CHAPTER IV

DERIVATIVES OF FUNCTIONS ON THE SIERPINSKI GASKET

Thoughout the rest of this chapter, we denote by $\mathcal{U} = \{\diagdown, \diagup, \_\}$ the set of directions on SG and set p_0, p_1 and p_2 be the boundary points of SG. In this chapter we will define a natural derivative of real-valued functions on SG that is consistent with the existing theory of analysis on fractals and investigate its properties. By the symmetry of SG, it suffices to prove our works only for direction $\diagup$.

4.1 Definition of derivatives

Definition 4.1. Let f be a function on SG, $p \in V_*$ and m be the smallest value for which $p = p_i(\omega) \in V_m$ for some $i = 0, 1, 2$ and $\omega \in W_m$. For $n \in \mathbb{N}_0, n \geq m$, $j \in \{0, 1, 2\}$ and $a_n = \left(\frac{3}{5}\right)^n$, define the **pre-derivatives** of f at $p_j(\omega)$ by

$$D^+_{\diagup,n} f(p_0(\omega)) = \frac{f(p_1(\omega \overbrace{000\cdots0}^{n-m \text{ times}})) - f(p_0(\omega))}{a_n},$$

$$D^-_{\diagdown,n} f(p_0(\omega)) = \frac{f(p_0(\omega)) - f(p_2(\omega \overbrace{000\cdots0}^{n-m \text{ times}}))}{a_n},$$

$$D^+_{\_,n} f(p_1(\omega)) = \frac{f(p_2(\omega \overbrace{111\cdots1}^{n-m \text{ times}})) - f(p_1(\omega))}{a_n},$$

$$D^-_{\diagup,n} f(p_1(\omega)) = \frac{f(p_1(\omega)) - f(p_0(\omega \overbrace{111\cdots1}^{n-m \text{ times}}))}{a_n},$$

and

$$D^{+}_{\diagdown,n}f(p_2(\omega)) = \frac{f(p_0(\omega\overbrace{222\cdots2}^{n-m\ times})) - f(p_2(\omega))}{a_n},$$

$$D^{-}_{\_,n}f(p_2(\omega)) = \frac{f(p_2(\omega)) - f(p_1(\omega\overbrace{222\cdots2}^{n-m\ times}))}{a_n}.$$

See the approximating sequence of $D^{+}_{\diagup,n}f(p_0(\omega))$ and $D^{-}_{\diagup,n}f(p_1(\omega))$ in Figure (5).

Remark 4.2. By the definition above, for $n \in \mathbb{N}_0$, (see Figure(6).)

$$Dom\left(D^{+}_{\diagup,n}f\right) = Dom\left(D^{-}_{\diagdown,n}f\right)$$
$$= \{p \in V_*|p = p_0(\omega) \text{ for some } \omega \in W_m, m \le n\},$$

$$Dom\left(D^{+}_{\_,n}f\right) = Dom\left(D^{-}_{\diagup,n}f\right)$$
$$= \{p \in V_*|p = p_1(\omega) \text{ for some } \omega \in W_m, m \le n\},$$

$$Dom\left(D^{+}_{\diagdown,n}f\right) = Dom\left(D^{-}_{\_,n}f\right)$$
$$= \{p \in V_*|p = p_2(\omega) \text{ for some } \omega \in W_m, m \le n\}.$$

Proposition 4.3. *Let f be a function on SG. For $n \in \mathbb{N}$, let*

$$A_{\diagup,n} = \{p \in V_n|p = p_0(\omega 1) \text{ for some } \omega \in W_{n-1}\}$$

$$A_{\_,n} = \{p \in V_n|p = p_1(\omega 2) \text{ for some } \omega \in W_{n-1}\}$$

$$A_{\diagdown,n} = \{p \in V_n|p = p_2(\omega 0) \text{ for some } \omega \in W_{n-1}\}.$$

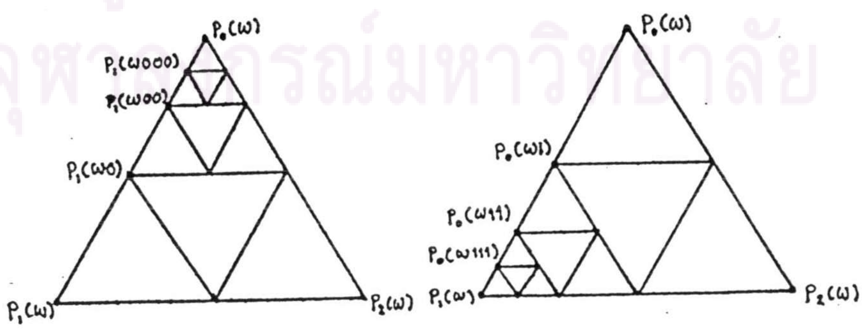

Figure (5). Approximating sequence of $D^{+}_{\diagup,n}f(p_0(\omega))$ and $D^{-}_{\diagup,n}f(p_1(\omega))$.

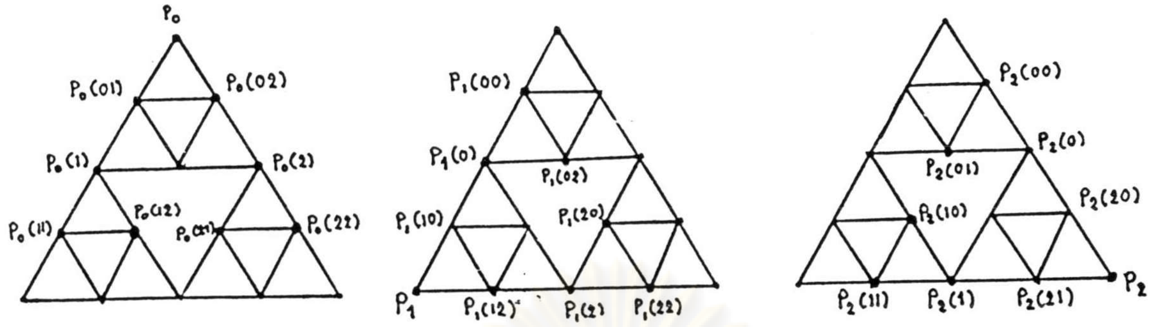

Figure (6). All of the points in domains of $D^{+}_{\diagup,2}$, $D^{+}_{\diagdown,2}$ *and* $D^{+}_{\_,2}$, *respectively.*

Then for each $n \in \mathbb{N}_0$,

$$Dom\left(D^{+}_{\diagup,n}f\right) = \left[\bigcup_{j=1}^{n}\left(A_{\diagup,j} \cup A_{\diagdown,j}\right)\right] \cup \{p_0\} = Dom\left(D^{-}_{\diagdown,n}f\right),$$

$$Dom\left(D^{+}_{\diagdown,n}f\right) = \left[\bigcup_{j=1}^{n}\left(A_{\diagdown,j} \cup A_{\_,j}\right)\right] \cup \{p_2\} = Dom\left(D^{-}_{\_,n}f\right),$$

$$Dom\left(D^{+}_{\_,n}f\right) = \left[\bigcup_{j=1}^{n}\left(A_{\diagup,j} \cup A_{\_,j}\right)\right] \cup \{p_1\} = Dom\left(D^{-}_{\diagup,n}f\right).$$

Proof. We will show that $Dom\left(D^{+}_{\diagup,n}f\right) = \left[\bigcup_{j=1}^{n}\left(A_{\diagup,j} \cup A_{\diagdown,j}\right)\right] \cup \{p_0\}$ for $n \in \mathbb{N}_0$. It is clear that $Dom\left(D^{+}_{\diagup,n}f\right) = \{p_0\}$ if $n = 0$. Assume that $n \geq 1$. By Remark 4.2, it follows that $\left[\bigcup_{j=1}^{n}\left(A_{\diagup,j} \cup A_{\diagdown,j}\right)\right] \cup \{p_0\} \subseteq Dom\left(D^{+}_{\diagup,n}f\right)$. Let $p \in Dom\left(D^{+}_{\diagup,n}f\right)$. Then there exists the smallest non-negative integer $m \leq n$ and $\omega_1\omega_2...\omega_m = \omega \in W_m$ such that $p = p_0(\omega)$. Note that $p = p_0\left(\omega_1\omega_2...\omega_m\right) = p_{\omega_m}\left(\omega_1\omega_2...\omega_{m-1}0\right)$. If $m = 0$, then $p = p_0$. Consider the case $m \geq 1$, by the smallest of m, we get that $w_m \neq 0$. Then $p \in A_{\diagup,m} \cup A_{\diagdown,m}$ and hence $Dom\left(D^{+}_{\diagup,n}f\right) \subseteq \left[\bigcup_{j=1}^{n}\left(A_{\diagup,j} \cup A_{\diagdown,j}\right)\right] \cup \{p_0\}$. The other statements can be shown by the same way. $\square$

Definition 4.4. Let f be a function on $SG, L \in \mathbb{R}$. Fix $m \in \mathbb{N}_0$. For $n \geq m$ and $p \in V_m \cap D^{+}_{u,n}f$ for some $u \in \{\diagdown, \diagup, \_\}$, we call L the **right-hand derivative** of f

at p in direction u if $\lim_{n\to\infty} D^+_{u,n}f(p) = L$. L is denoted by $D^+_u f(p)$, i.e.,

$$D^+_u f(p) = \lim_{n\to\infty} D^+_{u,n} f(p)$$

if the limit exists. Similarly, for $n \geq m$ and $p \in V_m \cap D^-_{u,n}f$ for some $u \in \{\diagdown, \diagup, \_\}$, we call L the **left-hand derivative** of f at p in direction u if $\lim_{n\to\infty} D^-_{u,n}f(p) = L$. L is denoted by $D^-_u f(p)$, i.e.,

$$D^-_u f(p) = \lim_{n\to\infty} D^-_{u,n} f(p)$$

if the limit exists.

Proposition 4.5. *Let f be a function on SG. Set $A_u = \bigcup_{j=1}^{\infty} A_{u,j}, u \in \mathcal{U}$ Then*

$$Dom\left(D^+_{\diagup} f\right) = A_{\diagup} \cup A_{\diagdown} \cup \{p_0\} = Dom\left(D^-_{\diagdown} f\right),$$
$$Dom\left(D^+_{\diagdown} f\right) = A_{\_} \cup A_{\diagdown} \cup \{p_2\} = Dom\left(D^-_{\_} f\right),$$
$$Dom\left(D^+_{\_} f\right) = A_{\diagup} \cup A_{\_} \cup \{p_1\} = Dom\left(D^-_{\diagup} f\right).$$

Moreover, each A_u is dense in SG and hence each of the domains $Dom\left(D^+_u f\right)$ and $Dom\left(D^-_u f\right)$ is dense in SG for any $u \in \mathcal{U}$.

Proof. We will show that $Dom\left(D^+_{\diagup} f\right) = A_{\diagup} \cup A_{\diagdown} \cup \{p_0\}$. Clearly that $A_{\diagup} \cup A_{\diagdown} \cup \{p_0\} \subseteq Dom\left(D^+_{\diagup} f\right)$. Let $p \in Dom\left(D^+_{\diagup} f\right)$. There exists the smallest non-negative integer m and $\omega \in W_m, m \in \mathbb{N}_0$ such that $p = p_0(\omega)$. Thus $p \in A_{\diagup} \cup A_{\diagdown} \cup \{p_0\}$.

