ON METRIC-PRESERVING FUNCTIONS

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ฟังก์ชัน $f:[0,\infty)\to[0,\infty)$ จะได้ชื่อว่าเป็นฟังก์ชันที่คงสภาพความเป็นเมตริก ถ้าทุกๆปริภูมิเมตริก (X,d) เราได้ $f\circ d$ เป็นเมตริกบน X และฟังก์ชันที่คงสภาพความเป็นเมตริก f จะได้ชื่อว่าเป็นฟังก์ชันที่คงสภาพความเป็นเมตริกอย่างเข้ม ถ้า $f\circ d$ เป็นเมตริกที่สมมูลเชิงโทโพโลยีกับเมตริก d

ในงานวิจัยนี้เราศึกษาสมบัติที่สำคัญบางประการของฟังก์ชันที่คงสภาพความเป็นเมตริก และฟังก์ชันที่คงสภาพความเป็นเมตริกอย่างเข้ม โดยเฉพาะที่เกี่ยวกับความบริบูรณ์และความมีขอบเขตเบ็ดเสร็จ

ภาควิชา **คณิตศาสตร์** ลายมือชื่อนิสิต..
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A function $f:[0,\infty)\to[0,\infty)$ is said to be metric-preserving if for every metric space (X,d), $f\circ d$ is a metric on X and a metric-preserving function f is said to be strongly metric-preserving if $f\circ d$ is topologically equivalent to d.

In our investigation, we study some important properties of metric-preserving functions and strongly metric-preserving functions, especially those concerning completeness and totally boundedness.

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CONTENTS

CHAPTER I

INTRODUCTION

A function $f : [0,\infty) \to [0,\infty)$ is said to be **metric-preserving** if for every metric space (X,d), $f \circ d$ is a metric on X and it is said to be **strongly metric-preserving** if for every metric space (X,d), $f \circ d$ is a metric on X that is topologically equivalent to d.

These functions were first studied by Sreenivasan ([6]) in 1947 and after that, a significant literature has developed on the subject of metric-preserving functions by many mathematicians. In 1999, Paul Corazza ([3]) proved the relationship between strongly metric-preserving functions and continuity and surveyed some of the results on differentiability in the context of metric-preserving functions.

In our investigation, we study some other important properties of metric-preserving functions and strongly metric-preserving functions.

The next chapter consists of basic definitions, examples, theorems and some interesting properties that will be used in our investigation. In chapter III, we consider some basic properties of metric-preserving functions and strongly metric-preserving functions. In the final chapter, we give theorems concerning the completeness and totally boundedness of the metric d and the metric $f \circ d$, and also some other important properties of these functions.

CHAPTER II

PRELIMINARIES

In this chapter, we consider briefly fundamental definitions, examples, theorems and some interesting properties that will be used in the proceeding chapters.

Definition 2.1. A **metric space** is a nonempty set X together with a function $d : X \times X \to [0, \infty)$ satisfying the following three conditions:

(M1) For all $x, y \in X$, $d(x, y) = 0$ if and only if $x = y$,

(M2) for all $x, y \in X$, $d(x, y) = d(y, x)$, and

(M3) for all $x, y, z \in X$, $d(x, y) + d(y, z) \geq d(x, z)$.

A function d with the above properties is called a **metric** on X. We denote the metric space X with the metric d on X by (X, d).

Example 2.2. The function $d_e : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ defined by

$$d_e(x, y) = [\sum_{i=1}^{n} (x_i - y_i)^2]^{1/2}$$

where $x = (x_1, x_2, ..., x_n)$, $y = (y_1, y_2, ..., y_n)$, is a metric on $\mathbb{R}^n$ called the **Euclidean metric**.

For each $n \in \mathbb{N}$, if the metric on $\mathbb{R}^n$ is the Euclidean metric d_e we will write $\mathbb{R}^n$ instead of $(\mathbb{R}^n, d_e)$.

Example 2.3. For any nonempty set X, the metric d on X defined by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y \end{cases}$$

is called the **discrete metric** on X.

