CHAPTER II

Preliminaries

In this chapter, we give some basic knowledges in probability which will be used in our work. The proof is omited but can be found in [4].

2.1 Probability space and Random variables

A **probability space** is a measure space $(\Omega, \mathcal{F}, P)$ for which $P(\Omega) = 1$. The measure P is called a **probability measure.** The set Ω will be referred to as a **sample space** and its elements are called **points** or **elementary events.** The elements of $\mathcal{F}$ are called **events.** For any event A, the value $P(A)$ is called the **probability of** A.

Let $(\Omega, \mathcal{F}, P)$ be a probability space. A function $X : \Omega \to \mathbb{R}$ is called a **random variable** if for every Borel set B in $\mathbb{R}$, $X^{-1}(B)$ belongs to $\mathcal{F}$. We shall use the notation $P(X \in B)$ in place of $P(\{\omega \in \Omega | X(\omega) \in B\})$. In the case where $B = (-\infty, a]$ or $[a, b]$, $P(X \in B)$ is denoted by $P(X \leq a)$ or $P(a \leq X \leq b)$, respectively. Let X be a random variable. A function $F : \mathbb{R} \to [0, 1]$ which is defined by

$$F(x) = P(X \leq x)$$

is called the **distribution function** of X.

Let X be a random variable with the distribution function F. X is said to be a **discrete random variable** if the image of X is countable and X is called a

continuous random variable if F can be written in the form

$$F(x) = \int_{-\infty}^{x} f(t)dt$$

for some nonnegative integrable function f on $\mathbb{R}$. In this case, we say that f is the **probability function** of X.

Now we will give some examples of random variables.

We say that X is a **normal** random variable with parameter μ and σ^2, written as $X \sim N(\mu, \sigma^2)$, if its probability function is defined by

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} exp\Big(- \frac{1}{2\sigma^2}(x-\mu)^2\Big).$$

Moreover, if $X \sim N(0,1)$ then X is said to be a **standard normal** random variable.

We say that X is a **uniform** random variable with parameter n if there exist $x_1, x_2, ..., x_n$ such that $P(X = x_i) = \frac{1}{n}$ for any $i = 1, 2, ..., n$ and denoted by $X \sim U(n)$.

2.2 Independence

Let $(\Omega, \Im, P)$ be a probability space and $\mathcal{F}_\alpha$ are sub σ-algebra of $\mathcal{F}$ for all $\alpha \in \Lambda$. We say that $\{\mathcal{F}_\alpha | \alpha \in \Lambda\}$ is **independent** if and only if for any subset $J = \{1, 2, ..., k\}$ of Λ,

$$P(\bigcap_{m=1}^{k} A_m) = \prod_{m=1}^{k} P(A_m)$$

where $A_m \in \mathcal{F}_m$ for $m = 1, ..., k$.

Let $\mathcal{E}_\alpha \subseteq \mathcal{F}$ for all $\alpha \in \Lambda$. We say that $\{\mathcal{E}_\alpha | \alpha \in \Lambda\}$ is **independent** if and only if $\{\sigma(\mathcal{E}_\alpha) | \alpha \in \Lambda\}$ is independent where $\sigma(\mathcal{E}_\alpha)$ is the smallest σ-algebra with

$\mathcal{E}_\alpha \subseteq \sigma(\mathcal{E}_\alpha)$.

We say that the set of random variables $\{X_\alpha |\ \alpha \in \Lambda\}$ is **independent** if $\{\sigma(X_\alpha)|\ \alpha \in \Lambda\}$ is independent, where $\sigma(X) = \{X^{-1}(B) \mid B$ is a Borel subset of $\mathbb{R}\}$.

Theorem 2.1 Random variables $X_1, X_2, ..., X_n$ are **independent** if for any Borel sets $B_1, B_2, ..., B_n$ we have

$$P(\bigcap_{i=1}^{n}\{X_i \in B_i\}) = \prod_{i=1}^{n} P(X_i \in B_i).$$

Proposition 2.2 If $X_{ij}\ ; i = 1, 2, ..., n,\ j = 1, 2, ..., m_i$ are independent and $f_i : \mathbb{R}^{m_i} \to \mathbb{R}$ are measurable, then $\{f_i(X_{i1}, X_{i2}, ..., X_{im_i})\ \ i = 1, 2, ..., n\}$ is independent.

2.3 Expectation, Variance and Conditional expectation

Let X be any random variable on a probability space $(\Omega, \mathcal{F}, P)$.
If $\int_\Omega |X| dP < \infty$, then we define its **expected value** to be

$$E(X) = \int_\Omega X dP.$$

Proposition 2.3

1. If X is a discrete random variable, then $E(X) = \sum_{x \in \mathrm{ImX}} xP(X = x)$.
2. If X is a continuous random variable with probability function f, then

$$E(X) = \int_\mathbb{R} xf(x)dx.$$

Proposition 2.4 Let X and Y be random variables such that $E(|X|) < \infty$ and $E(|Y|) < \infty$ and $a, b \in R$. Then we have the followings:

1. $E(aX + bY) = aE(X) + bE(Y)$.
2. If $X \leq Y$, then $E(X) \leq E(Y)$.
3. $|E(X)| \leq E(|X|)$.
4. If X and Y are independent , then $E(XY) = E(X)E(Y)$.

Let X be a random variable which $E(|X|^k) < \infty$. Then $E(|X|^k)$ is called the **k-th moment** of X about the origin and call $E[(X - E(X))^k]$ the **k-th moment** of X about the mean.

