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GENERALIZED HYERS-ULAM-RASSIAS STABILITY OF A PENTIC FUNCTIONAL EQUATION

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ในช่วงหลายทศวรรษที่ผ่านมา ได้มีการศึกษาปัญหาเสถียรภาพของสมการเชิงฟังก์ชันหลากหลายประเภทกันอย่างกว้างขวาง วิทยานิพนธ์นี้เราเริ่มด้วยการศึกษาผลเฉลยทั่วไปและเสถียรภาพไฮเออร์-อูแลม-แรสเซียสนัยทั่วไปของสมการเชิงฟังก์ชันกำลังสี่

$$f(3x+y)+f(x+3y)=64f(x)+64f(y)+24f(x+y)-6f(x-y)$$

แล้วขยายแนวคิดไปสู่การศึกษาผลเฉลยทั่วไปและเสถียรภาพไฮเออร์-อูแลม-แรสเซียสนัยทั่วไปของสมการเชิงฟังก์ชันกำลังห้า

$$f(x+5y)-5f(x+4y)+10f(x+3y)-10f(x+2y)+5f(x+y)-f(x)=120f(y)$$

ภาควิชา............คณิตศาสตร์.......... ลายมือชื่อนิสิต..

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During the last decades, the stability problems of several functional equations have been widely studied. In this thesis we first study the general solution and the generalized Hyers-Ulam-Rassias stability of the quartic functional equation

$$f(3x+y)+f(x+3y)=64f(x)+64f(y)+24f(x+y)-6f(x-y)$$

and then extend the idea to investigate the general solution and the generalized Hyers-Ulam-Rassias stability of the pentic functional equation

$$f(x+5y)-5f(x+4y)+10f(x+3y)-10f(x+2y)+5f(x+y)-f(x)=120f(y).$$

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CONTENTS

CHAPTER I

INTRODUCTION

Functional equations are equations in which the unknowns (or unknown) are functions. A function satisfying a functional equation on a given domain is called a *solution* of the equation on that domain. The set of all such solutions is called the *general solution* of the equation [2]. Functional equations have substantially grown to become an important branch of mathematics. Particularly during the last two decades, with its special methods, there are a number of interesting results and several applications [6].

One of the most famous functional equations is the *additive equation*, or the *Cauchy equation*, defined as follows:

$$f(x+y) = f(x) + f(y). \tag{1.1}$$

A function that satisfies the equation (1.1) will be called an *additive function* [2]. First, we will consider some properties of a function $f : \mathbb{R} \to \mathbb{R}$ satisfying the equation (1.1) for all $x, y \in \mathbb{R}$.

Putting $x = y = 0$ in the equation (1.1), we have $f(0) = 0$. Substituting y by $-x$ in the equation (1.1), we obtain $f(-x) = -f(x)$ for all $x \in \mathbb{R}$, i.e. f is an odd function. By a mathematical induction, we can extend the equation (1.1) to the equation

$$f(x_1 + \cdots + x_n) = f(x_1) + \cdots + f(x_n),$$

for all $n \in \mathbb{N}$ and $n \geq 2$, and then substitute x_i by x for all $i = 1, \ldots, n$, it follows directly that $f(nx) = nf(x)$ for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$. If $x = \frac{m}{n}t$ where $n, m \in \mathbb{N}$, then $nx = mt$. We have $f(nx) = f(mt)$, and also $nf(x) = mf(t)$. That is $f(\frac{m}{n}t) = f(x) = \frac{m}{n}f(t)$. Since f is an odd function, we obtain $f(rt) = rf(t)$ for all $r \in \mathbb{Q}, t \in \mathbb{R}$. Let $t = 1$ and $f(1) = c$. Then $f(r) = cr$ for all $r \in \mathbb{Q}$.

Moreover, it was proved [1] that if f is continuous everywhere, then $f(x) = cx$ for all $x \in \mathbb{R}$.

The so-called *quadratic functional equation* is the equation of the form

$$f(x+y) + f(x-y) = 2f(x) + 2f(y). \tag{1.2}$$

We have that the continuous solution to the equation (1.2) is of the form $f(x) = cx^2$ for all $x \in \mathbb{R}$. Moreover, every solution of the equation (1.2) is called a *quadratic function* [6].

The problem of stability originated from the question of S.M. Ulam [15] in 1940. He gave a wide ranging talk before the mathematics club of the University of Wisconsin in which he discussed a number of important unsolved problems. Among those was the question concerning the stability of homomorphism:

Let G_1 be a group and let G_2 be a metric group with a metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality

$$d(h(xy), h(x)h(y)) < \delta$$

for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \to G_2$ with

$$d(h(x), H(x)) < \epsilon$$

for all $x \in G_1$?

In other words, we are looking for situation when the homomorphisms are stable. If we turn our attention to the case of functional equations, we can ask the question: When the solutions of an equation differing slightly from a given one must be close to the true solution of the given equation [5]. In the next year, D.H. Hyers [7] has excellently answered the question of Ulam for the case of approximately additive mapping $f : E \to E'$ where E and E' are Banach spaces:

Let $f : E \to E'$ be a mapping between Banach spaces E, E' such that

$$\|f(x+y) - f(x) - f(y)\| \le \varepsilon \quad \textit{for all } x, y \in E$$

for some $\varepsilon > 0$. *Then there exists exactly one additive mapping* $L : E \to E'$ *such that*

$$\|f(x) - L(x)\| \le \varepsilon$$

for all $x \in E$ *given by the formula* $L(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$, $x \in E$.

In 1978, a generalized version of the theorem of Hyers for approximately linear mapping was given by Th.M. Rassias [14]:

Let $f : E \to E'$ *be a mapping between Banach spaces* E, E' *such that*

$$\|f(x+y) - f(x) - f(y)\| \le \varepsilon\big(\|x\|^p + \|y\|^p\big) \quad \textit{for all } x, y \in E$$

for some $\varepsilon > 0$ *and some* $0 \le p < 1$. *Then there exists exactly one additive mapping* $L : E \to E'$ *such that*

$$\|f(x) - L(x)\| \le \frac{2\varepsilon}{2 - 2^p}\|x\|^p$$

for all $x \in E$ *given by the formula* $L(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$, $x \in E$.

Th.M. Rassias Theorem stimulated several mathematicians working in functional equations to investigated this kind of stability for many important functional equations [6]. Finally, there exists the generalized Hyers-Ulam-Rassias stability which considers the inequality controlled by the function of variables x and y instead of the term $\varepsilon\big(\|x\|^p + \|y\|^p\big)$ in the theorem of Rassias.