To show $A_{\diagup}$ is dense in SG, let $p \in SG, m \in \mathbb{N}$ and $SG_{m,p} = \bigcup_{\substack{\omega\in W_m,\\ p\in SG_\omega}} SG_\omega$ be a neighborhood of p. Then $p \in SG_\omega$ for some $\omega \in W_m$. Thus $q = p_0(\omega 1) \in SG_\omega \subseteq SG_{m,p}$. Hence $q \in A_{\diagup} \cap SG_{m,p}$, i.e, $A_* \cap SG_{m,p}$ is nonempty.

Therefore, $A_{\diagup}$ is dense in SG and it is the same for $A_{\_}$ and $A_{\diagdown}$. □

Observe that there is a relation between our derivative and the normal derivative.

In fact, WLOG, let $p = p_0(\omega)$ and $\omega \in W_N$. For $k \geq N$,

$$
\begin{aligned}
\partial_n f(p) &= \lim_{m\to\infty}\left(\frac{5}{3}\right)^{N+m}\left[2f(p_0(\omega)) - f(p_1(\omega\overbrace{0\cdots0}^{m})) - f(p_2(\omega\overbrace{0\cdots0}^{m}))\right] \\
&= \lim_{m\to\infty}\frac{f(p_0(\omega)) - f(p_1(\omega\overbrace{000\cdots0}^{m\ times}))}{a_{N+m}} + \lim_{m\to\infty}\frac{f(p_0(\omega)) - f(p_2(\omega\overbrace{000\cdots0}^{m\ times}))}{a_{N+m}} \\
&= \lim_{k\to\infty}\frac{f(p_0(\omega)) - f(p_1(\omega\ \overbrace{000\cdots0}^{k-N\ times}))}{a_k} + \lim_{k\to\infty}\frac{f(p_0(\omega)) - f(p_2(\omega\ \overbrace{000\cdots0}^{k-N\ times}))}{a_k} \\
&= -\lim_{k\to\infty} D^{+}_{\diagup,k} f(p_0(\omega)) + \lim_{k\to\infty} D^{-}_{\diagdown,k} f(p_0(\omega)) \\
&= -D^{+}_{\diagup} f(p_0(\omega)) + D^{-}_{\diagdown} f(p_0(\omega)).
\end{aligned}
$$

By Proposition 4.5 and definition of the derivative of f at any point p, we have $Dom(D_{\diagup} f) \subseteq A_{\diagup}$, $Dom(D_{\_} f) \subseteq A_{\_}$, $Dom(D_{\diagdown} f) \subseteq A_{\diagdown}$. Moreover, for each $p \in V_* \setminus V_0$ there is at most one direction u for which $D_u f(p)$ exists. This is shown in the following:

Lemma 4.6. *Given $p \in V_* \setminus V_0$ there exists a unique $u \in \mathcal{U}$ such that $p \in A_u$. We call u the* **admissible direction** *of p.*

Proof. Applying the equation $V_* \setminus V_0 = A_{\diagup} \dot\cup A_{\_} \dot\cup A_{\diagdown}$, the statement is straightforward. □

Definition 4.7. Let f be a function on $SG, L \in \mathbb{R}$. For $p \in A_u$ for some $u \in \{\diagdown, \diagup, \_\}$, we call $D_u f(p) = L$ the **derivative** of f at p in admissible direction u if

$$D_u^{+} f(p) = L = D_u^{-} f(p).$$

Now we will prove the linearity property of differentiation of any functions on SG.

Theorem 4.8. *Let f and g be functions on SG and a and b be fixed real numbers. If f and g have derivative at $p \in V_* \setminus V_0$,then $af + bg$ has derivatives and*

$$D_u[(af + bg)(p)] = aD_u f(p) + bD_u g(p)$$

where u is the admissible direction of p.

Proof. Let $p \in A_{\diagup}$. Then $p = p_0(\omega 1)$ for some $m \in \mathbb{N}$ and $\omega \in W_{m-1}$. Then

$$\begin{aligned}
&D^+_{\diagup}(af+bg)(p_0(\omega 1)) \\
&= \lim_{n\to\infty} D^+_{\diagup,n}(af+bg)(p_0(\omega 1)) \\
&= \lim_{n\to\infty} \frac{(af+bg)(p_1(\omega 1\overbrace{000\cdots 0}^{n-m\ times})) - (af+bg)(p_0(\omega 1))}{a_n} \\
&= a\lim_{n\to\infty} \frac{f(p_1(\omega 1\overbrace{000\cdots 0}^{n-m\ times})) - f(p_0(\omega 1))}{a_n} \\
&\quad + b\lim_{n\to\infty} \frac{g(p_1(\omega 1\overbrace{000\cdots 0}^{n-m\ times})) - g(p_0(\omega 1))}{a_n} \\
&= aD^+_{\diagup} f(p_0(\omega 1)) + bD^+_{\diagup} g(p_0(\omega 1)).
\end{aligned}$$

Similary, $D^-_{\diagup}(af+bg)(p_0(\omega 1)) = aD^-_{\diagup} f(p_0(\omega 1)) + bD^-_{\diagup} g(p_0(\omega 1))$.

Hence

$$D_{\diagup}[(af+bg)(p)] = aD_{\diagup} f(p) + bD_{\diagup} g(p). \tag{4.9}$$

□

Now take $a = c$ and $b = 0$ in (4.9). It implies that

$$D_u[cf(p)] = cD_u f(p).$$

Next, take $a = b = 1$ in (4.9). The result is

$$D_u[(f+g)(p)] = D_u f(p) + D_u g(p). \tag{4.10}$$

Repeated application of (4.10) to a sum of a finite number of functions gives

$$D_u[f_1 + f_2 + \dots + f_n](p) = D_u f_1(p) + D_u f_2(p) + \dots + D_u f_n(p).$$

Theorem 4.11. *Let f and g be functions on SG. If f and g have derivatives at $p \in V_* \setminus V_0$, then fg has derivative and*

$$D_u[fg(p)] = f(p)D_u g(p) + g(p)D_u f(p)$$

where u is the admissible direction of p.

Proof. Let $p \in A_{\diagup}$. Then $p = p_0(\omega 1)$ for some $m \in \mathbb{N}$ and $\omega \in W_{m-1}$. Then

$$\begin{aligned}
&D^+_{\diagup}(fg)(p_0(\omega 1)) \\
&= \lim_{n\to\infty} \frac{(fg)(p_1(\omega 1 \overbrace{000\cdots 0}^{n-m \text{ times}})) - (fg)(p_0(\omega 1))}{a_n} \\
&= \lim_{n\to\infty} \frac{(fg)(p_1(\omega 1 \overbrace{000\cdots 0}^{n-m \text{ times}})) - f(p_1(\omega 1 \overbrace{000\cdots 0}^{n-m \text{ times}}))g(p_0(\omega 1))}{a_n} \\
&\qquad + \frac{f(p_1(\omega 1 \overbrace{000\cdots 0}^{n-m \text{ times}}))g(p_0(\omega 1)) - (fg)(p_0(\omega 1))}{a_n} \\
&= \lim_{n\to\infty} f(p_1(\omega 1 \overbrace{000\cdots 0}^{n-m \text{ times}})) \left[\frac{g(p_1(\omega 1 \overbrace{000\cdots 0}^{n-m \text{ times}})) - g(p_0(\omega 1))}{a_n} \right] \\
&\qquad + \lim_{n\to\infty} g(p_0(\omega 1)) \left[\frac{f(p_1(\omega 1 \overbrace{000\cdots 0}^{n-m \text{ times}})) - f(p_0(\omega 1))}{a_n} \right] \\
&= \lim_{n\to\infty} f(p_1(\omega 1 \overbrace{000\cdots 0}^{n-m \text{ times}})) \lim_{n\to\infty} \frac{g(p_1(\omega 1 \overbrace{000\cdots 0}^{n-m \text{ times}})) - g(p_0(\omega 1))}{a_n} \\
&\qquad + g(p_0(\omega 1)) \lim_{n\to\infty} \frac{f(p_1(\omega 1 \overbrace{000\cdots 0}^{n-m \text{ times}})) - f(p_0(\omega 1))}{a_n}.
\end{aligned}$$

Since $\lim_{n\to\infty} \frac{f(p_1(\omega 1 \overbrace{000\cdots 0}^{n-m \text{ times}})) - f(p_0(\omega 1))}{a_n}$ exists and $\lim_{n\to\infty} a_n = 0$, we get that

$$\lim_{n\to\infty} f(p_1(\omega 1 \overbrace{000\cdots 0}^{n-m \text{ times}})) - f(p_0(\omega 1)) = 0.$$

Then

$$\lim_{n\to\infty} f(p_1(\omega 1 \overbrace{000\cdots 0}^{n-m \text{ times}})) = \lim_{n\to\infty} f(p_1(\omega 1 \overbrace{000\cdots 0}^{n-m \text{ times}})) - f(p_0(\omega 1)) + f(p_0(\omega 1)) = f(p_0(\omega 1)).$$

Hence

$$D^+_{\diagup}(fg)(p_0(\omega 1)) = f(p_0(\omega 1))D^+_{\diagup}g(p_0(\omega 1)) + g(p_0(\omega 1))D^+_{\diagup}f(p_0(\omega 1)).$$

Similarly,

$$D^-_{\diagup}(fg)(p_1(\omega 0)) = f(p_1(\omega 0))D^-_{\diagup}g(p_1(\omega 0)) + g(p_1(\omega 0))D^-_{\diagup}f(p_1(\omega 0)).$$

□

4.2 Derivative of a harmonic function

Lemma 4.12. *Let f be a harmonic funtion on SG. Then pre-derivatives at the three points in $F_\omega(V_0)$, $\omega \in W_m$ and $m \in \mathbb{N}_0$ satisfy a system of equations written in the matrix form as follows:*

$$\begin{pmatrix} D^+_{\diagup,n}f(p_0(\omega)) \\ D^+_{\_\_,n}f(p_1(\omega)) \\ D^+_{\diagdown,n}f(p_2(\omega)) \end{pmatrix} = \frac{1}{5^{n-m}a_n} \begin{pmatrix} -3^{n-m} & b_{n-m} & b_{n-m}-1 \\ b_{n-m}-1 & -3^{n-m} & b_{n-m} \\ b_{n-m} & b_{n-m}-1 & -3^{n-m} \end{pmatrix} \begin{pmatrix} f(p_0(\omega)) \\ f(p_1(\omega)) \\ f(p_2(\omega)) \end{pmatrix}$$

and

$$\begin{pmatrix} D^-_{\diagdown,n}f(p_0(\omega)) \\ D^-_{\diagup,n}f(p_1(\omega)) \\ D^-_{\_\_,n}f(p_2(\omega)) \end{pmatrix} = \frac{-1}{5^{n-m}a_n} \begin{pmatrix} -3^{n-m} & b_{n-m}-1 & b_{n-m} \\ b_{n-m} & -3^{n-m} & b_{n-m}-1 \\ b_{n-m}-1 & b_{n-m} & -3^{n-m} \end{pmatrix} \begin{pmatrix} f(p_0(\omega)) \\ f(p_1(\omega)) \\ f(p_2(\omega)) \end{pmatrix}$$

where $n \geq m, a_n = (\frac{3}{5})^n$ and $b_n = \frac{3^n+1}{2}$.

