# CHAPTER III

# SOME DERIVATIVES ON THE SIERPINSKI GASKET

In this chapter, we will introduce four derivatives of functions on the Sierpinski gasket. They are the Neumann derivative, the normal derivative, the transverse derivative and the derivatives of Stichartz.

## 3.1 The Neumann derivative

**Definition 3.1.** ([3]) The **Neumann derivative** of $u$ at any point $p$ in $V_0$ is the limit of $-\left(\frac{5}{3}\right)^m (H_m u)(p)$ as $m \to \infty$, denote this limit by $(du)_p$, i.e., $(du)_p = \lim_{m\to\infty} -\left(\dfrac{5}{3}\right)^m (H_m u)(p)$ where $H_m$ is defined in definition 2.3.

**Lemma 3.2.** *Let $u \in Dom\Delta$. Then $(du)_p$ exist for all $p \in V_0$.*

*Proof.* It is enough to prove the lemma for $p = p_0$. By Lemma 2.4, we get that

$$\begin{aligned}&\frac{3}{5}(H_m u)(p_0)\\ &= (H_{m+1}u)(p_0) + \frac{2}{5}\sum_{k=1,2}(H_{m+1}u)(q_k(\overbrace{0\cdots 0}^{m})) + \frac{1}{5}(H_{m+1}u)(q_0(\overbrace{0\cdots 0}^{m})).\end{aligned}$$

Since $u \in Dom\Delta$, we have

$$\lim_{m\to\infty}(\Delta_m u)(q) = \frac{3}{2}\lim_{m\to\infty} 5^m (H_m u)(q)$$

exists for all $q \in V_* \setminus V_0$. Then $5^m(H_m u)$ is bounded and there exists $c \in \mathbb{R}$ such that for every integer $m \geq 1$ and $q \in V_m \setminus V_0$,

$$|5^m(H_m u)(q)| \leq c.$$

Thus

$$\begin{aligned}
&\left|\left(\frac{5}{3}\right)^{m}(H_m u)(p_0)-\left(\frac{5}{3}\right)^{m+1}(H_{m+1}u)(p_0)\right| \\
&=\left(\frac{5}{3}\right)^{m+1}\left|\frac{3}{5}(H_m u)(p_0)-(H_{m+1}u)(p_0)\right| \\
&=\left(\frac{5}{3}\right)^{m+1}\left|\frac{2}{5}\sum_{k=1,2}(H_{m+1}u)(q_k(\overbrace{0\cdots 0}^{m}))+\frac{1}{5}(H_{m+1}u)(q_0(\overbrace{0\cdots 0}^{m}))\right| \\
&\le \frac{1}{3^{m+1}}\left[\frac{2}{5}\sum_{k=1,2}\left|5^{m+1}(H_{m+1}u)(q_k(\overbrace{0\cdots 0}^{m}))\right|+\frac{1}{5}\left|5^{m+1}(H_{m+1}u)(q_0(\overbrace{0\cdots 0}^{m}))\right|\right] \\
&\le \frac{1}{3^{m+1}}c.
\end{aligned}$$

Therefore, the sequence $\left\{\left(\frac{5}{3}\right)^{m}(H_m u)(p_0)\right\}$ is Cauchy sequense in $\mathbb{R}$, and so it converges as $m\to\infty$. □

## 3.2 The Normal and transverse derivative

**Definition 3.3.** ([2], [5]) Let $p$ be any element in $V_*$ such that $p=p_i(\omega)$, $\omega\in W_N$ and $i\in\{0,1,2\}$. We define the **normal derivative** at $p$ of function $f$, if limit exists, by

$$\partial_n f(p)=\lim_{m\to\infty}\left(\frac{5}{3}\right)^{N+m}\left[2f(p_i(\omega))-f(p_j(\omega\overbrace{i\cdots i}^{m}))-f(p_k(\omega\overbrace{i\cdots i}^{m}))\right] \tag{3.4}$$

where $j,k\in\{0,1,2\}$ and $i,\ j,\ k$ are not all equal.