During the last decades, the stability problems of several functional equations have been proved by several reseachers (see further [3],[4],[9],[10],[13]). In 2002, I-S Chang, H-M Kim [5] studied the quadratic functional equation

$$f(2x+y) + f(2x-y) = f(x+y) + f(x-y) + 6f(x),$$

which is somewhat different from (1.2), and proved its generalized Hyers-Ulam-Rassias stability. In this year, K-W Jun, H-M Kim [8] studied the general solution and the generalized Hyers-Ulam-Rassias stability of the cubic functional equation

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x).$$

In 2005, S.H. Lee, S.M. Im and I.S. Hwang [11] studied the general solution of the quartic functional equation

$$f(2x+y)+f(2x-y)=4f(x+y)+4f(x-y)+24f(x)-6f(y) \tag{1.3}$$

and proved its stability in the sense of Hyers-Ulam. After that, the generalized Hyers-Ulam-Rassias stability of the quartic functional equation (1.3) was proved by A. Najati [12] in 2008.

In this thesis, we start by using a different approach from S.H. Lee, S.M. Im and I.S. Hwang to study the general solution of the new quartic functional equation

$$f(3x+y)+f(x+3y)=64f(x)+64f(y)+24f(x+y)-6f(x-y) \tag{1.4}$$

and prove its generalized Hyers-Ulam-Rassias stability. Our main result deals with the following new functional equation

$$f(x+5y)-5f(x+4y)+10f(x+3y)-10f(x+2y)+5f(x+y)-f(x)=120f(y) \tag{1.5}$$

which is a pentic functional equation. Also, we study the general solution and the generalized Hyers-Ulam-Rassias stability of the equation (1.5) which are more complicated than those of (1.4).

CHAPTER II

PRELIMINARIES

In this chapter, we collect some relevant definitions and theorems from the book *Functional Equations and Inequalities in Several Variables* by S. Czerwik [6]. The difference operator and its related theorems are given in Section 2.1. In Section 2.2, we give a connection between a polynomial function and an n-additive symmetric function which plays a role in studying the general solution.

2.1 The difference operator

Definition 2.1. Let X and Y be two linear spaces over $\mathbb{R}$, and let $f : X \to Y$ be an arbitrary function. The *difference operator* Δ_x with the span x is defined by

$$\Delta_x f(y) := f(y+x) - f(y) \tag{2.1}$$

for all $x, y \in X$. The iterates Δ_x^s, $s = 0, 1, 2, \ldots$, are defined by the natural recurrence

$$\Delta_x^0 f = f,\ \Delta_x^{s+1} f = \Delta_x \left(\Delta_x^s f\right). \tag{2.2}$$

The superposition of several difference operators will be written shortly

$$\Delta_{x_1 \cdots x_s} f = \Delta_{x_1} \cdots \Delta_{x_s} f, \quad s \in \mathbb{N}. \tag{2.3}$$

Theorem 2.2. *For arbitrary functions $f_1, f_2 : X \to Y$ and for arbitrary constants $\alpha, \beta \in \mathbb{R}$, we have*

$$\Delta_x(\alpha f_1 + \beta f_2) = \alpha \Delta_x f_1 + \beta \Delta_x f_2. \tag{2.4}$$

Note that the set of all above functions $f : X \to Y$ is a real vector space under the operations of ordinary addition and scalar multiplication of functions.

Furthermore, Theorem 2.2 tells us that the difference operator is a linear operator on this set.

Theorem 2.3. *For arbitrary* $x_1, x_2 \in X$ *the operators* $\Delta_{x_1}, \Delta_{x_2}$ *commute:*

$$\Delta_{x_1}\Delta_{x_2}f = \Delta_{x_2}\Delta_{x_1}f. \tag{2.5}$$

By Theorem 2.3, we can see that operator (2.3) is symmetric under the permutation of $x_1, ..., x_s$.

Theorem 2.4. *For arbitrary* $x_1, x_2 \in X$*,*

$$\Delta_{x_1+x_2}f - \Delta_{x_1}f - \Delta_{x_2}f = \Delta_{x_1x_2}f. \tag{2.6}$$

Theorem 2.5. *Let* $s \in \mathbb{N}$*, then*

$$\Delta_x^s f(y) = \sum_{n=0}^{s}(-1)^{s-n}\binom{s}{n}f(y+nx). \tag{2.7}$$

2.2 Polynomial functions

In this thesis we classify some functional equations by using the following definition.

Definition 2.6. Let $s \in \mathbb{N}$. A function $f : X \to Y$ fulfilling the condition

$$\Delta_x^{s+1} f(y) = 0 \tag{2.8}$$

for all $x, y \in X$ is called a *polynomial function of order* s.

Theorem 2.7. *If* $f : X \to Y$ *is a polynomial function of order* s*, then*

$$\Delta_{x_1 \cdots x_{s+1}} f(y) = 0 \tag{2.9}$$

for all $x_1, ..., x_{s+1}, y \in X$.

In order to investigate the general solution of functional equations of polynomial types, we need some properties of n -additive symmetric functions.

Definition 2.8. Suppose that $n \in \mathbb{N}$. A function $A_n : X^n \to Y$ is called n-*additive* if for every r, $1 \leq r \leq n$, and for every $x_1, ..., x_n, y_r \in X$,

$$A_n(x_1, ..., x_{r-1}, x_r + y_r, x_{r+1}, ..., x_n) = A_n(x_1, ..., x_n) + A_n(x_1, ..., x_{r-1}, y_r, x_{r+1}, ..., x_n).$$

That is, A_n is additive with respect to each of its variable $x_r \in X$, $r = 1, ..., n$.

A function A_n is called *symmetric* if

$$A_n(x_1, ..., x_n) = A_n(x_{\pi(1)}, ..., x_{\pi(n)})$$

for every permutation $\{\pi(1), ..., \pi(n)\}$ of $1, 2, ..., n$.

Given a function $A_n : X^n \to Y$, by the *diagonalization* of A_n we understand the function $A^n : X \to Y$ given by the formula

$$A^n(x) := A_n(x, ..., x), \quad x \in X.$$

For convenience, any constant function will be called a 0-additive function.

Theorem 2.9. *Let $A_n : X^n \to Y$ be a symmetric n-additive function and $A^n : X \to Y$ be the diagonalization of A_n. Then, for every $m \geq n$ and for every $x_1, ..., x_m, y \in X$, we have*

$$\Delta_{x_1 \ldots x_m} A^n(y) = \begin{cases} n! A_n(x_1, ..., x_n) & \text{if } m = n \\ 0 & \text{if } m > n. \end{cases} \tag{2.10}$$

Example 2.10. Let $A_2 : \mathbb{R}^2 \to \mathbb{R}$ defined by $A_2(x, y) = xy$, $x, y \in \mathbb{R}$.
It is easy to see that A_2 is symmetric and biadditive. By Theorem 2.9, we have $\Delta_{x_1 x_2} A^2(y) = 2A_2(x_1, x_2) = 2x_1 x_2$ for all $x_1, x_2, y \in \mathbb{R}$. In particular, $A^2(y + 2x) - 2A^2(y + x) + A^2(y) = \Delta_x^2 A^2(y) = 2x^2$. Moreover, $\Delta_{x_1 x_2 x_3} A^2(y) = 0$ for all $x_1, x_2, x_3, y \in \mathbb{R}$.

