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FRÉCHET FUNCTIONAL EQUATION ON RESTRICTED DOMAINS

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ในวิทยานิพนธ์นี้ เราจะแสดงว่าฟังก์ชันค่าจริง $f:\mathbb{R}\to\mathbb{R}$ ทุกฟังก์ชันที่สอดคล้องสมการเชิงฟังก์ชันเฟรเชทั้งในรูปสมมาตร และอสมมาตร เมื่อช่วงแผ่ถูกกำกัดลงบนช่วงเปิดใดๆ คือ ฟังก์ชันพหุนามทั่วไป

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In this dissertation, we will show that all real-valued functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the Fréchet functional equation, either in the symmetric form or non-symmetric form, when the spans are restricted to arbitrary open intervals on $\mathbb{R}$ are generalized polynomial functions.

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CHAPTER I

LITERATURE REVIEWS

1.1 Functional Equations

A functional equation is an equation in which unknowns are functions. Solutions of a functional equation are all functions that satisfy the functional equation.

Example 1.1. In order to determine all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the functional equation

$$f(x+y) = f(x) + y \text{ for all } x, y \in \mathbb{R}, \tag{1.1}$$

we will start with setting $x = 0$ in Eq.(1.1). Then

$$f(y) = f(0) + y \text{ for all } y \in \mathbb{R}.$$

which implies that the function f must be given by $f(x) = x + c$ where c is a constant.

On the other hand, if a function f is defined by $f(x) = x + c$ for all $x \in \mathbb{R}$ where c is a constant, then we obtain

$$f(x+y) = x + y + c = (x+c) + y = f(x) + y.$$

That is, $f(x) = x + c$ satisfies Eq.(1.1). Thus, all solutions of Eq.(1.1) are in the form $f(x) = x + c$ where c is a constant. $\square$

Determining the general solutions of a functional equation is a challenging problem as applicable methods will generally rely on the form of the equations.

A classical example of functional equations is the additive functional equation given as follows:

$$f(x+y) = f(x) + f(y) \tag{1.2}$$

for all $x, y, x + y$ in the domain of f. There were a number of research papers devoted to this additive problems but significant result regarding the solution was discovered by A.L. Cauchy [3] in 1821. Hence, the additive functional equation is commonly known as the *Cauchy functional equation*. Cauchy proved that all continuous solutions of Eq.(1.2) on $\mathbb{R}$ are linear functions given by $f(x) = cx$ for all $x \in \mathbb{R}$ and c is a constant. Nevertheless, an existence of a nonlinear additive function on $\mathbb{R}$ remained open for almost a century until 1905 when G. Hamel [7] succeeded in constructing the general solution of Eq.(1.2) as a linear combination of the elements of Hamel basis over $\mathbb{Q}$. Moreover, an intriguing property of a discontinuous additive function is that its graph, $G(f) = \{(x, f(x)) : x \in \mathbb{R}\}$, is a dense subset of $\mathbb{R}^2$ (see [8]), that is, for all $\varepsilon > 0$ and for all $(x, y) \in \mathbb{R}^2$, there exists a point $(a, f(a)) \in G(f)$ such that $(x-a)^2 + (y-f(a)) < \varepsilon^2$, which simply tells us that the graph of any nonlinear additive functions must possess wild oscillations.

Another functional equation that is closely related to the Cauchy functional equation is the Jensen functional equation:

$$f\left(\frac{x+y}{2}\right) = \frac{f(x) + f(y)}{2}, \tag{1.3}$$

which, under an appropriate choice of translation, is equivalent to Cauchy functional equation. The continuous solutions $f : \mathbb{R} \to \mathbb{R}$ of Eq.(1.3) are

$$f(x) = a + bx \text{ for all } x \in \mathbb{R} \tag{1.4}$$

where a and b are constants.

In 1909, a generalization of the Cauchy functional equation which characterizes polynomials was studied by M. Fréchet [6]. He focused on the functional equation of the form

$$\Delta_h^{m+1} f(x) = 0 \tag{1.5}$$

where Δ_h^{m+1} is the $(m+1)$-iteratives of a difference operator with a span h and found that the continuous solutions of $\Delta_h^{m+1} f(x) = 0$ are polynomial functions of degree not exceeding m. Eq.(1.5) was consequently known as the *Fréchet functional equation*. In 1967, M.A. McKiernan [11] gave an important result that the general solution of the Fréchet functional equation is a generalized polynomial function of order m. Recently, in 2007, J.M. Almira and A.J. Ĺopes-Moreno [1] proved a remarkable result on the continuous solutions on $\mathbb{R}$ of the Fréchet functional equation when some regularity assumptions are imposed on the function as in the following theorem.

Theorem 1.2. *Given $k \in \mathbb{N}$ and $f : \mathbb{R} \to \mathbb{R}$ such that $\Delta_h^{k+1} f = 0$, the following claims are equivalent:*

(i) *f is a polynomial function of degree less than or equal to k.*

(ii) *f is continuous at certain $x \in \mathbb{R}$.*

(iii) *$f\,|_{(a,b)}$ is bounded for certain nonempty interval $(a, b) \subseteq \mathbb{R}$.*

In particular, they also proved that the graph $G(f) = \{(x, f(x)) : \Delta_h^{m+1} f(x) = 0$ for all discontinuous functions $f : \mathbb{R} \to \mathbb{R}\}$ on $\mathbb{R}^2$ has an unbounded open subset.

1.2 Functional Equations with Restricted Domain

Functional equations are normally studied on the entire domain of the equation; that is, no additional restriction is imposed on each variable in the functional equation except for that all variables should ensure the validity of the function values. It challenges us to investigate solutions of the functional equation when the domain of variables of an equation is restricted to certain regions. The restricted domain is sometimes called *domains of validity* ([13] and [15]) Let us reconsider the classical example of Cauchy functional equation,

$$f(x + y) = f(x) + f(y), \tag{1.6}$$

when (x, y) is restricted to a region Ω. Its solutions may depend on the domain Ω as well as regularity assumptions of f such as the continuity and the boundedness.

Example 1.3. As we have already known, when $f : \mathbb{R} \to \mathbb{R}$ is an arbitrary function and the domain of validity, Ω, is the entire domain $\mathbb{R}^2$, all continuous solutions will be linear functions. On the other hand, consider all functions f satisfying

$$f(x + y) = f(x) + f(y) \text{ for all } (x, y) \in \mathbb{R} \times \{0\}. \tag{1.7}$$

That is, the variable y is restricted to be only zero. It is obvious that all solutions of Eq.(1.6) when $\Omega = \mathbb{R}^2$ are also solution of Eq.(1.7). However, there exists a function, for instance, $f(x) = x^2$, which satisfies Eq.(1.7), but does not satisfy Eq.(1.6) when $\Omega = \mathbb{R}^2$. As a consequence, it can be concluded that the set of solutions of Eq.(1.6) when $\Omega = \mathbb{R}^2$ is contained in the set of solutions of Eq.(1.7).
$\square$

The previous example is a good one to confirm the fact that new solutions may generally occur if the domain of variables is restricted. It is worth noting that some authors ([2], [13], for example) used the term *conditional functional equations* to describe the functional equation with restricted domains and there is number of research ([2], [5], [9], [13], and [14], for example) that address this type of problems. The Fréchet functional equation with restricted domain is one of interesting frameworks. However, the Fréchet functional equation with restricted domain has not yet been treated in literature. Therefore, in this dissertation, the Fréchet functional equation with restricted spans as well as its general solution will be investigated.

1.3 Proposed Problem

The objective of this dissertation is to determine the general solution $f : \mathbb{R} \to \mathbb{R}$ of the Fréchet functional equation in following two forms:

(i) the *nonsymmetric* form $\Delta_{h_1} ... \Delta_{h_{m+1}} f(x) = 0$, and

(ii) the *symmetric* form $\Delta_h^{m+1} f(x) = 0$,

when the spans $h, h_1, ..., h_{m+1}$ are restricted to arbitrary open intervals on $\mathbb{R}$.

CHAPTER II

GENERALIZED POLYNOMIAL FUNCTIONS

In this chapter, essential background knowledge related to the Fréchet functional equation will be provided. We will start with additive functions as well as their important properties. Then we will extend the concepts to multi-additive functions. Finally, we will investigate the results of applying difference operators to multi-additive functions. Throughout this chapter, X and Y will denote linear spaces over $\mathbb{Q}$.

2.1 Additive Functions

Let us start with the definition of additive functions.

Definition 2.1. If a function $A : X \to Y$ satisfies the property

$$A(x+y) = A(x) + A(y) \tag{2.1}$$

for all $x, y \in X$, then A will be called an *additive function*.

The additive function has many interesting properties as in the following theorem.

Theorem 2.2. *Let $A : X \to Y$ be an additive function, then*

(i) $A(0) = 0$.

(ii) $A(-x) = -A(x)$ *for all* $x \in X$.

(iii) $A(nx) = nA(x)$ *for all* $x \in X$ *and for all* $n \in \mathbb{Z}$.

(iv) $A(rx) = rA(x)$ *for all* $x \in X$ *and for all* $r \in \mathbb{Q}$.

