CHAPTER IV

A FINITE INVERSE BIPREFIX CODE WHOSE SYNTACTIC MONOID HAS n $\mathcal{D}$- CLASSES

In this chapter, we shall construct for any given positive integer $n \geq 3$, a finite inverse biprefix code C whose syntactic monoid $M(C^*)$ has n nonzero $\mathcal{D}$- classes.

We begin the construction with an alphabet set $A = \{ a_1, a_2, \ldots, a_n \}$. For each $i \in \{ 2, 3, \ldots, n \}$, let

$$A_i = \{ a_i, a_{i+1}, \ldots, a_n \}.$$

For each $j \in \{ 2, 3, \ldots, n-2 \}$, let

$$C_j = A_2 A_3 \ldots A_j a_1 a_2 \ldots a_{n-j},$$

$$C_{n-1} = A_2 A_3 \ldots A_{n-1} a_1 \cup A_2 A_3 \ldots A_{n-1} a_n.$$

Let $C = \{ a_1 a_2 \ldots a_n a_1 a_2 \ldots a_{n-1} \} \bigcup \; (\bigcup_{j=2}^{n-1} C_j)$.

Then C is a finite prefix code.

Observe that the code C defined above has the following properties :

(i) For each $u_k, v_k \in A_2 A_3 \ldots A_k$, $\quad u_k^{-1} C = v_k^{-1} C \quad (k = 2, 3, \ldots, n-1)$.

(ii) For each $u_{k,l}, v_{k,l} \in A_2 A_3 \ldots A_k a_1 a_2 \ldots a_l$, $\quad (u_{k,l})^{-1} C = (v_{k,l})^{-1} C$ $(k = 2, 3, \ldots, n-2$ and $l = 1, 2, \ldots, n-k-1)$.

Now, we will label the tree representation of C^*.

By (i), for each $u_k, v_k \in A_2 A_3 \ldots A_k$ $(k = 2, 3, \ldots, n-1)$, the node associated with u_k and v_k are labelled with the same name.

By (ii), for each $u_{k,l}, v_{k,l} \in A_2A_3 \dots A_k a_1 a_2 \dots a_l$ ($k = 2, 3, \dots, n-2$ and $l = 1, 2, \dots, n-k-1$), the node associated with $u_{k,l}$ and $v_{k,l}$ are labelled with the same name.

From the above observation, it suffices to label only the nodes associated with $a_1 w$ and $a_2 w'$ where $w, w' \in A^*$.

We label the tree representation of C^* as follows :

The top and the end points of the tree are labelled 1.

The node associated with $a_1 a_2 \dots a_n$ is labelled 2.

For each $j \in \{1, 2, \dots, n-1\}$, the node associated with $a_1 a_2 \dots a_j$ is labelled

$$1 + \sum_{l=0}^{j-1} (n - l).$$

For each $k \in \{1, 2, \dots, n-2\}$, the node associated with $a_1 a_2 \dots a_n a_1 a_2 \dots a_k$ is labelled

$$2 + \sum_{l=0}^{k-1} (n - l).$$

For each $j \in \{2, 3, \dots, n-1\}$, the node associated with $a_2 a_3 \dots a_j$ is labelled

$$j + 1.$$

For each $k \in \{2, 3, \dots, n-2\}, l \in \{1, 2, \dots, n-k\}$, the node associated with $a_2 a_3 \dots a_k a_1 a_2 \dots a_l$ is labelled

$$(k + 1) + \sum_{s=0}^{l-1} (n - s).$$

Since each of the remaining unlabelled nodes has the same subtree as one of the above labelled nodes, they must have the same name. Hence $P_{C^*}^{(r)}$ has been constructed.

The corresponding syntactic monoid $M(C^*)$ is generated by

$$\{ \tau(a_i) \mid i = 1, 2, \dots, n \}.$$

Notations:

(i) For each i, let $\triangle_i$ denote the domain of $\tau(a_i)$ and

∇_i denote the image of $\tau(a_i)$.

(ii) Let

$$A(1) = \{1, 2, \ldots, n\},$$

$$A(i) = \begin{cases} \varnothing & \text{if } i = 2 \\ \{3, 4, \ldots, i\} & \text{otherwise,} \end{cases}$$

$$B(i) = \{\, k + \sum_{l=0}^{i-2}(n-l) \mid k = 1, 2, \ldots, n-i+1 \,\}$$

for all $i \geq 2$.