Theorem 2.11. *Let $f : X \to Y$ be a polynomial function of order s. Then there exist n-additive symmetric mappings $A_n : X^n \to Y$, $n = 0, ..., s$, such that*

$$f(x) = \sum_{n=0}^{s} A^n(x), \; x \in X$$

where $A^n : X \to Y$ is the diagonalization of A_n, for each $n = 0, ..., s$.

Example 2.12. Let $f : \mathbb{R} \to \mathbb{R}$ satisfying the equation (1.2).
If we substitute x by $x + 2y$ and $x + y$, respectively, in the equation (1.2), then we obtain the new equations, respectively,

$$f(x+3y) - 2f(x+2y) + f(x+y) = 2f(y) \tag{2.11}$$

$$f(x+2y) - 2f(x+y) + f(x) = 2f(y). \tag{2.12}$$

Subtracting (2.12) from (2.11), we obtain the following equation

$$f(x+3y) - 3f(x+2y) + 3f(x+y) - f(x) = 0$$

This shows that $\Delta_y^3 f(x) = 0$, i.e. f is a polynomial function of order 2. Then, by Theorem 2.11, there exist n-additive symmetric mappings $A_n : X^n \to Y$, $n = 0, 1, 2$, such that

$$f(x) = A^0 + A^1(x) + A^2(x), \ x \in \mathbb{R} \tag{2.13}$$

where $A^n : \mathbb{R} \to \mathbb{R}$ is the diagonalization of A_n, for each $n = 0, 1, 2$.
Actually, we can verify that terms in the right hand side of the above equation may vanish. Putting $x = y = 0$ in the equation (1.2), we have $f(0) = 0$. Replacing y by $-y$ in the equation (1.2), we can see that $f(y) = f(-y)$ for all $y \in \mathbb{R}$. That is, f is an even function. We observe that A^0 and A^2 are even, so the evenness of f forces that $A^1(x) = 0$. Since $f(0) = 0$, we also obtain $A^0 = 0$. Hence, the equation (2.13) becomes

$$f(x) = A^2(x), \ x \in \mathbb{R}. \tag{2.14}$$

CHAPTER III

A QUARTIC FUNCTIONAL EQUATION

In this chapter, we study the general solution of the quartic functional equation

$$f(3x+y)+f(x+3y)=64f(x)+64f(y)+24f(x+y)-6f(x-y) \tag{3.1}$$

and prove its generalized Hyers-Ulam-Rassias stability.

3.1 The general solution

In this section, we establish the general solution of the equation (3.1). Throughout this section X and Y will be real vector spaces.

Theorem 3.1. *A function $f : X \to Y$ satisfies the functional equation (3.1) if and only if there exists a 4-additive symmetric function $A_4 : X^4 \to Y$ such that $f(x) = A^4(x)$ for all $x \in X$ where A^4 is the diagonalization of A_4.*

Proof. Assume that f satisfies the functional equation (3.1).
Putting $x = y = 0$ in the equation (3.1), we have $f(0) = 0$. Replacing x and y by $x+y$ and $x-y$, respectively, in the equation (3.1), we obtain

$$f(4x+2y)+f(4x-2y)=64f(x+y)+64f(x-y)+24f(2x)-6f(2y). \tag{3.2}$$

Replacing y by $-y$ in the equation (3.2), we can see that

$$f(y)=f(-y)$$

for all $y \in X$. That is f is an even function. Replacing y by $-x$ in the equation (3.1) and using the evenness of f, we get

$$f(2x)=16f(x) \tag{3.3}$$

for all $x \in X$. Applying the equation (3.3) to the equation (3.2), we obtain

$$f(2x+y)+f(2x-y)=4f(x+y)+4f(x-y)+24f(x)-6f(y). \tag{3.4}$$

Replacing y by $y+3x$ and $y+2x$, respectively, in the equation (3.4), then taking the difference of the two newly obtained equations, we get

$$f(5x+y)-5f(4x+y)+10f(3x+y)-10f(2x+y)+5f(x+y)-f(y)=0$$

Hence, f satisfies the condition $\Delta_x^5 f(y)=0$. Consequently, f is a polynomial function of order 4. Then, by Theorem 2.11, there exist n-additive symmetric functions $A_n : X^n \to Y$, $n = 0, ..., 4$, such that

$$f(x)=A^0+A^1(x)+A^2(x)+A^3(x)+A^4(x) \tag{3.5}$$

where $A^n : X \to Y$ is the diagonalization of A_n, for each $n = 0, ..., 4$. We observe that A^n, $n = 0, 2, 4$, are even. Since f is an even function, $A^1(x)$ and $A^3(x)$ must vanish. Moreover, since $f(0) = 0$, we have $A^0 = 0$. Then the equation (3.5) is reduced to

$$f(x)=A^2(x)+A^4(x). \tag{3.6}$$

By using the symmetry and the additivity, one can verify that

$$A^n(mx)=m^n A^n(x) \tag{3.7}$$

for all $n \in \mathbb{N}, m \in \mathbb{Z}$. Substituting the equation (3.6) into the equation (3.3) and using the property (3.7), we obtain $A^2(x) = 0$. Hence, we conclude that $f(x) = A^4(x)$ for all $x \in X$.

Conversely, assume that there exists a 4-additive symmetric function $A_4 : X^4 \to Y$ such that $f(x) = A^4(x)$ for all $x \in X$. By Theorem 2.9, we have $\Delta_x^4 A^4(y) = 4! A^4(x)$. Thus, we obtain

$$A^4(4x+y)-4A^4(3x+y)+6A^4(2x+y)-4A^4(x+y)+A^4(y)=24A^4(x). \tag{3.8}$$

Replacing y by $y-x$ in the equation (3.8), we get

$$A^4(3x+y)-4A^4(2x+y)+6A^4(x+y)-4A^4(y)+A^4(y-x)=24A^4(x). \tag{3.9}$$

Replacing x and y by $x+y$ and $-2y$, respectively, in the equation (3.9), we obtain

$$A^4(3x+y)-4A^4(2x)+6A^4(x-y)-4A^4(-2y)+A^4(-3y-x) = 24A^4(x+y). \quad (3.10)$$

Since $A^4(nx) = n^4A^4(x)$ for all $n \in \mathbb{Z}$. Then we have

$$A^4(3x+y)+A^4(3y+x) = 64A^4(x)+64A^4(y)+24A^4(x+y)-6A^4(x-y). \quad (3.11)$$

By the assumption, we arrive at the functional equation (3.1). □

3.2 The generalized Hyers-Ulam-Rassias stability

Throughout this section X and Y will be a real normed space and a real Banach space, respectively. Given a function $f : X \to Y$, we set

$$Df(x,y) := f(3x+y) + f(x+3y) - 64f(x) - 64f(y) - 24f(x+y) + 6f(x-y)$$

for all $x, y \in X$.