In a particular case when the spaces X and Y are the sets of real numbers with the usual addition and multiplication, it follows from Theorem 2.2 that

$$A(r) = rA(1) = cr \tag{2.2}$$

for all $r \in \mathbb{Q}$; that is, A is linear at all rational numbers. Nonetheless, the function values at irrational points cannot be determined without additional information. The following theorem will give some regularity assumptions that will ensure the linearity of additive functions on the entire domain $\mathbb{R}$

Theorem 2.3. *Let $A : \mathbb{R} \to \mathbb{R}$ be an additive function. Then the following statements are equivalent:*

(i) $A(x) = cx$.

(ii) *A is monotonic.*

(iii) *A is bounded for certain nonempty interval (a, b).*

(iv) *A is continuous at a point $x_0 \in \mathbb{R}$.*

(v) *A is continuous on $\mathbb{R}$.*

The above theorem implies that if an additive function A is nonlinear, then A is not bounded as well. Thus, it comes with no surprise that the graph of discontinuous additive function is dense on the plan $\mathbb{R}^2$.

Theorem 2.4. *If $A : \mathbb{R} \to \mathbb{R}$ is a discontinuous additive function, then the graph, $G(A) = \{(x, A(x)) : x \in \mathbb{R}\}$, is dense in $\mathbb{R}^2$.*

A general form of additive functions can be given in terms of Hamel basis. Let H be a Hamel basis of $\mathbb{R}$ over $\mathbb{Q}$. Then, for each $x \in \mathbb{R}$, there exists a unique set $\{h_1, ..., h_k\}$ of elements of H and a unique set $\{r_1, ..., r_k\}$ of rational numbers such that

$$x = r_1 h_1 + r_2 h_2 + ... + r_k h_k.$$

Then the value of the additive function A at point x is given by

$$A(x) = r_1 A(h_1) + r_2 A(h_2) + ... + r_k A(h_k).$$

Thus, if the values of A can be determined for all elements of H, then we will be able to calculate the values of A at every point in the domain. In particular, an additive function will be discontinuous if and only if there exist $h_1, h_2 \in H$ such that

$$\frac{A(h_1)}{h_1} \neq \frac{A(h_2)}{h_2}.$$

We can construct an example of nonlinear additive functions as in the following example.

Example 2.5. For the Hamel basis $H = \{h_1, h_2, ...\}$, let $A : \mathbb{R} \to \mathbb{R}$ be an additive function defined by

$$A(h_i) = \begin{cases} h_2 & \text{if } i = 1 \\ h_1 & \text{if } i = 2 \\ h_i & \text{if } i \geq 3. \end{cases}$$

One can readily see that $\frac{A(h_1)}{h_1} = \frac{h_2}{h_1} \neq \frac{h_1}{h_2} = \frac{A(h_2)}{h_2}$. □

2.2 Bi-Additive Functions

In this section, we will extend the concept of additive functions to bi-additive functions.

Definition 2.6. A function $B : X^2 \to Y$ will be called a *bi-additive function* if and only if B satisfies the properties

$$B(x + z, y) = B(x, y) + B(z, y)$$

and

$$B(x, y + z) = B(x, y) + B(x, z)$$

for all $x, y, z \in X$.

Example 2.7. A function $B : \mathbb{R}^2 \to \mathbb{R}$ defined by $B(x, y) = cxy$, where c is a constant, is a bi-additive function. □

Example 2.8. Define a function $B : X^2 \to Y$ by $B(x, y) = A_1(x)A_2(y)$ where A_1, A_2 are additive functions. We can verify that B is a bi-additive function. □

The following theorem summarize some properties of bi-additive functions.

Theorem 2.9. *Let $B : X^2 \to Y$ be a bi-additive function. Then for all $x, y \in X$,*

(i) $B(0, y) = 0$ *and* $B(x, 0) = 0$.

(ii) $B(-x, y) = -B(x, y)$ *and* $B(x, -y) = -B(x, y)$.

(iii) $B(rx, y) = rB(x, y)$ *and* $B(x, ry) = rB(x, y)$ *for all* $r \in \mathbb{Q}$.

(iv) $B(x_1 + x_2, y_1 + y_2) = B(x_1, y_1) + B(x_1, y_2) + B(x_2, y_1) + B(x_2, y_2)$ *for all* $x_1, x_2, y_1, y_2 \in X$.

(iv) *In general,* $B\left(\sum_{i=1}^{m} x_i, \sum_{j=1}^{n} y_j\right) = \sum_{i=1}^{m} \sum_{j=1}^{n} B(x_i, y_j)$ *for all* x_i*'s and* y_j*'s in* X.

All continuous bi-additive functions on $\mathbb{R}^2$ assume a particular form as in the following theorem.

Theorem 2.10. *If a bi-additive function* $B : \mathbb{R}^2 \to \mathbb{R}$ *is continuous on* $\mathbb{R}^2$*, then* $B(x, y) = cxy$ *for all* $x, y \in \mathbb{R}$*, where* c *is a constant.*

Proof. In order to prove the theorem, assume that $B : \mathbb{R}^2 \to \mathbb{R}$ is a bi-additive function which is continuous for all $(x, y) \in \mathbb{R}^2$. Let $x, y \in \mathbb{R}$. Using the fact that $\mathbb{Q}$ is dense on $\mathbb{R}$, there exist sequences $\{r_n\}$ and $\{l_n\}$ of rational numbers converging to x and y, respectively. Since B is continuous on $\mathbb{R}^2$, we have

$$B(x, y) = \lim_{n \to \infty} B(r_n, l_n).$$

It then follows from Theorem 2.9 that $B(r_n, l_n) = r_n \cdot l_n B(1, 1)$. Thus

$$B(x, y) = \lim_{n \to \infty} r_n \cdot l_n B(1, 1) = xyB(1, 1).$$

□

Bi-additive functions can be described in terms of Hamel basis as well. For any real numbers x and y with unique representation, we have

$$x = r_1 h_1 + r_2 h_2 + ... + r_k h_k$$

and

$$y = s_1 h_1 + s_2 h_2 + ... + s_l h_l.$$

We will evaluate

$$B(x, y) = B\left(\sum_{i=1}^{k} r_i h_i, \sum_{j=1}^{l} s_j h_j\right) = \sum_{i=1}^{k} \sum_{j=1}^{l} r_i s_j B(h_i, h_j).$$

Thus, the function values $B(h_i, h_j)$, for all $h_i, h_j \in H$, are essential to the value of $B(x, y)$. An example of bi-additive functions on $\mathbb{R}^2$ that contains discontinuity can be easily given in the following example.

Example 2.11. A function $B : \mathbb{R}^2 \to \mathbb{R}$ defined by $B(x, y) = xA(y)$ for all $x, y \in \mathbb{R}$, where A is a discontinuous additive function, is obviously discontinuous.

□

2.3 Multi-Additive Functions

Now the concepts in the previous two sections will be generalized to multi-additive functions.

Definition 2.12. For a positive integer k, a function $A_k : X^k \to Y$ will be called a *k-additive function* if, for all $i = 1, ..., k$,

$$A_k(x_1, ..., x_{i-1}, x_i + y, x_{i+1}, ..., x_k) = A_k(x_1, ..., x_k) + A_k(x_1, ..., x_{i-1}, y, x_{i+1}, ..., x_k) \tag{2.3}$$

for all x_i's ,$y \in X$.

We could also say that A_k is additive in each variable $x_1, ..., x_k$.

Example 2.13. Define a function $A_3 : \mathbb{R}^3 \to \mathbb{R}$ by $A_3(x, y, z) = f(x)g(y)h(z)$, for all $x, y, z \in \mathbb{R}$ where f, g and h are additive functions. Then A_3 is 3-additive.

□

Definition 2.14. A function $A_k : X^k \to Y$ will be called *symmetric* if for each $i, j = 1, ..., k$ such that $i \neq j$

$$A_k(x_1, ..., x_k) = A_k(x_1, ..., x_{i-1}, x_j, x_{i+1}, ..., x_{j-1}, x_i, x_{j+1}, ..., x_k).$$

Example 2.15. A function $A_3 : \mathbb{R}^3 \to \mathbb{R}$ defined by $A_3(x, y, z) = A(x)A(y)A(z)$ for all $x, y, z \in \mathbb{R}$, where A is additive function, is 3-additive and symmetric. □

Example 2.16. A function $A_3 : \mathbb{R}^3 \to \mathbb{R}$ defined by

$$\begin{aligned} A_3(x, y, z) &= f(x)g(y)h(z) - f(x)g(z)h(y) + f(y)g(x)h(z) - f(y)g(z)h(x) \\ &\quad + f(z)g(x)h(y) - f(z)g(y)h(x) \end{aligned}$$

for all $x, y, z \in \mathbb{R}$, where f, g and h are additive functions, is not symmetric but it is 3-additive and skew-symmetric. □

Example 2.17. A function $A_k : \mathbb{R}^k \to \mathbb{R}$ defined by $A_k(x_1, ..., x_k) = cx_1 \cdots x_k$ for all $x_1, ..., x_k \in \mathbb{R}$, where c is a constant, is k-additive and symmetric. □

We will give some important properties of multi-additive functions in the following theorem.

