; 1. \tag{2}$$

Let $x_k = 0$ for all $k \in \{1, \ldots, n_1\}$. If $k > n_1$, then there exists unique $j \in \mathbb{N}$ such that $n_j < k \leq n_{j+1}$ and hence by (1), there exists $x_k \in [-\frac{\alpha}{j}, \frac{\alpha}{j}]$ such that

$$0 \leq B(k, \tfrac{\alpha}{j}) \;<\; |g(k, x_k)|^p + 2^{-k}, \tag{3}$$

By the choice of x_k, we have $\lim_{k \to \infty} x_k = 0$. Then $(x_k)_{k=1}^{\infty} \in c_0$, and for $m \in \mathbb{N}$,

$$\begin{aligned} m \quad &< \quad \sum_{j=1}^{m} \sum_{k=n_j+1}^{n_{j+1}} B(k, \tfrac{\alpha}{j}) \quad \text{(by (2))} \\ &< \quad \sum_{k=n_1+1}^{n_{m+1}} \{|g(k, x_k)|^p + 2^{-k}\} \quad \text{(by (3))} \\ &< \quad \sum_{k=1}^{n_{m+1}} |g(k, x_k)|^p + \sum_{k=1}^{\infty} 2^{-k}, \end{aligned}$$

which implies that $\left(\sum_{k=1}^{n}|g(k,x_k)|^p\right)_{n=1}^{\infty}$ is not bounded. Thus $P_g((x_k)_{k=1}^{\infty}) = (g(k,x_k))_{k=1}^{\infty} \notin \ell_p$. This is a contradiction since $P_g: c_0 \to \ell_p$.

Conversely, assume that there exist a neighborhood V_0 of 0 and $N \in \mathbb{N}$ such that $\sum_{k=N}^{\infty} \sup_{t \in V_0} |g(k,x_k)|^p < \infty$. Then there exists $\alpha > 0$ such that

$$\sum_{k=N}^{\infty} \sup_{t \in [-\alpha,\alpha]} |g(k,t)|^p < \infty \tag{4}$$

To show that $P_g: c_0 \to \ell_p$, let $x \in c_0$. Then $\lim_{k\to\infty} x_k = 0$, so there exists $N' \ge N$ such that $|x_k| < \alpha$ for all $k \ge N'$. It implies by (4) that $\sum_{k=1}^{\infty}|g(k,x_k)|^p < \infty$. Hence $P_g(x) = (g(k,x_k))_{k=1}^{\infty} \in \ell_p$. □

We recall that ℓ_p is solid. To characterize $P_g: cs \to \ell_p$, the following lemma is required.

Lemma 28. *If Y is a solid sequence space and $P_g: cs \to Y$, then $P_g: c_0 \to Y$.*

Proof. Let $x \in c_0$. Define the sequences $(y_k)_{k=1}^{\infty}$ and $(z_k)_{k=1}^{\infty}$ by

$$y_k = \begin{cases} x_k & \text{if } k \text{ is odd,} \\ -x_{k-1} & \text{if } k \text{ is even} \end{cases}$$

and

$$z_k = \begin{cases} -x_{k+1} & \text{if } k \text{ is odd,} \\ x_k & \text{if } k \text{ is even,} \end{cases}$$

that is, $(y_k)_{k=1}^{\infty} = (x_1, -x_1, x_3, -x_3, x_5, -x_5, \dots)$ and $(z_k)_{k=1}^{\infty} = (-x_2, x_2, -x_4, x_4, -x_6, x_6, \dots)$. Then for all $n \in \mathbb{N}$,

$$\sum_{k=1}^{n} y_k = \begin{cases} x_n & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

and

$$\sum_{k=1}^{n} z_k = \begin{cases} -x_{n+1} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

Since $x \in c_0$, $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} y_k = 0 = \lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} z_k$. Then $(y_k)_{k=1}^{\infty}, (z_k)_{k=1}^{\infty} \in cs$. Since $P_g: cs \to Y$, $(g(k, y_k))_{k=1}^{\infty} \in Y$ and $(g(k, z_k))_{k=1}^{\infty} \in Y$. Define the sequences $(u_k)_{k=1}^{\infty}$ and $(v_k)_{k=1}^{\infty}$ by

$$u_k = \begin{cases} g(k, y_k) & \text{if } k \text{ is odd,} \\ 0 & \text{if } k \text{ is even} \end{cases}$$

and

$$v_k = \begin{cases} 0 & \text{if } k \text{ is odd,} \\ g(k, z_k) & \text{if } k \text{ is even.} \end{cases}$$

Then $|u_k| \le |g(k, y_k)|$ and $|v_k| \le |g(k, z_k)|$ for all $k \in \mathbb{N}$. Since Y is solid, we have $(u_k)_{k=1}^{\infty}, (v_k)_{k=1}^{\infty} \in Y$. But $g(k, y_k) = g(k, x_k)$ if k is odd and $g(k, z_k) = g(k, x_k)$ if k is even, so $(g(k, x_k))_{k=1}^{\infty} = (u_k)_{k=1}^{\infty} + (v_k)_{k=1}^{\infty} \in Y$. □

Theorem 29. *$P_g: cs \to \ell_p$ if and only if there exist a neighborhood V_0 of* 0 *and $N \in \mathbb{N}$ such that* $\sum\limits_{k=N}^{\infty} \sup\limits_{t \in V_0} |g(k, t)|^p < \infty$.

Proof. It follows from Lemma 28, Theorem 27 and the facts that ℓ_p is solid and $cs \subseteq c_0$. □

The next theorem give a characterization of $P_g: X \to \ell_p$ where X is one of the sequence spaces c, bs and ℓ_∞.

Theorem 30. *If X is one of the sequence spaces c, bs and ℓ_∞, then $P_g: X \to \ell_p$ if and only if for every bounded subset S of $\mathbb{R}$, there exists $N \in \mathbb{N}$ such that* $\sum\limits_{k=N}^{\infty} \sup\limits_{t \in S} |g(k, t)|^p < \infty$.

Proof. To prove the theorem, we shall prove that the following statements are equivalent:

(*i*) $P_g: c \to \ell_p$.

(*ii*) For every bounded subset S of $\mathbb{R}$, there exists $N \in \mathbb{N}$ such that $\sum_{k=N}^{\infty} \sup_{t \in S} |g(k,t)|^p < \infty$.

(*iii*) $P_g: \ell_\infty \to \ell_p$.

(*iv*) $P_g: bs \to \ell_p$.

To show (*i*)$\Rightarrow$(*ii*), assume that $P_g: c \to \ell_p$. Claim that the following statement holds:

$$\text{For every } s \in \mathbb{R}, \text{ there exist } r > 0 \text{ and } N \in \mathbb{N} \text{ such that } \sum_{k=N}^{\infty} \sup_{t \in (s-r, s+r)} |g(k,t)|^p < \infty. \tag{1}$$

Let $s \in \mathbb{R}$. Since $P_g: c \to \ell_p$ and $(s) + c_0 \subseteq c$, $P_g: (s) + c_0 \to \ell_p$. Then $P_{g_s}: c_0 \to \ell_p$. It follows from Theorem 27 that there exist $r > 0$ and $N \in \mathbb{N}$ such that $\sum_{k=N}^{\infty} \sup_{t \in (-r,r)} |g_s(k,t)|^p < \infty$. But $\{ g_s(k,t) \mid |t| < r \} = \{ g(k,t) \mid |t-s| < r \}$ for all $k \in \mathbb{N}$, so $\sum_{k=N}^{\infty} \sup_{t \in (s-r,s+r)} |g(k,t)|^p < \infty$. Then we have the claim. To show (*ii*), it is equivalent to show that the following statement holds.

$$\text{For every } \alpha > 0, \text{ there exists } N \in \mathbb{N} \text{ such that } \sum_{k=N}^{\infty} \sup_{t \in [-\alpha,\alpha]} |g(k,t)|^p < \infty. \tag{2}$$

To prove (2), let $\alpha > 0$ be given. Then by (1), we have that for every $s \in \mathbb{R}$, there exist $r_s > 0$ and $N_s \in \mathbb{N}$ such that $\sum_{k=N_s}^{\infty} \sup_{t \in (s-r_s, s+r_s)} |g(k,t)|^p < \infty$. For each $s \in [-\alpha, \alpha]$, let $I(s) = (s - r_s, s + r_s)$. Then $[-\alpha, \alpha] \subseteq \bigcup_{s \in [-\alpha,\alpha]} I(s)$. Since $[-\alpha, \alpha]$ is compact, $[-\alpha, \alpha] \subseteq \bigcup_{i=1}^{n} I(s_i)$ for some $n \in \mathbb{N}$, $s_1, \ldots, s_n \in [-\alpha, \alpha]$. Let $N = \max\{ N_{s_1}, \ldots, N_{s_n} \}$. Then we have $\sum_{k=N}^{\infty} \sup_{t \in I(s_i)} |g(k,t)|^p < \infty$ for all $i \in \{1, \ldots, n\}$ and thus

$$\sum_{k=N}^{\infty} \Big(\sum_{i=1}^{n} \sup_{t \in I(s_i)} |g(k,t)|^p \Big) = \sum_{i=1}^{n} \Big(\sum_{k=N}^{\infty} \sup_{t \in I(s_i)} |g(k,t)|^p \Big) < \infty \tag{3}$$

If $t' \in [-\alpha, \alpha]$, then $t' \in I(s_j)$ for some $j \in \{1, \ldots, n\}$, which implies that for $k \geq N$, $|g(k,t')|^p \leq \sup_{t \in I(s_j)} |g(k,t)|^p \leq \sum_{i=1}^{n} \sup_{t \in I(s_i)} |g(k,t)|^p$. Thus for all $k \geq N$, $\sup_{t \in [-\alpha,\alpha]} |g(k,t)|^p \leq \sum_{i=1}^{n} \sup_{t \in I(s_i)} |g(k,t)|^p$. By (3), we have that $\sum_{k=N}^{\infty} \sup_{t \in [-\alpha,\alpha]} |g(k,t)|^p < \infty$. Hence (2) holds.