Theorem 3.2. *Let $\phi : X^2 \to [0,\infty)$ be a function such that*

$$\begin{cases} \displaystyle\sum_{i=0}^{\infty} \frac{\phi(3^i x, 0)}{81^i} \text{ converges and} \\ \displaystyle\lim_{n\to\infty} \frac{\phi(3^n x, 3^n y)}{81^n} = 0 \text{ for all } x, y \in X \end{cases} \quad (3.12)$$

or

$$\begin{cases} \displaystyle\sum_{i=1}^{\infty} 81^i \phi(\frac{x}{3^i}, 0) \text{ converges and} \\ \displaystyle\lim_{n\to\infty} 81^n \phi(\frac{x}{3^n}, \frac{y}{3^n}) = 0 \text{ for all } x, y \in X. \end{cases} \quad (3.13)$$

If a function $f : X \to Y$ satisfies

$$\|Df(x,y)\| \le \phi(x,y) \quad (3.14)$$

for all $x, y \in X$ and $f(0) = 0$, then there exists a unique function $T : X \to Y$ which satisfies the equation (3.1) and the inequality

$$\|f(x) - T(x)\| \le \begin{cases} \displaystyle\frac{1}{81}\sum_{i=0}^{\infty} \frac{\phi(3^i x, 0)}{81^i} & \text{if (3.12) holds} \\ \displaystyle\frac{1}{81}\sum_{i=1}^{\infty} 81^i \phi(\frac{x}{3^i}, 0) & \text{if (3.13) holds} \end{cases} \quad (3.15)$$

for all $x \in X$. *The function* T *is given by*

$$T(x) = \begin{cases} \lim_{n\to\infty} \dfrac{f(3^n x)}{81^n} & \text{if (3.12) holds} \\ \lim_{n\to\infty} 81^n f(\dfrac{x}{3^n}) & \text{if (3.13) holds} \end{cases} \tag{3.16}$$

for all $x \in X$.

Proof. First, we assume that the conditon (3.12) holds.

Putting $y = 0$ in the inequality (3.14) and then dividing by 81, we have

$$\|\frac{f(3x)}{81} - f(x)\| \le \frac{1}{81}\phi(x,0) \tag{3.17}$$

for all $x \in X$. Replacing x by $3x$ in the inequality (3.17) and dividing by 81, we obtain

$$\|\frac{f(3^2x)}{81^2} - \frac{f(3x)}{81}\| \le \frac{1}{81^2}\phi(3x,0) \tag{3.18}$$

for all $x \in X$. From inequalitys (3.17) and (3.18) , we have

$$\|\frac{f(3^2x)}{81^2} - f(x)\| \le \frac{1}{81}\left(\phi(x,0) + \frac{\phi(3x,0)}{81}\right) \tag{3.19}$$

for all $x \in X$. Using a mathematical induction, we can extend the inequality (3.19) to

$$\|\frac{f(3^nx)}{81^n} - f(x)\| \le \frac{1}{81}\sum_{i=0}^{n-1}\frac{\phi(3^ix,0)}{81^i} \le \frac{1}{81}\sum_{i=0}^{\infty}\frac{\phi(3^ix,0)}{81^i} \tag{3.20}$$

for all $x \in X$ and for all $n \in \mathbb{N}$.

For integers $m, n > 0$, we have

$$\begin{aligned} \|\frac{f(3^n3^mx)}{81^{n+m}} - \frac{f(3^mx)}{81^m}\| &= \frac{1}{81^m}\|\frac{f(3^n3^mx)}{81^n} - f(3^mx)\| \\ &\le \frac{1}{81^m}\cdot\frac{1}{81}\sum_{i=0}^{n-1}\frac{\phi(3^i3^mx,0)}{81^i} \\ &\le \frac{1}{81}\sum_{i=0}^{\infty}\frac{\phi(3^i3^mx,0)}{81^{i+m}}. \end{aligned}$$

Since the right-hand side of the inequality tends to 0 as $m \to \infty$, the sequence $\{81^{-n}f(3^nx)\}$ is a Cauchy sequence. Since Y is complete, there exists the limit function $T(x) = lim_{n\to\infty}81^{-n}f(3^nx)$ for all $x \in X$. By letting $n \to \infty$ in the inequality (3.20), we arrive at the formula (3.15). To show that T satisfies the equation (3.1), replace x and y by $3^n x$ and $3^n y$, respectively, in (3.14) and divide by 81^n, then it follows that

$$81^{-n}\|f(3^n(3x+y)) + f(3^n(x+3y)) - 64f(3^n x) - 64f(3^n y) - 24f(3^n(x+y)) \\ +6f(3^n(x-y))\| \leq 81^{-n}\phi(3^n x, 3^n y).$$

Taking the limit as $n \to \infty$, we find that T satisfies the equation (3.1) for all $x, y \in X$.

To prove the uniqueness of quartic function T subject to the inequality (3.15), assume that there exists a function $S : X \to Y$ which satisfies the equation (3.1) and the inequality (3.15) with T replaced by S. Note that Theorem 3.1 gives us $T(3^n x) = 81^n T(x)$ and $S(3^n x) = 81^n S(x)$ for all $x \in X$ and $n \in \mathbb{N}$. Then we have

$$\begin{aligned} \|T(x) - S(x)\| &= \frac{1}{81^n}\|T(3^n x) - S(3^n x)\| \\ &\leq \frac{1}{81^n}\left(\|T(3^n x) - f(3^n x)\| + \|f(3^n x) - S(3^n x)\|\right) \\ &\leq \frac{1}{81^n}\left(\frac{2}{81}\sum_{i=0}^{\infty}\frac{\phi(3^i 3^n x, 0)}{81^i}\right) \\ &= \frac{2}{81}\sum_{i=0}^{\infty}\frac{\phi(3^i 3^n x, 0)}{81^{i+n}} \end{aligned}$$

for all $x \in X$. By letting $n \to \infty$ in the preceding inequality, we immediately find the uniqueness of T.