**Note 3.5.** If $N=0$, then the Neumann derivative and the normal derivative are the same.

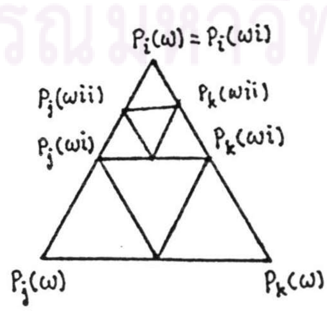


Figure (4). The boundary points of $SG_\omega$ and passing to $SG_{\omega i}$.

In addition, we define the **transverse(tangential) derivative**, if limit exists, by

$$\partial_T f(p) = \lim_{m\to\infty} 5^{N+m}\left[ f(p_j(\omega\overbrace{i\cdots i}^{m})) - f(p_k(\omega\overbrace{i\cdots i}^{m}))\right] \tag{3.6}$$

where $j,k \in \{0,1,2\}$ and $i,\ j,\ k$ are not all equal.

The exponent is $N+m$ because the points $p_i(\omega)$, $p_j(\omega\overbrace{i\cdots i}^{m})$ and $p_k(\omega\overbrace{i\cdots i}^{m})$ are the boundary points of $F_\omega F_i^m(SG)$. Moreover, the explanation of the factor 5 comes from the matrix

$$M_0 = \begin{pmatrix} 1 & 0 & 0 \\ 2/5 & 2/5 & 1/5 \\ 2/5 & 1/5 & 2/5 \end{pmatrix}$$

which describes the algorithm for extending a harmonic function

$$\begin{pmatrix} h(F_0(p_0)) \\ h(F_0(p_1)) \\ h(F_0(p_2)) \end{pmatrix} = M_0 \begin{pmatrix} h(p_0) \\ h(p_1) \\ h(p_2) \end{pmatrix}$$

form the boundary of $SG$ to the boundary of $F_0(SG)$. Similarly, for $F_1$ and $F_2$ we get that

$$M_1 = \begin{pmatrix} 2/5 & 2/5 & 1/5 \\ 0 & 1 & 0 \\ 1/5 & 2/5 & 2/5 \end{pmatrix} \text{ and } M_2 = \begin{pmatrix} 2/5 & 1/5 & 2/5 \\ 1/5 & 2/5 & 2/5 \\ 0 & 0 & 1 \end{pmatrix}.$$

**Note 3.7.** The eigenvalues of $M_i, i = 0,1,2$ are the same that are $1, \frac{3}{5}$ and $\frac{1}{5}$. The factors $\frac{5}{3}$ and 5 in (3.4) and (3.6) are the reciprocals of the nontrivial eigenvalues (the eigenvalue 1 corresponds to extending a constant function). The existence of two derivatives for any harmonic function will be shown in the next section.

## 3.3 The Derivatives of Strichartz

In this section, we will introduce the derivative of Strichartz that extend the normal and transverse derivative. See[2] for more details.

*Hypothesis*

(1). Each point $p_j, j = 0,1,2$ in $V_0$ is the fixed point of $F_j$, we assume that for any $F_j$ and $F_l, j \neq l$, the intersection $F_j(SG) \cap F_l(SG)$ consists of at most one point $x$ with $x = F_j(p_m) \cap F_l(p_n)$ for some point $p_m$ and $p_n$ in $V_0$.

(2). For each $p_j$ in $V_0$, recall that $M_j$ are the $3 \times 3$ matrix that transforms the value $f|_{V_0}$ to $f|_{F_jV_0}$ for hamonic functions $f$, i.e,

$$f(F_j(p_k)) = \sum_{l=0}^{2}(M_j)_{kl}f(p_l).$$

We assume that for each $M_j$ has a set of real left eigenvectors $\beta_{jk}$ with real nonzero eigenvalues $\lambda_{jk}$,