To prove $(ii) \Rightarrow (iii)$, assume that (ii) holds. Then (2) holds. To prove that $P_g: \ell_\infty \to \ell_p$, let $x \in \ell_\infty$. Then there exists $\alpha > 0$ such that $|x_k| \le \alpha$ for all $k \in \mathbb{N}$. By (2), there exists $N \in \mathbb{N}$ such that $\sum_{k=N}^{\infty} \sup_{t \in [-\alpha, \alpha]} |g(k,t)|^p < \infty$. This implies that $\sum_{k=1}^{\infty} |g(k, x_k)|^p < \infty$. Hence $P_g(x) = (g(k, x_k))_{k=1}^{\infty} \in \ell_p$.

The implications $(iii) \Rightarrow (i)$ and $(iii) \Rightarrow (iv)$ hold since $c \subseteq \ell_\infty$ and $bs \subseteq \ell_\infty$, respectively. Since ℓ_p is solid, $(iv) \Rightarrow (iii)$ is obtained by Lemma 6. □

The last theorem of this section gives a necessary and sufficient condition for $P_g: \omega \to \ell_p$.

Theorem 31. *$P_g: \omega \to \ell_p$ if and only if there exists $N \in \mathbb{N}$ such that $\sum_{k=N}^{\infty} \sup_{t \in \mathbb{R}} |g(k,t)|^p < \infty$.*

Proof. Assume that $P_g: \omega \to \ell_p$. Then $P_g: \omega \to \ell_\infty$. By Theorem 9, there exists $N \in \mathbb{N}$ such that $(g(k,\cdot))_{k=N}^{\infty}$ is uniformly bounded on $\mathbb{R}$. Then for all $k \ge N$, $\sup_{t \in \mathbb{R}} |g(k,t)|^p < \infty$, and hence for each $k \ge N$, there exists $x_k \in \mathbb{R}$ such that

$$0 \le \sup_{t \in \mathbb{R}} |g(k,t)|^p < |g(k, x_k)|^p + 2^{-k}. \qquad (*)$$

Let $x_k = 0$ for all $k \in \{1, \dots, N-1\}$. Then $(x_k)_{k=1}^{\infty} \in \omega$. Since $P_g: \omega \to \ell_p$, $\sum_{k=1}^{\infty} |g(k, x_k)|^p < \infty$. It implies by $(*)$ that $\sum_{k=N}^{\infty} \sup_{t \in \mathbb{R}} |g(k,t)|^p < \infty$.

The converse of the theorem is obvious. □

Superposition Operators into Φ

By Lemma 1 and the definition of Φ, we have

Theorem 32. *$P_g: \Phi \to \Phi$ if and only if $(g(k,0))_{k=1}^{\infty} = \sum_{k=1}^{N} g(k,0) e^{(k)}$ for some $N \in \mathbb{N}$.*

The next theorem characterizes $P_g: X \to \ell_\infty$ where X is one of the sequence spaces ℓ_p, cs and c_0. The theorem follows directly from the following lemma.

Lemma 33. *Let X be a sequence space such that $\ell_1 \subseteq X \subseteq c_0$. Then the following statements are equivalent:*

(*i*) $P_g: c_0 \to \Phi$.

(*ii*) $P_g: X \to \Phi$.

(*iii*) $P_g: \ell_1 \to \Phi$.

(*iv*) *There exist a neighborhood V_0 of 0 and $N \in \mathbb{N}$ such that $g(k, \cdot) = 0$ on V_0 for all $k \geq N$.*

Proof. Since $\ell_1 \subseteq X \subseteq c_0$, the implications $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (iii)$ are directly obtained.

To show $(iii) \Rightarrow (iv)$, suppose that (iv) is not true. This implies that there exist a subsequence $(n_k)_{k=1}^{\infty}$ of $(n)_{n=1}^{\infty}$ and a sequence $(x_{n_k})_{k=1}^{\infty}$ such that $x_{n_k} \in [-2^{-k}, 2^{-k}]$ and $g(n_k, x_{n_k}) \neq 0$ for all $k \in \mathbb{N}$. Then $(x_{n_k})_{k=1}^{\infty} \in \ell_1$ and $(g(n_k, x_{n_k}))_{k=1}^{\infty} \notin \Phi$. Let $(y_n)_{n=1}^{\infty}$ be a sequence defined by

$$y_n = \begin{cases} x_{n_k} & \text{if } n = n_k \text{ for some } k \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Then $(y_n)_{n=1}^{\infty} \in \ell_1$ and $(g(n, y_n))_{n=1}^{\infty} \notin \Phi$. Hence $P_g: \ell_1 \not\to \Phi$.

Finally, to show $(iv) \Rightarrow (i)$, assume that (iv) holds. Then there exist $\alpha > 0$ and $N \in \mathbb{N}$ such that for all $k \geq N$,

$$g(k, \cdot) = 0 \text{ on } [-\alpha, \alpha]. \tag{$*$}$$

Let $x\in c_0$. Then there exists $N'\geq N$ such that $|x_k|\leq\alpha$ for all $k\geq N'$ which implies by (*) that $g(k,x_k)=0$ for all $k\geq N'$. Hence $(g(k,x_k))_{k=1}^{\infty}\in\Phi$. This proves that $P_g: c_0\to\Phi$. □

Theorem 34. *If X is one of the sequence spaces ℓ_p, cs and c_0, then $P_g:X\to\Phi$ if and only if there exist a neighborhood V_0 of 0 and $N\in\mathbb{N}$ such that $g(k,\cdot)=0$ on V_0 for all $k\geq N$.*

A characterizations of $P_g:X\to\Phi$ where X is one of the sequence spaces c, bs and ℓ_∞ is as follows:

Theorem 35. *If X is one of the sequence spaces c, bs and ℓ_∞, then $P_g:X\to\Phi$ if and only if for every bounded subset S of $\mathbb{R}$, there exists $N\in\mathbb{N}$ such that $g(k,\cdot)=0$ on S for all $k\geq N$.*

Proof. To prove the theorem, we shall prove that the following statements are equivalent:

(*i*) $P_g: c\to\Phi$.

(*ii*) For every bounded subset S of $\mathbb{R}$, there exists $N\in\mathbb{N}$ such that $g(k,\cdot)=0$ on S for all $k\geq N$.

(*iii*) $P_g:\ell_\infty\to\Phi$.

(*iv*) $P_g: bs\to\Phi$.

To show $(i)\Rightarrow(ii)$, assume that $P_g: c\to\Phi$. Since every bounded subset of $\mathbb{R}$ is contained in a compact subset of $\mathbb{R}$, we have that (*ii*) is equivalent to the following statement:

$$\begin{array}{c}\text{For every } s\in\mathbb{R}, \text{ there exist } r>0 \text{ and } N\in\mathbb{N} \text{ such that}\\ g(k,\cdot)=0 \text{ on } (s-r,s+r) \quad \text{for all } k\geq N.\end{array}\tag{*}$$

Next, to prove that (*) holds, let $s \in \mathbb{R}$. Since $P_g: c \to \Phi$ and $(s)+c_0 \subseteq c$, $P_g: (s)+c_0 \to \Phi$. Then $P_{g_s}: c_0 \to \Phi$. It follows from Theorem 34 that there exist $r>0$ and $N \in \mathbb{N}$ such that $g_s(k,\cdot)=0$ on $(-r,r)$ for all $k \geq N$. But $\{g_s(k,t) \mid |t|<r\} = \{g(k,t) \mid |t-s|<r\}$ for all $k \in \mathbb{N}$, so $g(k,\cdot)=0$ on $(s-r,s+r)$ for all $k \geq N$.

To prove $(ii) \Rightarrow (iii)$, assume that (ii) holds. To show that $P_g: \ell_\infty \to \Phi$, let $x \in \ell_\infty$. Then there exists $\alpha>0$ such that $|x_k| \leq \alpha$ for all $k \in \mathbb{N}$. By (ii), there exists $N \in \mathbb{N}$ such that $g(k,\cdot)=0$ on $[-\alpha,\alpha]$ for all $k \geq N$. Then $g(k,x_k)=0$ for all $k \geq N$. This shows that $(g(k,x_k))_{k=1}^{\infty} \in \Phi$. Hence (iii) holds.

The implications $(iii) \Rightarrow (i)$ and $(iii) \Rightarrow (iv)$ hold since $c \subseteq \ell_\infty$ and $bs \subseteq \ell_\infty$, respectively. Since Φ is solid, $(iv) \Rightarrow (iii)$ follows from Lemma 6. □

We end this section by characterizing $P_g: \omega \to \Phi$.

Theorem 36. *$P_g: \omega \to \Phi$ if and only if there exists $N \in \mathbb{N}$ such that $g(k,\cdot)=0$ on $\mathbb{R}$ for all $k \geq N$.*

Proof. Suppose that for every $n \in \mathbb{N}$, there exist $k>n$ and $t \in \mathbb{R}$ such that $g(k,t) \neq 0$. Then there exist a subsequence $(n_k)_{k=1}^{\infty}$ of $(n)_{n=1}^{\infty}$ and a sequence $(x_{n_k})_{k=1}^{\infty}$ such that $g(n_k, x_{n_k}) \neq 0$ for all $k \in \mathbb{N}$. Then $(g(n_k,x_{n_k}))_{k=1}^{\infty} \notin \Phi$. Let $(y_n)_{n=1}^{\infty}$ be a sequence defined by

$$y_n = \begin{cases} x_{n_k} & \text{if } n = n_k \text{ for some } k \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Then $(y_n)_{n=1}^{\infty} \in \omega$ and $(g(n,y_n))_{n=1}^{\infty} \notin \Phi$ since $(g(n_k,x_{n_k}))_{k=1}^{\infty}$ is a subsequence of $(g(n,y_n))_{n=1}^{\infty}$. This proves that if $P_g: \omega \to \Phi$, then there exists $N \in \mathbb{N}$ such that $g(k,\cdot)=0$ on $\mathbb{R}$ for all $k \geq N$. The converse of this statement is obvious. □

CHAPTER III

CONTINUITY OF SUPERPOSITION OPERATORS

The work on continuity of superposition operators we have seen has been done by J. Robert [1]. Under the conditions of $g: \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ that $g(k, 0) = 0$ and $g(k, \cdot)$ is continuous at 0 for all $k \in \mathbb{N}$, he has given a characterization determining when the superposition operator P_g between any two Orlicz sequence spaces is continuous at (0). By making use of our results in Chapter II, without any additional conditions of g, we characterize $P_g: X \to Y$ which is continuous at every point of X where $X \in \{\Phi, \ell_p, cs, c_0, c, bs, \ell_\infty, \omega\}$ and $Y \in \{\Phi, \ell_p, c_0, c, \ell_\infty, \omega\}$. The topologies for these sequence spaces are standard ones given in Chapter I.