For the case when the condition (3.13) holds, we replace x by $3^{-1}x$ in the inequality (3.17) and then consider the sequence $\{81^n f(3^{-n}x)\}$. We can see that the limit $T(x) = lim_{n\to\infty} 81^n f(3^{-n}x)$ exists for all $x \in X$ which is the unique function satisfying the equation (3.1) and the inequality (3.15). This completes the proof of the theorem. □

Remark 3.3. *In case of condition (3.12) a function f which satisfies the inequality (3.14) needs not to be zero at $x = 0$. By using the same argument, we can find a unique quartic function $T : X \to Y$ defined by $T(x) = \lim_{n\to\infty} 81^{-n} f(3^n x)$*

which satisfies the equation (3.1) and the inequality

$$\|f(x) - T(x) - \frac{4}{5}f(0)\| \le \frac{1}{81}\sum_{i=0}^{\infty}\frac{\phi(3^i x, 0)}{81^i} \tag{3.21}$$

for all $x \in X$.

Corollary 3.4. *If a function* $f : X \to Y$ *satisfies the inequality*

$$\|Df(x,y)\| \le \varepsilon \tag{3.22}$$

for all $x, y \in X$ *for some real number* $\varepsilon > 0$, *then there exists a unique function* $T : X \to Y$ *such that* T *satisfies the equation (3.1) and*

$$\|f(x) - T(x) - \frac{4}{5}f(0)\| \le \frac{\varepsilon}{80}$$

for all $x \in X$. *The function* T *is given by* $T(x) = \lim_{n\to\infty} 81^{-n} f(3^n x)$ *for all* $x \in X$.

Proof. Taking $\phi(x,y) = \varepsilon$ for all $x, y \in X$. Being in accordance with (3.12) in Remark 3.3, we obtain

$$\|f(x) - T(x) - \frac{4}{5}f(0)\| \le \frac{1}{81}\sum_{i=0}^{\infty}\frac{\varepsilon}{81^i} = \frac{\varepsilon}{80}$$

for all $x \in X$, as desired. □

Corollary 3.5. *Given positive real numbers* ε *and* p *with* $p \ne 4$. *If a function* $f : X \to Y$ *satisfies the inequality*

$$\|Df(x,y)\| \le \varepsilon\big(\|x\|^p + \|y\|^p\big) \tag{3.23}$$

for all $x, y \in X$, *then there exists a unique function* $T : X \to Y$ *such that* T *satisfies the equation (3.1) and*

$$\|f(x) - T(x)\| \le \frac{\varepsilon}{|3^4 - 3^p|}\|x\|^p$$

for all $x \in X$.

Proof. Taking $\phi(x,y) = \varepsilon\big(\|x\|^p + \|y\|^p\big)$ for all $x, y \in X$.

Putting $x = y = 0$ in the inequality (3.23), we obtain $\|f(0)\| \leq 0$. Hence,we have $f(0) = 0$.

If $0 < p < 4$, then the condition (3.12) in Theorem 3.2 holds. It follows that

$$\begin{aligned} \|f(x) - T(x)\| &\leq \frac{\varepsilon}{81}\sum_{i=0}^{\infty}\frac{(3^{ip}\|x\|^p)}{81^i} \\ &= \frac{\varepsilon}{3^4 - 3^p}\|x\|^p \end{aligned}$$

for all $x \in X$. If $p > 4$, then the condition (3.13) in Theorem 3.2 holds. It follows that

$$\begin{aligned} \|f(x) - T(x)\| &\leq \frac{\varepsilon}{81}\sum_{i=1}^{\infty}81^i \cdot \frac{\|x\|^p}{3^{ip}} \\ &= \frac{\varepsilon}{3^p - 3^4}\|x\|^p \end{aligned}$$

for all $x \in X$. □

CHAPTER IV
A PENTIC FUNCTIONAL EQUATION

In this chapter, we study the general solution of the pentic functional equation

$$\Delta_y^5 f(x) = 120 f(y) \tag{4.1}$$

or, equivalently,

$$f(x+5y)-5f(x+4y)+10f(x+3y)-10f(x+2y)+5f(x+y)-f(x) = 120f(y) \tag{4.2}$$

and prove its generalized Hyers-Ulam-Rassias stability.

4.1 The general solution

In this section, we establish the general solution of the equation (4.1). Throughout this section X and Y will be real vector spaces.

Theorem 4.1. *A function $f : X \to Y$ satisfies the functional equation (4.1) if and only if there exists a 5-additive symmetric function $A_5 : X^5 \to Y$ such that $f(x) = A^5(x)$ for all $x \in X$ where A^5 is the diagonalization of A_5.*

Proof. Assume that f satisfies the functional equation (4.1). Note that

$$\Delta_y^6 f(x) = \Delta_y^5 (\Delta_y f(x)) = \Delta_y^5 f(x+y) - \Delta_y^5 f(x).$$

By assumption, we have $\Delta_y^5 f(x) = 120 f(y) = \Delta_y^5 f(x+y)$, so f fulfills the condition $\Delta_y^6 f(x) = 0$ for all $x, y \in X$. Consequently, f is a polynomial function of order 5.

Define the function $A_5 : X^5 \to Y$ by

$$A_5(x_1, x_2, x_3, x_4, x_5) = \frac{1}{5!} \Delta_{x_1 x_2 x_3 x_4 x_5} f(0) \tag{4.3}$$

for all $x_i \in X$, $i = 1, ..., 5$. We have that A_5 is symmetric since the operator is symmetric under the permutation of $x_1, ..., x_5$. We have for each $i = 1, ..., 5$ and $x_1, ..., x_5, y_i \in X$,

$$A_5(x_1, ..., x_{i-1}, x_i + y_i, x_{i+1}, ..., x_5) - A_5(x_1, ..., x_5) - A_5(x_1, ..., x_{i-1}, y_i, x_{i-1}, ..., x_5)$$
$$= \frac{1}{5!}\Delta_{x_1 ... x_{i-1} x_{i+1} ... x_5}(\Delta_{x_i + y_i} f(0) - \Delta_{x_i} f(0) - \Delta_{y_i} f(0)).$$

By Theorem 2.4, we obtain $\Delta_{x_i+y_i} f(0) - \Delta_{x_i} f(0) - \Delta_{y_i} f(0) = \Delta_{x_i y_i} f(0)$, so

$$\begin{aligned} A_5(x_1, ..., x_{i-1}, x_i + y_i, x_{i+1}, ..., x_5) - A_5(x_1, ..., x_5) - A_5(x_1, ..., x_{i-1}, y_i, x_{i-1}, ..., x_5) \\ = \frac{1}{5!}\Delta_{x_1 ... x_{i-1} x_{i+1} ... x_5}(\Delta_{x_i y_i} f(0)) \\ = \frac{1}{5!}\Delta_{x_1 ... x_5 y_i} f(0). \end{aligned}$$

Since f is a polynomial function of order 5, by Theorem 2.7, we have

$$\frac{1}{5!}\Delta_{x_1 ... x_5 y_i} f(0) = 0.$$

This shows that A_5 is 5-additive.