A few remarks on $\tau(a_i)$'s are given here :

Remark 4.1. (i) $\triangle_1 = A(1)$.

$$\triangle_i = \{1\} \cup A(i) \cup B(i) \qquad \textit{for all } i \geq 2.$$

Since $\max A(i) \leq i$ *and* $\min B(i) \geq n+1$ *for all* i,

$$A(i) \cap B(j) = \varnothing \qquad \textit{for all } i, j.$$

Thus $\triangle_i \cap \triangle_j = \{1\} \cup A(i)$ *for all* i, j *with* $i < j$.

(ii) $\nabla_i = \triangle_{i+1}$ *for all* $i \geq 1$.

It follows that

$$\textit{Dom}\ \tau(a_i a_{i+1} \ldots a_j) = \triangle_i \quad \textit{and} \quad \textit{Im}\ \tau(a_i a_{i+1} \ldots a_j) = \nabla_j$$

for all $i, j \in \mathbb{N}$ (*the indices are integers modulo* n).

(iii) $\tau(a_i)$*'s are defined as follows:*

$$s\tau(a_1) = \begin{cases} 1 & \textit{if } s = n \\ s+n & \textit{otherwise,} \end{cases}$$

$$s\tau(a_n) = \begin{cases} 3 & \textit{if } s = 1 \\ 1 & \textit{if } s = n \\ s+1 & \textit{if } s \in A(n) - \{\ n\ \} \\ 2 & \textit{if } s \in B(n), \end{cases}$$

$$s\tau(a_i) = \begin{cases} 3 & \textit{if } s = 1 \\ s+1 & \textit{if } s \in A(i) \\ 1 & \textit{if } s = (n-i+1) + \sum\limits_{l=0}^{i-2}(n-l) \\ s+n-i+1 & \textit{if } s \in B(i) - \{(n-i+1) + \sum\limits_{l=0}^{i-2}(n-l)\}. \end{cases}$$

Hence

$$A(1)\tau(a_1) = \{\ 1\ \} \cup B(2).$$

For each $i \in \{2, 3, \ldots, n\}$,

$$(\ \{1\} \cup A(i)\ )\tau(a_i) = \begin{cases} A(1) - \{\ 2\ \} & \text{if } i = n \\ A(i+1) & \text{otherwise,} \end{cases}$$

$$B(i)\tau(a_i) = \begin{cases} \{\ 2\ \} & \text{if } i = n \\ B(i+1) \cup \{\ 1\ \} & \text{otherwise.} \end{cases}$$

Lemma 4.2. *For each* $i \in \{\ 1, 2, \ldots, n\ \}$,

$$Dom\ \tau(a_i a_{i+1} \ldots a_n a_1 a_2 \ldots a_{i-1}) = Im\ \tau(a_i a_{i+1} \ldots a_n a_1 a_2 \ldots a_{i-1}) = \triangle_i.$$

Moreover, there exists $l \in \mathbb{N}$ *such that*

$$\tau(a_i a_{i+1} \ldots a_n a_1 a_2 \ldots a_{i-1})^l = 1_{\triangle_i}.$$

Proof. It follows from Remark 4.1 (ii). □

Theorem 4.3. *The code* $C = \{\ a_1 a_2 \ldots a_n a_1 a_2 \ldots a_{n-1}\ \} \bigcup\ (\bigcup_{j=2}^{n-1} C_j)$ *is a finite inverse biprefix code.*

Proof. To show that $M(C^*)$ is an inverse semigroup, it suffices to show that for each $i \in \{1, 2, \ldots, n\}, \tau(a_i)$ has an inverse in $M(C^*)$.

By Lemma 4.2, for each $i \in \{1, 2, \ldots, n\}$, there exists $l \in \mathbb{N}$ such that

$$\tau(a_i a_{i+1} a_{i+2} \ldots a_n a_1 a_2 \ldots a_{i-1})^l = 1_{\triangle_i}.$$

Let

$$\alpha_i = \tau(a_{i+1} a_{i+2} \ldots a_n a_1 a_2 \ldots a_{i-1}) \tau(a_i a_{i+1} \ldots a_n a_1 a_2 \ldots a_{i-1})^{l-1}.$$