$$\beta_{jk}M_j = \lambda_{jk}\beta_{jk}.$$

We will assume that for each $j$ the eigenvalues $\lambda_{jk}$ are labeled in decreasing order of absolute value,i.e., $\lambda_{j0} = 1, \lambda_{j1} = \frac{3}{5}, \lambda_{j2} = \frac{1}{5}$ for $j = 0,1,2$. Moreover, let $\tilde{M_j}$ denote the matrix obtain from $M_j$ by deleting the $j^{th}$ row and column. Then the largest eigenvalue of $\tilde{M_j}$ is $\lambda_{j1}$ of $M_j$. Observe that the $j^{th}$ row of $M_j$ is $\delta_{jk}$ since $F_jv_j = v_j$. Other rows, all the entries of $M_j$ are strictly positive. Next we will find the eigenvectors $\beta_{jk}$ for the eigenvalue $\lambda_{jk}$.

Let $\beta_{jk} = \begin{pmatrix} a & b & c \end{pmatrix}$ be eigenvector for the eigenvalue $\lambda_{jk}$ for all $j = 0,1,2$. Then $\beta_{00}M_0 = \lambda_{00}\beta_{00}$. We get the linear system

$$5a + 2b + 2c = 5a$$

$$2b + c = 5b$$

$$b + 2c = 5c.$$

Thus $a \in \mathbb{R}$, $b = 3c$ and $c = 3b$ and hence $\beta_{00} = \begin{pmatrix} a & 0 & 0 \end{pmatrix}, a \in \mathbb{R}^*$. Moreover, if $\beta_{01}M_0 = \lambda_{01}\beta_{01}$,

$$5a + 2b + 2c = 3a$$

$$2b + c = 3b$$

$$b + 2c = 3c$$

then $a = -2b, b = c$ and $c \in \mathbb{R}$ and hence $\beta_{01} = \begin{pmatrix} -2b & b & b \end{pmatrix}, b \in \mathbb{R}^*$. Finally, if $\beta_{02}M_0 = \lambda_{02}\beta_{02}$,

$$5a + 2b + 2c = a$$

$$2b + c = b$$

$$b + 2c = c$$

then $a = 0, c = -b$ and $b \in \mathbb{R}$ and hence $\beta_{02} = \begin{pmatrix} 0 & b & -b \end{pmatrix}, b \in \mathbb{R}^*$. Similarly, $\beta_{10} = \begin{pmatrix} 0 & a & 0 \end{pmatrix}$, $\beta_{11} = \begin{pmatrix} b & -2b & b \end{pmatrix}$, $\beta_{12} = \begin{pmatrix} c & 0 & -c \end{pmatrix}$ for $a, b, c$ in $\mathbb{R}*$ and $\beta_{20} = \begin{pmatrix} 0 & 0 & a \end{pmatrix}$, $\beta_{21} = \begin{pmatrix} b & b & -2b \end{pmatrix}$, $\beta_{22} = \begin{pmatrix} c & -c & 0 \end{pmatrix}$ for all $a, b, c \in \mathbb{R}^*$.

Next, we will also define derivatives associated to all $\beta_{jk}$ with $k \geq 0$.

**Definition 3.8.** Let $f$ be a continuous function defined in a neighborhood of a boundary point $p_j$ for some $j \in \{0, 1, 2\}$. Then the **derivatives** $d_{jk}f(p_j)$ for $k = 1, 2$ are defined by the following limits, if they exist,

$$d_{jk}f(p_j) = \lim_{m \to \infty} \lambda_{jk}^{-m} \beta_{jk} f|_{F_j^m V_0}, \tag{3.9}$$

where $\beta_{jk} f|_{F_j^m V_0}$ means

$$\sum_{l=0}^{2} (\beta_{jk})_l f(F_j^m(p_l)).$$

**Note 3.10.** The derivative associated with $\beta_{j1}$ and $\beta_{j2}$ will just be a multiple of the normal derivative and transverse derivative, respectively, at $p_j$.

*proof of note.* Recall that $\beta_{j1}$ have $-2b$ in the $j^{th}$ place and the others are $b, b \in \mathbb{R}^*$.