Next, interchanging y and x in the equation (4.1) and then putting $y = 0$, we obtain

$$\Delta_x^5 f(0) = 120 f(x). \tag{4.4}$$

Then

$$A^5(x) = \Delta_x^5 f(0)/5! = f(x).$$

Conversely, assume that there exists a 5-additive symmetric function $A_5 : X^5 \to Y$ such that $f(x) = A^5(x)$ for all $x \in X$. By Theorem 2.9, we have $\Delta_y^5 A^5(x) = 5! A^5(y)$. By assumption, we evidently arrive at the functional equation (4.1). □

4.2 The generalized Hyers-Ulam-Rassias stability

Throughout this section X and Y will be a real normed space and a real Banach space, respectively. Given a function $f : X \to Y$, we set

$$Df(x, y) := \Delta_y^5 f(x) - 120 f(y)$$

for all $x, y \in X$.

Lemma 4.2. *Let $x \in X$, $n \in \mathbb{N}$, and $m_0, m_1, ..., m_n \in \mathbb{Z}$ such that $\sum_{i=0}^{n} m_i = 0$. Then for any function $f : X \to Y$*

$$\sum_{i=0}^{n} m_i f((n-i)x) = \Delta_x \left(\sum_{i=1}^{n} M_i f((n-i)x) \right), \tag{4.5}$$

where $M_i = m_0 + m_1 + \cdots + m_{i-1}$ for all $i = 1, ..., n$.

Proof. By the definition of Δ_x and Theorem 2.2, we have

$$\begin{aligned}
\Delta_x \left(\sum_{i=1}^{n} M_i f((n-i)x) \right) &= \sum_{i=1}^{n} M_i \Delta_x f((n-i)x) \\
&= \sum_{i=1}^{n} M_i f((n-i+1)x) - \sum_{i=1}^{n} M_i f((n-i)x) \\
&= \sum_{i=0}^{n-1} M_{i+1} f((n-i)x) - \sum_{i=1}^{n} M_i f((n-i)x) \\
&= M_1 f(nx) + \sum_{i=1}^{n-1} (M_{i+1} - M_i) f((n-i)x) - M_n f(0).
\end{aligned}$$

Since $M_1 = m_0$ and $M_n = m_0 + m_1 + \cdots + m_{n-1} = -m_n$, we can conclude that

$$\begin{aligned}
\Delta_x \left(\sum_{i=1}^{n} M_i f((n-i)x) \right) &= m_0 f(nx) + \sum_{i=1}^{n-1} m_i f((n-i)x) + m_n f(0) \\
&= \sum_{i=0}^{n} m_i f((n-i)x).
\end{aligned}$$

□

Lemma 4.3. *Let $f : X \to Y$ and $\phi : X^2 \to [0, \infty)$. Assume $\| \Delta_y^5 f(x) - 5! f(y) \| \leq \phi(x, y)$ for all $x, y \in X$. Then*

$$\begin{aligned}
&\| A^5(2x) - 32A^5(x) \| \\
&\qquad \leq \frac{1}{5!} \left[\phi(5x, x) + 5\phi(4x, x) + 10\phi(3x, x) + 10\phi(2x, x) + 5\phi(x, x) + 31\phi(0, x) \right].
\end{aligned}$$

where A^5 is the diagonalization of $A_5 : X^5 \to Y$ defined in the equation (4.3).

Proof. Define

$$B^{m,n}(x) = A_5(\underbrace{x, ..., x}_{m}, \underbrace{2x, ..., 2x}_{n}) \quad \text{and}$$

$$\nabla_x^{m,n} f(0) = \Delta_x^m \Delta_{2x}^n f(0) \quad \text{where } m, n \in \mathbb{N} \cup \{0\}\,.$$

Actually, there is no meaning if both m and n are zero and, in this work, we use m and n such that $m+n$ equals 5 or 6.

Note that, for all $m = 0, 1, 2, 3, 4$,

$$\begin{aligned} B^{4-m,m+1}(x) - 2B^{5-m,m}(x) &= A_5(\underbrace{x, ..., x}_{4-m}, \underbrace{2x, ..., 2x}_{m+1}) - 2A_5(\underbrace{x, ..., x}_{5-m}, \underbrace{2x, ..., 2x}_{m}) \\ &= \frac{1}{5!}\left(\Delta_x^{4-m}\Delta_{2x}^{m+1} f(0) - 2\Delta_x^{5-m}\Delta_{2x}^{m} f(0)\right). \end{aligned}$$

By Theorem 2.2, we have

$$B^{4-m,m+1}(x) - 2B^{5-m,m}(x) = \frac{1}{5!}\Delta_x^{4-m}\Delta_{2x}^{m}\left(\Delta_{2x} f(0) - 2\Delta_x f(0)\right).$$

Thus, by Theorem 2.4, the above equation becomes

$$\begin{aligned} B^{4-m,m+1}(x) - 2B^{5-m,m}(x) &= \frac{1}{5!}\Delta_x^{4-m}\Delta_{2x}^{m}\left(\Delta_x^2 f(0)\right) \\ &= \frac{1}{5!}\Delta_x^{6-m}\Delta_{2x}^{m} f(0) \\ &= \frac{1}{5!}\nabla_x^{6-m,m} f(0) \end{aligned} \tag{4.6}$$

for all $x \in X$. Then

$$\begin{aligned} \frac{1}{5!}\sum_{m=0}^{4} 2^{4-m}\nabla_x^{6-m,m} f(0) &= \sum_{m=0}^{4} 2^{4-m}\left(B^{4-m,m+1}(x) - 2B^{5-m,m}(x)\right) \\ &= \sum_{m=0}^{4} 2^{4-m}B^{4-m,m+1}(x) - \sum_{m=0}^{4} 2^{5-m}B^{5-m,m}(x) \\ &= \sum_{m=1}^{5} 2^{5-m}B^{5-m,m}(x) - \sum_{m=0}^{4} 2^{5-m}B^{5-m,m}(x) \\ &= B^{0,5}(x) - 32B^{5,0}(x). \end{aligned}$$

By Theorem 2.5, we have

$$\begin{aligned} \nabla_x^{2,4} f(0) &= \Delta_x^2\Delta_{2x}^4 f(0) \\ &= \Delta_x^2(f(8x) - 4f(6x) + 6f(4x) - 4f(2x) + f(0)). \end{aligned}$$

Using Lemma 4.2 three times, we obtain

$$\nabla_x^{2,4} f(0) = \Delta_x^5(f(5x) + 3f(4x) + 2f(3x) - 2f(2x) - 3f(x) - f(0)).$$