Proof. If f is a constant function, by Definition 4.1 we get that $D^{+}_{\diagup,n}f \equiv D^{-}_{\diagdown,n}f \equiv 0$ for all $n \geq m$. Assume that $f(p_0), f(p_1)$ and $f(p_2)$ are not all equal real numbers and $p \in Dom(D^{+}_{\diagup,n}f) \cap Dom(D^{-}_{\diagdown,n}f)$. We can write $p = p_0(\omega)$ for some $\omega \in W_m$ and $m \in \mathbb{N}_0$. Let $n \geq m$. By definition of $D^{+}_{\diagup,n}f(p_0(\omega))$ and $D^{-}_{\diagdown,n}f(p_0(\omega))$, we have

$$\begin{aligned} D^{+}_{\diagup,m}f(p_0(\omega)) &= \frac{f(p_1(\omega)) - f(p_0(\omega))}{a_m} \\ &= \frac{-3^{m-m}f(p_0(\omega)) + b_{m-m}f(p_1(\omega)) + (b_{m-m}-1)\,f(p_2(\omega))}{5^{m-m}a_m} \end{aligned}$$

and

$$\begin{aligned} D^{-}_{\diagdown,m}f(p_0(\omega)) &= \frac{f(p_0(\omega)) - f(p_2(\omega))}{a_m} \\ &= \frac{3^{m-m}f(p_0(\omega)) - (b_{m-m}-1)\,f(p_1(\omega)) - b_{m-m}f(p_2(\omega))}{5^{m-m}a_m} \end{aligned}$$

Assume that

$$D^{+}_{\diagup,n}f(p_0(\omega)) = \frac{1}{5^{n-m}a_n}\left[-3^{n-m}f(p_0(\omega)) + b_{n-m}f(p_1(\omega)) + (b_{n-m}-1)f(p_2(\omega))\right]$$

and

$$D^{-}_{\diagdown,n}f(p_0(\omega)) = \frac{-1}{5^{n-m}a_n}\left[-3^{n-m}f(p_0(\omega)) + (b_{n-m}-1)f(p_1(\omega)) + b_{n-m}f(p_2(\omega))\right]$$

where $n \geq m$. Then

$$\begin{aligned} &D^{+}_{\diagup,n+1}f(p_0(\omega)) \\ &= \frac{f(p_1(\omega\overbrace{000\cdots0}^{(n+1)-m\ times})) - f(p_0(\omega))}{a_{n+1}} \\ &= \frac{f(p_2(\omega\overbrace{00\cdots0}^{n-m\ times})) + 2f(p_1(\omega\overbrace{00\cdots0}^{n-m\ times})) + 2f(p_0(\omega)) - 5f(p_0(\omega))}{5a_{n+1}} \end{aligned}$$

$$= \frac{2}{3}\left[\frac{f(p_1(\omega\overbrace{000\cdots0}^{n-m\ times})) - f(p_0(\omega))}{a_n}\right] - \frac{1}{3}\left[\frac{f(p_0(\omega)) - f(p_2(\omega\overbrace{000\cdots0}^{n-m\ times}))}{a_n}\right]$$

$$= \frac{2D^{+}_{\diagup,n}f(p_0(\omega))}{3} - \frac{D^{-}_{\diagdown,n}f(p_0(\omega))}{3}$$

$$= \frac{-3^{(n+1)-m}f(p_0(\omega)) + (3b_{n-m}-1)f(p_1(\omega)) + (3b_{n-m}-2)f(p_2(\omega))}{3\cdot 5^{n-m}a_n}.$$

Note that

$$3b_{n-m} - 1 = 3\cdot\left(\frac{3^{n-m}+1}{2}\right) - 1 = \frac{3^{n+1-m}+1}{2} = b_{(n+1)-m}.$$

Then

$$D^{+}_{\diagup,n+1}f(p_0(\omega))$$

$$= \frac{-3^{(n+1)-m}f(p_0(\omega)) + b_{(n+1)-m}f(p_1(\omega)) + (b_{(n+1)-m}-1)f(p_2(\omega))}{5^{(n+1)-m}a_{n+1}}.$$

Moreover, we get that

$$D^{-}_{\diagdown,n+1}f(p_0(\omega))$$

$$= \frac{f(p_0(\omega)) - f(p_2(\omega\overbrace{000\cdots0}^{(n+1)-m\ times}))}{a_{n+1}}$$

$$= \frac{-f(p_1(\omega\overbrace{00\cdots0}^{n-m\ times})) - 2f(p_2(\omega\overbrace{00\cdots0}^{n-m\ times})) - 2f(p_0(\omega)) + 5f(p_0(\omega))}{5a_{n+1}}$$

$$= \frac{2}{3}\left[\frac{f(p_0(\omega)) - f(p_2(\omega\overbrace{000\cdots0}^{n-m\ times}))}{a_n}\right] - \frac{1}{3}\left[\frac{f(p_1(\omega\overbrace{000\cdots0}^{n-m\ times})) - f(p_0(\omega))}{a_n}\right]$$

$$= \frac{2D^-_{\diagdown,n}f(p_0(\omega))}{3} - \frac{D^+_{\diagup,n}f(p_0(\omega))}{3}$$

$$= \frac{3^{(n+1)-m}f(p_0(\omega)) - (3b_{n-m}-2)f(p_1(\omega)) - (3b_{n-m}-1)f(p_2(\omega))}{3\cdot 5^{n-m}a_n}$$

$$= \frac{3^{(n+1)-m}f(p_0(\omega)) - (b_{(n+1)-m}-1)f(p_1(\omega)) - b_{(n+1)-m}f(p_2(\omega))}{5^{(n+1)-m}a_n}.$$

By induction, for $n \geq m$,

$$D^+_{\diagup,n}f(p_0(\omega)) = \tfrac{1}{5^{n-m}a_n}\left[-3^{n-m}f(p_0(\omega)) + b_{n-m}f(p_1(\omega)) + (b_{n-m}-1)f(p_2(\omega))\right]$$

and

$$D^-_{\diagdown,n}f(p_0(\omega)) = \tfrac{-1}{5^{n-m}a_n}\left[-3^{n-m}f(p_0(\omega)) + (b_{n-m}-1)f(p_1(\omega)) + b_{n-m}f(p_2(\omega))\right].$$

The other equations involving $D^+_{\_\_,n}f$, $D^-_{\diagup,n}f$ and $D^+_{\diagdown,n}f$, $D^-_{\_\_,n}f$ can be proved similarly. □

By Lemma 4.12, we derive the following equation. For $\omega \in W_m$ and $n \geq m$,

$$\frac{f(p_1(\omega\overbrace{000\cdots 0}^{n-m\ times})) - f(p_0(\omega))}{a_n}$$

$$= D^+_{\diagup,n}f(p_0(\omega))$$

$$= \frac{1}{5^{n-m}a_n}\left[-3^{n-m}f(p_0(\omega)) + b_{n-m}f(p_1(\omega)) + (b_{n-m}-1)f(p_2(\omega))\right].$$

This implies that

$$f(p_1(\omega\overbrace{000\cdots 0}^{n-m\ times}))$$

$$= \frac{1}{5^{n-m}}\left[(5^{n-m}-3^{n-m})f(p_0(\omega)) + b_{n-m}f(p_1(\omega)) + (b_{n-m}-1)f(p_2(\omega))\right].$$

Then we have the following Lemma:

Lemma 4.13. *Let f be a harmonic function on SG. If $\omega \in W_m$ and $m,\ n \in \mathbb{N}_0$, then we have a matrix form*

$$\begin{pmatrix} f(p_1(\omega\overbrace{000\cdots0}^{n\ times})) \\ f(p_2(\omega\overbrace{000\cdots0}^{n\ times})) \\ f(p_0(\omega\overbrace{111\cdots1}^{n\ times})) \\ f(p_2(\omega\overbrace{111\cdots1}^{n\ times})) \\ f(p_0(\omega\overbrace{222\cdots2}^{n\ times})) \\ f(p_1(\omega\overbrace{222\cdots2}^{n\ times})) \end{pmatrix} = \frac{1}{5^n}\begin{pmatrix} 5^n-3^n & b_n & b_n-1 \\ 5^n-3^n & b_n-1 & b_n \\ b_n & 5^n-3^n & b_n-1 \\ b_n-1 & 5^n-3^n & b_n \\ b_n & b_n-1 & 5^n-3^n \\ b_n-1 & b_n & 5^n-3^n \end{pmatrix}\begin{pmatrix} f(p_0(\omega)) \\ f(p_1(\omega)) \\ f(p_2(\omega)) \end{pmatrix}$$

where $b_n = \frac{3^n+1}{2}$.

Lemma 4.14. *Let f be a harmonic function on SG. The left and right derivatives in direction $u \in \mathcal{U}$ are given by*

$$\begin{pmatrix} D^+_{\diagup} f(p_0(\omega)) \\ D^+_{\_} f(p_1(\omega)) \\ D^+_{\diagdown} f(p_2(\omega)) \end{pmatrix} = \left(\frac{5}{3}\right)^m \begin{pmatrix} -1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -1 \end{pmatrix}\begin{pmatrix} f(p_0(\omega)) \\ f(p_1(\omega)) \\ f(p_2(\omega)) \end{pmatrix}$$

and

$$\begin{aligned} D^-_{\diagdown} f(p_0(\omega)) &= -D^+_{\diagup} f(p_0(\omega)), \\ D^-_{\diagup} f(p_1(\omega)) &= -D^+_{\_} f(p_1(\omega)), \\ D^-_{\_} f(p_2(\omega)) &= -D^+_{\diagdown} f(p_2(\omega)) \end{aligned}$$

where $\omega \in W_m,\ m \in \mathbb{N}_0$.

Proof. If f is a constant function, then $D^+_u f \equiv D^-_u f \equiv 0$ for all $u \in \mathcal{U}$. Assume that $f(p_0), f(p_1)$ and $f(p_2)$ are not all equal real numbers. Let $\omega \in W_m$ and $m \in \mathbb{N}_0$. By definition of $D^{+}_{\diagup}f(p_0(\omega))$, $D^{-}_{\diagdown}f(p_0(\omega))$ and Lemma 4.12, we get that

$$\begin{aligned}
&D^{+}_{\diagup}f(p_0(\omega))\\
&= \lim_{n\to\infty} D^{+}_{\diagup,n}f(p_0(\omega))\\
&= \lim_{n\to\infty} \frac{-3^{n-m}f(p_0(\omega)) + b_{n-m}f(p_1(\omega)) + (b_{n-m}-1)f(p_2(\omega))}{5^{n-m}a_n}\\
&= \lim_{n\to\infty} \frac{5^m}{3^n}\left[\frac{-3^n}{3^m}f(p_0(\omega)) + \frac{3^{n-m}+1}{2}f(p_1(\omega)) + \frac{3^{n-m}-1}{2}f(p_2(\omega))\right]\\
&= -\lim_{n\to\infty}\left(\frac{5}{3}\right)^m f(p_0(\omega)) + \lim_{n\to\infty}\left(\frac{5^m}{2\cdot 3^m} + \frac{5^m}{2\cdot 3^n}\right)f(p_1(\omega))\\
&\quad + \lim_{n\to\infty}\left(\frac{5^m}{2\cdot 3^m} - \frac{5^m}{2\cdot 3^n}\right)f(p_2(\omega))\\
&= -\left(\frac{5}{3}\right)^m f(p_0(\omega)) + \left(\frac{5^m}{2\cdot 3^m}\right)f(p_1(\omega)) + \left(\frac{5^m}{2\cdot 3^m}\right)f(p_2(\omega))
\end{aligned}$$

and

$$\begin{aligned}
&D^{-}_{\diagdown}f(p_0(\omega))\\
&= \lim_{n\to\infty} D^{-}_{\diagdown,n}f(p_0(\omega))\\
&= \lim_{n\to\infty} \frac{-1}{5^{n-m}a_n}\left[-3^{n-m}f(p_0(\omega)) + (b_{n-m}-1)f(p_1(\omega)) + b_{n-m}f(p_2(\omega))\right]\\
&= -D^{+}_{\diagup}f(p_0(\omega)).
\end{aligned}$$