*Case* $k=1$. For any positive integer $m$, and $b$ is fixed,

$$\begin{aligned}\lambda_{j1}^{-m}\beta_{j1}f|_{F_j^m V_0} &= \left(\frac{3}{5}\right)^{-m}\sum_{l=0}^{2}(\beta_{j1})_l\, f\left(F_j^m(p_l)\right)\\ &= \left(\frac{5}{3}\right)^{m}\left[-2bf(p_j)+bf(p_s(\overbrace{j\cdots j}^{m}))+bf(p_t(\overbrace{j\cdots j}^{m}))\right],\end{aligned}$$

where $s,t\in\{0,1,2\}$ and $s,\ t,\ j$ are not all equal. Then

$$\begin{aligned}d_{j1}f(p_j) &= \lim_{m\to\infty}\lambda_{j1}^{-m}\beta_{j1}f|_{F_j^m V_0}\\ &= -b\lim_{m\to\infty}\left(\frac{5}{3}\right)^{m}\left[2f(p_j)-f(p_s(\overbrace{j\cdots j}^{m}))-f(p_t(\overbrace{j\cdots j}^{m}))\right]\\ &= -b\partial_n f(p_j).\end{aligned}$$

Thus $d_{j1}f(p_j)$ is a multiple of the normal derivative at $p_j$.

*Case* $k=2$. For any positive integer $m$, and $c$ is fixed,

$$\begin{aligned}\lambda_{j2}^{-m}\beta_{j2}f|_{F_j^m V_0} &= \left(\frac{1}{5}\right)^{-m}\sum_{l=0}^{2}(\beta_{j2})_l\, f\left(F_j^m(p_l)\right)\\ &= 5^m\left[-cf(p_s(\overbrace{j\cdots j}^{m}))+cf(p_t(\overbrace{j\cdots j}^{m}))\right],\end{aligned}$$

where $s,t\in\{0,1,2\}$ and $s,\ t,\ j$ are not all equal. Then

$$\begin{aligned}d_{j2}f(p_j) &= \lim_{m\to\infty}\lambda_{j2}^{-m}\beta_{j2}f|_{F_j^m V_0}\\ &= -c\lim_{m\to\infty}5^m\left[f(p_t(\overbrace{j\cdots j}^{m}))-f(p_s(\overbrace{j\cdots j}^{m}))\right]\\ &= -c\partial_T f(p_j).\end{aligned}$$

Thus $d_{j2}f(p_j)$ is a multiple of the transverse derivative at $p_j$.

**Lemma 3.11.** *If $f$ is harmonic in a neighborhood of $p_j$ then all the derivatives $d_{jk}f(p_j)$ exist and may be evaluated without taking the limit in (3.9). In fact, $d_{jk}f(p_j)=\beta_{jk}f|_{V_0}$.*

*Proof.* Let $f$ be a harmonic function on $SG$ and set $A_m=\lambda_{jk}^{-m}\sum_{l=0}^{2}(\beta_{jk})_l\, f\left(F_j^m(p_l)\right)$.

WLOG, let $j = 0$ and $b \in \mathbb{R}^*$ If $k = 1$, then

$$\begin{aligned}
A_1 &= \lambda_{01}^{-1} \sum_{l=0}^{2} (\beta_{01})_l \, f\left(F_0^m(p_l)\right) \\
&= \frac{5b}{3} \left[-2f(p_0) + f(p_1(0)) + f(p_2(0))\right] \\
&= \frac{5b}{3} \left[-2f(p_0) + \frac{f(p_2) + 2f(p_0) + 2f(p_1)}{5} + \frac{f(p_1) + 2f(p_0) + 2f(p_2)}{5}\right] \\
&= b\left[-2f(p_0) + f(p_1) + f(p_2)\right] \\
&= A_0.
\end{aligned}$$

If $k = 2$, then

$$\begin{aligned}
A_1 &= \lambda_{02}^{-1} \sum_{l=0}^{2} (\beta_{02}) \, l f\left(F_0^m(p_l)\right) \\
&= 5b\left[f(p_1(0)) - f(p_2(0))\right] \\
&= 5b\left[\frac{f(p_2) + 2f(p_0) + 2f(p_1)}{5} - \frac{f(p_1) + 2f(p_0) + 2f(p_2)}{5}\right] \\
&= b\left[f(p_1) - f(p_2)\right] \\
&= A_0.
\end{aligned}$$

Then the $m = 0$ and $m = 1$ terms on the right side of (3.9) are equal.