Similarly, we have

$$\begin{aligned}
\nabla_x^{3,3} f(0) &= \Delta_x^5(f(4x) + 2f(3x) - 2f(x) - f(0)),\\
\nabla_x^{4,2} f(0) &= \Delta_x^5(f(3x) + f(2x) - f(x) - f(0)),\\
\nabla_x^{5,1} f(0) &= \Delta_x^5(f(2x) - f(0)) \text{ and}\\
\nabla_x^{6,0} f(0) &= \Delta_x^5(f(x) - f(0)).
\end{aligned}$$

Hence,

$$\begin{aligned}
B^{0,5}(x) - 32B^{5,0}(x) &= \frac{1}{5!}\sum_{m=0}^{4} 2^{4-m}\nabla_x^{6-m,m} f(0)\\
&= \frac{1}{5!}\Delta_x^5\left(f(5x) + 5f(4x) + 10f(3x) + 10f(2x) + 5f(x) - 31f(0)\right)\\
&= \frac{1}{5!}[(\Delta_x^5 f(5x) - 5!f(x)) + 5(\Delta_x^5 f(4x) - 5!f(x)) + 10(\Delta_x^5 f(3x) - 5!f(x))\\
&\quad + 10(\Delta_x^5 f(2x) - 5!f(x)) + 5(\Delta_x^5 f(x) - 5!f(x)) - 31(\Delta_x^5 f(0) - 5!f(x))].
\end{aligned}$$

Since $\|\Delta_y^5 f(x) - 5!f(y)\| \le \phi(x,y)$ for all $x, y \in X$, we thus conclude that

$$\begin{aligned}
\|A^5(2x) - 32A^5(x)\| &= \|A_5(2x, ..., 2x) - 32A_5(x, ..., x)\|\\
&= \|B^{0,5}(x) - 32B^{5,0}(x)\|\\
&\le \frac{1}{5!}\left[\phi(5x,x) + 5\phi(4x,x) + 10\phi(3x,x) + 10\phi(2x,x) + 5\phi(x,x) + 31\phi(0,x)\right].
\end{aligned}$$

□

Theorem 4.4. *Let* $\phi : X^2 \to [0,\infty)$ *be a function such that*

$$\begin{cases} \displaystyle\sum_{i=0}^{\infty} \frac{\tau(2^i x)}{32^i} \text{ converges and} \\ \displaystyle\lim_{n\to\infty} \frac{\phi(2^n x, 2^n y)}{32^n} = 0 \text{ for all } x, y \in X \end{cases} \tag{4.7}$$

or

$$\begin{cases} \displaystyle\sum_{i=1}^{\infty} 32^i \tau(\frac{x}{2^i}) \text{ converges and} \\ \displaystyle\lim_{n\to\infty} 32^n \phi(\frac{x}{2^n}, \frac{y}{2^n}) = 0 \text{ for all } x, y \in X. \end{cases} \tag{4.8}$$

where

$$\tau(x) = \left[\phi(0,2x) + 2^5\phi(0,x)\right] + \\ [\phi(5x,x) + 5\phi(4x,x) + 10\phi(3x,x) + 10\phi(2x,x) + 5\phi(x,x) + 31\phi(0,x)]$$

for all $x \in X$. If a function $f : X \to Y$ satisfies

$$\|Df(x,y)\| \leq \phi(x,y) \tag{4.9}$$

for all $x, y \in X$, then there exists a unique function $T : X \to Y$ which satisfies the equation (4.1) and the inequality

$$\|f(x) - T(x)\| \leq \begin{cases} \dfrac{1}{32 \cdot 5!}\displaystyle\sum_{i=0}^{\infty}\frac{\tau(2^i x)}{32^i} & \text{if (4.7) holds} \\ \dfrac{1}{32 \cdot 5!}\displaystyle\sum_{i=1}^{\infty}32^i\tau(\frac{x}{2^i}) & \text{if (4.8) holds} \end{cases} \tag{4.10}$$

for all $x \in X$. The function T is given by

$$T(x) = \begin{cases} \displaystyle\lim_{n\to\infty}\frac{f(2^n x)}{32^n} & \text{if (4.7) holds} \\ \displaystyle\lim_{n\to\infty}32^n f(\frac{x}{2^n}) & \text{if (4.8) holds} \end{cases} \tag{4.11}$$

for all $x \in X$.

Proof. First, we assume that the conditon (4.7) holds.

For all $x \in X$, we have

$$\|f(x) - A^5(x)\| = \|f(x) - \frac{1}{5!}\Delta_x^5 f(0)\| \\ = \frac{1}{5!}\|5!f(x) - \Delta_x^5 f(0)\|$$

where $A_5 : X^5 \to Y$ is a function defined in the equation (4.3). Then, by assumption, we obtain

$$\|f(x) - A^5(x)\| \leq \frac{1}{5!}\phi(0,x) \tag{4.12}$$

for all $x \in X$. Note that

$$\|f(2x) - 2^5 f(x)\| = \|\left[f(2x) - A^5(2x)\right] - 2^5\left[f(x) - A^5(x)\right] + \left[A^5(2x) - 2^5 A^5(x)\right]\| \\ \leq \|f(2x) - A^5(2x)\| + 2^5\|f(x) - A^5(x)\| + \|A^5(2x) - 2^5 A^5(x)\|. \tag{4.13}$$

By the inequality (4.12) and Lemma 4.3, we obtain

$$\begin{aligned}\|f(2x)-2^5f(x)\| &\le \|f(2x)-A^5(2x)\|+2^5\|f(x)-A^5(x)\|+\|A^5(2x)-2^5A^5(x)\|\\ &\le \frac{1}{5!}\left[\phi(0,2x)+2^5\phi(0,x)\right]+\\ &\frac{1}{5!}\left[\phi(5x,x)+5\phi(4x,x)+10\phi(3x,x)+10\phi(2x,x)+5\phi(x,x)+31\phi(0,x)\right]\\ &=\frac{\tau(x)}{5!} \qquad (4.14)\end{aligned}$$

where

$$\begin{aligned}\tau(x)=&\left[\phi(0,2x)+2^5\phi(0,x)\right]+\\ &\left[\phi(5x,x)+5\phi(4x,x)+10\phi(3x,x)+10\phi(2x,x)+5\phi(x,x)+31\phi(0,x)\right]\end{aligned}$$

for all $x\in X$. Dividing the inequality (4.14) by 32, we have

$$\left\|\frac{f(2x)}{32}-f(x)\right\|\le\frac{\tau(x)}{32\cdot 5!} \qquad (4.15)$$

We can show the following relation by induction on n together with the inequality (4.15)

$$\left\|\frac{f(2^nx)}{32^n}-f(x)\right\|\le\frac{1}{32\cdot 5!}\sum_{i=0}^{n-1}\frac{\tau(2^ix)}{32^i}\le\frac{1}{32\cdot 5!}\sum_{i=0}^{\infty}\frac{\tau(2^ix)}{32^i} \qquad (4.16)$$

for all $x\in X$ and for all $n\in\mathbb{N}$.