Theorem 2.18. *Let k be a positive integer and let $A_k : X^k \to Y$ be a multi-additive function. For all $x_1, ..., x_k \in X$ and for all $i = 1, ..., k$,*

(i) $A_k(x_1, ..., x_{i-1}, 0, x_{i+1}, ..., x_k) = 0$.

(ii) $A_k(x_1, ..., x_{i-1}, -x_i, x_{i+1}, ..., x_k) = -A_k(x_1, ..., x_k)$.

(iii) $A_k(x_1, ..., x_{i-1}, rx_i, x_{i+1}, ..., x_k) = rA_k(x_1, ..., x_k)$ *for all* $r \in \mathbb{Q}$.

(iv) $A_k(x_1 + y_1, ..., x_k + y_k) = \displaystyle\sum_{\varepsilon_1, ..., \varepsilon_k \in \{0,1\}} A_k(\varepsilon_1 x_1 + (1 - \varepsilon_1)y_1, ..., \varepsilon_k x_k + (1 - \varepsilon_k)y_k)$ *for all x_i's and y_i's in X .*

Moreover, a particular form of all continuous multi-additive functions will be given as in the following theorem.

Theorem 2.19. *Let k be a positive integer. If a k-additive function $A_k : \mathbb{R}^k \to \mathbb{R}$ is continuous on $\mathbb{R}^k$, then*

$$A_k(x_1, ..., x_k) = cx_1 \cdots x_k$$

for all $x_1, ..., x_k \in \mathbb{R}$, where c is a constant.

We can prove this theorem in a similar fashion as Theorem 2.10. Thus, details of the proof are leaved.

Definition 2.20. Let $A_k : X^k \to Y$ be an arbitrary function. A function $A^k : X \to Y$ will be called a *diagonalization of* A_k if

$$A^k(x) = A_k(\underbrace{x, ..., x}_{k}) \text{ for all } x \in X.$$

From Example 2.17, the diagonalization A^k is given in the following example.

Example 2.21. Define a function $A_k : \mathbb{R}^k \to \mathbb{R}$ by $A_k(x_1, ..., x_k) = cx_1 \cdots x_k$ for all $x_1, ..., x_k \in \mathbb{R}$ where c is constant. The diagonalization of A_k is given by

$$A^k(x) = cx^k \text{ for all } x \in \mathbb{R}.$$

□

From Theorem 2.18, we obtain the proposition of A^k as follows.

Proposition 2.22. *Let $A_k : X^k \to Y$ be a k-additive function and $A^k : X \to Y$ is a diagonalization of A_k. Then for all rational numbers r,*

$$A^k(rx) = r^k A^k(x) \text{ for all } x \in X. \tag{2.4}$$

Moreover, a diagonalization $A^k : X \to Y$ of $A_k : X^k \to Y$ at a point $x + y$ can be described in terms of A_k as in the following theorem.

Theorem 2.23. *Let $A_k : X^k \to Y$ be a k-additive function and $A^k : X \to Y$ is a diagonalization of A_k. Then*

$$A^k(x + y) = \sum_{i=0}^{k} \binom{k}{i} A_k(\underbrace{x, ..., x}_{k-i}, \underbrace{y, ..., y}_{i}) \tag{2.5}$$

for all $x, y \in X$.

2.4 Difference Operators

In this section, we will give the definition of the difference operators and some properties. Let X and Y be linear spaces over $\mathbb{Q}$ and let $f : X \to Y$ be an arbitrary function.

Definition 2.24. The *difference operator* with a *span* $h \in X$, denoted by Δ_h, is defined by

$$\Delta_h f(x) = f(x+h) - f(x) \qquad \text{for all} \quad x \in X.$$

The difference operator is a linear operator; that is, for arbitrary function $f_1, f_2 : X \to Y$ and arbitrary constants α, β, we have

$$\Delta_h \left(\alpha f_1(x) + \beta f_2(x)\right) = \alpha \Delta_h f_1(x) + \beta \Delta_h f_2(x).$$

For all $h_1, h_2 \in X$, the iteratives $\Delta_{h_2}\Delta_{h_1} f(x)$ is given as in the following example.

Example 2.25.

$$\begin{aligned}
\Delta_{h_2}\Delta_{h_1} f(x) &= \Delta_{h_2}(f(x+h_1) - f(x)) \\
&= \Delta_{h_2} f(x+h_1) - \Delta_{h_2} f(x) \\
&= f(x+h_1+h_2) - f(x+h_1) - f(x+h_2) + f(x) \\
&= (f(x+h_2+h_1) - f(x+h_2)) - (f(x+h_1) - f(x)) \\
&= \Delta_{h_1} f(x+h_2) - \Delta_{h_1} f(x) \\
&= \Delta_{h_1}(f(x+h_2) - f(x)) \\
&= \Delta_{h_1}\Delta_{h_2} f(x).
\end{aligned}$$

□

Example 2.26. For all $h_1, h_2, h_3 \in X$,

$$\begin{aligned}
\Delta_{h_3}\Delta_{h_2}\Delta_{h_1} f(x) &= f(x+h_1+h_2+h_3) - f(x+h_1+h_2) - f(x+h_1+h_3) \\
&\quad -f(x+h_2+h_3) + f(x+h_1) + f(x+h_2) + f(x+h_3) - f(x).
\end{aligned}$$

□

Moreover, from Example 2.25, we can see that the difference operators are commutative.

Theorem 2.27. *For arbitrary spans h_1 and h_2 in X, we obtain*

$$\Delta_{h_1}\Delta_{h_2} f(x) = \Delta_{h_2}\Delta_{h_1} f(x)$$

where $f : X \to Y$ is an arbitrary function.

That is, the difference operators are invariant under a permutation.

Remark 2.28. $\Delta_{h_1}\Delta_{h_2}...\Delta_{h_m}$ *will be denoted succinctly by* $\Delta_{h_1,...,h_m}$*.*

Thus, we can expand the iteratives $\Delta_{h_1,...,h_m}$ by the following theorem.

Theorem 2.29. *Let m be a positive integer and let $f : X \to Y$ be a function*

$$\Delta_{h_1,...,h_m} f(x) = \sum_{\varepsilon_1,...,\varepsilon_m \in \{0,1\}} (-1)^{m-(\varepsilon_1+...+\varepsilon_m)} f(x + \varepsilon_1 h_1 + ... + \varepsilon_m h_m) \quad (2.6)$$

for all $h_1, ..., h_m \in X$.

Furthermore, the iteratives with the same spans $h \in X$ can be written shortly by

$$\Delta_h^m f(x) = \Delta_h \left(\Delta_h^{m-1} f(x)\right)$$

where m is a positive integer. In the case of $m = 0$, we will define $\Delta_h^0 f(x)$ by $f(x)$. Then $\Delta_h^m f(x)$ can be also expanded as the following:

Theorem 2.30. *Let m be a positive integer and let $f : X \to Y$ be a function*

$$\Delta_h^m f(x) = \sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} f(x + ih) \quad (2.7)$$

for all $h \in X$.

Moreover, there are some properties of the difference operators which will be useful later on. For more details, please refer to S.Czerwik [4] and M. Kuczma [10].

Lemma 2.31. *Let $A_k : X^k \to Y$ be a symmetric k-additive function and let $A^k : X \to Y$ be a diagonalization of A_k. Then for arbitrary $m \geq k$ and for every $x, h_1, ..., h_m \in X$, we have*

$$\Delta_{h_1,...,h_m} A^k(x) = \begin{cases} k! A_k(h_1, ..., h_k) & \text{if } m = k \\ 0 & \text{if } m > k. \end{cases}$$

Lemma 2.32. *Let $f : X \to Y$ be an arbitrary function. For all $h_1, h_2 \in X$,*

$$\Delta_{h_1+h_2} f(x) - \Delta_{h_1} f(x) - \Delta_{h_2} f(x) = \Delta_{h_1,h_2} f(x).$$

Theorem 2.33. *Let $f : X \to Y$ be a function and let $h_1, \ldots, h_m \in X$ be arbitrary. For $\varepsilon_1, \ldots, \varepsilon_m \in \{0, 1\}$, define*

$$\alpha_{\varepsilon_1 \ldots \varepsilon_m} = -\sum_{r=1}^{m} \frac{\varepsilon_r h_r}{r}$$

and

$$b_{\varepsilon_1 \ldots \varepsilon_m} = -\sum_{r=1}^{m} \varepsilon_r h_r.$$

Then for every $x \in X$,

$$\Delta_{h_1,\ldots,h_m} f(x) = \sum_{\varepsilon_1,\ldots,\varepsilon_m \in \{0,1\}} (-1)^{\varepsilon_1 + \ldots + \varepsilon_m} \Delta^m_{\alpha_{\varepsilon_1 \ldots \varepsilon_m}} f\left(x + b_{\varepsilon_1 \ldots \varepsilon_m}\right).$$

The last theorem shows that $\Delta_{h_1,\ldots,h_m} f(x)$ can be written in terms of $\Delta^m_h f(x)$ for various values of x and h.