The other derivative formulars can be verified similarly. □

Theorem 4.15. *Let f be a harmonic funtion on SG. Then f has derivative at any point $p \in V_* \setminus V_0$ in the admissible direction.*

Proof. Recall that f has derivative at p in the direction $u \in \mathcal{U}$ means $D^{+}_{u}f(p), D^{-}_{u}f(p)$ exist and $D^{+}_{u}f(p) = D^{-}_{u}f(p)$. Assume that $p \in A_{\diagup}$. Then $p = p_0(\omega 1) = p_1(\omega 0)$ for some $\omega \in W_m$ and m is a nonnegative integer. If f is a constant function, the derivative of f at p is zero. Suppose $f(p_0), f(p_1)$ and $f(p_2)$ are not all equal. Thus

$$
\begin{aligned}
D^{+}_{\diagup} f(p_0(\omega 1)) &= \left(\frac{5}{3}\right)^{m+1}\left[-f(p_0(\omega 1)) + \frac{f(p_1(\omega))}{2} + \frac{f(p_2(\omega 1))}{2}\right] \\
&= \frac{-5^{m+1}}{5\cdot 3^{m+1}}\left[f(p_2(\omega)) + 2f(p_0(\omega)) + 2f(p_1(\omega))\right] + \frac{5^{m+1}}{2\cdot 3^{m+1}} f(p_1(\omega)) \\
&\quad + \frac{5^{m+1}}{5\cdot 2\cdot 3^{m+1}}\left[f(p_0(\omega)) + 2f(p_1(\omega)) + 2f(p_2(\omega))\right] \\
&= \left(\frac{5}{3}\right)^{m}\left(\frac{f(p_1(\omega)) - f(p_0(\omega))}{2}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
D^{-}_{\diagup} f(p_1(\omega 0)) &= -\left(\frac{5}{3}\right)^{m+1}\left[\frac{f(p_0(\omega))}{2} - f(p_1(\omega 0)) + \frac{f(p_2(\omega 0))}{2}\right] \\
&= \frac{-5^{m+1}}{2\cdot 3^{m+1}} f(p_0(\omega)) + \frac{5^{m+1}}{5\cdot 3^{m+1}}\left[f(p_2(\omega)) + 2f(p_0(\omega)) + 2f(p_1(\omega))\right] \\
&\quad - \frac{5^{m+1}}{5\cdot 2\cdot 3^{m+1}}\left[f(p_1(\omega)) + 2f(p_2(\omega)) + 2f(p_0(\omega))\right] \\
&= \left(\frac{5}{3}\right)^{m}\left(\frac{f(p_1(\omega)) - f(p_0(\omega))}{2}\right)
\end{aligned}
$$

Similarly, if $D^{+}_{\diagdown} f(p_2(\omega 0))$ and $D^{-}_{\diagdown} f(p_0(\omega 2))$ exist, then

$$
\begin{aligned}
D^{+}_{\diagdown} f(p_2(\omega 0)) &= D^{-}_{\diagdown} f(p_0(\omega 2)) \\
&= \left(\frac{5}{3}\right)^{m}\left(\frac{f(p_0(\omega)) - f(p_2(\omega))}{2}\right)
\end{aligned}
$$

and if $D^{+}_{-} f(p_1(\omega 2))$ and $D^{-}_{-} f(p_2(\omega 1))$ exist, then

$$
\begin{aligned}
D^{+}_{-} f(p_1(\omega 2)) &= D^{-}_{-} f(p_2(\omega 1)) \\
&= \left(\frac{5}{3}\right)^{m}\left(\frac{f(p_2(\omega)) - f(p_1(\omega))}{2}\right).
\end{aligned}
$$

Then f has derivative at every point p in $V_* \setminus V_0$. □

Corollary 4.16. *Let f be a harmonic function on SG. For $\omega \in W_m, m \in \mathbb{N}_0$, then*

$$
\begin{aligned}
-D^-_{\diagdown} f(p_0(\omega 1)) = D^+_{\diagup} f(p_0(\omega 1)) &= \left(\frac{5}{3}\right)^m \left[\frac{f(p_1(\omega)) - f(p_0(\omega))}{2}\right] \\
&= D^-_{\diagup} f(p_0(\omega 1)) = -D^+_{\_} f(p_0(\omega 1)),
\end{aligned}
$$

$$
\begin{aligned}
-D^-_{\_} f(p_2(\omega 0)) = D^+_{\diagdown} f(p_2(\omega 0)) &= \left(\frac{5}{3}\right)^m \left[\frac{f(p_0(\omega)) - f(p_2(\omega))}{2}\right] \\
&= D^-_{\diagdown} f(p_2(\omega 0)) = -D^+_{\diagup} f(p_2(\omega 0)),
\end{aligned}
$$

and

$$
\begin{aligned}
-D^-_{\diagup} f(p_1(\omega 2)) = D^+_{\_} f(p_1(\omega 2)) &= \left(\frac{5}{3}\right)^m \left[\frac{f(p_2(\omega)) - f(p_1(\omega))}{2}\right] \\
&= D^-_{\_} f(p_1(\omega 2)) = -D^+_{\diagdown} f(p_2(\omega 1)).
\end{aligned}
$$

Proof. This corollary is straightforward by Lemma 4.14 and Theorem 4.15. □

Lemma 4.17. *Let f be any harmonic function on SG and $\omega \in W_m$, $m \in \mathbb{N}_0$. For each $n \in \mathbb{N}$,*

$$
\begin{aligned}
0 = & D_{\diagup} f(p_1(\omega 1 \overbrace{000\cdots 0}^{n \text{ times}})) - D_{\diagup} f(p_0(\omega 0 \overbrace{111\cdots 1}^{n \text{ times}})) \\
& + D_{\_} f(p_2(\omega 0 \overbrace{111\cdots 1}^{n \text{ times}})) - D_{\diagdown} f(p_2(\omega 1 \overbrace{000\cdots 0}^{n \text{ times}})), \\
0 = & D_{\diagdown} f(p_0(\omega 0 \overbrace{222\cdots 2}^{n \text{ times}})) - D_{\diagdown} f(p_2(\omega 2 \overbrace{000\cdots 0}^{n \text{ times}})) \\
& + D_{\diagup} f(p_1(\omega 2 \overbrace{000\cdots 0}^{n \text{ times}})) - D_{\_} f(p_1(\omega 0 \overbrace{222\cdots 2}^{n \text{ times}}))
\end{aligned}
$$

and

$$
\begin{aligned}
0 = & D_{\_} f(p_2(\omega 2 \overbrace{111\cdots 1}^{n \text{ times}})) - D_{\_} f(p_1(\omega 1 \overbrace{222\cdots 2}^{n \text{ times}})) \\
& + D_{\diagdown} f(p_0(\omega 1 \overbrace{222\cdots 2}^{n \text{ times}})) - D_{\diagup} f(p_0(\omega 2 \overbrace{111\cdots 1}^{n \text{ times}})).
\end{aligned}
$$

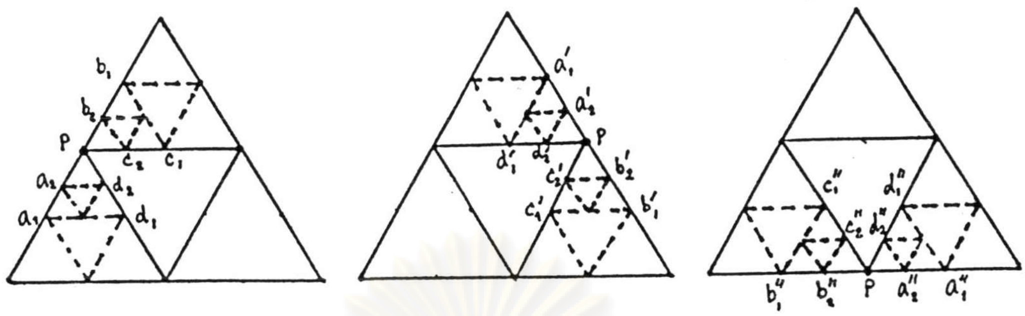

Figure (7). The sequence of points converging to p.

Proof. For each $n \in \mathbb{N}_0$, set

$$a_n = p_1(\omega 1\overbrace{000\cdots 0}^{n\ times}),\ b_n = p_0(\omega 0\overbrace{111\cdots 1}^{n\ times}),$$
$$c_n = p_2(\omega 0\overbrace{111\cdots 1}^{n\ times}),\ d_n = p_2(\omega 1\overbrace{000\cdots 0}^{n\ times}).$$

For each $n \in \mathbb{N}$, we get that

$$\begin{aligned}
&D_{\diagup}f(p_1(\omega 1\overbrace{000\cdots 0}^{n\ times})) - D_{\diagup}f(p_0(\omega 0\overbrace{111\cdots 1}^{n\ times}))\\
&+ D_{_}f(p_2(\omega 0\overbrace{111\cdots 1}^{n\ times})) - D_{\diagdown}f(p_2(\omega 1\overbrace{000\cdots 0}^{n\ times}))\\
&= \left(\frac{5}{3}\right)^{m+n}\left[\frac{f(a_{n-1}) - f(p_0(\omega 1))}{2}\right] - \left(\frac{5}{3}\right)^{m+n}\left[\frac{f(p_0(\omega 1)) - f(b_{n-1})}{2}\right]\\
&\quad + \left(\frac{5}{3}\right)^{m+n}\left[\frac{f(c_{n-1}) - f(p_0(\omega 1))}{2}\right] - \left(\frac{5}{3}\right)^{m+n}\left[\frac{f(p_0(\omega 1)) - f(d_{n-1})}{2}\right]\\
&= \frac{1}{2}\left(\frac{5}{3}\right)^{m+n}[f(a_{n-1}) + f(b_{n-1}) + f(c_{n-1}) + f(d_{n-1}) - 4f(p_0(\omega 1))]\\
&= \frac{1}{2}\left(\frac{5}{3}\right)^{m+n} H_{m+n}f(p_0(\omega 1)) = 0.
\end{aligned}$$

Similarly, for each $n \in \mathbb{N}_0$, set

$$a_n' = p_0(\omega 0\overbrace{222\cdots 2}^{n\ times}),\ b_n' = p_2(\omega 2\overbrace{000\cdots 0}^{n\ times}),$$
$$c_n' = p_1(\omega 2\overbrace{000\cdots 0}^{n\ times}),\ d_n' = p_1(\omega 0\overbrace{222\cdots 2}^{n\ times}).$$