For integers $m,n>0$, we have

$$\begin{aligned}\left\|\frac{f(2^n2^mx)}{32^{n+m}}-\frac{f(2^mx)}{32^m}\right\| &= \frac{1}{32^m}\left\|\frac{f(2^n2^mx)}{32^n}-f(2^mx)\right\|\\ &\le\frac{1}{32\cdot 5!}\sum_{i=0}^{\infty}\frac{\tau(2^{i+m}x)}{32^{i+m}}.\end{aligned}$$

Since the right-hand side of the inequality tends to 0 as $m\to\infty$, the sequence $\{32^{-n}f(2^nx)\}$ is a Cauchy sequence. Since Y is complete, there exists the limit function $T(x)=lim_{n\to\infty}32^{-n}f(2^nx)$ for all $x\in X$. By letting $n\to\infty$ in the inequality (4.16), we arrive at the formula (4.10). To show that T satisfies the equation (4.1), replace x and y by 2^nx and 2^ny, respectively, in the inequality (4.9) and divide by 32^n, then it follows that

$$\begin{aligned}32^{-n}\|f(2^n(x+5y))-5f(2^n(x+4y))+10f(2^n(x+3y))-10f(2^n(x+2y))\\ +5f(2^n(x+y))-f(2^nx)-120f(2^ny)\|\le 32^{-n}\phi(2^nx,2^ny)\end{aligned}$$

Taking the limit as $n \to \infty$, we find that T satisfies the equation (4.1) for all $x, y \in X$.

To prove the uniqueness of pentic function T subject to the inequality (4.10), assume that there exists a function $S : X \to Y$ which satisfies the equation (4.1) and the inequality (4.10) with T replaced by S. Note that Theorem 4.1 gives us $T(2^n x) = 32^n T(x)$ and $S(2^n x) = 32^n S(x)$ for all $x \in X$ and $n \in \mathbb{N}$. Then we have

$$\begin{aligned} \|T(x) - S(x)\| &= \frac{1}{32^n}\|T(2^n x) - S(2^n x)\| \\ &\leq \frac{1}{32^n}\left(\|T(2^n x) - f(2^n x)\| + \|f(2^n x) - S(2^n x)\|\right) \\ &\leq \frac{1}{32^n}\left(\frac{1}{16 \cdot 5!}\sum_{i=0}^{\infty}\frac{\tau(2^{i+n}x)}{32^{i+n}}\right) \end{aligned}$$

for all $x \in X$. By letting $n \to \infty$ in the preceding inequality, we immediately find the uniqueness of T.

For the case when the condition (4.8) holds, we replace x by $2^{-1}x$ in the inequality (4.14) and then show the following relation

$$\|f(x) - 32^n f(\frac{x}{2^n})\| \leq \frac{1}{32 \cdot 5!}\sum_{i=1}^{\infty} 32^i \tau(\frac{x}{2^i}). \tag{4.17}$$

Using the same argument in the case when the condition (4.7) holds, we have that the limit $T(x) = lim_{n \to \infty} 32^n f(2^{-n}x)$ exists for all $x \in X$ which is the unique function satisfying the equation (4.1) and the inequality (4.10). This completes the proof of the theorem. □

Corollary 4.5. *If a function $f : X \to Y$ satisfies the inequality*

$$\|Df(x,y)\| \leq \varepsilon \tag{4.18}$$

for all $x, y \in X$ for some real number $\varepsilon > 0$, then there exists a unique function $T : X \to Y$ such that T satisfies the equation (4.1) and

$$\|f(x) - T(x)\| \leq \frac{95\varepsilon}{31 \cdot 5!}$$

for all $x \in X$. The function T is given by $T(x) = \lim_{n \to \infty} 32^{-n} f(2^n x)$ for all $x \in X$.

Proof. Taking $\phi(x,y) = \varepsilon$ for all $x, y \in X$. Being in accordance with (4.7) in Theorem 4.4, we obtain

$$\|f(x) - T(x)\| \leq \frac{1}{32 \cdot 5!} \sum_{i=0}^{\infty} \frac{95\varepsilon}{32^i} = \frac{95\varepsilon}{31 \cdot 5!}$$

for all $x \in X$, as desired. □

Corollary 4.6. *Given positive real numbers ε and p with $p \neq 5$. If a function $f : X \to Y$ satisfies the inequality*

$$\|Df(x,y)\| \leq \varepsilon\big(\|x\|^p + \|y\|^p\big) \tag{4.19}$$

for all $x, y \in X$, then there exists a unique function $T : X \to Y$ such that T satisfies the equation (4.1) and

$$\|f(x) - T(x)\| \leq \frac{\varepsilon(5^p + 5 \cdot 4^p + 10 \cdot 3^p + 11 \cdot 2^p + 99)\|x\|^p}{5! \cdot |2^5 - 2^p|}$$

for all $x \in X$.

Proof. Taking $\phi(x,y) = \varepsilon\big(\|x\|^p + \|y\|^p\big)$ for all $x, y \in X$.
If $0 < p < 5$, then the condition (4.7) in Theorem 4.4 holds. It follows that

$$\begin{aligned}\|f(x) - T(x)\| &\leq \frac{\varepsilon}{32 \cdot 5!} \sum_{i=0}^{\infty} \frac{(5^p + 5 \cdot 4^p + 10 \cdot 3^p + 11 \cdot 2^p + 99)\, 2^{ip}\|x\|^p}{32^i} \\ &= \frac{\varepsilon(5^p + 5 \cdot 4^p + 10 \cdot 3^p + 11 \cdot 2^p + 99)\|x\|^p}{5! \cdot (2^5 - 2^p)}\end{aligned}$$

for all $x \in X$. If $p > 5$, then the condition (4.8) in Theorem 4.4 holds. It follows that

$$\begin{aligned}\|f(x) - T(x)\| &\leq \frac{\varepsilon}{32 \cdot 5!} \sum_{i=1}^{\infty} \frac{(5^p + 5 \cdot 4^p + 10 \cdot 3^p + 11 \cdot 2^p + 99)\, 32^i\|x\|^p}{2^{ip}} \\ &= \frac{\varepsilon(5^p + 5 \cdot 4^p + 10 \cdot 3^p + 11 \cdot 2^p + 99)\|x\|^p}{5! \cdot (2^p - 2^5)}\end{aligned}$$

for all $x \in X$. □

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