CHAPTER III

POLYNOMIAL-TYPE FUNCTIONAL EQUATIONS

The characterization of functional equations is crucial for the study of their solutions. One of the most studied type is the polynomial-type functional equation. In this chapter, we will start with the polynomial-type functional equation of order 1, and then move on to the polynomial-type functional equation of order 2 and of higher orders. Finally, we will conclude this chapter with the Fréchet functional equation, which is a particular example of polynomial-type functional equations.

3.1 Polynomial-Type Functional Equations of order 1

The additive functional equation,

$$f(x+y) = f(x) + f(y), \tag{3.1}$$

and the Jensen functional equation,

$$f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2},$$

are classical examples of the polynomial-type functional equations of order 1. Both equations are closely related to each other.

Theorem 3.1. *A function $f : X \to Y$ is a solution of the Jensen functional equation,*

$$f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2} \text{ for all } x, y \in X \tag{3.2}$$

if and only if there exists a constant $A^0 \in X$ and an additive function $A : X \to Y$ such that

$$f(x) = A^0 + A(x) \text{ for all } x \in X. \tag{3.3}$$

Clearly, any solution of the additive functional equation is also a solution of the Jensen functional equation. In addition, the graph of a discontinuous solution of the Jensen functional equation must also be a dense subset of $\mathbb{R}^2$.

We can describe the Jensen functional equation in terms of difference operators. If we let $h = \frac{y-x}{2}$, then Eq.(3.2) becomes

$$f(x+2h) - 2f(x+h) + f(x) = 0 \text{ for all } x, h \in X.$$

That is,

$$\Delta_h^2 f(x) = 0 \text{ for all } x, h \in X.$$

Likewise, the additive functional equation, $f(x+y) - f(x) = f(y)$, can be written as

$$\Delta_y f(x) = f(y)$$

which follows that

$$\Delta_y^2 f(x) = 0.$$

The following example shows that there are functional equations that can be reduced to the Cauchy functional equation or the Jensen functional equation.

Example 3.2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the equation

$$f(x+y) = f(x) + f(y) - a \text{ for all } x, y \in \mathbb{R} \tag{3.4}$$

where a is a constant.

Proof. Define a function $A : \mathbb{R} \to \mathbb{R}$ by

$$A(x) = f(x) - a \text{ for all } x \in \mathbb{R}.$$

Let $x, y \in \mathbb{R}$. We then obtain

$$\begin{aligned} A(x+y) &= f(x+y) - a \\ &= f(x) + f(y) - 2a \\ &= A(x) + A(y). \end{aligned}$$

Thus, A is an additive function. As a consequence, we conclude that $f(x) = A(x) + a$.

Conversely, we can verify that $f(x) = A(x) + a$ indeed satisfies Eq.(3.4). $\square$

3.2 Polynomial-Type Functional Equations of order 2

A well-known polynomial-type functional equation of order 2 is the *classical* quadratic functional equation,

$$f(x+y) + f(x-y) = 2f(x) + 2f(y). \tag{3.5}$$

Symmetric and bi-additive functions will be very useful to the study of the general solution of the quadratic functional equation.

Theorem 3.3. *A function $f : X \to Y$ satisfies Eq.(3.5) for all $x, y \in X$ if and only if there exists a unique symmetric bi-additive function $A_2 : X^2 \to Y$ such that*

$$f(x) = A_2(x,x) \text{ for all } x \in X. \tag{3.6}$$

The function A_2 is given by $A_2(x,y) = \frac{1}{2}(f(x+y) - f(x) - f(y))$ for all $x, y \in X$. That is, the general solution of a quadratic functional equation is a diagonalization A^2 of a symmetric bi-additive function A_2. The continuous solutions of Eq.(3.5) on $\mathbb{R}$ is a polynomial function of degree 2, $f(x) = cx^2$ where c is a constant. If we replace x by $x+y$ in Eq.(3.5) and rearrange the result, then we obtain

$$f(x+2y) - 2f(x+y) + f(x) = 2f(y)$$

which can be simplified to

$$\Delta_y^2 f(x) = 2f(y).$$

Thus,

$$\Delta_y^3 f(x) = 0.$$

The following theorem gives an example of polynomial-type functional equations of order 2 in a non-classical form and was established by W. Towanlong and P. Nakmahachalasint [16].

Theorem 3.4. *Let X and Y be real vector spaces. A function $f : X \to Y$ satisfies*

$$f(3x+y) + f(3x-y) = f(x+y) + f(x-y) + 16f(x) \tag{3.7}$$

for all $x, y \in X$ if and only if f satisfies Eq.(3.5) for all $x, y \in X$.

Proof. Let us start the proof by assuming that a function $f : X \to Y$ satisfies Eq.(3.7) for all $x, y \in X$. Putting $(x, y) = (0, 0)$ in Eq.(3.7), we get $f(0) = 0$. Setting $y = x$ in Eq.(3.7), we obtain

$$f(4x) = 16f(x) \tag{3.8}$$

for all $x \in X$. Putting $(x, y) = (x, x - y)$ in Eq.(3.7) again, we have

$$f(4x - y) + f(2x + y) = f(2x - y) + f(y) + 16f(x). \tag{3.9}$$

Reverse the sign of y in Eq.(3.9),

$$f(4x + y) + f(2x - y) = f(2x + y) + f(-y) + 16f(x).$$

Adding the previous resulting equation to Eq.(3.9), it yields

$$f(4x - y) + f(4x + y) = f(y) + f(-y) + 32f(x). \tag{3.10}$$

Replacing y with $4y$ in Eq.(3.10) and using Eq.(3.8), we will be left with

$$f(x + y) + f(x - y) = f(y) + f(-y) + 2f(x). \tag{3.11}$$

Putting $y = x$ in Eq.(3.11), we get

$$f(2x) = 3f(x) + f(-x). \tag{3.12}$$

Reversing the sign of x in Eq.(3.12) gives us $f(-2x) = 3f(-x) + f(x)$. Replacing x with $2x$ in Eq.(3.12) and taking into account Eq.(3.8), we obtian

$$16f(x) = 3f(2x) + f(-2x) = 3\big(3f(x) + f(-x)\big) + \big(3f(-x) + f(x)\big)$$

which simplifies to $f(x) = f(-x)$ for all $x \in X$. Hence, Eq.(3.11) reduces to Eq.(3.5).

Suppose a function $f : X \to Y$ satisfies Eq.(3.5). It then follows from Eq.(2.4) and Eq.(3.6) that f possesses a quadratic property; i.e., $f(nx) = A^2(nx) = n^2A^2(x) = n^2f(x)$ for all integers n and for all $x \in X$. Thus,

$$\begin{aligned} f(3x + y) + f(3x - y) &= 2f(3x) + 2f(y) \\ &= 18f(x) + 2f(y). \end{aligned}$$

Applying Eq.(3.5) again, we can see that Eq.(3.7) immediately follows, that is,

$$f(3x + y) + f(3x - y) = f(x + y) + f(x - y) + 16f(x).$$

□

The following example gives an interesting polynomial-type functional equation of order 2. Its solution is a mix of an additive function and a quadratic function.

Example 3.5. P. Nakmahachalasint [12] investigated that the general solution of mixed-type functional equation,

$$f\left(\sum_{i=1}^{n} x_i\right) + (n-2)\sum_{i=1}^{n} f(x_i) = \sum_{1\le i<j\le n} f(x_i + x_j) \text{ for all } x_1, ..., x_n \in X \quad (3.13)$$

where $f : X \to Y$ is any function and X, Y are linear spaces over $\mathbb{Q}$, consists of an additive function and a quadratic function, namely,

$$f(x) = A^1(x) + A^2(x) \text{ for all } x \in X.$$

3.3 Polynomial-Type Functional Equations of Higher Orders

Let m be a positive integer. A functional equation

$$\Delta_h^m f(x) = m! f(h) \quad (3.14)$$

is a classical example of polynomial-type functional equations of order m. Letting x by $x + y$ in Eq.(3.14) and then taking the difference between the result and Eq.(3.14), we obtain

$$\Delta_h^m f(x + y) - \Delta_h^m f(x) = 0,$$

that is,

$$\Delta_h^{m+1} f(x) = 0. \quad (3.15)$$

The general solution of Eq.(3.14) is given in the following theorem.

Theorem 3.6. *A function $f : X \to Y$ satisfies Eq.(3.14) for all $x, h \in X$ if and only if there exists a symmetric m-additive function $A_m : X^m \to Y$ such that*

$$f(x) = A^m(x) \text{ for all } x \in X$$

where A^m is a diagonalization of A_m.