We recall that all classical sequence spaces we consider contain Φ. For each $t \in \mathbb{R}$, we have $te^{(k)} \in \Phi$ for all $k \in \mathbb{N}$. Then for a sequence space X containing Φ, we have $te^{(k)} \in X$ for all $t \in \mathbb{R}$ and $k \in \mathbb{N}$. An important consequence of this fact is as follows: Let X and Y be topological sequence spaces, $\Phi \subseteq X$ and $P_g: X \to Y$. Then (1) implies (2) where

(1) for $x \in X$, P_g is continuous at x if and only if $g(k, \cdot)$ is continuous at x_k for all $k \in \mathbb{N}$ and

(2) P_g is continuous on X if and only if $g(k, \cdot)$ is continuous on $\mathbb{R}$ for all $k \in \mathbb{N}$.

Continuous Superposition Operators into ω

We recall that ω is a metric K-space with the metric d_ω defined by

$$d_\omega(x,y) \quad = \quad \sum_{k=1}^{\infty} \frac{|x_k - y_k|}{2^k(1 + |x_k - y_k|)}$$

and for $x, x^{(n)} \in \omega$ $(n \in \mathbb{N})$, $\lim\limits_{n\to\infty} x^{(n)} = x$ in ω if and only if $\lim\limits_{n\to\infty} x_k^{(n)} = x_k$ in $\mathbb{R}$ for all $k \in \mathbb{N}$. The following two lemmas are useful for this section.

Lemma 1. *Let X be a metric K-space. For $x \in X$, if $g(k,\cdot)$ is continuous at x_k for all $k \in \mathbb{N}$, then $P_g : X \to \omega$ is continuous at x.*

Proof. Let $x \in X$ and assume that $g(k,\cdot)$ is continuous at x_k for all $k \in \mathbb{N}$. To show that $P_g : X \to \omega$ is continuous at x, it is equivalent to show that for any sequence $(x^{(n)})_{n=1}^{\infty}$ in X, $\lim\limits_{n\to\infty} x^{(n)} = x$ in X implies $\lim\limits_{n\to\infty} P_g(x^{(n)}) = P_g(x)$ in ω. Let $(x^{(n)})_{n=1}^{\infty}$ be a sequence in X such that $\lim\limits_{n\to\infty} x^{(n)} = x$ in X. Since X is a K-space, we have that $\lim\limits_{n\to\infty} x_k^{(n)} = x_k$ in $\mathbb{R}$ for all $k \in \mathbb{N}$. Then by the continuity of each $g(k,\cdot)$ at x_k, $\lim\limits_{n\to\infty} g(k, x_k^{(n)}) = g(k, x_k)$ in $\mathbb{R}$ for all $k \in \mathbb{N}$ which implies that $\lim\limits_{n\to\infty} (g(k, x_k^{(n)}))_{k=1}^{\infty} = (g(k, x_k))_{k=1}^{\infty}$ in ω. Hence $\lim\limits_{n\to\infty} P_g(x^{(n)}) = P_g(x)$ in ω. □

Lemma 2. *Let X be a normed sequence space containing Φ. Assume that there exists $\alpha > 0$ such that $\|e^{(n)}\|_X \le \alpha$ for all $n \in \mathbb{N}$. For $x \in X$, if $P_g : X \to \omega$ is continuous at x, then $g(k,\cdot)$ is continuous at x_k for all $k \in \mathbb{N}$.*

Proof. Let $x \in X$ and assume that $P_g : X \to \omega$ is continuous at x. Let $k \in \mathbb{N}$. To prove that $g(k,\cdot)$ is continuous at x_k, let $\varepsilon > 0$ be given. Set $\beta = \min\{\frac{1}{2^{k+1}}, \frac{\varepsilon}{2^k(1+\varepsilon)}\}$. Then $0 < \beta \le \frac{\varepsilon}{2^k(1+\varepsilon)}$ which implies that $2^k\beta \le \varepsilon(1 - 2^k\beta)$. But $\beta \le \frac{1}{2^{k+1}}$, so $1 - 2^k\beta > 0$. Then $\frac{2^k\beta}{(1 - 2^k\beta)} \le \varepsilon$. Since P_g is continuous at x, there exists $\delta > 0$ such that for each $z \in X$,

$$\|z-x\|_X < \delta \text{ implies } d_\omega(P_g(z), P_g(x)) < \beta. \qquad (*)$$

Let $t \in \mathbb{R}$ be such that $|t-x_k| < \frac{\delta}{\alpha}$. Let $u = (t-x_k)e^{(k)} + x$. Then $u_k = t$. Since $\Phi \subseteq X$ and $x \in X$, we have $u \in X$. Then $\|u-x\|_X = \|(t-x_k)e^{(k)}\|_X = |t-x_k|\,\|e^{(k)}\|_X < \frac{\delta}{\alpha}\alpha = \delta$, so by (*), we have $\frac{|g(k,t)-g(k,x_k)|}{2^k(1+|g(k,t)-g(k,x_k)|)} \le d_\omega(P_g(u), P_g(x)) < \beta$. It follows that $|g(k,t)-g(k,x_k)| < \frac{2^k\beta}{(1-2^k\beta)} \le \varepsilon$. □

The sequence spaces Φ, ℓ_p, cs, c_0, c, bs and ℓ_∞ are normed K-spaces containing Φ and in these spaces, the norm of $e^{(n)}$ is 1 for every $n \in \mathbb{N}$. Then the following theorem follows directly from Lemma 1 and Lemma 2.

Theorem 3. *Let X be one of the sequence spaces Φ, ℓ_p, cs, c_0, c, bs and ℓ_∞. Then under considering $P_g: X \to \omega$, the following statements hold:*

(i) For $x \in X$, P_g is continuous at x if and only if $g(k,\cdot)$ is continuous at x_k for all $k \in \mathbb{N}$.

(ii) P_g is continuous on X if and only if $g(k,\cdot)$ is continuous on $\mathbb{R}$ for all $k \in \mathbb{N}$.

In order to characterize the continuous superposition operator $P_g: \omega \to \omega$, we need Lemma 1 and the next lemma.

Lemma 4. *Let Y be a metric K-space and assume that $P_g: \omega \to Y$. For $x \in \omega$, if P_g is continuous at x, then $g(k,\cdot)$ is continuous at x_k for all $k \in \mathbb{N}$.*

Proof. Let $x \in \omega$ and assume that P_g is continuous at x. Let $k \in \mathbb{N}$. To show that $g(k,\cdot)$ is continuous at x_k, let $(y_n)_{n=1}^{\infty}$ be a sequence such that $\lim_{n\to\infty} y_n = x_k$ in $\mathbb{R}$. For each $n \in \mathbb{N}$, define the sequence $z^{(n)}$ by

$$z_i^{(n)} = \begin{cases} y_n & \text{if } i = k, \\ x_i & \text{if } i \neq k, \end{cases}$$

that is, $z^{(n)} = (x_1, \ldots, x_{k-1}, y_n, x_{k+1}, \ldots)$. Then $\lim_{n\to\infty} z_k^{(n)} = \lim_{n\to\infty} y_n = x_k$ in $\mathbb{R}$ and if $i \neq k$, we have $\lim_{n\to\infty} z_i^{(n)} = \lim_{n\to\infty} x_i = x_i$ in $\mathbb{R}$. This implies that $\lim_{n\to\infty} z^{(n)} = x$ in ω. Since P_g is continuous at x, $\lim_{n\to\infty} P_g(z^{(n)}) = P_g(x)$ in Y. But $(P_g(z^{(n)}))_k = g(k, z_k^{(n)}) = g(k, y_n)$ and $(P_g(x))_k = g(k, x_k)$, so $\lim_{n\to\infty} g(k, y_n) = g(k, x_k)$ in $\mathbb{R}$ since Y is a K-space. This shows that $g(k, \cdot)$ is continuous at x_k. □

Since ω is a metric K-space, by Lemma 1 and Lemma 4 we have

Theorem 5. *Under considering $P_g: \omega \to \omega$, the following statements hold:*

(*i*) *For $x \in \omega$, P_g is continuous at x if and only if $g(k, \cdot)$ is continuous at x_k for all $k \in \mathbb{N}$.*

(*ii*) *P_g is continuous on ω if and only if $g(k, \cdot)$ is continuous on $\mathbb{R}$ for all $k \in \mathbb{N}$.*

Continuous Superposition Operators into ℓ_∞

We begin this section by recalling that $\|\cdot\|_{\ell_\infty} \leq \|\cdot\|_{\ell_p}$ on ℓ_p and $\|\cdot\|_{\ell_\infty} \leq 2\|\cdot\|_{bs}$ on bs. Then $\|\cdot\|_{\ell_\infty} \leq 2\|\cdot\|_{cs}$ on cs since cs is a normed subspace of bs. Since the norm in each of Φ, c_0, and c is the sup-norm, we have that $\|\cdot\|_{\ell_\infty} = \|\cdot\|_X$ on X if X is one of Φ, c_0, and c.

Lemma 6. *Let X be a normed sequence space containing Φ and Y a normed sequence space such that $Y \subseteq \ell_\infty$. Assume that*

(*i*) $P_g: X \to Y$,

(*ii*) *there exists $\alpha > 0$ such that $\|e^{(n)}\|_X \leq \alpha$ for all $n \in \mathbb{N}$ and* (*iii*) $\|\cdot\|_{\ell_\infty} \le \beta\|\cdot\|_Y$ on Y *for some* $\beta>0$.