We call the second moment of X about the mean, the **variance** of X, denoted by $Var(X)$. Then

$$Var(X) = E[X - E(X)]^2.$$

We note that

1. $Var(X) \;=\; E(X^2) - E^2(X)$.
2. If $X \sim N(\mu, \sigma^2)$ then $E(X) = \mu$ and $Var(X) = \sigma^2$.

Proposition 2.5 If $X_1, ..., X_n$ are independent and $E|X_i| < \infty$ for $i = 1, 2, ..., n$, then

1. $E(X_1 X_2 ... X_n) = E(X_1)E(X_2)...E(X_n)$,
2. $Var(a_1X_1 + \cdots + a_nX_n) \;=\; a_1^2 Var(X_1) + \cdots + a_n^2 Var(X_n)$ for any real number $a_1, ..., a_n$.

The following inequalities are useful in our work.

1. **Hölder's inequality :**

$$E(|XY|) \leq E^{\frac{1}{p}}(|X|^p)E^{\frac{1}{q}}(|X|^q)$$

where $0 < p, q < 1$, $\dfrac{1}{p} + \dfrac{1}{q} = 1$ and $E(|X|^p) < \infty$,$E(|Y|^q) < \infty$.

2. **Cauchy-Schwarz's inequality :**

$$E^2(|XY|) \leq E(X^2)E(Y^2)$$

where $E(X^2) < \infty$ and $E(Y^2) < \infty$.

3. **Chebyshev's inequality :**

$$P(\{|X - E(X)| \geq \varepsilon\}) \leq \frac{Var(X)}{\varepsilon^2} \quad \text{for all} \quad \varepsilon > 0$$

where $E(X^2) < \infty$.

Let X be a finite expected value random variable on a probability space $(\Omega, \mathcal{F}, P)$ and $\mathcal{D}$ be a sub σ-algebra of $\mathcal{D}$. Define a probability measure $P_{\mathcal{D}} : \mathcal{D} \to [0, 1]$ by

$$P_{\mathcal{D}}(E) = P(E)$$

and sign-measure $\mathcal{Q}_X : \mathcal{D} \to \mathbb{R}$ by

$$\mathcal{Q}_X(E) = \int_E X dP.$$

Then, by Radon-Nikodym theorem we have $\mathcal{Q}_X \ll P_{\mathcal{D}}$ and there exists a unique measurable function $E^{\mathcal{D}}(X)$ on $(\Omega, \mathcal{F}, P)$ such that

$$\int_E E^{\mathcal{D}}(X) dP_{\mathcal{D}} = \mathcal{Q}_X(E) = \int_E X dP \quad \text{for any } E \in \mathcal{D}.$$

We will say that $E^{\mathcal{D}}(X)$ is the **conditional expectation** of X with respect to $\mathcal{D}$.

Moreover, for any random variables X and Y on the same probability space $(\Omega, \mathcal{F}, P)$ such that $E(|X|) < \infty$, we will denote $E^{\sigma(Y)}(X)$ by $E^Y(X)$.

Theorem 2.9 Let X be a random variable on probability space $(\Omega, \mathcal{F}, P)$ such that $E(|X|) < \infty$, then the followings hold for any sub σ-algebra $\mathcal{D}$ of $\mathcal{F}$.

1. If X is random variable on $(\Omega, \mathcal{D}, P_{\mathcal{D}})$, then $E^{\mathcal{D}}(X) = X$ $a.s.[P_{\mathcal{D}}]$.

2. $E^{\mathcal{F}}(X) = X$ $a.s.[P]$.

3. If $\sigma(X)$ and $\mathcal{D}$ are independent, then $E^{\mathcal{D}}(X) = E(X)$ $a.s.[P_{\mathcal{D}}]$.

Theorem 2.10 Let X and Y be random variables on the same probability space $(\Omega, \mathcal{F}, P)$ such that $E(|X|)$ and $E(|Y|)$ are finite . Then for any sub σ-algebra $\mathcal{D}$ of $\mathcal{F}$ the followings hold.

1. If $X \leq Y$, then $E^{\mathcal{D}}(X) \leq E^{\mathcal{D}}(Y)$ $a.s.$ $[P_{\mathcal{D}}]$.

2. $E^{\mathcal{D}}(aX + bY) = aE^{\mathcal{D}}(X) + bE^{\mathcal{D}}(X)$ $a.s.$ $[P_{\mathcal{D}}]$ for any $a,\ b \in \mathbb{R}$.

Theorem 2.11 Let X and Y be random variables on the same probability space $(\Omega, \mathcal{F}, P)$ such that $E(|XY|)$ and $E(|Y|)$ are finite and $\mathcal{D}_1, \mathcal{D}_2$ be any sub σ-algebra of $\mathcal{F}$. If X is a random variable with respect to $\mathcal{D}_1$, then

1. $E^{\mathcal{D}_1}(XY) = XE^{\mathcal{D}_1}(Y)$ $a.s.$ $[P_{\mathcal{D}_1}]$.

2. $E^{\mathcal{D}_2}(XY) = E^{\mathcal{D}_2}(XE^{\mathcal{D}_1}(Y))$ $a.s.$ $[P_{\mathcal{D}_2}]$.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathcal{D}$ be a sub σ-algebra of $\mathcal{F}$. For any event A on $\mathcal{F}$, we defined the **conditional probability of A given $\mathcal{D}$** by

$$P(A|\mathcal{D}) = E^{\mathcal{D}}(I_A).$$