For each $n \in \mathbb{N}$, we get that

$$
\begin{aligned}
&D_{\diagdown} f(p_0(\omega 0\overbrace{222\cdots 2}^{n\ times})) - D_{\diagdown} f(p_2(\omega 2\overbrace{000\cdots 0}^{n\ times})) \\
&+ D_{\diagup} f(p_1(\omega 2\overbrace{000\cdots 0}^{n\ times})) - D_{\_} f(p_1(\omega 0\overbrace{222\cdots 2}^{n\ times})) \\
&= \left(\frac{5}{3}\right)^{m+n}\left[\frac{f(a'_{n-1}) - f(p_0(\omega 2))}{2}\right] - \left(\frac{5}{3}\right)^{m+n}\left[\frac{f(p_0(\omega 2)) - f(b'_{n-1})}{2}\right] \\
&\quad + \left(\frac{5}{3}\right)^{m+n}\left[\frac{f(c'_{n-1}) - f(p_0(\omega 2))}{2}\right] - \left(\frac{5}{3}\right)^{m+n}\left[\frac{f(p_0(\omega 2)) - f(d'_{n-1})}{2}\right] \\
&= \frac{1}{2}\left(\frac{5}{3}\right)^{m+n}\left[f(a'_{n-1}) + f(b'_{n-1}) + f(c'_{n-1}) + f(d'_{n-1}) - 4f(p_0(\omega 2))\right] \\
&= \frac{1}{2}\left(\frac{5}{3}\right)^{m+n} H_{m+n} f(p_0(\omega 2)) = 0.
\end{aligned}
$$

Finally, for each $n \in \mathbb{N}_0$, set

$$
a''_n = p_2(\omega 2\overbrace{111\cdots 1}^{n\ times}),\ b''_n = p_1(\omega 1\overbrace{222\cdots 2}^{n\ times}),
$$
$$
c''_n = p_0(\omega 1\overbrace{222\cdots 2}^{n\ times}),\ d''_n = p_0(\omega 2\overbrace{111\cdots 1}^{n\ times}).
$$

For each $n \in \mathbb{N}$, we get that

$$
\begin{aligned}
&D_{\_} f(p_2(\omega 2\overbrace{111\cdots 1}^{n\ times})) - D_{\_} f(p_1(\omega 1\overbrace{222\cdots 2}^{n\ times})) \\
&+ D_{\diagdown} f(p_0(\omega 1\overbrace{222\cdots 2}^{n\ times})) - D_{\diagup} f(p_0(\omega 2\overbrace{111\cdots 1}^{n\ times})) \\
&= \left(\frac{5}{3}\right)^{m+n}\left[\frac{f(a''_{n-1}) - f(p_1(\omega 2))}{2}\right] - \left(\frac{5}{3}\right)^{m+n}\left[\frac{f(p_1(\omega 2)) - f(b''_{n-1})}{2}\right] \\
&\quad + \left(\frac{5}{3}\right)^{m+n}\left[\frac{f(c''_{n-1}) - f(p_1(\omega 2))}{2}\right] - \left(\frac{5}{3}\right)^{m+n}\left[\frac{f(p_1(\omega 2)) - f(d''_{n-1})}{2}\right] \\
&= \frac{1}{2}\left(\frac{5}{3}\right)^{m+n}\left[f(a''_{n-1}) + f(b''_{n-1}) + f(c''_{n-1}) + f(d''_{n-1}) - 4f(p_1(\omega 2))\right] \\
&= \frac{1}{2}\left(\frac{5}{3}\right)^{m+n} H_{m+n} f(p_1(\omega 2)) = 0.
\end{aligned}
$$

□

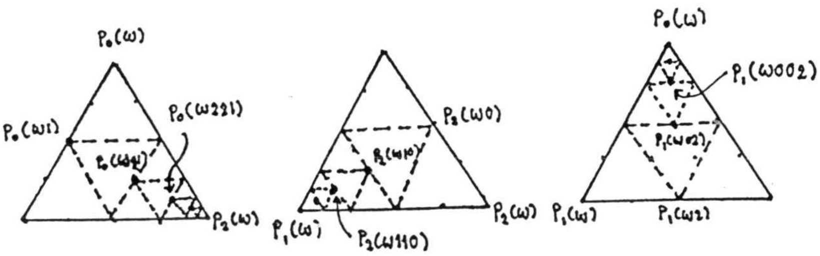

Figure (8). Sequence $p_0(\omega\overbrace{2\cdots2}^{n}1)$, $p_2(\omega\overbrace{1\cdots1}^{n}0)$ and $p_1(\omega\overbrace{0\cdots0}^{n}2)$, $n=1,2$.

Proposition 4.18. *Let f be a nonconstant harmonic function on SG. For $\omega \in W_m, m \in \mathbb{N}_0$, the sequences*

$$\left\{D_{\diagup}f(p_0(\omega\overbrace{2\cdots2}^{n}1))\right\},\ \left\{D_{\diagdown}f(p_2(\omega\overbrace{1\cdots1}^{n}0))\right\}\ \text{and}\ \left\{D_{\_}f(p_1(\omega\overbrace{0\cdots0}^{n}2))\right\}$$

all converge to zero.

Proof. Fix $m \in \mathbb{N}_0$ and $\omega \in W_m$. Corollary 4.16 implies that

$$\begin{aligned}
&D^+_{\diagup}f(p_0(\omega21))\\
&= \left(\frac{5}{3}\right)^{m+1}\left[\frac{f(p_1(\omega2)) - f(p_0(\omega2))}{2}\right]\\
&= \left(\frac{5}{3}\right)^{m+1}\left[\frac{f(p_0(\omega)) + 2f(p_1(\omega)) + 2f(p_2(\omega)) - f(p_1(\omega)) - 2f(p_0(\omega)) - 2f(p_2(\omega))}{2\cdot5}\right]\\
&= \left(\frac{5}{3}\right)^{m+1}\left[\frac{f(p_1(\omega)) - f(p_0(\omega))}{2\cdot5}\right]\\
&= \frac{D^+_{\diagup}f(p_0(\omega1))}{3}.
\end{aligned}$$

For $n \in \mathbb{N}$, assume that

$$D^+_{\diagup}f(p_0(\omega\overbrace{2\cdots2}^{n}1) = \frac{D^+_{\diagup}f(p_0(\omega1))}{3^n}.$$

Then

$$D^{+}_{\diagup} f(p_0(\omega \overbrace{2\cdots2}^{n+1} 1) = \left(\frac{5}{3}\right)^{m+n+1} \left[\frac{f(p_1(\omega \overbrace{2\cdots2}^{n+1})) - f(p_0(\omega \overbrace{2\cdots2}^{n+1}))}{2} \right]$$

$$= \left(\frac{5}{3}\right)^{m+n+1} \left[\frac{f(p_1(\omega \overbrace{2\cdots2}^{n})) - f(p_0(\omega \overbrace{2\cdots2}^{n}))}{2\cdot 5} \right]$$

$$= \frac{D^{+}_{\diagup} f(p_0(\omega \overbrace{2\cdots2}^{n} 1))}{3}$$

$$= \frac{D^{+}_{\diagup} f(p_0(\omega 1))}{3^{n+1}}.$$

By induction, it implies that for $n \in \mathbb{N}$,

$$D^{+}_{\diagup} f(p_0(\omega \overbrace{2\cdots2}^{n} 1)) = \frac{D^{+}_{\diagup} f(p_0(\omega 1))}{3^{n}}.$$

Since $D^{+}_{\diagup} f(p_0(\omega \overbrace{2\cdots2}^{n} 1)) = D_{\diagup} f(p_0(\omega \overbrace{2\cdots2}^{n} 1))$, we get that $(D_{\diagup} f(p_0(\omega \overbrace{2\cdots2}^{n} 1))$ converge to zero. □

Lemma 4.19. *Let f be a nonconstant harmonic function on SG. For $m \in \mathbb{N}_0$ and $\omega \in W_m$, the sequences*

$$\left\{ D^{+}_{\diagup} f(p_1(\omega 1 \overbrace{0\cdots0}^{n})) \right\}, \left\{ D^{+}_{\diagup} f(p_2(\omega 1 \overbrace{0\cdots0}^{n})) \right\},$$

$$\left\{ D^{-}_{\diagup} f(p_0(\omega 0 \overbrace{1\cdots1}^{n})) \right\}, \left\{ D^{-}_{\diagup} f(p_2(\omega 0 \overbrace{1\cdots1}^{n})) \right\}$$

converge to the same point that is $\frac{D^{+}_{\diagup} f(p_0(\omega 1))}{2}$. Moreover, the sequences

$$\left\{ D^{+}_{\diagdown} f(p_0(\omega 0 \overbrace{2\cdots2}^{n})) \right\}, \left\{ D^{+}_{\diagdown} f(p_1(\omega 0 \overbrace{2\cdots2}^{n})) \right\},$$

$$\left\{ D^{-}_{\diagdown} f(p_2(\omega 2 \overbrace{0\cdots0}^{n})) \right\}, \left\{ D^{-}_{\diagdown} f(p_1(\omega 2 \overbrace{0\cdots0}^{n})) \right\}$$

converge to $\frac{D^{+}_{\diagdown} f(p_2(\omega 0))}{2}$ and the sequences

$$\left\{ D^{+}_{-} f(p_2(\omega 2 \overbrace{1 \cdots 1}^{n})) \right\}, \left\{ D^{+}_{-} f(p_0(\omega 2 \overbrace{1 \cdots 1}^{n})) \right\},$$
$$\left\{ D^{-}_{-} f(p_1(\omega 1 \overbrace{2 \cdots 2}^{n})) \right\}, \left\{ D^{-}_{-} f(p_0(\omega 1 \overbrace{2 \cdots 2}^{n})) \right\}$$

converge to $\frac{D^{+}_{-} f(p_1(\omega 2))}{2}$.

Proof. By Corollary 4.16, it implies that $D^{+}_{/} f\,(p_0(1)) = \frac{f(p_1) - f(p_0)}{2}$.

Claim that for $n \in \mathbb{N}$,

$$\begin{aligned} &D^{+}_{/} f(p_1(\omega 1 \overbrace{0 \cdots 0}^{n})) \\ &= \frac{5^m}{2 \cdot 3^{n+m}} \left[\left(3 + \sum_{i=1}^{n-1} 3^i \right) f(p_1(\omega)) - f(p_2(\omega)) - \left(2 + \sum_{i=1}^{n-1} 3^i \right) f(p_0(\omega)) \right], \end{aligned}$$

and

$$\begin{aligned} &D^{+}_{/} f(p_2(\omega 1 \overbrace{0 \cdots 0}^{n})) \\ &= \frac{5^m}{2 \cdot 3^{n+m}} \left[\left(\sum_{i=1}^{n-1} 3^i \right) f(p_1(\omega)) + f(p_2(\omega)) - \left(1 + \sum_{i=1}^{n-1} 3^i \right) f(p_0(\omega)) \right]. \end{aligned}$$

It easy to see that

$$\begin{aligned} D^{+}_{/} f(p_1(\omega 10)) &= \left(\frac{5}{3}\right)^{m+1} \left[\frac{f(p_1(\omega 1)) - f(p_0(\omega 1))}{2} \right] \\ &= \left(\frac{5}{3}\right)^{m+1} \left[\frac{f(p_1(\omega))}{2} - \frac{f(p_2(\omega)) + 2f(p_0(\omega)) + 2f(p_1(\omega))}{2 \cdot 5} \right] \\ &= \frac{5^m}{2 \cdot 3^{m+1}} \left[3f(p_1(\omega)) - f(p_2(\omega)) - 2f(p_0(\omega)) \right], \end{aligned}$$

$$\begin{aligned} D^{+}_{/} f(p_2(\omega 10)) &= \left(\frac{5}{3}\right)^{m+1} \left[\frac{f(p_2(\omega 1)) - f(p_0(\omega 1))}{2} \right] \\ &= \left(\frac{5}{3}\right)^{m+1} \left[\frac{f(p_0(\omega)) + 2f(p_1(\omega)) + 2f(p_2(\omega))}{2 \cdot 5} \right] \\ &\quad - \left(\frac{5}{3}\right)^{m+1} \left[\frac{f(p_2(\omega)) + 2f(p_0(\omega)) + 2f(p_1(\omega))}{2 \cdot 5} \right] \\ &= \frac{5^m}{2 \cdot 3^{m+1}} \left[f(p_2(\omega)) - f(p_0(\omega)) \right]. \end{aligned}$$