For $x\in X$, *if* P_g *is continuous at* x, *then for any* $\varepsilon>0$, *there exists* $\delta>0$ *such that for all* $k\in\mathbb{N}$, $t\in\mathbb{R}$,

$$|t-x_k|<\delta \quad \textit{implies} \quad |g(k,t)-g(k,x_k)|<\varepsilon.$$

Proof. Assume that P_g is continuous at $x\in X$. Let $\varepsilon>0$ be given. Then there exists $\delta>0$ such that for each $z\in X$,

$$\|z-x\|_X<\delta \quad \text{implies} \quad \|P_g(z)-P_g(x)\|_Y<\frac{\varepsilon}{\beta}. \qquad (*)$$

Let $k\in\mathbb{N}$ and $t\in\mathbb{R}$ be such that $|t-x_k|<\frac{\delta}{\alpha}$. Let $u=(t-x_k)e^{(k)}+x$. Then $u_k=t$. Since $\Phi\subseteq X$, $u\in X$. Then by (*ii*), $\|u-x\|_X=\|(t-x_k)e^{(k)}\|_X=|t-x_k|\,\|e^{(k)}\|_X<\frac{\delta}{\alpha}\alpha=\delta$. By (*), we have that $\|P_g(u)-P_g(x)\|_Y<\frac{\varepsilon}{\beta}$. Hence by (*iii*), $|g(k,t)-g(k,x_k)|<\|P_g(u)-P_g(x)\|_{\ell_\infty}\le\beta\|P_g(u)-P_g(x)\|_Y<\varepsilon$. □

Lemma 7. *Let* Y *be a normed subspace of* ℓ_∞ *and* X *a normed sequence space such that* $X\subseteq\ell_\infty$ *and* $\|\cdot\|_{\ell_\infty}\le\alpha\|\cdot\|_X$ on X *for some* $\alpha>0$. *Assume that* $P_g\colon X\to Y$. *Then for* $x\in X$, P_g *is continuous at* x *if the following statement holds:*

$$\begin{gathered}\textit{For any } \varepsilon>0, \textit{ there exists } \delta>0 \textit{ such that for all } k\in\mathbb{N},\ t\in\mathbb{R},\\ |t-x_k|<\delta \quad \textit{implies} \quad |g(k,t)-g(k,x_k)|<\varepsilon.\end{gathered} \qquad (*)$$

Proof. First, we note that $\|\cdot\|_Y=\|\cdot\|_{\ell_\infty}$ on Y. Assume that (*) holds. To show that P_g is continuous at x, given $\varepsilon>0$. Then by (*), there exists $\delta>0$ such that for all $k\in\mathbb{N}$, $t\in\mathbb{R}$, $|t-x_k|<\delta$ implies $|g(k,t)-g(k,x_k)|<\varepsilon$. Let $z\in X$ be such that $\|z-x\|_X<\frac{\delta}{\alpha}$. Then by assumption, we have that $|z_k-x_k|<\|z-x\|_{\ell_\infty}\le\alpha\|z-x\|_X<\delta$ for all $k\in\mathbb{N}$. This implies that $|g(k,z_k)-g(k,x_k)|<\varepsilon$ for all $k\in\mathbb{N}$. Hence $\|P_g(z)-P_g(x)\|_Y=\|P_g(z)-P_g(x)\|_{\ell_\infty}\le\varepsilon$. This shows that P_g is continuous at x. □

The following theorem is an immediate consequence of Lemma 6 and Lemma 7.

Theorem 8. *If X is one of the sequence spaces Φ, ℓ_p, cs, c_0, c, bs and ℓ_∞ and $P_g: X \to \ell_\infty$, then for $x \in X$, P_g is continuous at x if and only if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for all $k \in \mathbb{N}$, $t \in \mathbb{R}$,*

$$|t - x_k| < \delta \quad \text{implies} \quad |g(k,t) - g(k,x_k)| < \varepsilon.$$

Characterizing when P_g is continuous at every point of ω where $P_g: \omega \to \ell_\infty$ is a part of the next lemma. The lemma is also referred in the next section.

Lemma 9. *Let Y be a normed subspace of ℓ_∞ and assume that $P_g: \omega \to Y$. Then for $x \in \omega$, P_g is continuous at x if and only if*

(*i*) *$g(k,\cdot)$ is continuous at x_k for all $k \in \mathbb{N}$ and*

(*ii*) *for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $k \in \mathbb{N}$, $t \in \mathbb{R}$,*

$$k \ge N \quad \text{implies} \quad |g(k,t) - g(k,x_k)| < \varepsilon.$$

Proof. Assume that P_g is continous at $x \in \omega$. Since ℓ_∞ is a normed K-space and Y is a normed subspace of ℓ_∞, Y is a normed K-space. Then by Lemma 4, (*i*) holds. To show that (*ii*) holds, suppose not. Then there exists $\varepsilon > 0$ such that *for all* $n \in \mathbb{N}$ there exist $k \ge n$ and $t \in \mathbb{R}$ such that

$$|g(k,t) - g(k,x_k)| \ \ge \ \varepsilon. \tag{1}$$

Let $\delta > 0$ be given. Then there exists $m \in \mathbb{N}$ such that $2^{-m} < \delta$, so by (1), there exist $j \ge m$ and $s \in \mathbb{R}$ such that $|g(j,s) - g(j,x_j)| \ge \varepsilon$. Let $z = (s - x_j)e^{(j)} + x$. Then we have that $z \in \omega$, $d_\omega(z,x) = \dfrac{|s - x_j|}{2^j(1 + |s - x_j|)} \le 2^{-j} \le 2^{-m} < \delta$ and $\|P_g(z) - P_g(x)\|_Y \ge$

$|g(j,s)-g(j,x_j)| \geq \varepsilon$. This proves that P_g is not continous at x, a contradiction. Hence (ii) holds.

Conversely, assume that (i) and (ii) are true. To show that P_g is continous at x, let $\varepsilon > 0$ be given. Then by (ii), there exists $N \in \mathbb{N}$ such that for all $k \in \mathbb{N}$, $t \in \mathbb{R}$,

$$k \geq N \quad implies \quad |g(k,t)-g(k,x_k)| < \varepsilon. \tag{2}$$

By (i), there exists $\delta > 0$ such that for all $k \in \{1,\ldots,N-1\}$, $t \in \mathbb{R}$,

$$|t-x_k| < \delta \quad implies \quad |g(k,t)-g(k,x_k)| < \varepsilon. \tag{3}$$

Let $\delta' = \min\{\frac{1}{2^{N+1}}, \frac{\delta}{2^N(1+\delta)}\}$. Then $\frac{2^N\delta'}{(1-2^N\delta')} \leq \delta$. Let $z \in \omega$ be such that $d_\omega(z,x) < \delta'$. Then by (2), $|g(k,z_k)-g(k,x_k)| < \varepsilon$ for all $k \geq N$. For $k \in \{1,\ldots,N-1\}$, we have that $\frac{|z_k-x_k|}{2^k(1+|z_k-x_k|)} \leq d_\omega(z,x) < \delta'$. Then for $k \in \{1,\ldots,N-1\}$, $\frac{|z_k-x_k|}{1+|z_k-x_k|} < 2^k\delta' \leq 2^N\delta'$ which implies that $|z_k-x_k| < \frac{2^N\delta'}{(1-2^N\delta')} \leq \delta$. It follows from (3) that $|g(k,z_k)-g(k,x_k)| < \varepsilon$ for all $k \in \{1,\ldots,N-1\}$. Hence $|g(k,z_k)-g(k,x_k)| < \varepsilon$ for all $k \in \mathbb{N}$. This implies that $\|P_g(z)-P_g(x)\|_Y = \|P_g(z)-P_g(x)\|_{\ell_\infty} \leq \varepsilon$. This shows that P_g is continuous at x. □

Theorem 10. *If $P_g: \omega \to \ell_\infty$, then for $x \in \omega$, P_g is continuous at x if and only if*

(i) $g(k,\cdot)$ is continuous at x_k for all $k \in \mathbb{N}$ and

(ii) for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $k \in \mathbb{N}$, $t \in \mathbb{R}$,

$$k \geq N \quad implies \quad |g(k,t)-g(k,x_k)| < \varepsilon.$$

Continuous Superposition Operators into c

Since $\|e^{(n)}\|_\Phi = 1$ for all $n \in \mathbb{N}$ and $\|\cdot\|_{\ell_\infty} = \|\cdot\|_\Phi$ on Φ, we have the following theorem by Lemma 6 and Lemma 7.

Theorem 11. *If $P_g: \Phi \to c$, then for $x \in \Phi$, P_g is continuous at x if and only if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for all $k \in \mathbb{N}$, $t \in \mathbb{R}$,*

$$|t - x_k| < \delta \quad \textit{implies} \quad |g(k,t) - g(k,x_k)| < \varepsilon.$$

Our next step is to determine a necessary and sufficient condition for $P_g: X \to c$ to be continuous on X where X is one of ℓ_p, cs and c_0. It is an immediate consequence of Lemma 6, the relationships of norms mentioned in the beginning of the section entitled "Continuous Superposition Operators into ℓ_∞" and the following lemma.

Lemma 12. *Let Y be a normed subspace of c and X a normed sequence space such that $\ell_1 \subseteq X \subseteq c_0$ and $\|\cdot\|_{\ell_\infty} \le \alpha \|\cdot\|_X$ on X for some $\alpha > 0$. Assume that $P_g: X \to Y$. Then for $x \in X$, if $g(k,\cdot)$ is continuous at x_k for all $k \in \mathbb{N}$, then P_g is continuous at x.*

Proof. Let $x \in X$ and assume that $g(k,\cdot)$ is continuous at x_k for all $k \in \mathbb{N}$. To show that P_g is continuous at x, by Lemma 7, it suffices to show that for any $\varepsilon > 0$, there exists $\delta > 0$ such that for all $k \in \mathbb{N}$, $t \in \mathbb{R}$,

$$|t - x_k| < \delta \quad \text{implies} \quad |g(k,t) - g(k,x_k)| < \varepsilon. \tag{1}$$

Let $\varepsilon > 0$ be given. Since $P_g: X \to Y$, $\ell_1 \subseteq X \subseteq c_0$ and $Y \subseteq c$, by Lemma II.12, $\lim\limits_{k\to\infty} g(k,0)$ exists and $(g(k,\cdot))_{k=1}^{\infty}$ converges continuously at 0 to $\lim\limits_{k\to\infty} g(k,0)$. Let $L = \lim\limits_{k\to\infty} g(k,0)$. Then there exist $\delta > 0$ and $N \in \mathbb{N}$ such that for all $k \in \mathbb{N}$, $t \in \mathbb{R}$,

$$k \ge N \text{ and } |t| < \delta \quad \text{imply} \quad |g(k,t) - L| < \tfrac{\varepsilon}{2}. \tag{2}$$