Then $\alpha_i \in M(C^*)$ and

$$\tau(a_i)\alpha_i = \tau(a_i a_{i+1} \ldots a_n a_1 a_2 \ldots a_{i-1})^l = 1_{\text{Dom}\alpha_i}.$$

This yields

$$\tau(a_i)\alpha_i\tau(a_i) = 1_{\triangle_i}\tau(a_i) = \tau(a_i)$$

and

$$\alpha_i\tau(a_i)\alpha_i = \alpha_i 1_{\triangle_i} = \alpha_i.$$

Therefore, $M(C^*)$ is an inverse semigroup. □

Corollary 4.4. *The syntactic monoid of C^*, $M(C^*)$, contains* $1_{\triangle_i}$ *for all i.*

Lemma 4.5. *$\tau(a_1a_2\ldots a_n)$ is a cycle of length n on $\triangle_1$.*

Moreover, $s\tau(a_1a_2\ldots a_n)^l = s + l$ *for all l, and $s \in \triangle_1$.*

Proof. By Remark 4.1 (ii), Dom $\tau(a_1a_2\ldots a_n) = \triangle_1 = \nabla_n = \text{Im } \tau(a_1a_2\ldots a_n)$. To prove $\tau(a_1a_2\ldots a_n)$ is a cycle, let $s \in \triangle_1$. We divide the proof into three cases depending on s.

Case 1 : $s = n$.

$$\begin{aligned} n\tau(a_1a_2\ldots a_n) &= (1\tau(a_2a_3\ldots a_n) \\ &= n\tau(a_n) \\ &= 1. \end{aligned}$$

Case 2 : $s = 1$.

$$\begin{aligned} s\tau(a_1a_2\ldots a_n) &= (3\tau(a_2a_3\ldots a_{n-1}))\tau(a_n) \\ &= (1 + \sum_{l=0}^{n-2}(n-l)))\tau(a_n) \\ &= 2. \end{aligned}$$

Case 3 : $2 \leq s \leq n-1$.

$$\begin{aligned} s\tau(a_1a_2\ldots a_n) &= (s\tau(a_1a_2\ldots a_{n-s}))\tau(a_{n-s+1}\ldots a_n) \\ &= (s+\sum_{l=0}^{n-s-1}(n-l))\tau(a_{n-s+1}\ldots a_n) \\ &= 1\tau(a_{n-s+2}\ldots a_n) \\ &= \begin{cases} 3 & \text{if } s=2 \\ 3\tau(a_{n-s+3}\ldots a_n) & \text{otherwise} \end{cases} \\ &= s+1. \end{aligned}$$

□

The next theorem shows, for any $i \in \{1,2,\ldots,n\}$, the existence of a cycle of length i. This theorem is a key to prove Lemma 4.9.

Theorem 4.6. *The syntactic monoid of C^*, $M(C^*)$, contains a cycle of length i on $\{\ 1\ \} \cup A(i+1)$ for all $i \leq n$.*

Proof. One can choose

$$\theta_1 = \tau(a_2)\tau(a_2)^{-1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\theta_n = \tau(a_1a_2\ldots a_n) = (\ 1\ \ 2\ \ 3\ \ldots\ n\ ).$$

For each i, $2 \leq i \leq n-1$, let

$$\theta_i = (\ 1\ \ 3\ \ 4\ \ 5\ \ldots\ i\ \ i+1\ ).$$

We will show that $\theta_i \in M(C^*)$.

In order to obtain θ_i, we first verify the following statements :

(a) $1_{\nabla_{i+2}}\tau(a_{i+1})$,

(b) $(\tau(a_na_n)\theta^{n-2})^k$ for all k, $1 \le k \le n-3$.

(a) : Dom $1_{\nabla_{i+2}}\tau(a_{i+1}) = \triangle_{i+3} \cup \triangle_{i+1} = \{\ 1\ \} \cup A(i+1)$.

Observe that

$$(\ \{\ 1\ \} \cup A(i+1)\ )1_{\nabla_{i+2}}\tau(a_{i+1}) = \tau(a_{i+1})|_{\{\ 1\ \} \cup A(i+1)}.$$

This yields

$$s1_{\nabla_{i+2}}\tau(a_{i+1}) = \begin{cases} 3 & \text{if } s = 1 \\ s+1 & \text{otherwise.} \end{cases}$$

(b) : Dom $\tau(a_n)\theta^{n-2} = \triangle_n = \{\ 1\ \} \cup A(n) \cup B(n)$ and Im $\tau(a_n)\theta^{n-2} = \triangle_1$.