By applying the same argument to $f \circ F_0^m$, thus

$$\begin{aligned}
A_{m+1} &= \lambda_{0k}^{-m-1} \left[\sum_{l=0}^{2} (\beta_{0k})_l \, f\left(F_0^{m+1}(p_l)\right)\right] \\
&= \lambda_{0k}^{-m} \left[\sum_{l=0}^{2} (\beta_{0k})_l \, (f \circ F_0^m)\left(F_0(p_l)\right)\right] \\
&= \lambda_{0k}^{-m} \sum_{l=0}^{2} (\beta_{0k}) \, l \, (f \circ F_0^m)(p_l) \\
&= A_m.
\end{aligned}$$

It implies that all term on the right side of (3.9) are equal. If $f$ is a harmonic in a neighborhood of $p_0$, we can choose the sufficiently large $m$ to begin the argument with $f \circ F_0^m$ such that $F_0^m K$ is contained in that neighborhood. $\square$

**Lemma 3.12.** *Fix* $p_j$

*(a). A harmonic function* $f$ *is uniquely determined by the value of* $f(p_j)$ *and* $d_{jk}f(p_j)$, $k = 1, 2$ *and any values may be assigned.*

*(b). Let* $f$ *be a harmonic function satisfying*

$$\beta_{jk} f|_{F_j^m V_0} = o\left((\lambda_{jk})^m\right) \text{ as } m \to \infty$$

*for* $k = 1, 2$ *and* $f(p_j) = 0$. *Then* $f$ *is identically zero.*

*Proof.* (a) Recall that a harmonic function $f$ is uniquely determined by the value $f|_{V_0}$. Then it suffices to find $f(p_k)$ and $f(p_l)$ such that $j,\ k,\ l$ are not all equal in $\{0, 1, 2\}$. By the previous Lemma,

$$d_{jk} f(p_j) = \beta_{jk} f|_{V_0} = \sum_{l=0}^{2} (\beta_{jk})_l \, f(p_l),$$

it implies that

$$d_{j1} f(p_j) = -2b f(p_j) + b f(p_l) + b f(p_k),$$

$$d_{j2} f(p_j) = c f(p_k) - c f(p_l)$$

where $j,\ k,\ l$ are not all equal in $\{0, 1, 2\}$ and $b, c \in \mathbb{R}^*$. Since $d_{j1} f(p_j), d_{j2} f(p_j)$ and $f(p_j)$ are known, then we can find all of the values $f|_{V_0}$ which is determine by the constants.

(b) Note that

$$\beta_{jk} f|_{F_j^m V_0} = o\left((\lambda_{jk})^m\right) \text{ as } m \to \infty \text{ if and only if } \lim_{m \to \infty} \frac{\beta_{jk} f|_{F_j^m V_0}}{(\lambda_{jk})^m} = 0.$$

Then

$$d_{jk} f(p_j) = \lim_{m \to \infty} \lambda_{jk}^{-m} \beta_{jk} f|_{F_j^m V_0} = 0 \quad \text{for all} \quad k = 1, 2.$$

By (a) and $f(p_j) = 0$, we get that $f$ is identically zero. □

**Definition 3.13.** Suppose $n$ is the first value for which $p \in V_n$. We say that $p$ is a **junction point** if there are exactly $\omega$ and $\omega'$ in $W_n$ such that

$$p = F_\omega(p_j) = F_{\omega'}(p_k) \qquad \text{for} \qquad j \neq k \in \{0, 1, 2\}.$$

Let $p$ be a junction point in $V_1$ and $J(p)$ denote the set of indices $j$ such that there exists $j'$ in $\{0,1,2\}$ with $p = F_j(p_{j'})$. Moreover, if $p$ is a junction point in $V_n$, then $p = F_\omega x'$ for $x'$ a junction point in $V_1$ and $\omega \in W_{n-1}$ and we set $J(p) = J(x')$. Then $p = F_\omega F_j p_j{}'$ for $j \in J(p)$.