Assume that the claim holds for $n \in \mathbb{N}$. Then

$$
\begin{aligned}
&D^{+}_{\diagup} f(p_1(\omega 1 \overbrace{0\cdots 0}^{n+1}))\\
&= \left(\frac{5}{3}\right)^{n+m+1} \left[\frac{f(p_1(\omega 1 \overbrace{0\cdots 0}^{n})) - f(p_0(\omega 1))}{2} \right]\\
&= \left(\frac{5}{3}\right)^{n+m+1} \left[\frac{f(p_2(\omega 1 \overbrace{0\cdots 0}^{n-1})) + 2f(p_1(\omega 1 \overbrace{0\cdots 0}^{n-1})) + 2f(p_0(\omega 1))}{2\cdot 5} - \frac{f(p_0(\omega 1))}{2} \right]\\
&= \frac{1}{3}\left(\frac{5}{3}\right)^{n+m} \left[\frac{f(p_2(\omega 1 \overbrace{0\cdots 0}^{n-1})) - f(p_0(\omega 1))}{2} \right]\\
&\quad + \frac{2}{3}\left(\frac{5}{3}\right)^{n+m} \left[\frac{f(p_1(\omega 1 \overbrace{0\cdots 0}^{n-1})) - f(p_0(\omega 1))}{2} \right]\\
&= \frac{-D^{-}_{\diagdown} f(p_2(\omega 1 \overbrace{0\cdots 0}^{n}))}{3} + \frac{2D^{+}_{\diagup} f(p_1(\omega 1 \overbrace{00\cdots 0}^{n}))}{3}\\
&= \frac{D^{+}_{\diagup} f(p_2(\omega 1 \overbrace{0\cdots 0}^{n}))}{3} + \frac{2D^{+}_{\diagup} f(p_1(\omega 1 \overbrace{00\cdots 0}^{n}))}{3}\\
&= \frac{5^m}{2\cdot 3^{n+m+1}} \left[\left(6 + 3\sum_{i=1}^{n-1} 3^i\right) f(p_1(\omega)) - f(p_2(\omega)) - \left(5 + 3\sum_{i=1}^{n-1} 3^i\right) f(p_0(\omega)) \right]\\
&= \frac{5^m}{2\cdot 3^{n+m+1}} \left[\left(3 + \sum_{i=1}^{n} 3^i\right) f(p_1(\omega)) - f(p_2(\omega)) - \left(2 + \sum_{i=1}^{n} 3^i\right) f(p_0(\omega)) \right]
\end{aligned}
$$

and

$$
D^{+}_{\diagup} f(p_2(\omega 1 \overbrace{0\cdots 0}^{n+1})) = \left(\frac{5}{3}\right)^{n+m+1} \left[\frac{f(p_2(\omega 1 \overbrace{0\cdots 0}^{n})) - f(p_0(\omega 1))}{2} \right]
$$

$$= \left(\frac{5}{3}\right)^{n+m+1}\left[\frac{f(p_1(\omega 1\overbrace{0\cdots0}^{n-1})) + 2f(p_2(\omega 1\overbrace{0\cdots0}^{n-1})) + 2f(p_0(\omega 1))}{2\cdot 5} - \frac{f(p_0(\omega 1))}{2}\right]$$

$$= \frac{1}{3}\left(\frac{5}{3}\right)^{n+m}\left[\frac{f(p_1(\omega 1\overbrace{0\cdots0}^{n-1})) - f(p_0(\omega 1))}{2}\right]$$

$$+ \frac{2}{3}\left(\frac{5}{3}\right)^{n+m}\left[\frac{f(p_2(\omega 1\overbrace{0\cdots0}^{n-1})) - f(p_0(\omega 1))}{2}\right]$$

$$= \frac{D^{+}_{\swarrow} f(p_1(\omega 1\overbrace{0\cdots0}^{n}))}{3} - \frac{2D^{-}_{\searrow} f(p_2(\omega 1\overbrace{0\cdots0}^{n}))}{3}$$

$$= \frac{D^{+}_{\swarrow} f(p_1(\omega 1\overbrace{0\cdots0}^{n}))}{3} + \frac{2D^{+}_{\swarrow} f(p_2(\omega 1\overbrace{0\cdots0}^{n}))}{3}$$

$$= \frac{5^m}{2\cdot 3^{n+m+1}}\left[\left(3+\sum_{i=1}^{n-1}3^i\right) f(p_1(\omega)) + f(p_2(\omega)) - \left(4 + 3\sum_{i=1}^{n-1}3^i\right) f(p_0(\omega))\right]$$

$$= \frac{5^m}{2\cdot 3^{n+m+1}}\left[\left(\sum_{i=1}^{n}3^i\right) f(p_1(\omega)) + f(p_2(\omega)) - \left(1 + \sum_{i=1}^{n}3^i\right) f(p_0(\omega))\right].$$

It follows that

$$\lim_{n\to\infty} D^{+}_{\swarrow} f(p_1(\omega 1\overbrace{0\cdots0}^{n}))$$

$$= \lim_{n\to\infty}\frac{5^m}{2\cdot 3^{n+m}}\left[\left(3+\sum_{i=1}^{n-1}3^i\right) f(p_1(\omega)) - f(p_2(\omega)) - \left(2 + \sum_{i=1}^{n-1}3^i\right) f(p_0(\omega))\right]$$

$$= \frac{f(p_1(\omega)) - f(p_0(\omega))}{2}\left(\frac{5}{3}\right)^m \lim_{n\to\infty}\frac{1}{3}\sum_{i=1}^{n-1}3^i$$

$$= \frac{f(p_1(\omega)) - f(p_0(\omega))}{2}\left(\frac{5}{3}\right)^m \lim_{n\to\infty}\left(\frac{1}{3}+\frac{1}{3^2}+\cdots+\frac{1}{3^{n-1}}\right)$$

$$= \frac{f(p_1(\omega)) - f(p_0(\omega))}{2}\left(\frac{5}{3}\right)^m\cdot\frac{1}{2}$$

$$= \frac{D^{+}_{\swarrow} f(p_0(\omega 1))}{2}$$

and

$$
\begin{aligned}
&\lim_{n\to\infty} D^{+}_{\diagup} f(p_2(\omega 1 \overbrace{0\cdots 0}^{n})) \\
&= \lim_{n\to\infty} \frac{5^m}{2\cdot 3^{n+m}} \left[\left(\sum_{i=1}^{n-1} 3^i \right) f(p_1(\omega)) + f(p_2(\omega)) - \left(1 + \sum_{i=1}^{n-1} 3^i \right) f(p_0(\omega)) \right] \\
&= \frac{f(p_1(\omega)) - f(p_0(\omega))}{2} \left(\frac{5}{3}\right)^m \lim_{n\to\infty} \left(\frac{1}{3} + \frac{1}{3^2} + \cdots + \frac{1}{3^{n-1}} \right) \\
&= \frac{D^{+}_{\diagup} f(p_0(\omega 1))}{2}.
\end{aligned}
$$

Similarly, we get that for $n \in \mathbb{N}_0$,

$$
\begin{aligned}
&D^{-}_{\diagup} f(p_0(\omega 0 \overbrace{1\cdots 1}^{n})) \\
&= \frac{5^m}{2\cdot 3^{n+m}} \left[\left(2 + \sum_{i=1}^{n-1} 3^i \right) f(p_1(\omega)) + f(p_2(\omega)) - \left(3 + \sum_{i=1}^{n-1} 3^i \right) f(p_0(\omega)) \right]
\end{aligned}
$$

and

$$
\begin{aligned}
&D^{-}_{\diagup} f(p_2(\omega 0 \overbrace{1\cdots 1}^{n})) \\
&= \frac{5^m}{2\cdot 3^{n+m}} \left[\left(1 + \sum_{i=1}^{n-1} 3^i \right) f(p_1(\omega)) - f(p_2(\omega)) - \left(\sum_{i=1}^{n-1} 3^i \right) f(p_0(\omega)) \right].
\end{aligned}
$$

Moreover,

$$
\lim_{n\to\infty} D^{-}_{\diagup} f(p_0(\omega 0 \overbrace{1\cdots 1}^{n})) = \lim_{n\to\infty} D^{-}_{\diagup} f(p_2(\omega 0 \overbrace{1\cdots 1}^{n})) = \frac{D^{+}_{\diagup} f(p_0(\omega 1))}{2}.
$$

□

Theorem 4.20. *Let f be a harmonic function on SG and $u \in \mathcal{U}$. Then TFAE:*

(a) f is a constant function on SG,

(b) $D^{+}_{u} f \equiv 0$,

(c) $D^{+}_{u} f$ is continuous on its domain.

Moreover, this also holds for $D^{-}_{u} f$.

Proof. Let u be $\diagup$.

$(a) \Rightarrow (b)$. Let $p \in Dom(D^+_{\diagup} f) = A_{\diagup} \cup A_{\diagdown} \cup \{p_0\}$. Recall that

$$A_{\diagup} = \bigcup_{n=1}^{\infty} \{p \in V_n | p = p_0(\omega 1) \text{ for some } \omega \in \{0,1,2\}^{n-1}\}$$
$$A_{\diagdown} = \bigcup_{n=1}^{\infty} \{p \in V_n | p = p_0(\omega 2) \text{ for some } \omega \in \{0,1,2\}^{n-1}\}.$$

Then $p = p_0(\omega)$ for some W_m and $m \in \mathbb{N}_0$.

By lemma 4.14, we get that

$$D^+_{\diagup} f(p_0(\omega)) = \left(\frac{5}{3}\right)^m \left[-f(p_0(\omega)) + \frac{f(p_1(\omega))}{2} + \frac{f(p_2(\omega))}{2}\right] = 0$$

since f is a constant function. Hence $D^+_{\diagup} f(p) = 0$.

$(b) \Rightarrow (c)$. Let $p \in Dom(D^+_{\diagup} f), m \in \mathbb{N}$ and $SG_{m,p} = \bigcup_{\substack{\omega \in W_m \\ p \in SG_\omega}} SG_\omega$ be a neighborhood of p. For each $q \in SG_{m,p}$ and $q \in Dom(D^+_{\diagup} f)$,

$$D^+_{\diagup} f(q) = 0 = D^+_{\diagup} f(p).$$

Hence $D^+_{\diagup} f$ is continuous on $Dom(D^+_{\diagup} f)$.

$(c) \Rightarrow (a)$. By lemma 4.18, the sequence $p_0(\omega \overbrace{2 \cdots 2}^{n} 1)$ converge to $p_2(\omega)$ and $(D_{\diagup} f(p_0(\omega \overbrace{2 \cdots 2}^{n} 1))$ converge to 0 for all $W_m, m \in \mathbb{N}$. Since $D^+_{\diagup} f$ is continuous, we have $D^+_{\diagup} f(p_2(\omega)) = 0$ for all $\omega \in W_m, m \in \mathbb{N}$.