Proof. In order to prove the theorem, assume that a function $f : X \to Y$ satisfies Eq.(3.14) for all $x, h \in X$. Substitute (x, h) by $(0, x)$ and rearrange the result in the form

$$f(x) = \frac{1}{m!}\Delta_x^m f(0). \tag{3.16}$$

Define a function $A_m : X^m \to Y$ by

$$A_m(x_1, ..., x_m) = \frac{1}{m!}\Delta_{x_1,...,x_m} f(0).$$

We claim that A_m is symmetric m-additive. Since the difference operators are commutative, A_m is symmetric. For all $h, x_1, ..., x_m \in X$, consider

$$\begin{aligned} & A_m(x_1 + h, ..., x_m) - A_m(x_1, ..., x_m) - A_m(h, x_2, ..., x_m) \\ = \; & \frac{1}{m!}\left(\Delta_{(x_1+h),...,x_m} f(0) - \Delta_{x_1,...,x_m} f(0) - \Delta_{h,x_2,...,x_m} f(0)\right) \\ = \; & \frac{1}{m!}\left(\Delta_{x_2,...,x_m}(\Delta_{x_1+h} f(0) - \Delta_{x_1} f(0) - \Delta_h f(0))\right). \end{aligned} \tag{3.17}$$

From Lemma 2.32, we then obtain that

$$\begin{aligned} \frac{1}{m!}\left(\Delta_{x_2,...,x_m}(\Delta_{x_1+h} f(0) - \Delta_{x_1} f(0) - \Delta_h f(0))\right) &= \frac{1}{m!}\Delta_{x_2,...,x_m}\Delta_{x_1,h} f(0) \\ &= \frac{1}{m!}\Delta_{x_1,...,x_m,h} f(0). \end{aligned}$$

By applying Theorem 2.33 and Eq.(3.15) together, then the previous equation becomes

$$\frac{1}{m!}\Delta_{x_1,...,x_m,h} f(0) = 0.$$

Thus, Eq.(3.17) is reduced to

$$A_m(x_1 + h, ..., x_m) = A_m(x_1, ..., x_m) + A_m(h, x_2, ..., x_m).$$

That is, A_m is additive in the first variable. Since A_m is symmetric, it will follow that A_m is additive for each variable. Thus, A_m is symmetric m-additive. Hence, from Eq.(3.16), we have $f(x) = A^m(x)$.

On the other hand, assume that there exists a symmetric m-additive function $A_m : X^m \to Y$ such that

$$f(x) = A^m(x) \text{ for all } x \in X.$$

It then follows from Lemma 2.31 that for all $y \in X$,

$$\Delta_h^m f(x) = \Delta_y^m A^m(x) = m!A^m(y) = m!f(y).$$

□

The following section, we will introduce a particular form of polynomial-type functional equation which will be of our main interest in this dissertation.

3.4 Fréchet Functional Equations

The generalization of polynomial functional equations was studied since 1909 by M. Fréchet [6]. He determined the continuous solutions of the Fréchet functional equation on $\mathbb{R}$ and found the following result (see [1]).

Theorem 3.7. *(Fréchet, 1909)*
Let us consider the operator

$$\begin{aligned}\mathcal{F}_{m+1}(f)(h_1, ..., h_{m+1}) &= f(h_1 + ... + h_{m+1}) + \sum_{i=1}^{m}(-1)^t \\ &\sum_{\{i_1,...,i_{m+1-t}\}\in\mathcal{P}_t(m+1)} f(h_{i_1} + ... + h_{i_{m+1}}) + (-1)^{m+1}f(0),\end{aligned}$$

where $h_1, ..., h_{m+1}$ are real variables and

$$\mathcal{P}_t(m+1) = \{A \subset \{1, 2, ..., m+1\} : A = m+1-t\}, t = 1, 2, ..., m.$$

A continuous function $f : \mathbb{R} \to \mathbb{R}$ is a polynomial of degree less than or equal to m if and only if $\mathcal{F}_{m+1}(f)$ vanishes identically on $\mathbb{R}^{m+1}$.

The functional equation mentioned in Fréchet's Theorem can be conveniently interpreted by the difference operators, namely,

$$\mathcal{F}_m(f)(h_1, ..., h_{m+1}) = \Delta_{h_1,...,h_m} f(x).$$

Definition 3.8. The *Fréchet functional equation* is an equation of the form

$$\Delta_{h_1,...,h_{m+1}} f(x) = 0. \tag{3.18}$$

Eq.(3.18) was named after M. Fréchet. In addition, if the spans h_i's are constrained to the same value, then Eq.(3.18) reduced to

$$\Delta_h^{m+1} f(x) = 0. \tag{3.19}$$

Definition 3.9. Eq.(3.18) will be called a *nonsymmetric Fréchet functional equation*, and Eq.(3.19) will be called a *symmetric Fréchet functional equation*.

Definition 3.10. A function $f : X \to Y$ satisfying Eq.(3.19) for all $x, h \in X$ will be called a *generalized polynomial function of order m*.

The general solution of the symmetric Fréchet functional equation was given in 1967 by M.A. McKiernan [11] and takes the form as stated in the following theorem.

Theorem 3.11. *Let m be a nonnegative integer. A function $f : X \to Y$ is a polynomial function of order m if and only if there exist k-additive symmetric functions $A_k : X^k \to Y, k = 0, 1, 2, \ldots, m$ such that*

$$f(x) = A^0 + A^1(x) + A^2(x) + \ldots + A^m(x) \tag{3.20}$$

for all $x \in X$ where A^0 is a constant and $A^k : X \to Y, k = 1, 2, \ldots, m$ is the diagonalization of A_k.

For a nonsymmetric Fréchet functional equation, the Theorem 2.33 tells us that the general solution of the equation is still a generalized polynomial function of order m.

Theorem 3.12. *A function $f : X \to Y$ is a polynomial function of order m if and only if $\Delta_{h_1,...,h_{m+1}} f(x) = 0$ for every $x, h_1, ..., h_m \in X$.*

CHAPTER IV

FRÉCHET FUNCTIONAL EQUATIONS WITH RESTRICTED SPANS

In this chapter, we will give the general solution of the Fréchet functional equation on the set of real numbers, in both nonsymmetric and symmetric form, when the spans are restricted to any open intervals.

4.1 Functional Equation with Restricted Spans

It is quite natural to investigate the functional equation on the entire domain of function, but it would be considerably questionable to restrict the domain of variables; that is, the domains of variables will not assumed in full all possible values of the variables. As a matter of fact, we could possibly obtain other solutions of the equation if the domains of variables are restricted. For example, we can define a set $S(\Omega)$, where $\Omega \subseteq \mathbb{R}^2$, of solutions of the Cauchy functional equation, by

$$S(\Omega) = \{ f : \mathbb{R} \to \mathbb{R} \,|\, f(x+y) = f(x) + f(y) \text{ for all } (x,y) \in \Omega \} . \tag{4.1}$$

In general, if $\Omega_1 \subseteq \Omega_2 \subseteq \mathbb{R}^2$, then $S(\Omega_1) \supseteq S(\Omega_2)$. There exist some $\Omega \subset \mathbb{R}^2$ such that $S(\Omega) = S(\mathbb{R}^2)$ as in the following example.

Example 4.1. Let $a \in \mathbb{R}$ be a constant and let $\Omega = \mathbb{R} \times (a, \infty)$. It is readily seen that $S(\Omega) \supseteq S(\mathbb{R}^2)$; thus, it remains to prove that $S(\Omega) \subseteq S(\mathbb{R}^2)$. Let x and y be arbitrary real numbers. Choose $z \in \mathbb{R}$ such that $z > \max\{a - y, a\}$. Thus, $y + z > a$ and $z > a$. Since $(x + y, z) \in \Omega$, we have

$$f(x+y+z) = f(x+y) + f(z). \tag{4.2}$$

Similary, since $(x, y+z) \in \Omega$ and $(y, z) \in \Omega$,

$$f(x+y+z) = f(x) + f(y+z) = f(x) + f(y) + f(z). \tag{4.3}$$

It follows from Eq.(4.2) and Eq.(4.3) that $f(x+y) = f(x) + f(y)$.

That is, $S(\Omega) \subseteq S(\mathbb{R}^2)$ as desired. □

4.2 Nonsymmetric Fréchet Functional Equation with Restricted Spans

In this section, we will determine the general solution of the nonsymmetric Fréchet functional equation, $\Delta_{h_1,\ldots,h_{m+1}} f(x) = 0$, when the spans, h_i's, are restricted to arbitrary open intervals (a_i, b_i) for each $i = 1, \ldots, m+1$. The following proofs are given by W. Towanlong and P. Nakmahachalasint [17]. We will define some notations that will be used throughout this section. Let m be a nonnegative integer and let $\varepsilon_i \in \{0, 1\}$ for every $i = 1, \ldots, m+1$. Denote

$$E_{m+1} = \{\varepsilon_1, \ldots, \varepsilon_{m+1}\}$$

and the sum

$$\sum_{E_{m+1}} = \sum_{\varepsilon_1,\ldots,\varepsilon_{m+1} \in \{0,1\}}.$$

For any finite set A of integers, we define a function S of A by

$$S(A) = \prod_{\varepsilon \in A} (-1)^{1-\varepsilon}.$$

Adopting these notations, Eq.(2.6) can be expressed as

$$\Delta_{h_1,\ldots,h_m} f(x) = \sum_{E_{m+1}} S(E_{m+1}) f(x + \varepsilon_1 h_1 + \ldots + \varepsilon_m h_m). \tag{4.4}$$

Lemma 4.2. *Let m be a nonnegative integer and let $a_i, b_i \in \mathbb{R}$ with $a_i < b_i$ for each $i = 1, \ldots, m+1$. If a function $f : \mathbb{R} \to \mathbb{R}$ satisfies the Fréchet functional equation*

$$\Delta_{h_1,\ldots,h_{m+1}} f(x) = 0 \tag{4.5}$$

for all $x \in \mathbb{R}$ and for all $h_i \in (a_i, b_i), i = 1, \ldots, m+1$, then f also satisfies Eq.(4.5) for all $x \in \mathbb{R}$ and for all $h_i \in (n_i a_i, n_i b_i)$ for all positive integers $n_1, n_2, \ldots, n_{m+1}$.