By Lemma 4.5 and Remark 4.1 (iii),

$$s\tau(a_n)\theta^{n-2} = \begin{cases} 1 & \text{if } s = 1 \\ s-1 & \text{if } s \in A(n) \\ n & \text{if } s \in B(n). \end{cases}$$

Dom $(\ \tau(a_n)\theta^{n-2}\ )^2 = \triangle_n - \{\ 3\ \}$.

Observe that

$$(\ \triangle_n - \{\ 3\ \}\ )(\tau(a_n)\theta^{n-2})^2 = [(\tau(a_n)\theta^{n-2})|_{(\triangle_n - \{\ 3\ \})}]\tau(a_n)\theta^{n-2}.$$

Consequently,

$$(\tau(a_n)\theta^{n-2})^2 = \begin{cases} 1 & \text{if } s = 1 \\ s-2 & \text{if } s \in A(n) \\ n-1 & \text{if } s \in B(n). \end{cases}$$

It can be shown by induction on k that

$$\text{Dom } (\tau(a_n)\theta^{n-2})^k = (\triangle_1 \cap \triangle_n)(\tau(a_n)\theta^{n-2})^{-1} = \triangle_n - \{\ 3, 4, \ldots, k+1\ \},$$

and

$$(\tau(a_n)\theta^{n-2})^k = \begin{cases} 1 & \text{if } s = 1 \\ s-k & \text{if } s \in A(n) - \{\ 3, 4, \ldots, k+1\ \} \\ n-(k-1) & \text{if } s \in B(n). \end{cases}$$

for all k, $1 \leq k \leq n-3$.

Since

$$\text{Dom } 1_{\nabla_{i+2}}\tau(a_{i+1})\theta^{n-i-1}(\tau(a_n)\theta^{n-2})^{n-i-1} = \{\ 1\ \} \cup A(i+1),$$

it follows from (a) and (b) and Lemma 4.5 that

$$1_{\nabla_{i+2}}\tau(a_{i+1})\theta^{n-i-1}(\tau(a_n)\theta^{n-2})^{n-i-1} = (\ 1\ \ 3\ \ 4\ \ 5\ \ldots\ i\quad i+1\ ) = \theta_i.$$

Therefore, $\theta_i \in M(C^*)$. □

Remark 4.7. *For any $\alpha \in M(C^*)$, α^0 means the identity map on Dom α.*

Lemma 4.8. *There is a subset A_1 of $\triangle_1$ such that for any subset A of $\triangle_1$ of cardinality $n-1$, there is $\alpha_A \in M(C^*)$ such that Dom $\alpha_A = A_1$ and Im $\alpha_A = A$.*

Proof. In order to obtain A_1, we first investigate the domain of $\tau(a_na_n)$.

$$\text{Dom } \tau(a_na_n) = \text{Im } \tau(a_na_n) = \triangle_1 - \{\ 2\ \}.$$

Let $A_1 = \triangle_1 - \{\ 2\ \}$.

Writing $A = \triangle_1 - \{\ k\ \}$ for some $k \in \triangle_1$,

we choose $\alpha_A = \tau(a_na_n)\theta^{k-2}$ where $\theta = \tau(a_1a_2\ldots a_n)$. Then $\alpha_A \in M(C^*)$.

To prove that Dom $\alpha_A = A_1$, observe that

$$\text{Im } \tau(a_na_n) \subseteq \text{Dom } \theta^{k-2}.$$

Consequently, Dom $\alpha_A = A_1$.

Note that

$$\text{Im } \alpha_A = A_1\tau(a_n a_n)\theta^{k-2} = A_1\theta^{k-2}.$$

By Lemma 4.5,

$$\begin{aligned} A_1\theta^{k-2} &= \triangle_1 - \{\ 2\theta^{k-2}\ \} \\ &= \triangle_1 - \{\ 2 + (k-2)\ \} \\ &= A. \end{aligned}$$

□

Lemma 4.9. *If A and B are subsets of $\triangle_1$ of the same cardinality, then there exists η in $M(C^*)$ with the domain A and image B.*

Proof. Let A and B be subsets of $\triangle_1$ of cardinality $n-k$ for some $1 \le k \le n-1$. We will prove the lemma by induction on k.