If $\omega = 0$, then $0 = D^+_{\diagup} f(p_2(0)) = \left[\frac{f(p_2) - f(p_0)}{2}\right]$ so that

$$f(p_0) = f(p_2). \tag{4.21}$$

If $\omega = 00$, then

$$0 = D^+_{\diagup} f(p_2(00)) = \frac{5}{3} \cdot \left[\frac{f(p_2(0)) - f(p_0)}{2}\right] = \frac{1}{2 \cdot 3}\left[f(p_1) + 2f(p_2) - 3f(p_0)\right]$$

so that

$$f(p_1) = 3f(p_0) - 2f(p_2). \tag{4.22}$$

By the equations 4.21 and 4.22, $f(p_1) = 3f(p_0) - 2f(p_2) = f(p_0)$. Hence f is a constant on SG. $\square$

Return to Theorem 4.8, we can reproof the statement for any harmonic functions by our formulas in the following theorem:

Theorem 4.23. *Let f and g be harmonic functions on SG and a and b be fixed real numbers. Then*

$$D_u[af(p) + bg(p)] = aD_u f(p) + bD_u g(p) \tag{4.24}$$

where p is a nonboundary point in V_* and u is an admissible direction of u.

Proof. Let p be an element in $A_{\diagup}$. Then $p = p_0(\omega 1)$ for some $m \in \mathbb{N}$ and $\omega \in W_{m-1}$. Recall that $af + bg$ is a harmonic function. By Theorem 4.15, we obtain that

$$D_{\diagup} h(p) = D^+_{\diagup} h(p) = D^-_{\diagup} h(p)$$

for all harmonic functions on SG. By applying Corollary 4.16, we get that

$$\begin{aligned}
&D_{\diagup}[af + bg](p) \\
&= D^+_{\diagup}[af + bg](p_0(\omega 1)) \\
&= \left(\frac{5}{3}\right)^m \left[\frac{(af + bg)(p_1(\omega)) - (af + bg)(p_0(\omega))}{2}\right] \\
&= a\left(\frac{5}{3}\right)^m \left[\frac{f(p_1(\omega)) - f(p_0(\omega))}{2}\right] + b\left(\frac{5}{3}\right)^m \left[\frac{g(p_1(\omega)) - g(p_0(\omega))}{2}\right] \\
&= aD^+_{\diagup} f(p_0(\omega 1)) + bD^+_{\diagup} g(p_0(\omega 1)) \\
&= aD_{\diagup} f(p) + bD_{\diagup} g(p).
\end{aligned}$$

$\square$

However, we can not prove Theorem 4.11 for harmonic functions by our formula because for any harmonic function f and g, fg is not necessary a harmonic function. In general, fg is a harmonic function only when one of them is constant.

Lemma 4.25. *Let f be a nonconstant harmonic function and g be any harmonic function on SG. Then fg is a harmonic function if and only if g is a constant.*

Proof. Clearly, if g is a constant then fg is a harmonic function.
Assume that fg is a harmonic function. Set

$$f(p_0) = x,\ f(p_1) = y,\ f(p_2) = z,\ g(p_0) = a,\ g(p_1) = b,\ \text{and } g(p_2) = c.$$

Since fg is a harmonic function, we obtain that $(fg)(p_1(2)) = f(p_1(2))g(p_1(2))$. Then

$$\frac{ax + 2by + 2cz}{5} = \left[\frac{x + 2y + 2z}{5}\right]\left[\frac{a + 2b + 2c}{5}\right],$$

$$5ax + 10by + 10cz = ax + 2ay + 2az + 2bx + 4by + 4bz + 2cx + 4cy + 4cz,$$

$$2ax + 3by + 3cz = ay + az + bx + 2bz + cx + 2cy.$$

Morever, $(fg)(p_0(2)) = f(p_0(2))g(p_0(2))$. Then

$$\frac{by + 2ax + 2cz}{5} = \left[\frac{y + 2x + 2z}{5}\right]\left[\frac{b + 2a + 2c}{5}\right],$$

$$5by + 10ax + 10cz = by + 2bx + 2bz + 2ay + 4ax + 4az + 2cy + 4cx + 4cz,$$

$$3ax + 2by + 3cz = ay + 2az + bx + bz + 2cx + cy.$$

Finally, $(fg)(p_0(1)) = f(p_0(1))g(p_0(1))$. Then

$$\frac{cz + 2ax + 2by}{5} = \left[\frac{z + 2x + 2y}{5}\right]\left[\frac{c + 2a + 2b}{5}\right],$$

$$5cz + 10ax + 10by = cz + 2cx + 2cy + 2az + 4ax + 4ay + 2bz + 4bx + 4by,$$

$$3ax + 3by + 2cz = 2ay + az + 2bx + bz + cx + cy.$$

Step1 WLOG, we will consider that $z = 0$.

Case $z = 0, x \neq 0, y \neq 0$. The above equations imply that

$$
\begin{aligned}
&& 6ax - 3bx - 3cx + 9by - 3ay - 6cy &= 0, && (1)\\
&& 6ax - 2bx - 4cx + 4by - 2ay - 2cy &= 0, && (2)\\
&& 6ax - 4bx - 2cx + 6by - 4ay - 2cy &= 0, && (3)\\
(1)-(2); && -bx + cx + 5by - ay - 4cy &= 0, && (4)\\
(1)-(3); && bx - cx + 3by + ay - 4cy &= 0, && (5)\\
(4)+(5); && 8by - 8cy &= 0.
\end{aligned}
$$

Then $b = c$. In stead of $b = c$ in (5), we get that $a = b = c$. Hence $a = b = c$.

Case $z = 0, x \neq 0, y = 0$. The result is

$$
\begin{aligned}
2ax - bx - cx &= 0,\\
3ax - bx - 2cx &= 0,\\
3ax - 2bx - cx &= 0.
\end{aligned}
$$

It easy to see that $a = b = c$. By two cases, g is a constant function.

Step2 If $z \neq 0$, then $f - z$ is a nonconstant harmonic function and

$$(f - z)(p_2) = 0.$$

Apply the first step, g is a constant function.

By two steps, g is a constant function. □

Definition 4.26. A real value $f(p)$ is a **local maximum value** of the function f if $f(q) \leq f(p)$ for all q sufficiently closed to p. Similarly, the real value $f(p)$ is a **local minimum value** of f if $f(q) \geq f(p)$ for all q sufficiently closed to p.

Proposition 4.27. *Let f be a nonconstant harmonic function on SG and p is any point in $V_* \setminus V_0$. If $f(p)$ is either a local maximum value or a local minimum value of f, then $D_u f(p) = 0$ where u is the admissible direction of p.*

Proof. Assume that $f(p)$ is a local maximum value of f. Note that p take's one of the three form $p_0(\omega 1)$, $p_0(\omega 2)$, or $p_1(\omega 2)$. Let p be $p_0(\omega 1)$ for some $m \in \mathbb{N}$ and $\omega \in W_{m-1}$. For $n \in \mathbb{N}$ is large enough,

$$\frac{f(p_1(\omega 1 \overbrace{000\cdots 0}^{n-m-1\ times})) - f(p_0(\omega 1))}{a_n} \leq 0.$$

Then

$$D^{+}_{\diagup} f(p) = \lim_{n\to\infty} \frac{f(p_1(\omega 1 \overbrace{000\cdots 0}^{n-m-1\ times})) - f(p_0(\omega 1))}{a_n} \leq 0.$$

Moreover,

$$\frac{f(p_1(\omega 0)) - f(p_0(\omega 0 \overbrace{111\cdots 1}^{n-m-1\ times}))}{a_n} \geq 0.$$

Then

$$D^{-}_{\diagup} f(p) = \lim_{n\to\infty} \frac{f(p_1(\omega 0)) - f(p_0(\omega 0 \overbrace{111\cdots 1}^{n-m-1\ times}))}{a_n} \geq 0.$$

Since $p \in V_* \setminus V_0$, by Theorem 4.15, f has derivative at p, i.e., $D^{+}_{\diagup} f(p) = D^{-}_{\diagup} f(p)$, and we get that

$$Df = D^{+}_{\diagup} f(p) = D^{-}_{\diagup} f(p) = 0.$$

□

4.3 Derivatives and the Laplacian

In this section, we will define a new derivative satisfying Definition 4.1 at every nonboundary point for any harmonic function for which the second derivative is zero. To this purpose, we will replace $\left(\frac{3}{5}\right)^n$ in Definition 4.1 by $\frac{8}{5^{n+1}}\left(\frac{3}{5}\right)^m$.

Definition 4.28. Fix f, a function on SG. Define $\mathcal{D}f : SG \to \mathbb{R}$ by

$\mathcal{D}f(p_0(\omega 1))$

$$= \lim_{n\to\infty} \frac{f(p_1(\omega 1\overbrace{000\cdots 0}^{n\ times})) - f(p_0(\omega 0\overbrace{111\cdots 1}^{n\ times})) + f(p_2(\omega 0\overbrace{111\cdots 1}^{n\ times})) - f(p_2(\omega 1\overbrace{000\cdots 0}^{n\ times}))}{a_n},$$

$\mathcal{D}f(p_0(\omega 2))$

$$= \lim_{n\to\infty} \frac{f(p_0(\omega 0\overbrace{222\cdots 2}^{n\ times})) - f(p_2(\omega 2\overbrace{000\cdots 0}^{n\ times})) + f(p_1(\omega 2\overbrace{000\cdots 0}^{n\ times})) - f(p_1(\omega 0\overbrace{222\cdots 2}^{n\ times}))}{a_n},$$

and

$\mathcal{D}f(p_1(\omega 2))$

$$= \lim_{n\to\infty} \frac{f(p_2(\omega 2\overbrace{111\cdots 1}^{n\ times})) - f(p_1(\omega 1\overbrace{222\cdots 2}^{n\ times})) + f(p_0(\omega 1\overbrace{222\cdots 2}^{n\ times})) - f(p_0(\omega 2\overbrace{111\cdots 1}^{n\ times}))}{a_n}$$

where $a_n = \frac{8}{5^{n+1}}\left(\frac{3}{5}\right)^m$ and $\omega \in W_m$ for some $m \in \mathbb{N}_0$ if limit exist.

Theorem 4.29. *Let f be a harmonic function on SG. If p is a nonboundary point in V_*, then*

$$\mathcal{D}f(p) = D_u f(p)$$

where u is the admissible direction of p. Moreover, $\mathcal{D}^2 f \equiv \mathcal{D}(\mathcal{D}f)$ exists at every $p \in V_* \setminus V_0$ and

$$\mathcal{D}^2 f(p) = 0.$$

Proof. Note that for $p \in V_* \setminus V_0$, we can write $p = p_0(\omega 1), p_0(\omega 2)$ or $p_1(\omega 2)$ for some $\omega \in W_m$, $m \in \mathbb{N}_0$.