Proof. Suppose that a function $f:\mathbb{R}\to\mathbb{R}$ satisfies the hypothesis of the theorem. For any spans $y_i \in (a_i,b_i), i=1,...,m+1$. We can see that for a nonnegative integer j,

$$\Delta_{y_1,\dots,y_{m+1}} f(x+jy_1)=0$$

for all $x\in\mathbb{R}$. Let $x\in\mathbb{R}$. Let n_1 be a positive integer. Then

$$\sum_{j=0}^{n_1-1}\Delta_{y_1,\dots,y_{m+1}} f(x+jy_1)=0. \tag{4.6}$$

Applying Eq.(4.4) to Eq.(4.6), it yields

$$\sum_{j=0}^{n_1-1}\sum_{E_{m+1}} S(E_{m+1})f\left(x+(\varepsilon_1+j)y_1+\sum_{k=2}^{m+1}\varepsilon_k y_k\right)=0.$$

Note that in the case $m=0$, $\sum_{k=2}^{m+1}\varepsilon_k y_k$ is understood to be zero. If we evaluate the sum over ε_1 in E_{m+1}, then we get

$$\sum_{j=0}^{n_1-1}\sum_{E_{m+1}\setminus\{\varepsilon_1\}} S(E_{m+1}\setminus\{\varepsilon_1\}) \left(f\left(x+(1+j)y_1+\sum_{k=2}^{m+1}\varepsilon_k y_k\right)-f\left(x+jy_1+\sum_{k=2}^{m+1}\varepsilon_k y_k\right)\right)=0. \tag{4.7}$$

Swapping the order of two summations, it can be observed that

$$\sum_{j=0}^{n_1-1}\left(f\left(x+(1+j)y_1+\sum_{k=2}^{m+1}\varepsilon_k y_k\right)-f\left(x+jy_1+\sum_{k=2}^{m+1}\varepsilon_k y_k\right)\right) = f\left(x+n_1y_1+\sum_{k=2}^{m+1}\varepsilon_k y_k\right)-f\left(x+\sum_{k=2}^{m+1}\varepsilon_k y_k\right).$$

Then Eq.(4.7) is reduced to

$$\sum_{E_{m+1}\setminus\{\varepsilon_1\}} S(E_{m+1}\setminus\{\varepsilon_1\})\left(f\left(x+n_1y_1+\sum_{k=2}^{m+1}\varepsilon_k y_k\right)-f\left(x+\sum_{k=2}^{m+1}\varepsilon_k y_k\right)\right)=0$$

which can be rewritten as

$$\sum_{E_{m+1}\setminus\{\varepsilon_1\}} S(E_{m+1}\setminus\{\varepsilon_1\})\left(\sum_{\varepsilon_1\in\{0,1\}}(-1)^{1-\varepsilon_1}f\left(x+\varepsilon_1 n_1y_1+\sum_{k=2}^{m+1}\varepsilon_k y_k\right)\right)=0.$$

That is,

$$\sum_{E_{m+1}} S(E_{m+1}) f\left(x + \varepsilon_1 n_1 y_1 + \sum_{k=2}^{m+1} \varepsilon_k y_k\right) = 0.$$

Using Eq.(4.4) again, it becomes

$$\Delta_{n_1 y_1, y_2, \ldots, y_{m+1}} f(x) = 0.$$

Therefore, we can see that the domain of validity of y_1 can be extended from the open interval (a_1, b_1) to $(n_1 a_1, n_1 b_1)$. Since the order of the difference operators can be permuted; that is, $\Delta_{y_1, y_2, \ldots, y_{m+1}} f(x) = \Delta_{y_2, y_1, y_3, \ldots, y_{m+1}} f(x)$ for all $i = 1, \ldots, m+1$, the domain of validity of each variable y_i can be extended from (a_i, b_i) to $(n_i a_i, n_i b_i)$ for each i and for every positive integers n_i. Thus, we obtain

$$\Delta_{n_1 y_1, \ldots, n_{m+1} y_{m+1}} f(x) = 0.$$

Therefore,

$$\Delta_{h_1, \ldots, h_{m+1}} f(x) = 0$$

for all $x \in \mathbb{R}$ and for all $h_i \in (n_i a_i, n_i b_i)$ when $n_1, n_2, \ldots, n_{m+1}$ are arbitrary positive integers. □

The next lemma will extend the domain of the spans from half-open infinite intervals to $\mathbb{R}$.

Lemma 4.3. *Let m and n be nonnegative integers with $n \leq m+1$. Let $p_1, \ldots, p_{m+1} \in \mathbb{R}$. If a function $f : \mathbb{R} \to \mathbb{R}$ satisfies Eq.(4.5) for all $x \in \mathbb{R}$ and for all $h_i > p_i$ when $i = 1, \ldots, n$ and for all $h_i < p_i$ when $i = n+1, \ldots, m+1$, then Eq.(4.5) also holds for all $x, h_1, \ldots, h_{m+1} \in \mathbb{R}$.*

Proof. Suppose a function $f : \mathbb{R} \to \mathbb{R}$ satisfies equation

$$\Delta_{h_1, \ldots, h_{m+1}} f(x) = 0 \tag{4.8}$$

for all $x \in \mathbb{R}$ and for all $h_i > p_i$ when $i = 1, \ldots, n$ and for all $h_i < p_i$ when $i = n+1, \ldots, m+1$. Let $x, h_1, \ldots, h_{m+1} \in \mathbb{R}$. Choose

$$t_i = \begin{cases} \max\{p_i, p_i + h_i\} + 1 & \text{if} \quad i = 1, \ldots, n, \\ \min\{p_i, p_i + h_i\} - 1 & \text{if} \quad i = n+1, \ldots, m+1. \end{cases}$$

Then for each $i = 1, ..., m+1$, we obtain that for all $t_i - \varepsilon_i h_i > p_i$ for $\varepsilon_i \in \{0,1\}$. Similarly, for each $i = n+1, ..., m+1$, $t_i - \varepsilon_i h_i < p_i$. Consider Eq.(4.8) at the point $x + \sum_{i=1}^{m+1} \varepsilon_i h_i$ with spans $t_i - \varepsilon_i h_i$'s. It follows that

$$\Delta_{(t_1-\varepsilon_1 h_1),...,(t_{m+1}-\varepsilon_{m+1}h_{m+1})} f\left(x + \sum_{i=1}^{m+1} \varepsilon_i h_i\right) = 0$$

and then for $E_{m+1} = \{\varepsilon_1, ..., \varepsilon_{m+1}\}$,

$$\sum_{E_{m+1}} S(E_{m+1}) \Delta_{(t_1-\varepsilon_1 h_1),...,(t_{m+1}-\varepsilon_{m+1}h_{m+1})} f\left(x + \sum_{i=1}^{m+1} \varepsilon_i h_i\right) = 0 \qquad (4.9)$$

where $S(E_{m+1}) = \prod_{\varepsilon_i \in E_{m+1}} (-1)^{1-\varepsilon_i}$. Using Eq.(4.4), we can rewrite Eq.(4.9) as follows:

$$\sum_{E_{m+1}} S(E_{m+1}) \sum_{D_{m+1}} S(D_{m+1}) f\left(x + \sum_{i=1}^{m+1} \varepsilon_i h_i + \sum_{i=1}^{m+1} \delta_i (t_i - \varepsilon_i h_i)\right) = 0$$

where we have used $D_{m+1} = \{\delta_1, ..., \delta_{m+1}\}$ and $\delta_1, ..., \delta_{m+1} \in \{0,1\}$ in order to avoid a conflict of notations. Swapping two summations, it yields

$$\sum_{D_{m+1}} S(D_{m+1}) \sum_{E_{m+1}} S(E_{m+1}) f\left(x + \sum_{i=1}^{m+1} \varepsilon_i h_i + \sum_{i=1}^{m+1} \delta_i (t_i - \varepsilon_i h_i)\right) = 0. \qquad (4.10)$$

We first consider the term $\sum_{E_{m+1}} S(E_{m+1}) f\left(x + \sum_{i=1}^{m+1} \varepsilon_i h_i + \sum_{i=1}^{m+1} \delta_i (t_i - \varepsilon_i h_i)\right)$ when there exists $\delta_k = 1$ for some $k \in \{1, ..., m+1\}$. Let j be the minimum of such k. We obtain

$$\sum_{E_{m+1}} S(E_{m+1}) f\left(x + \sum_{i=1}^{m+1} \varepsilon_i h_i + \sum_{i=1}^{m+1} \delta_i (t_i - \varepsilon_i h_i)\right) =$$
$$\sum_{E_{m+1}} S(E_{m+1}) f\left(x + t_j + \sum_{\substack{i=1 \\ i \neq j}}^{m+1} ((1-\delta_i)\varepsilon_i h_i + \delta_i t_i)\right).$$