Since $x \in X \subseteq c_0$, there exists $N' \ge N$ such that $|x_k| < \frac{\delta}{2}$ for all $k \ge N'$. By (2), we have $|g(k,x_k) - L| < \frac{\varepsilon}{2}$ for all $k \ge N'$. Since $g(k,\cdot)$ is continuous at x_k for all

$k \in \{1, \ldots, N'-1\}$, there exists $\delta' \in \mathbb{R}$ with $0 < \delta' \leq \frac{\delta}{2}$ such that for all $k \in \{1, \ldots, N'-1\}$, $t \in \mathbb{R}$,

$$|t - x_k| < \delta' \text{ implies } |g(k,t) - g(k,x_k)| < \varepsilon. \tag{3}$$

Let $k \in \mathbb{N}$ and $t \in \mathbb{R}$ be such that $|t - x_k| < \delta'$. If $k \geq N'$, then $|g(k,x_k) - L| < \frac{\varepsilon}{2}$ and $|t| \leq |t - x_k| + |x_k| < \delta' + \frac{\delta}{2} \leq \frac{\delta}{2} + \frac{\delta}{2} = \delta$, so by (2), we have that $|g(k,t) - L| < \frac{\varepsilon}{2}$ and thus $|g(k,t) - g(k,x_k)| \leq |g(k,t) - L| + |L - g(k,x_k)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. If $k \in \{1, \ldots, N'-1\}$, then by (3), $|g(k,t) - g(k,x_k)| < \varepsilon$. This proves that (1) holds, as required. □

Theorem 13. *If X is one of the sequence spaces ℓ_p, cs and c_0 and $P_g : X \to c$, then the following statements hold:*

(i) For $x \in X$, P_g is continuous at x if and only if $g(k,\cdot)$ is continuous at x_k for all $k \in \mathbb{N}$.

(ii) P_g is continuous on X if and only if $g(k,\cdot)$ is coniinuous on $\mathbb{R}$ for all $k \in \mathbb{N}$.

To study the continuity of $P_g : c \to c$, we prove the following lemma. The lemma yields the next theorem directly.

Lemma 14. *Let Y be a normed subspace of c and assume that $P_g : c \to Y$. Then for $x \in c$, P_g is continuous at x if and only if $g(k,\cdot)$ is continuous at x_k for all $k \in \mathbb{N}$.*

Proof. It follows from Lemma 6 that if P_g is continuous at $x \in c$, then $g(k,\cdot)$ is continuous at x_k for all $k \in \mathbb{N}$ since $\|e^{(n)}\|_c = 1$ for all $n \in \mathbb{N}$ and $\|\cdot\|_{\ell_\infty} = \|\cdot\|_Y$ on Y.

Conversely, let $x \in c$ and assume that $g(k,\cdot)$ is continuous at x_k for all $k \in \mathbb{N}$. Since $x \in c \subseteq \ell_\infty$, there exists $\alpha > 0$ such that $|x_k| \leq \alpha$ for all $k \in \mathbb{N}$. Since

$P_g: c \to Y$ and $Y \subseteq c$, by Theorem II.14, we have $\lim_{k\to\infty} g(k,t)$ exists for all $t \in \mathbb{R}$, $h \equiv \lim_{k\to\infty} g(k,\cdot)$ is continuous on $\mathbb{R}$ and $(g(k,\cdot))_{k=1}^{\infty}$ converges uniformly to h on $[-2\alpha, 2\alpha]$. Since h is continuous on $\mathbb{R}$ and $[-2\alpha, 2\alpha]$ is a compact subset of $\mathbb{R}$, h is uniformly continuous on $[-2\alpha, 2\alpha]$. To prove that P_g is continuous at x, by Lemma 7, it is enough to show that for any $\varepsilon > 0$, there exists $\delta > 0$ such that for all $k \in \mathbb{N}$, $t \in \mathbb{R}$,

$$|t - x_k| < \delta \quad \text{implies} \quad |g(k,t) - g(k,x_k)| < \varepsilon. \tag{1}$$

Let $\varepsilon > 0$ be given. Since $(g(k,\cdot))_{k=1}^{\infty}$ converges uniformly to h on $[-2\alpha, 2\alpha]$, there exists $N \in \mathbb{N}$ such that

$$|g(k,t) - h(t)| < \tfrac{\varepsilon}{3} \quad \text{for all } k \geq N \text{ and } t \in [-2\alpha, 2\alpha]. \tag{2}$$

Then $|g(k,x_k) - h(x_k)| < \frac{\varepsilon}{3}$ for all $k \geq N$ because $|x_k| \leq \alpha$ for all $k \in \mathbb{N}$. By uniform continuity of h on $[-2\alpha, 2\alpha]$, there exists $\delta \in \mathbb{R}$ with $0 < \delta \leq \alpha$ such that for all $t_1, t_2 \in [-2\alpha, 2\alpha]$,

$$|t_1 - t_2| < \delta \quad \text{implies} \quad |h(t_1) - h(t_2)| < \tfrac{\varepsilon}{3}. \tag{3}$$

Since $g(k,\cdot)$ is continuous at x_k for all $k \in \{1, \ldots, N-1\}$, there exists $\delta' \in \mathbb{R}$ with $0 < \delta' \leq \delta$ such that for all $k \in \{1, \ldots, N-1\}$, $t \in \mathbb{R}$,

$$|t - x_k| < \delta' \quad \text{implies} \quad |g(k,t) - g(k,x_k)| < \varepsilon. \tag{4}$$

Let $k \in \mathbb{N}$ and $t \in \mathbb{R}$ be such that $|t - x_k| < \delta'$. Then $|t| \leq |t - x_k| + |x_k| < \delta' + \alpha \leq \alpha + \alpha = 2\alpha$, so by (3), we have $|h(t) - h(x_k)| < \frac{\varepsilon}{3}$. If $k \geq N$, then by (2), we have that $|g(k,t) - h(t)| < \frac{\varepsilon}{3}$ and hence $|g(k,t) - g(k,x_k)| \leq |g(k,t) - h(t)| + |h(t) - h(x_k)| + |h(x_k) - g(k,x_k)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$. If $k \in \{1, \ldots, N-1\}$, then by (4), we have $|g(k,t) - g(k,x_k)| < \varepsilon$. This proves that (1) holds. □

Theorem 15. *If $P_g: c \to c$, then the following statements hold:*

(*i*) *For $x \in c$, P_g is continuous at x if and only if $g(k,\cdot)$ is continuous at x_k for all $k \in \mathbb{N}$.*

(*ii*) *P_g is continuous on c if and only if $g(k,\cdot)$ is continuous on $\mathbb{R}$ for all $k \in \mathbb{N}$.*

With the help of the statements mentioned at the beginning of the section entitled "Continuous Superposition Operators into ℓ_∞", Lemma 6 and the next lemma characterize when P_g is continuous on X where X is any one of bs and ℓ_∞ and $P_g: X \to c$.

Lemma 16. *Let Y be a normed subspace of c and X a normed sequence space such that $bs \subseteq X \subseteq \ell_\infty$ and $\|\cdot\|_{\ell_\infty} \le \alpha\|\cdot\|_X$ on X for some $\alpha > 0$. Assume that $P_g: X \to Y$. Then for $x \in X$, if $g(k,\cdot)$ is continuous at x_k for all $k \in \mathbb{N}$, then P_g is continuous at x.*

Proof. Let $x \in X$ and assume that $g(k,\cdot)$ is continuous at x_k for all $k \in \mathbb{N}$. To show that P_g is continuous at x, by Lemma 7, it suffices to show that for any $\varepsilon > 0$, there exists $\delta > 0$ such that for all $k \in \mathbb{N}$, $t \in \mathbb{R}$,

$$|t - x_k| < \delta \quad \text{implies} \quad |g(k,t) - g(k,x_k)| < \varepsilon. \tag{1}$$

Let $\varepsilon > 0$ be given. Since $x \in X \subseteq \ell_\infty$, there exists $\beta > 0$ such that $|x_k| \le \beta$ for all $k \in \mathbb{N}$. Since $P_g: X \to Y$, $bs \subseteq X \subseteq \ell_\infty$ and $Y \subseteq c$, we have $P_g: bs \to c$. Then by Theorem II.18, we have that $\lim\limits_{k\to\infty} g(k,0)$ exists and $(g(k,\cdot))_{k=1}^{\infty}$ converges uniformly to $\lim\limits_{k\to\infty} g(k,0)$ on $[-2\beta, 2\beta]$. Let $L = \lim\limits_{k\to\infty} g(k,0)$. Then there exists $N \in \mathbb{N}$ such that for all $k \in \mathbb{N}$, $t \in \mathbb{R}$,

$$|g(k,t) - L| < \frac{\varepsilon}{2} \quad \text{for all } k \ge N \text{ and } t \in [-2\beta, 2\beta]. \tag{2}$$

Since $|x_k| \leq \beta$ for all $k \in \mathbb{N}$, by (2) we have $|g(k,x_k)-L| < \frac{\varepsilon}{2}$ for all $k \geq N$. Since $g(k,\cdot)$ is continuous at x_k for all $k \in \{1,\ldots,N-1\}$, there exists $\delta \in \mathbb{R}$ with $0 < \delta \leq \beta$ such that for all $k \in \{1,\ldots,N-1\}$, $t \in \mathbb{R}$,

$$|t-x_k| < \delta \quad \text{implies} \quad |g(k,t)-g(k,x_k)| < \varepsilon. \tag{3}$$

Let $k \in \mathbb{N}$ and $t \in \mathbb{R}$ be such that $|t-x_k| < \delta$. Then $|t| \leq |t-x_k| + |x_k| < \beta + \beta \leq 2\beta$. If $k \geq N$, then by (2), we have that $|g(k,t)-L| < \frac{\varepsilon}{2}$ and thus $|g(k,t)-g(k,x_k)| \leq |g(k,t)-L| + |L-g(k,x_k)| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon$. If $k \in \{1,\ldots,N-1\}$, then by (3) $|g(k,t)-g(k,x_k)| < \varepsilon$. This proves that (1) holds. □

Theorem 17. *If X is one of the sequence spaces bs and ℓ_∞ and $P_g : X \to c$, then the following statements hold:*

(i) For $x \in X$, P_g is continuous at x if and only if $g(k,\cdot)$ is continuous at x_k for all $k \in \mathbb{N}$.