Case $\mathbf{p = p_0(\omega 1)}$. By Lemma 4.13, we get that

$\mathcal{D}f(p_0(\omega 1))$

$$= \lim_{n\to\infty} \frac{f(p_1(\omega 1\overbrace{000\cdots 0}^{n\ times})) - f(p_0(\omega 0\overbrace{111\cdots 1}^{n\ times})) + f(p_2(\omega 0\overbrace{111\cdots 1}^{n\ times})) - f(p_2(\omega 1\overbrace{000\cdots 0}^{n\ times}))}{\frac{8}{5^{n+1}}\left(\frac{3}{5}\right)^m}$$

$$= \frac{5}{8}\left(\frac{5}{3}\right)^m \lim_{n\to\infty} [(5^n - 3^n) f(p_0(\omega 1)) + b_n f(p_1(\omega 1)) + (b_n - 1) f(p_2(\omega 1))]$$

$$- [b_n f(p_0(\omega 0)) + (5^n - 3^n) f(p_1(\omega 0)) + (b_n - 1) f(p_2(\omega 0))]$$

$$+ [(b_n - 1) f(p_0(\omega 0)) + (5^n - 3^n) f(p_1(\omega 0)) + b_n f(p_2(\omega 0))]$$

$$- [(5^n - 3^n) f(p_0(\omega 1)) + (b_n - 1) f(p_1(\omega 1)) + b_n f(p_2(\omega 1))]$$

$$= \frac{5}{8}\left(\frac{5}{3}\right)^m \lim_{n\to\infty} [f(p_1(\omega 1)) - f(p_2(\omega 1)) - f(p_0(\omega 0)) + f(p_2(\omega 0))]$$

$$= \frac{5}{8}\left(\frac{5}{3}\right)^m [f(p_1(\omega)) - f(p_0(\omega))] - \frac{5}{8}\left(\frac{5}{3}\right)^m \left[\frac{f(p_0(\omega)) + 2f(p_2(\omega)) + 2f(p_1(\omega))}{5}\right]$$

$$+ \frac{5}{8}\left(\frac{5}{3}\right)^m \left[\frac{f(p_1(\omega)) + 2f(p_2(\omega)) + 2f(p_0(\omega))}{5}\right]$$

$$= \frac{5}{8}\left(\frac{5}{3}\right)^m \left[\frac{4f(p_1(\omega)) - 4f(p_0(\omega))}{5}\right]$$

$$= \left(\frac{5}{3}\right)^m \frac{f(p_1(\omega)) - f(p_0(\omega))}{2}$$

$$= D_{\diagup} f(p_0(\omega 1)).$$

Moreover, by Lemma 4.17 we get that for $a_n = \frac{8}{5^{n+1}}\left(\frac{3}{5}\right)^m$

$$\mathcal{D}^2 f(p_0(\omega 1)) = \lim_{n\to\infty} \frac{\mathcal{D} f(p_1(\omega 1 \overbrace{000\cdots 0}^{n\ times}))}{a_n} - \frac{\mathcal{D} f(p_0(\omega 0 \overbrace{111\cdots 1}^{n\ times}))}{a_n}$$

$$+ \frac{\mathcal{D} f(p_2(\omega 0 \overbrace{111\cdots 1}^{n\ times}))}{a_n} - \frac{\mathcal{D} f(p_2(\omega 1 \overbrace{000\cdots 0}^{n\ times}))}{a_n}$$

$$= \lim_{n\to\infty} \frac{D_{\diagup} f(p_1(\omega 1 \overbrace{000\cdots 0}^{n\ times}))}{a_n} - \frac{D_{\diagup} f(p_0(\omega 0 \overbrace{111\cdots 1}^{n\ times}))}{a_n}$$

$$+ \frac{D_{\_} f(p_2(\omega 0 \overbrace{111\cdots 1}^{n\ times}))}{a_n} - \frac{D_{\diagdown} f(p_2(\omega 1 \overbrace{000\cdots 0}^{n\ times}))}{a_n}$$

$$= 0.$$

Case $\mathbf{p} = \mathbf{p_0}(\omega \mathbf{2})$.

$$\mathcal{D}f(p_0(\omega 2))$$

$$= \lim_{n\to\infty} \frac{f(p_0(\omega 0\overbrace{222\cdots 2}^{n\ times})) - f(p_2(\omega 2\overbrace{000\cdots 0}^{n\ times})) + f(p_1(\omega 2\overbrace{000\cdots 0}^{n\ times})) - f(p_1(\omega 0\overbrace{222\cdots 2}^{n\ times}))}{\frac{8}{5^{n+1}}\left(\frac{3}{5}\right)^m}$$

$$= \frac{5}{8}\left(\frac{5}{3}\right)^m \lim_{n\to\infty} [b_n f(p_0(\omega 0)) + (b_n - 1)f(p_1(\omega 0)) + (5^n - 3^n)f(p_2(\omega 0))]$$

$$- [(5^n - 3^n)f(p_0(\omega 2)) + (b_n - 1)f(p_1(\omega 2)) + b_n f(p_2(\omega 2))]$$

$$+ [(5^n - 3^n)f(p_0(\omega 2)) + b_n f(p_1(\omega 2)) + (b_n - 1)f(p_2(\omega 2))]$$

$$- [(b_n - 1)f(p_0(\omega 0)) + b_n f(p_1(\omega 0)) + (5^n - 3^n)f(p_2(\omega 0))]$$

$$= \frac{5}{8}\left(\frac{5}{3}\right)^m \lim_{n\to\infty} [f(p_0(\omega 0)) - f(p_1(\omega 0)) + f(p_1(\omega 2)) - f(p_2(\omega 2))]$$

$$= \frac{5}{8}\left(\frac{5}{3}\right)^m [f(p_0(\omega)) - f(p_2(\omega))] - \frac{5}{8}\left(\frac{5}{3}\right)^m \left[\frac{f(p_2(\omega)) + 2f(p_1(\omega)) + 2f(p_0(\omega))}{5}\right]$$

$$+ \frac{5}{8}\left(\frac{5}{3}\right)^m \left[\frac{f(p_0(\omega)) + 2f(p_1(\omega)) + 2f(p_2(\omega))}{5}\right]$$

$$= \frac{5}{8}\left(\frac{5}{3}\right)^m \left[\frac{4f(p_0(\omega)) - 4f(p_2(\omega))}{5}\right]$$

$$= \left(\frac{5}{3}\right)^m \frac{f(p_0(\omega)) - f(p_2(\omega))}{2}$$

$$= D_{\searrow} f(p_0(\omega 2)).$$

Moreover, for $a_n = \frac{8}{5^{n+1}}\left(\frac{3}{5}\right)^m$

$$\mathcal{D}^2 f(p_0(\omega 1)) = \lim_{n\to\infty} \frac{\mathcal{D}f(p_0(\omega 0\overbrace{222\cdots 2}^{n\ times}))}{a_n} - \frac{\mathcal{D}f(p_2(\omega 2\overbrace{000\cdots 0}^{n\ times}))}{a_n}$$

$$+ \frac{\mathcal{D}f(p_1(\omega 2\overbrace{000\cdots 0}^{n\ times}))}{a_n} - \frac{\mathcal{D}f(p_1(\omega 0\overbrace{222\cdots 2}^{n\ times}))}{a_n}$$

$$= \lim_{n\to\infty} \frac{D_{\diagdown} f(p_0(\omega 0 \overbrace{222\cdots 2}^{n\ times}))}{a_n} - \frac{D_{\diagdown} f(p_2(\omega 2 \overbrace{000\cdots 0}^{n\ times}))}{a_n}$$
$$+ \frac{D_{\diagup} f(p_1(\omega 2 \overbrace{000\cdots 0}^{n\ times}))}{a_n} - \frac{D_{\_} f(p_1(\omega 0 \overbrace{222\cdots 2}^{n\ times}))}{a_n}$$
$$= 0.$$

Case $\mathbf{p} = \mathbf{p_1}(\omega \mathbf{2})$.

$\mathcal{D}f(p_1(\omega 2))$

$$= \lim_{n\to\infty} \frac{f(p_2(\omega 2 \overbrace{111\cdots 1}^{n\ times})) - f(p_1(\omega 1 \overbrace{222\cdots 2}^{n\ times})) + f(p_0(\omega 1 \overbrace{222\cdots 2}^{n\ times})) - f(p_0(\omega 2 \overbrace{111\cdots 1}^{n\ times}))}{\frac{8}{5^{n+1}}\left(\frac{3}{5}\right)^m}$$

$$= \frac{5}{8}\left(\frac{5}{3}\right)^m \lim_{n\to\infty} [(b_n - 1)f(p_0(\omega 2)) + (5^n - 3^n)f(p_1(\omega 2)) + b_n f(p_2(\omega 2))]$$
$$- [(b_n - 1)f(p_0(\omega 1)) + b_n f(p_1(\omega 1)) + (5^n - 3^n)f(p_2(\omega 1))]$$
$$+ [b_n f(p_0(\omega 1)) + (b_n - 1)f(p_1(\omega 1)) + (5^n - 3^n)f(p_2(\omega 1))]$$
$$- [b_n f(p_0(\omega 2)) + (5^n - 3^n)f(p_1(\omega 2)) + (b_n - 1)f(p_2(\omega 2))]$$
$$= \frac{5}{8}\left(\frac{5}{3}\right)^m \lim_{n\to\infty} [-f(p_0(\omega 2)) + f(p_2(\omega 2)) + f(p_0(\omega 1)) - f(p_1(\omega 1))]$$
$$= \frac{5}{8}\left(\frac{5}{3}\right)^m [f(p_2(\omega)) - f(p_1(\omega))] - \frac{5}{8}\left(\frac{5}{3}\right)^m \left[\frac{f(p_1(\omega)) + 2f(p_0(\omega)) + 2f(p_2(\omega))}{5}\right]$$
$$+ \frac{5}{8}\left(\frac{5}{3}\right)^m \left[\frac{f(p_2(\omega)) + 2f(p_0(\omega)) + 2f(p_1(\omega))}{5}\right]$$
$$= \frac{5}{8}\left(\frac{5}{3}\right)^m \left[\frac{4f(p_2(\omega)) - 4f(p_1(\omega))}{5}\right]$$
$$= \left(\frac{5}{3}\right)^m \frac{f(p_2(\omega)) - f(p_1(\omega))}{2}$$
$$= D_{\_} f(p_1(\omega 2)).$$

Moreover, for $a_n = \frac{8}{5^{n+1}}\left(\frac{3}{5}\right)^m$

$$\begin{aligned}
\mathcal{D}^2 f(p_1(\omega 2)) &= \lim_{n\to\infty} \frac{\mathcal{D}f(p_2(\omega 2\overbrace{111\cdots 1}^{n\ times}))}{a_n} - \frac{\mathcal{D}f(p_1(\omega 1\overbrace{222\cdots 2}^{n\ times}))}{a_n} \\
&\qquad + \frac{\mathcal{D}f(p_0(\omega 1\overbrace{222\cdots 2}^{n\ times}))}{a_n} - \frac{\mathcal{D}f(p_0(\omega 2\overbrace{111\cdots 1}^{n\ times}))}{a_n} \\
&= \lim_{n\to\infty} \frac{D_{_}f(p_2(\omega 2\overbrace{111\cdots 1}^{n\ times}))}{a_n} - \frac{D_{_}f(p_1(\omega 1\overbrace{222\cdots 2}^{n\ times}))}{a_n} \\
&\qquad + \frac{D_{\diagdown}f(p_0(\omega 1\overbrace{222\cdots 2}^{n\ times}))}{a_n} - \frac{D_{\diagup}f(p_0(\omega 2\overbrace{111\cdots 1}^{n\ times}))}{a_n} \\
&= 0.
\end{aligned}$$

Therefore, $\mathcal{D}^2 f \equiv \mathcal{D}(\mathcal{D}f)$ exists at every $p \in V_* \setminus V_0$ and $\mathcal{D}^2 f(p) = 0$. $\square$