We then separate the sum over ε_j to get that

$$\sum_{E_{m+1}} S(E_{m+1}) f\left(x + t_j + \sum_{\substack{i=1 \\ i \neq j}}^{m+1} ((1-\delta_i)\varepsilon_i h_i + \delta_i t_i)\right)$$
$$= \sum_{E_{m+1} \setminus \{\varepsilon_j\}} S(E_{m+1} \setminus \{\varepsilon_j\}) \sum_{\varepsilon_j \in \{0,1\}} (-1)^{1-\varepsilon_j} f\left(x + t_j + \sum_{\substack{i=1 \\ i \neq j}}^{m+1} ((1-\delta_i)\varepsilon_i h_i + \delta_i t_i)\right).$$

Since the term $f\left(x+t_j+\sum_{\substack{i=1\\i\neq j}}^{m+1}((1-\delta_i)\varepsilon_i h_i+\delta_i t_i)\right)$ does not depend on ε_j,

$$\sum_{\varepsilon_j\in\{0,1\}}(-1)^{1-\varepsilon_j}f\left(x+t_j+\sum_{\substack{i=1\\i\neq j}}^{m+1}((1-\delta_i)\varepsilon_i h_i+\delta_i t_i)\right)=0.$$

Therefore, we can conclude that

$$\sum_{E_{m+1}}S(E_{m+1})f\left(x+\sum_{i=1}^{m+1}\varepsilon_i h_i+\sum_{i=1}^{m+1}\delta_i(t_i-\varepsilon_i h_i)\right)=0 \tag{4.11}$$

whenever there exists $\delta_k=1$ for some $k\in\{1,...,m+1\}$. Therefore, the sum over D_{m+1} in Eq.(4.10) will reduce to a single term where $\delta_1=...=\delta_{m+1}=0$; that is,

$$\sum_{E_{m+1}}(-1)^{(\varepsilon_1+...+\varepsilon_{m+1})}f\left(x+\sum_{i=1}^{m+1}\varepsilon_i h_i\right)=0,$$

which simply concludes that, $\Delta_{h_1,...,h_{m+1}}f(x)=0$. □

Having accomplished the previous two lemmas, we are now ready to extend the domain of validity of spans h_i's from any open interval (a_i,b_i) to the set of real numbers $\mathbb{R}$.

Theorem 4.4. *Let m be a nonnegative integer and let $a_i,b_i\in\mathbb{R}$ with $a_i<b_i$ for each $i=1,...,m+1$. If $f:\mathbb{R}\to\mathbb{R}$ is a function fulfilling Eq.(4.5) for all $x\in\mathbb{R}$ and all $h_i\in(a_i,b_i), i=1,...,m+1$, then Eq.(4.5) also holds for all $x,h_1,...,h_{m+1}\in\mathbb{R}$.*

Proof. Suppose a function $f:\mathbb{R}\to\mathbb{R}$ satisfies Eq.(4.5) for all $x,h_1,...,h_{m+1}\in\mathbb{R}$. We apply Lemma 4.2 to get that Eq.(4.5) holds for all $x\in\mathbb{R}$ and for all $h_i\in(n_ia_i,n_ib_i), i=1,...,m+1$ for every positive integers $n_1,n_2,...,n_{m+1}$. Define $H_i=\bigcup_{k=1}^{\infty}(ka_i,kb_i), i=1,...,m+1$. Then f also fulfills

$$\Delta_{h_1,...,h_{m+1}}f(x)=0 \tag{4.12}$$

for all $x\in\mathbb{R}$ and for all $h_i\in H_i, i=1,...,m+1$. Each interval (a_i,b_i) will satisfy one of the following 3 cases;

case 1: $a_i<0<b_i$. Then $H_i=(-\infty,\infty)$.

case 2: $0 \leq a_i < b_i$. Choose an integer $M_i > \frac{a_i}{b_i - a_i}$. Then $jb_i - (j+1)a_i = j(b_i - a_i) - a_i > 0$ for all $j > M_i$. It follows that $(M_i a_i, \infty) \subset H_i$.

case 3: $a_i < b_i \leq 0$. Choose an integer $m_i > \frac{-b_i}{b_i - a_i}$. Thus, we get that $(j+1)b_i - ja_i = j(b_i - a_i) + b_i > 0$ for all $j > m_i$; consequently, $(-\infty, m_i b_i) \subset H_i$.

We note that if $a_i < 0 < b_i$ for all $i = 1, ..., m+1$, then f satisfies Eq.(4.5) for all $x, h_i \in \mathbb{R}, i = 1, ..., m+1$. For each $i = 1, ..., m+1$, let

$$p_i = \begin{cases} 0 & \text{if} \quad a_i < 0 < b_i, \\ M_i a_i & \text{if} \quad 0 \leq a_i < b_i, \\ m_i b_i & \text{if} \quad a_i < b_i \leq 0. \end{cases}$$

Thus, it follows from Eq.(4.12) and the cases that Eq.(4.5) holds for all $x \in \mathbb{R}$ and for each $i = 1, ..., m+1$, for all $h_i > p_i$ or for all $h_i < p_i$. Since the order of the difference operators can be permuted, we can rearrange the order of the difference operators to

$$\Delta_{h_{j_1}, ..., h_{j_{m+1}}} f(x) = 0$$

where $h_{j_i} > p_{j_i}$ if $i = 1, ..., n$ and $h_{j_i} < p_{j_i}$ if $i = n+1, ..., m+1$ for some nonnegative integer n with $n \leq m+1$. Therefore, we can apply Lemma 4.3, and then it yields that f also fulfills Eq.(4.5) for all $x, h_i \in \mathbb{R}, i = 1, ..., m+1$ as desired. □

The following corollary will show as that the general solution of the Fréchet functional equation when the spans are restricted to open intervals $(a_i, b_i), i = 1, ..., m+1$, is still a generalized polynomial function of order m.

Corollary 4.5. *Let m be a nonnegative integer and let $a_i, b_i \in \mathbb{R}$ with $a_i < b_i$ for each $i = 1, ..., m+1$. A function $f : \mathbb{R} \to \mathbb{R}$ satisfies Eq.(4.5) for all $x \in \mathbb{R}$ and all $h_i \in (a_i, b_i), i = 1, ..., m+1$ if and only if the function f is a generalized polynomial function of order m given by Eq.(3.20).*

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ satisfy an Eq.(4.5) for all $x \in \mathbb{R}$ and all $h_i \in (a_i, b_i), i = 1, ..., m+1$. Applying Theorem 4.4, we obtain that Eq.(4.5) also holds for all

$x, h_i \in \mathbb{R}, i = 1, ..., m+1$. Thus, it follows from Theorem 3.12 that f takes the form as Eq.(3.20). Conversely, from Theorems 3.11 and 3.12, it is obvious that f satisfies Eq.(4.5) for all $x \in \mathbb{R}$ and all $h_i \in (a_i, b_i), i = 1, ..., m+1$. □

4.3 Symmetric Fréchet Functional Equation with Restricted Span

In the previous section, we have determined that the general solution of the nonsymmetric Fréchet functional equation, $\Delta_{h_1} ... \Delta_{h_{m+1}} f(x) = 0$ with restricted spans is a generelized polynomial function of order m. In this section, we will treat a relatively challenging situation when all spans h_i's are constrained to the same value; that is, we will determine the general solution of the symmetric Fréchet functional equation, $\Delta_h^{m+1} f(x) = 0$, when the span h is restricted to an arbitrary open interval $(a, b) \subseteq \mathbb{R}$.

We will first prove that the domain of validity of the span can be extended from any nonempty open interval $(a, b) \subseteq \mathbb{R}$ to the open interval (na, nb) for every positive integer n.

Lemma 4.6. *Let $a, b \in \mathbb{R}$ with $a < b$ and let m be a nonnegative integer. If a function $f : \mathbb{R} \to \mathbb{R}$ satisfies the equation*

$$\Delta_h^{m+1} f(x) = 0 \tag{4.13}$$

for all $x \in \mathbb{R}$ and for all $h \in (a, b)$, then Eq.(4.13) also holds for all $x \in \mathbb{R}$ and for all $h \in (na, nb)$ where n is any positive integer.