(ii) P_g is continuous on X if and only if $g(k,\cdot)$ is continuous on $\mathbb{R}$ for all $k \in \mathbb{N}$.

The last theorem of this section is a special case of the following lemma.

Lemma 18. *Let Y be a normed subspace of c and assume that $P_g : \omega \to Y$. Then for $x \in \omega$, P_g is continuous at x if and only if $g(k,\cdot)$ is continuous at x_k for all $k \in \mathbb{N}$.*

Proof. According to Lemma 9, we have that if P_g is continuous at $x \in \omega$, then $g(k,\cdot)$ is continuous at x_k for all $k \in \mathbb{N}$.

Conversely, let $x \in \omega$ assume that $g(k,\cdot)$ is continuous at x_k for all $k \in \mathbb{N}$. To show that P_g is continuous at x, by Lemma 9, it suffices to show that for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|g(k,t)-g(k,x_k)| < \varepsilon \quad \text{for all } k \geq N \text{ and } t \in \mathbb{R}.$$

Let $\varepsilon > 0$ be given. Since $P_g: \omega \to Y$ and $Y \subseteq c$, $P_g: \omega \to c$. Then by Theorem II.19, we have that $\lim_{k\to\infty} g(k,0)$ exists and $(g(k,\cdot))_{k=1}^{\infty}$ converges uniformly to $\lim_{k\to\infty} g(k,0)$ on $\mathbb{R}$. Let $L = \lim_{k\to\infty} g(k,0)$. Then there exists $N \in \mathbb{N}$ such that

$$|g(k,t)-L| < \frac{\varepsilon}{2} \quad \text{for all } k \geq N \text{ and } t \in \mathbb{R}. \qquad (*)$$

In particular, $|g(k,x_k)-L| < \frac{\varepsilon}{2}$ for all $k \geq N$. By this inequality and (*), we have that $|g(k,t)-g(k,x_k)| \leq \varepsilon$ for all $k \geq N$ and $t \in \mathbb{R}$. □

Theorem 19. *If $P_g: \omega \to c$, the following statements hold:*

(i) For $x \in \omega$, P_g is continuous at x if and only if $g(k,\cdot)$ is continuous at x_k for all $k \in \mathbb{N}$.

(ii) P_g is continuous on ω if and only if $g(k,\cdot)$ is continuous on $\mathbb{R}$ for all $k \in \mathbb{N}$.

Continuous Superposition Operators into c_0

The first theorem follows directly from Lemma 6 and Lemma 7.

Theorem 20. *If $P_g: \Phi \to c_0$, then for $x \in \Phi$, P_g is continuous at x if and only if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for all $k \in \mathbb{N}$, $t \in \mathbb{R}$,*

$$|t - x_k| < \delta \quad \textit{implies} \quad |g(k,t)-g(k,x_k)| < \varepsilon.$$

Lemma 6, Lemma 12, Lemma 14, Lemma 16 and Lemma 18 lead us to have the following theorem.

Theorem 21. *If X is one of the sequence spaces ℓ_p, cs, c_0, c, bs, ℓ_∞ and ω and $P_g: X \to c_0$, then the following statements hold:*

(*i*) *For $x \in X$, P_g is continuous at x if and only if $g(k,\cdot)$ is continuous at x_k for all $k \in \mathbb{N}$.*

(*ii*) *P_g is continuous on X if and only if $g(k,\cdot)$ is continuous on $\mathbb{R}$ for all $k \in \mathbb{N}$.*

Proof. By Lemma 14 and Lemma 18, the theorem holds for the case that $X = c$ or ω.

Recall that if X is any one of ℓ_p, cs, c_0, bs and ℓ_∞, then $\|e^{(n)}\|_X = 1$ for all $n \in \mathbb{N}$. Then by Lemma 6 we have that for $X = \ell_p$, cs, c_0, bs and ℓ_∞, if P_g is continuous at $x \in X$, then $g(k,\cdot)$ is continuous at x_k for all $k \in \mathbb{N}$. The converse is true by Lemma 12 and Lemma 16 since $\|\cdot\|_{\ell_\infty} \le \|\cdot\|_{\ell_p}$ on ℓ_p, $\|\cdot\|_{\ell_\infty} \le 2\|\cdot\|_{cs}$ on cs, $\|\cdot\|_{\ell_\infty} = \|\cdot\|_{c_0}$ on c_0 and $\|\cdot\|_{\ell_\infty} \le 2\|\cdot\|_{bs}$ on bs. □

Continuous Superposition Operators into ℓ_p

We first prove a lemma. It is useful for the next two theorems in characterizing when $P_g: X \to \ell_p$ is continuous on X where X is any one of Φ, cs and c_0.

Lemma 22. *Let X be a normed sequence space such that $cs \subseteq X \subseteq c_0$ and $\|\cdot\|_{\ell_\infty} \le \alpha\|\cdot\|_X$ on X for some $\alpha > 0$. Assume that $P_g: X \to \ell_p$. Then for $x \in X$, if $g(k,\cdot)$ is continuous at x_k for all $k \in \mathbb{N}$, then P_g is continuous at x.*

Proof. Let $x \in X$ and assume that $g(k,\cdot)$ is continuous at x_k for all $k \in \mathbb{N}$. Since $P_g: X \to \ell_p$ and $cs \subseteq X$, we have $P_g: cs \to \ell_p$. By Theorem II.29, there exist $\beta > 0$ and $m_1 \in \mathbb{N}$ such that $\sum_{k=m_1}^{\infty} \sup_{t \in [-\beta,\beta]} |g(k,t)|^p < \infty$. Since $x \in X \subseteq c_0$, there exists $m_2 \ge m_1$

such that $|x_k| < \frac{\beta}{2}$ for all $k \geq m_2$. To show that P_g is continuous at x, let $\varepsilon > 0$ be given. Since $\sum_{k=m_1}^{\infty} \sup_{t\in[-\beta,\beta]} |g(k,t)|^p < \infty$, $\lim_{n\to\infty} \sum_{k=n}^{\infty} \sup_{t\in[-\beta,\beta]} |g(k,t)|^p = 0$. Then there exists $N \geq m_2$ such that

$$\sum_{k=N}^{\infty} \sup_{t\in[-\beta,\beta]} |g(k,t)|^p \ < \ \frac{\varepsilon^p}{2^{p+1}}. \tag{1}$$

Since $g(k,\cdot)$ is continuous at x_k for all $k \in \{1,\ldots,N-1\}$, there exists $\delta \in \mathbb{R}$ with $0 < \delta \leq \frac{\beta}{2}$ such that for all $k \in \{1,\ldots,N-1\}$, $t \in \mathbb{R}$,

$$|t - x_k| < \delta \quad \text{implies} \quad |g(k,t) - g(k,x_k)| < \left(\frac{\varepsilon^p}{2N}\right)^{1/p}. \tag{2}$$

Let $z \in X$ be such that $\|z-x\|_X < \frac{\delta}{\alpha}$. Then $|z_k - x_k| \leq \|z-x\|_{\ell_\infty} \leq \alpha\|z-x\|_X < \delta$ for all $k \in \mathbb{N}$. By (2), $|g(k,z_k) - g(k,x_k)|^p < \frac{\varepsilon^p}{2N}$ for all $k \in \{1,\ldots,N-1\}$. For $k \geq N$, we have that $|z_k| \leq |z_k - x_k| + |x_k| < \delta + \frac{\beta}{2} \leq \frac{\beta}{2} + \frac{\beta}{2} = \beta$. Then for $k \geq N$,

$$\begin{aligned}
|g(k,z_k) - g(k,x_k)|^p &\leq (|g(k,z_k)| + |g(k,x_k)|)^p \\
&\leq 2^p \max\{|g(k,z_k)|^p, |g(k,x_k)|^p\} \\
&\leq 2^p \sup_{t\in[-\beta,\beta]} |g(k,t)|^p.
\end{aligned}$$

By (1), we have

$$\begin{aligned}
\sum_{k=N}^{\infty} |g(k,z_k) - g(k,x_k)|^p &\leq 2^p \sum_{k=N}^{\infty} \sup_{t\in[-\beta,\beta]} |g(k,t)|^p \\
&< 2^p \frac{\varepsilon^p}{2^{p+1}} = \frac{\varepsilon^p}{2}.
\end{aligned}$$

Hence

$$\begin{aligned}
\sum_{k=1}^{\infty} |g(k,z_k) - g(k,x_k)|^p &= \sum_{k=1}^{N-1} |g(k,z_k) - g(k,x_k)|^p + \sum_{k=N}^{\infty} |g(k,z_k) - g(k,x_k)|^p \\
&< (N-1)\frac{\varepsilon^p}{2N} + \frac{\varepsilon^p}{2} < \varepsilon^p.
\end{aligned}$$

This implies that $\|P_g(z) - P_g(x)\|_{\ell_p} = \left(\sum_{k=1}^{\infty} |g(k,z_k) - g(k,x_k)|^p\right)^{1/p} < \varepsilon$. □

Theorem 23. *If $P_g\colon \Phi \to \ell_p$, then for $x \in \Phi$, P_g is continuous at x if and only if*

(i) $g(k,\cdot)$ is continuous at x_k for all $k \in \mathbb{N}$ and

(*ii*) *there exist a neighborhood* V_0 *of* 0 *and* $N \in \mathbb{N}$ *such that* $\sum_{k=N}^{\infty} \sup_{t \in V_0} |g(k,t)|^p < \infty$.