For $k = 1$, there exist by Lemma 4.8, α_A, α_B in $M(C^*)$ such that

$$\text{Dom } \alpha_A = \text{Dom } \alpha_B = A_1 \text{ , Im } \alpha_A = A \text{ and Im } \alpha_B = B$$

for some subset A_1 of $\triangle_1$.

Set $\eta = \alpha_A^{-1}\alpha_B$. Then $\eta \in M(C^*)$. It can be shown that Dom $\eta = A$ and Im $\eta = B$.

Suppose $k \ge 1$ and assume that the statement of the lemma is true for all $A', B' \subseteq \triangle_1$ such that $|A'| = |B'| = n-k$.

Let $A, B \subseteq \triangle_1$ be such that $|A| = |B| = n-(k+1)$.

Choose $s \in \triangle_1 - A$, $t \in \triangle_1 - B$ with $s \neq t$.

Let $A_1 = A \cup \{s\}$, $B_1 = B \cup \{t\}$ and $B_2 = B \cup \{s\}$.

Then $| A_1 | = | B_1 | = | B_2 | = n-k$.

By Theorem 4.6 , there exists a cycle θ_{n-k} in $M(C^*)$ of length $n-k$ such that

$|\text{ Dom } \theta_{n-k}\ | = |\text{ Im } \theta_{n-k}\ | = n-k$. By induction hypothesis, there exist α_1, α_2 such that Dom $\alpha_1 = A_1$, Im $\alpha_1 =$ Dom θ_{n-k}, Dom $\alpha_2 =$ Im θ_{n-k} and Im $\alpha_2 = B_1$.
Since θ_{n-k} is a cycle, there is $l \in \mathbb{N}$ such that $s\alpha_1\theta_{n-k}^l = t\alpha_2^{-1}$.
Set $\phi = \alpha_1\theta_{n-k}^l\alpha_2$. Then $\phi \in M(C^*)$. It can be shown that Dom $\phi = A_1$, Im $\phi = B_1$ and $s\phi = t$.
Set $\eta = \phi 1_{B_2}$. Then $\eta \in M(C^*)$. Since $s \neq t$, Dom $\eta = A$ and Im $\eta = B$.

□

Proposition 4.10. *Let A and B be subsets of $\triangle_i$ and $\triangle_j$, respectively. If A and B have the same cardinality, then there exists η in $M(C^*)$ with the domain A and image B.*

Proof. Let $A \subseteq \triangle_i, B \subseteq \triangle_j$ be such that $|A| = |B|$.
Let $\alpha_i = \tau(a_i a_{i+1} \ldots a_n)|_A$ and $\alpha_j = \tau(a_j a_{j+1} \ldots a_n)|_B$.
Then Dom $\alpha_i = A$, Dom $\alpha_j = B$ and Im α_i, Im $\alpha_j \subseteq \triangle_1$.
We have $|$ Im $\alpha_i\ | = |$ Im $\alpha_j\ |$.
By Lemma 4.9 , there exists $\eta_{(i,j)} \in M(C^*)$ such that

$$\text{Dom } \eta_{(i,j)} = \text{Im } \alpha_i \quad \text{and} \quad \text{Im } \eta_{(i,j)} = \text{Im } \alpha_j.$$

Set $\eta = \alpha_i \eta_{(i,j)} \alpha_j^{-1}$. Then $\eta \in M(C^*)$ and rank $\eta = |A|$.
It can be shown that Dom $\eta = A$ and Im $\eta = B$.

□

Theorem 4.11. *Let $\alpha, \beta \in M(C^*)$.*

$$\alpha \mathcal{D} \beta \quad \textit{if and only if} \quad \textit{rank } \alpha = \textit{rank } \beta.$$

Proof. The sufficient part follows from Corollary 2.2.

For the necessary part , assume that rank $\alpha =$ rank $\beta = m$ for some $m \leq n$. Then $|$ Im $\alpha\ | = |$ Im $\beta\ | = |$ Dom $\beta\ |$. By Theorem 4.10 , there exists $\eta \in M(C^*)$

such that Dom η = Im α and Im η = Dom β. Set $\gamma = \alpha\eta\beta$. Then $\gamma \in M(C^*)$

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ank γ = rank β, Im γ = Im β. By Theorem 2.1 , $\alpha\mathcal{R}\gamma$

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Theorem 3.9.

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