Proof. Suppose that a function $f : \mathbb{R} \to \mathbb{R}$ satisfies Eq.(4.13) for all $x \in \mathbb{R}$ and for all $h \in (a, b)$. Let $x_0 \in \mathbb{R}$ and $h_0 \in (a, b)$. Applying Eq.(2.7) in Eq.(4.13) when $x = x_0$ and $h = h_0$, we will have

$$\sum_{i=0}^{m+1} (-1)^{m+1-i} \binom{m+1}{i} f(x_0 + ih_0) = 0. \tag{4.14}$$

For any nonnegative integer k, if we replace x_0 by $x_0 + kh_0$ in Eq.(4.14) and single out the term with $i = m+1$ in the sum, we then obtain that

$$f(x_0 + (m+1+k)h_0) = \sum_{i=0}^{m} (-1)^{m-i} \binom{m+1}{i} f(x_0 + (i+k)h_0). \tag{4.15}$$

For fixed x_0 and h_0, Eq.(4.15) can be regarded as a linear recurrence relation of order $m+1$ with all characteristic roots equal to 1. Thus,

$$f(x_0 + th_0) = A_0 + \sum_{j=1}^{m} A_j t^j \tag{4.16}$$

for all nonnegative integers t, where A_i's are constants (which may depend on x_0 and h_0). Let n be a positive integer. We will consider $\Delta_{nh_0}^{m+1} f(x_0)$. Again, using Eq.(2.7), we obtain

$$\Delta_{nh_0}^{m+1} f(x_0) = \sum_{i=0}^{m+1} (-1)^{m+1-i} \binom{m+1}{i} f(x_0 + inh_0). \tag{4.17}$$

Applying Eq.(4.16) to the term $f(x_0 + inh_0)$ in Eq.(4.17), it yields

$$\Delta_{nh_0}^{m+1} f(x_0) = \sum_{i=0}^{m+1} (-1)^{m+1-i} \binom{m+1}{i} A_0 + \sum_{i=0}^{m+1} (-1)^{m+1-i} \binom{m+1}{i} \sum_{j=1}^{m} A_j (in)^j.$$

We swap the order of the sums over i and j in the second term to get

$$\Delta_{nh_0}^{m+1} f(x_0) = A_0(1-1)^{m+1} + \sum_{j=1}^{m} A_j \sum_{i=0}^{m+1} (-1)^{m+1-i} \binom{m+1}{i} (in)^j. \tag{4.18}$$

Since each value of the index j of the sum in Eq.(4.18) is less than $m+1$, we can apply Lemma 2.31 to conclude that $\Delta_n^{m+1} x^j = 0$, which, by virtue of Eq.(2.7), is equivalent to

$$\sum_{i=0}^{m+1} (-1)^{m+1-i} \binom{m+1}{i} (x + in)^j = 0 \text{ for all } x \in \mathbb{R}.$$

Thus, if we replace x by 0, then

$$\sum_{i=0}^{m+1} (-1)^{m+1-i} \binom{m+1}{i} (in)^j = 0$$

for all $j = 1, ..., m$. By this fact, Eq.(4.18) is reduced to $\Delta_{nh_0}^{m+1} f(x_0) = 0$. Therefore, $\Delta_h^{m+1} f(x) = 0$ for all $x \in \mathbb{R}$ and for all $h \in (na, nb)$ for any positive integer n. □

The following lemma will extend the domain of validity from a half-open infinite interval to $\mathbb{R}$.

Lemma 4.7. *Let $p \in \mathbb{R}$ and let m be a nonnegative integer. If a function $f : \mathbb{R} \to \mathbb{R}$ satisfies Eq.(4.13) for all $x \in \mathbb{R}$ and for all $h > p$, then Eq.(4.13) also holds for all $x, h \in \mathbb{R}$.*

Proof. Let a function $f : \mathbb{R} \to \mathbb{R}$ satisfy

$$\Delta_h^{m+1} f(x) = 0 \tag{4.19}$$

for all $x \in \mathbb{R}$ and for all $h \in (p, \infty)$. Let $x_0, h_0 \in \mathbb{R}$. Choose $a > \max\{0, p, p - h_0\}$. We obtain that $(k+1)a > p$ for all $k = 0, ..., m+1$. Letting $h = (k+1)a > p$ and $x = x_0 + kh_0$, Eq.(4.19) becomes

$$\Delta_{(k+1)a}^{m+1} f(x_0 + kh_0) = 0 \tag{4.20}$$

where $k = 0, \ldots, m+1$. It follows that

$$\sum_{k=0}^{m+1} (-1)^{m+1-k} \binom{m+1}{k} \Delta_{(k+1)a}^{m+1} f(x_0 + kh_0) = 0.$$

Using Eq.(2.7), we arrive at

$$\sum_{k=0}^{m+1} (-1)^{m+1-k} \binom{m+1}{k} \sum_{j=0}^{m+1} (-1)^{m+1-j} \binom{m+1}{j} f(x_0 + kh_0 + j(k+1)a) = 0.$$

Swapping the order of the sums, it yields

$$\sum_{j=0}^{m+1} (-1)^{m+1-j} \binom{m+1}{j} \left(\sum_{k=0}^{m+1} (-1)^{m+1-k} \binom{m+1}{k} f(x_0 + ja + k(h_0 + ja)) \right) = 0.$$

By Eq.(2.7), the above equation can be written as

$$\sum_{j=0}^{m+1} (-1)^{m+1-j} \binom{m+1}{j} \Delta_{h_0+ja}^{m+1} f(x_0 + ja) = 0. \tag{4.21}$$

Recall that $a > \max\{0, p, p - h_0\}$; thus, $h_0 + ja > p$ for all $j = 1, \ldots, m+1$. Thus, from Eq.(4.19),

$$\Delta_{h_0+ja}^{m+1} f(x_0 + ja) = 0 \text{ for all } j = 1, \ldots, m+1.$$

Therefore, in Eq.(4.21), there remains only the term with $j = 0$, and Eq.(4.21) is reduced to

$$\Delta_{h_0}^{m+1} f(x_0) = 0.$$

□

In addition, we can also prove that if f satisfies Eq.(4.13) for all $x \in \mathbb{R}$ and all $h < p$ where p is a real number, then the domain of validity can be extended to $\mathbb{R}$ as in the following lemma:

Lemma 4.8. *Let $p \in \mathbb{R}$ and let m be a nonnegative integer. If a function $f : \mathbb{R} \to \mathbb{R}$ satisfies Eq.(4.13) for all $x \in \mathbb{R}$ and for all $h < p$, then Eq.(4.13) also holds for all $x, h \in \mathbb{R}$.*

Proof. It can be proved in a similar manner to that in Lemma 4.7. Choose $a < \min\{0, p, p - h_0\}$ where $h_0 \in \mathbb{R}$, and then conclude that $(k + 1)a < p$ for all $k = 0, \ldots, m + 1$ and $h_0 + ja < p$ for all $j = 1, \ldots, m + 1$. The rest of the proof can be carried out in a similar fashion to that in Lemma 4.7. □

The following theorem will extend the domain of validity from an open interval to the whole set $\mathbb{R}$.

Theorem 4.9. *Let $a, b \in \mathbb{R}$ with $a < b$ and let $f : \mathbb{R} \to \mathbb{R}$ fulfill Eq.(4.13) for all $x \in \mathbb{R}$ and for all $h \in (a, b)$, then Eq.(4.13) also holds for all $x, h \in \mathbb{R}$.*

Proof. Assume that a function $f : \mathbb{R} \to \mathbb{R}$ satisfies Eq.(4.13) for all $x \in \mathbb{R}$ and all $h \in (a, b)$. Using Lemma 4.6, we obtain that f satisfies Eq.(4.13) for all $x \in \mathbb{R}$ and all $h \in (na, nb)$ for every positive integer n. If we define $H = \bigcup_{k=1}^{\infty}(ka, kb)$, then Eq.(4.13) will hold for all $x \in \mathbb{R}$ and for all $h \in H$. Next, we will consider the interval (a, b) in the following 3 cases:

case 1: $a < 0 < b$. Then $H = (-\infty, \infty)$; consequently, f satisfies Eq.(4.13) for all $x, h \in \mathbb{R}$.

case 2: $0 \leq a < b$. Choose an integer $n_1 > \frac{a}{b-a}$. Then $jb-(j+1)a = j(b-a)-a > 0$ for all $j \geq n_1$. Consequently, $(n_1a, \infty) \subset H$. Let $p = n_1a$ in Lemma 4.7, we obtain that f satisfies Eq.(4.13) for all $x, h \in \mathbb{R}$.

case 3: $a < b \leq 0$. Choose an integer $n_2 > \frac{-b}{b-a}$. We have $(j+1)b - ja = j(b-a) + b > 0$ for all $j \geq n_2$ and then $(-\infty, n_2 b) \subset H$. Let $p = n_2 b$ in Lemma 4.8 to conclude that f fulfills Eq.(4.13) for all $x, h \in \mathbb{R}$. □

Corollary 4.10. *Let $a, b \in \mathbb{R}$ with $a < b$. A function $f : \mathbb{R} \to \mathbb{R}$ satisfies Eq.(4.13) for all $x \in \mathbb{R}$ and for all $h \in (a, b)$ if and only if f is given by Eq.(3.20).*

Proof. Suppose a function $f : \mathbb{R} \to \mathbb{R}$ satisfies Eq.(4.13) for all $x \in \mathbb{R}$ and for all $h \in (a, b)$. Then, by Theorem 4.9, Eq.(4.13) also holds for all $x, h \in \mathbb{R}$. It follows from Theorem 3.11 that the function f takes the form as in Eq.(3.20). The contrary is obvious. □

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