Proof. Let $x = \sum_{k=1}^{m} x_k e^{(k)} \in \Phi$. Assume that P_g is continuous at x. By Lemma 6, $g(k,\cdot)$ is continuous at x_k for all $k \in \mathbb{N}$. Since P_g is continuous at x, there exists $\delta > 0$ such that for each $z \in \Phi$,

$$\|z - x\|_{\Phi} < \delta \quad \text{implies} \quad \|P_g(z) - P_g(x)\|_{\ell_p} \leq 1. \tag{1}$$

Claim that

$$\sum_{k=m+1}^{\infty} \sup_{t \in [-\delta,\delta]} |g(k,t) - g(k,0)|^p \leq 1. \tag{2}$$

To prove (2), let $n \geq m+1$ and $t_{m+1}, \ldots, t_n \in [-\delta, \delta]$. Set $u = \sum_{k=m+1}^{n} t_k e^{(k)} + x$. Then $\|u - x\|_{\Phi} = \| \sum_{k=m+1}^{n} t_k e^{(k)} \|_{\Phi} \leq \delta$, so by (1), $\sum_{k=m+1}^{n} |g(k,t_k) - g(k,0)|^p \leq (\|P_g(u) - P_g(x)\|_{\ell_p})^p \leq 1$. This implies that $\sup \{ \sum_{k=m+1}^{n} |g(k,t_k) - g(k,0)|^p \mid t_{m+1}, \ldots, t_n \in [-\delta,\delta] \} \leq 1$ for all $n > m$. But $\sum_{k=m+1}^{n} \sup_{t \in [-\delta,\delta]} |g(k,t) - g(k,0)|^p = \sup \{ \sum_{k=m+1}^{n} |g(k,t_k) - g(k,0)|^p \mid t_{m+1}, \ldots, t_n \in [-\delta,\delta] \}$ for all $n > m$, so $\sum_{k=m+1}^{\infty} \sup_{t \in [-\delta,\delta]} |g(k,t) - g(k,0)|^p \leq 1$. Next, to show that there exist a neighborhood V_0 of 0 and $N \in \mathbb{N}$ such that $\sum_{k=N}^{\infty} \sup_{t \in V_0} |g(k,t)|^p < \infty$, by Theorem II.27, it is equivalent to show that $P_g : c_0 \to \ell_p$. Let $z \in c_0$. Then there exists $N > m$ such that $|z_k| < \delta$ for all $k \geq N$, so by (2), $\sum_{k=N}^{\infty} |g(k,z_k) - g(k,0)|^p \leq 1$. Since $P_g : \Phi \to \ell_p$ and $(0) \in \Phi$, $\sum_{k=1}^{\infty} |g(k,0)|^p < \infty$. Then for $n \geq N$,

$$\begin{aligned}
\sum_{k=N}^{n} |g(k,z_k)|^p &\leq \sum_{k=N}^{n} (|g(k,z_k) - g(k,0)| + |g(k,0)|)^p \\
&\leq \sum_{k=N}^{n} 2^p \max\{ |g(k,z_k) - g(k,0)|^p, |g(k,0)|^p \} \\
&\leq 2^p \sum_{k=N}^{n} (|g(k,z_k) - g(k,0)|^p + |g(k,0)|^p) \\
&\leq 2^p \left(\sum_{k=N}^{\infty} |g(k,z_k) - g(k,0)|^p + \sum_{k=1}^{\infty} |g(k,0)|^p \right) \\
&\leq 2^p \left(1 + \sum_{k=1}^{\infty} |g(k,0)|^p \right)
\end{aligned}$$

which implies that $\sum_{k=1}^{\infty} |g(k,z_k)|^p < \infty$. Then $P_g(z) = (g(k,z_k))_{k=1}^{\infty} \in \ell_p$. This proves that $P_g : c_0 \to \ell_p$.

Conversely, assume that (i) and (ii) hold. The assumption (ii) implies that $P_g: c_0 \to \ell_p$ by Theorem II.27 and hence P_g is continuous at x by (i) and Lemma 22. Since Φ is a normed subspace of c_0, it follows that P_g is continuous at x under the consideration of $P_g: \Phi \to \ell_p$. □

Theorem 24. *If X is one of the sequence spaces cs and c_0 and $P_g: X \to \ell_p$, then the following statements hold:*

(i) For $x \in X$, P_g is continuous at x if and only if $g(k, \cdot)$ is continuous at x_k for all $k \in \mathbb{N}$.

(ii) P_g is continuous on X if and only if $g(k, \cdot)$ is continuous on $\mathbb{R}$ for all $k \in \mathbb{N}$.

Proof. It follows from Lemma 6 and Lemma 22. □

The next theorem deals with the continuity of $P_g: \ell_q \to \ell_p$.

Theorem 25. *If $P_g: \ell_q \to \ell_p$, then the following statements hold:*

(i) For $x \in \ell_q$, P_g is continuous at x if and only if $g(k, \cdot)$ is continuous at x_k for all $k \in \mathbb{N}$.

(ii) P_g is continuous on ℓ_q if and only if $g(k, \cdot)$ is continuous on $\mathbb{R}$ for all $k \in \mathbb{N}$.

Proof. let $x \in \ell_q$. By Lemma 6, we have that if P_g is continuous at x, then $g(k, \cdot)$ is continuous at x_k for all $k \in \mathbb{N}$. To show the converse, assume that $g(k, \cdot)$ is continuous at x_k for all $k \in \mathbb{N}$. To show that P_g is continuous at x, given $\varepsilon > 0$. Since $P_g: \ell_q \to \ell_p$, by Theorem II.26, there exist $\alpha > 0$, $\beta > 0$, $m \in \mathbb{N}$ and $(c_k)_{k=1}^{\infty} \in \ell_1$ such that

$$|g(k, t)|^p \quad \leq \quad c_k + \beta |t|^q \quad \text{for all } k \geq m \text{ and } t \in [-\alpha, \alpha]. \qquad (1)$$

Since $x \in \ell_q$ and $(c_k)_{k=1}^{\infty} \in \ell_1$ we have that $\lim_{k\to\infty} x_k$, $\lim_{n\to\infty} \sum_{k=n}^{\infty} |x_k|^q$ and $\lim_{n\to\infty} \sum_{k=n}^{\infty} c_k$ are all 0. Then there exists $N \geq m$ such that

$$|x_k| \leq \alpha \quad \text{for all } k \geq N, \tag{2}$$

$$\sum_{k=N}^{\infty} |x_k|^q \leq \min\{(\frac{\alpha}{2})^q, \frac{1}{2^q}(\frac{\varepsilon^p}{\beta 2^{p+3}}), \frac{\varepsilon^p}{\beta 2^{p+3}}\} \quad \text{and} \tag{3}$$

$$\sum_{k=N}^{\infty} c_k \leq \frac{\varepsilon^p}{2^{p+1}}. \tag{4}$$

Then by (1) and (2), $|g(k,x_k)|^p \leq c_k + \beta |x_k|^q$ for all $k \geq N$. Since $g(k,\cdot)$ is continuous at x_k for all $k \in \{1,\ldots,N-1\}$, there exists $\delta \in \mathbb{R}$ with $0 < \delta \leq \min\{\frac{\alpha}{2}, \frac{1}{2}(\frac{\varepsilon^p}{\beta 2^{p+3}})^{1/q}\}$ such that for all $k \in \{1,\ldots,N-1\}$, $t \in \mathbb{R}$,

$$|t - x_k| < \delta \quad \text{implies} \quad |g(k,t) - g(k,x_k)| < (\frac{\varepsilon^p}{2N})^{1/p}. \tag{5}$$

Let $z \in \ell_q$ be such that $\|z - x\|_{\ell_q} < \delta$. Then $\|(z_k)_{k=N}^{\infty} - (x_k)_{k=N}^{\infty}\|_{\ell_q} \leq \|z - x\|_{\ell_q} < \delta$, so for $k \geq N$,

$$\begin{aligned} |z_k| &\leq \|(z_k)_{k=N}^{\infty}\|_{\ell_q} \\ &\leq \|(z_k)_{k=N}^{\infty} - (x_k)_{k=N}^{\infty}\|_{\ell_q} + \|(x_k)_{k=N}^{\infty}\|_{\ell_q} \\ &< \delta + \frac{\alpha}{2} \quad \text{(by (3))} \\ &\leq \frac{\alpha}{2} + \frac{\alpha}{2} = \alpha \end{aligned} \tag{6}$$

and

$$\begin{aligned} \|(z_k)_{k=N}^{\infty}\|_{\ell_q} &\leq \|(z_k)_{k=N}^{\infty} - (x_k)_{k=N}^{\infty}\|_{\ell_q} + \|(x_k)_{k=N}^{\infty}\|_{\ell_q} \\ &< \delta + \frac{1}{2}(\frac{\varepsilon^p}{\beta 2^{p+3}})^{1/q} \quad \text{(by (3))} \\ &\leq \frac{1}{2}(\frac{\varepsilon^p}{\beta 2^{p+3}})^{1/q} + \frac{1}{2}(\frac{\varepsilon^p}{\beta 2^{p+3}})^{1/q} \\ &= (\frac{\varepsilon^p}{\beta 2^{p+3}})^{1/q}. \end{aligned}$$

Then $\sum_{k=N}^{\infty} |z_k|^q = (\|(z_k)_{k=N}^{\infty}\|_{\ell_q})^q < \frac{\varepsilon^p}{\beta 2^{p+3}}$. By (1) and (6), we have that for $k \geq N$, $|g(k,z_k)|^p \leq c_k + \beta |z_k|^q$ and thus

$$\begin{aligned} |g(k,z_k) - g(k,x_k)|^p &\leq (|g(k,z_k)| + |g(k,x_k)|)^p \\ &\leq 2^p \max\{|g(k,z_k)|^p, |g(k,x_k)|^p\} \\ &\leq 2^p (|g(k,z_k)|^p + |g(k,x_k)|^p) \end{aligned}$$

$$\leq \quad 2^p\,(2c_k + \beta|x_k|^q + \beta|z_k|^q).$$

It then follows that

$$\begin{aligned}
\sum_{k=N}^{\infty}|g(k,z_k)-g(k,x_k)|^p &\leq 2^{p+1}\sum_{k=N}^{\infty}c_k + 2^p\beta\sum_{k=N}^{\infty}|x_k|^q + 2^p\beta\sum_{k=N}^{\infty}|z_k|^q \\
&< 2^{p+1}\frac{\varepsilon^p}{2^{p+3}}+2^p\beta\frac{\varepsilon^p}{\beta 2^{p+3}}+2^p\beta\frac{\varepsilon^p}{\beta 2^{p+3}} \quad \text{(by (3) and (4))} \\
&= \frac{\varepsilon^p}{2}.
\end{aligned}$$

For all $k\in\{1,\ldots,N-1\}$, we have that $|z_k-x_k|\leq\|z-x\|_{\ell_q}<\delta$ and thus by (5), $|g(k,z_k)-g(k,x_k)|^p<\frac{\varepsilon^p}{2N}$. Then

$$\begin{aligned}
\sum_{k=1}^{\infty}|g(k,z_k)-g(k,x_k)|^p &= \sum_{k=1}^{N-1}|g(k,z_k)-g(k,x_k)|^p+\sum_{k=N}^{\infty}|g(k,z_k)-g(k,x_k)|^p \\
&< (N-1)\frac{\varepsilon^p}{2N}+\frac{\varepsilon^p}{2} \quad < \quad \varepsilon^p.
\end{aligned}$$

Hence $\|P_g(z)-P_g(x)\|_{\ell_p}=\left(\sum_{k=1}^{\infty}|g(k,z_k)-g(k,x_k)|^p\right)^{1/p}<\varepsilon.$ □

The next theroem is obtained from Lemma 6 and the following lemma.