**Definition 3.14.** Let $f$ be a continuous function defined in a neighborhood of a junction point $p \in V_N$ (but $p \notin V_{N-1}$). Then $d_{j'k}f(p)$ for $j \in J(p)$ and $k = 1,2$ are definde by the following limit, if they exist,

$$d_{j'k}f(p) = \left(\frac{3}{5}\right)^N \lim_{m\to\infty} \lambda_{j'k}^{-m}\beta_{j'k}f|_{F_\omega F_j F_{j'}^m V_0}, \tag{3.15}$$

where $\beta_{j'k}f|_{F_\omega F_j F_{j'}^m V_0}$ means

$$\sum_{l=0}^{2}(\beta_{j'k})_l f(F_\omega F_j F_{j'}^m p_l).$$

Furthermore, the normal derivative $d_{j'2}f(p)$ are said to satisfy the **compatibility condition** if

$$\sum_{j\in J(p)} d_{j2}f(p) = 0.$$

The **gradient** of $f$ at $p, df(p)$, is the collection of all derivatives defined here.

**Lemma 3.16.** *If $f$ is harmonic in a neighborhood of a vertex $p$, then all the derivatives $d_{jk}f(p)$ or $d_{j'k}f(p)$ exist, and may be evaluated without taking the limit in (3.15). Furthermore, if $p$ is a junction point, then the compatibility condition for the normal derivative holds.*

*Proof.* Since $f\circ F_\omega$ is a harmonic function, the existence follows by Lemma 3.11 and applied to $f\circ F_\omega \circ F_j$. If $p$ is a junction point, then the compatibility condition for the normal derivative holds. □

By Proposition 1.10, it easy to see that neighborhoods of $p$ is $U_m(p) = F_j^m K$ where $p = p_j$ is a boundary point or $U_m(p) = \bigcup_{j\in J(p)} F_\omega F_j F_{j'}^m K$ where $p$ is a junction point. The boundary of $U_m(p)$ is taken to be $\left\{F_\omega F_j^m p_k\right\}, k \in \{0,1,2\}$ in the first case (including $p$), and $F_\omega F_j F_{j'}^m p_k, k \in \{0,1,2\}$ with $p$ deleted in the second case.

**Lemma 3.17.** *Fix a point $p$ in $SG$.*

*(a). A harmonic function $f$ on $U_m(p)$ is uniquely determined by the value of $f(p)$ and the gradient $df(p)$, and any values satisfying the compatibility condition ($p$ a junction point) may be freely assigned.*

*(b). Let $f$ be a harmonic function on some $U_{m_0}(p)$ satisfying $h(p) = 0$ and*

$$\beta_{jk} f|_{F_j^m V_0} = o\left((\lambda_{jk})^m\right) \text{ as } m \to \infty$$

*for $k = 1, 2$ ($p = p_j$ a boundary point) or*

$$\beta_{jk} f|_{F_\omega F_j F_{j'}^m V_0} = o\left((\lambda_{j'k})^m\right) \text{ as } m \to \infty$$

*for all $j \in J(p)$ and $k = 1, 2$ ($p$ a junction point). Then $f$ is identically zero on $U_{m_0}(p)$.*

*Proof.* (a) If $p$ is a boundary point, it obvious by Lemma 3.12. Assume that $p$ is a junction point, say $p = p_{j'}(\omega j), j \in J(p)$. We apply Lemma 3.12 with the harmonic functions $f \circ F_\omega \circ F_j$ and look $p$ as a boundary point. Each of them is uniquely determined by the value of $f \circ F_\omega \circ F_j(p_{j'})$ and $d_{j'k} f \circ F_\omega \circ F_j(p_{j'}), k = 1, 2$, and any values may be assigned. Hence, we have the unique harmonic function on $U_{m_0}(p)$.

(b) Similar with Lemma 3.12. □