Lemma 26. *Let X be a normed sequence space such that $X\subseteq\ell_\infty$ and X contains c or bs and $\|\cdot\|_{\ell_\infty}\leq\alpha\|\cdot\|_X$ on X for some $\alpha>0$. Assume that $P_g:X\to\ell_p$. Then for $x\in X$, if $g(k,\cdot)$ is continuous at x_k for all $k\in\mathbb{N}$, then P_g is continuous at x.*

Proof. Let $x\in X$ and assume that $g(k,\cdot)$ is continuous at x_k for all $k\in\mathbb{N}$. Since $x\in X\subseteq\ell_\infty$, there exists $\beta\geq 0$ such that $|x_k|\leq\beta$ for all $k\in\mathbb{N}$. Since $P_g:X\to\ell_p$ and X contain c or bs, by Theorem II.30, there exists $m\in\mathbb{N}$ such that $\sum_{k=m}^{\infty}\sup_{t\in[-2\beta,2\beta]}|g(k,t)|^p<\infty$. To show that P_g is continuous at x, let $\varepsilon>0$ be given. Since $\sum_{k=m}^{\infty}\sup_{t\in[-2\beta,2\beta]}|g(k,t)|^p<\infty$, $\lim_{n\to\infty}\sum_{k=n}^{\infty}\sup_{t\in[-2\beta,2\beta]}|g(k,t)|^p=0$. Then there exists $N\geq m$ such that

$$\sum_{k=N}^{\infty}\sup_{t\in[-2\beta,2\beta]}|g(k,t)|^p < \frac{\varepsilon^p}{2^{p+1}}. \tag{1}$$

Since $g(k,\cdot)$ is continuous at x_k for all $k\in\{1,\ldots,N-1\}$, there exists $\delta\in\mathbb{R}$ with $0<\delta\leq\beta$ such that for all $k\in\{1,\ldots,N-1\}$, $t\in\mathbb{R}$,

$$|t-x_k|<\delta \quad \text{implies} \quad |g(k,t)-g(k,x_k)|<\left(\tfrac{\varepsilon^p}{2N}\right)^{1/p}. \qquad (2)$$

Let $z\in X$ be such that $\|z-x\|_X<\frac{\delta}{\alpha}$. Then $|z_k-x_k|\leq\|z-x\|_{\ell_\infty}\leq\alpha\|z-x\|_X<\delta$ for all $k\in\mathbb{N}$. By (2), $|g(k,z_k)-g(k,x_k)|^p<\frac{\varepsilon^p}{2N}$ for all $k\in\{1,\ldots,N-1\}$. For $k\in\mathbb{N}$, we have that $|z_k|\leq|z_k-x_k|+|x_k|<\delta+\beta\leq\beta+\beta=2\beta$. Then for $k\geq N$,

$$\begin{aligned}
|g(k,z_k)-g(k,x_k)|^p &\leq (|g(k,z_k)|+|g(k,x_k)|)^p\\
&\leq 2^p\max\{|g(k,z_k)|^p,|g(k,x_k)|^p\}\\
&\leq 2^p\sup_{t\in[-2\beta,2\beta]}|g(k,t)|^p. \qquad (3)
\end{aligned}$$

By (1) and (3), we have that $\sum\limits_{k=N}^{\infty}|g(k,z_k)-g(k,x_k)|^p<\frac{\varepsilon^p}{2}$. Therefore, we have that

$$\begin{aligned}
\sum_{k=1}^{\infty}|g(k,z_k)-g(k,x_k)|^p &= \sum_{k=1}^{N-1}|g(k,z_k)-g(k,x_k)|^p+\sum_{k=N}^{\infty}|g(k,z_k)-g(k,x_k)|^p\\
&< (N-1)\frac{\varepsilon^p}{2N}+\frac{\varepsilon^p}{2} \quad < \quad \varepsilon^p,
\end{aligned}$$

so $\|P_g(z)-P_g(x)\|_{\ell_p}=\left(\sum\limits_{k=1}^{\infty}|g(k,z_k)-g(k,x_k)|^p\right)^{1/p}<\varepsilon.$ □

Theorem 27. *If X is one of the sequence spaces c, bs and ℓ_∞ and $P_g:X\to\ell_p$, then the following statements hold:*

(i) For $x\in X$, P_g is continuous at x if and only if $g(k,\cdot)$ is continuous at x_k for all $k\in\mathbb{N}$.

(ii) P_g is continuous on X if and only if $g(k,\cdot)$ is continuous on $\mathbb{R}$ for all $k\in\mathbb{N}$.

The last theorem of this section characterizes when $P_g:X\to\ell_p$ is continuous on ω.

Theorem 28. *If $P_g: \omega \to \ell_p$, then the following statements hold:*

(i) For $x \in \omega$, P_g is continuous at x if and only if $g(k,\cdot)$ is continuous at x_k for all $k \in \mathbb{N}$.

(ii) P_g is continuous on ω if and only if $g(k,\cdot)$ is continuous on $\mathbb{R}$ for all $k \in \mathbb{N}$.

Proof. Let $x \in \omega$. By Lemma 4, if P_g is continuous at x, then $g(k,\cdot)$ is continuous at x_k for all $k \in \mathbb{N}$. To prove the converse, assume that $g(k,\cdot)$ is continuous at x_k for all $k \in \mathbb{N}$. To show that P_g is continuous at x, given $\varepsilon > 0$. Since $P_g: \omega \to \ell_p$, by Theorem II.31, there exists $m \in \mathbb{N}$ such that $\sum_{k=m}^{\infty} \sup_{t\in\mathbb{R}} |g(k,t)|^p < \infty$. Then $\lim_{n\to\infty} \sum_{k=n}^{\infty} \sup_{t\in\mathbb{R}} |g(k,t)|^p = 0$, so there exists $N \geq m$ such that

$$\sum_{k=N}^{\infty} \sup_{t\in\mathbb{R}} |g(k,t)|^p < \frac{\varepsilon^p}{2^{p+1}}. \tag{1}$$

Since $g(k,\cdot)$ is continuous at x_k for all $k \in \{1,\dots,N-1\}$, there exists $\delta > 0$ such that for all $k \in \{1,\dots,N-1\}$, $t \in \mathbb{R}$,

$$|t - x_k| < \delta \quad \text{implies} \quad |g(k,t) - g(k,x_k)| < \left(\frac{\varepsilon^p}{2N}\right)^{1/p}. \tag{2}$$

Let $\delta' = \min\{\frac{1}{2^{N+1}}, \frac{\delta}{2^N(1+\delta)}\}$. Then $\frac{2^N\delta'}{(1-2^N\delta')} \leq \delta$. Let $z \in \omega$ be such that $d_\omega(z,x) < \delta'$. For $k \geq N$, we have that

$$\begin{aligned} |g(k,z_k) - g(k,x_k)|^p &\leq (|g(k,z_k)| + |g(k,x_k)|)^p \\ &\leq 2^p \max\{|g(k,z_k)|^p, |g(k,x_k)|^p\} \\ &\leq 2^p \sup_{t\in\mathbb{R}} |g(k,t)|^p. \end{aligned} \tag{3}$$

Then inequalities (1) and (3) imply that $\sum_{k=N}^{\infty} |g(k,z_k) - g(k,x_k)|^p < \frac{\varepsilon^p}{2}$. For $k \in \mathbb{N}$, we have that $\frac{|z_k - x_k|}{2^k(1 + |z_k - x_k|)} \leq d_\omega(z,x) < \delta'$. Then for $k \in \{1,\dots,N-1\}$, $\frac{|z_k - x_k|}{1 + |z_k - x_k|} < 2^k\delta' \leq 2^N\delta'$ which implies that $|z_k - x_k| \leq \frac{2^N\delta'}{(1-2^N\delta')} \leq \delta$ for all $k \in \{1,\dots,N-1\}$. By (2), $|g(k,z_k) - g(k,x_k)| < \frac{\varepsilon^p}{2N}$ for all $k \in \{1,\dots,N-1\}$. Hence

$$\sum_{k=1}^{\infty} |g(k,z_k)-g(k,x_k)|^p = \sum_{k=1}^{N-1} |g(k,z_k)-g(k,x_k)|^p + \sum_{k=N}^{\infty} |g(k,z_k)-g(k,x_k)|^p$$
$$< (N-1)\frac{\varepsilon^p}{2N} + \frac{\varepsilon^p}{2} < \varepsilon^p,$$

so $\|P_g(z)-P_g(x)\|_{\ell_p} = \left(\sum_{k=1}^{\infty} |g(k,z_k)-g(k,x_k)|^p\right)^{1/p} < \varepsilon.$ □

Continuous Superposition Operators into Φ

The results of this section are analogous to those of the section entitled "Continuous Superposition Operators into c_0".

Theorem 29. *If P_g: $\Phi \to \Phi$, then for $x \in \Phi$, P_g is continuous at x if and only if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for all $k \in \mathbb{N}$, $t \in \mathbb{R}$,*

$$|t-x_k| < \delta \quad \textit{implies} \quad |g(k,t)-g(k,x_k)| < \varepsilon.$$

Proof. It is obtained directly from Lemma 6 and Lemma 7. □

Theorem 30. *If X is one of the sequence spaces ℓ_p, cs, c_0, c, bs ℓ_∞, and ω and P_g:$X \to \Phi$, then the following statements hold:*

(i) For $x \in X$, P_g is continuous at x if and only if $g(k,\cdot)$ is continuous at x_k for all $k \in \mathbb{N}$.

(ii) P_g is continuous on X if and only if $g(k,\cdot)$ is continuous on $\mathbb{R}$ for all $k \in \mathbb{N}$.

Proof. By referring Lemma 6, Lemma 12, Lemma 14, Lemma 16 and Lemma 18, the proof is given similarly to that of Theorem 21. □

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