Definition 2.4. Let (X, d) be a metric space, $x \in X$ and ϵ be a positive real number. We call the set $\{y \in X | d(x, y) < \epsilon\}$ the **open ball** with center x and radius ϵ and denote it by $B_d(x, \epsilon)$, that is

$$B_d(x, \epsilon) = \{y \in X | d(x, y) < \epsilon\}.$$

Definition 2.5. Let (X, d) be a metric space. A subset G of X is said to be d-**open** or **open** in (X, d) if for any point x of G, there is a positive real number ϵ such that $B_d(x, \epsilon) \subseteq G$.

Definition 2.6. Two metrics d and d' on a space X are said to be **numerically equivalent** if there exist positive constants m, k such that for all $(x, y) \in X \times X$, we have

$$md(x, y) \leq d'(x, y) \leq kd(x, y).$$

Definition 2.7. Let (X, d) and (Y, ϱ) be metric spaces. A function $g : (X, d) \to (Y, \varrho)$ is said to be **continuous** if for every open set V in (Y, ϱ), $g^{-1}(V)$ is open in (X, d).

Definition 2.8. Metric spaces (X, d) and (Y, ϱ) are **homeomorphic** if there exists a 1-1, onto, continuous function $g : X \to Y$ such that g^{-1} is continuous. Such a function g is called a **homeomorphism** (from X to Y).

Definition 2.9. Two metrics d and d' on a space X are said to be **topologically equivalent** if the identity mapping of (X, d) onto (X, d') is a homeomorphism.

Note. 1. Two metrics d and d' on a space X are topologically equivalent if and only if the collections of all open sets of (X, d) , and of all open sets of (X, d') coincide.

2. If the collections of all open sets of (X, d) , and of all open sets of (X, d') coincide, then (X, d) and (X, d') are homeomorphic.

3. Two metrics d and d' on a space X are topologically equivalent if for each $x \in X$ and for each $\epsilon > 0$, there exists a $\delta > 0$ such that for all $y \in X$,

$$d(x,y) < \delta \textit{ implies } d'(x,y) < \epsilon$$

$$\text{and } \; d'(x,y) < \delta \textit{ implies } d(x,y) < \epsilon.$$

4. Any metric d on X which is equivalent to the discrete metric is also called a discrete metric on X.

Example 2.10. ([4], p.293) In $\mathbb{R}$, the Euclidean metric $d_e(x,y) = |x-y|$ is topologically equivalent to the metric $d_\varphi(x,y) = |\frac{x}{1+|x|} - \frac{y}{1+|y|}|$ since the latter is derived from the homeomorphism $x \mapsto x/(1+|x|)$ of $\mathbb{R}$ and $(-1,1)$.

Remark 2.11. Numerically equivalent metrics d and d' are topologically equivalent. The converse is not true, for example in $\mathbb{R}$ with $d(x,y) = |x-y|$ and $d'(x,y) = \min\{1, d(x,y)\}$, are topologically equivalent but not numerically equivalent.

Definition 2.12. A function $f : [0,\infty) \to \mathbb{R}$ is said to be **subadditive** if for all $x, y \geq 0,\ f(x+y) \leq f(x) + f(y)$.

Definition 2.13. A function $f : [0,\infty) \to [0,\infty)$ is said to be **convex** on $[0,c]$ if

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y) \quad \textit{for all } \alpha \in (0,1) \tag{2.1}$$

whenever $0 \leq x < y \leq c$.

Moreover, f is **strictly convex** if (2.1) holds when $\leq$ is replaced by $<$.

Definition 2.14. A function $f : [0,\infty) \to [0,\infty)$ is said to be **concave** on $[0,\infty)$ if $-f$ is convex on $[0,c]$ for all $c > 0$.

Definition 2.15. A metric space (X,d) is **complete** or we say that d is **complete** if every Cauchy sequence in X converges (to some point in X).

Definition 2.16. A metric space (X, d) is said to be **totally bounded** (or **precompact**) if for each $\epsilon > 0$, there exists a finite subset F of X say $F = \{x_1, x_2, ..., x_n\}$ such that $X = \bigcup_{i=1}^{n} B_d(x_i, \epsilon)$. We sometime say that d is totally bounded (on X), instead of saying that (X, d) is totally bounded.

Definition 2.17. A space X is **compact** if and only if every open cover of X has a finite subcover. That is, for any collection $\mathcal{G} = \{G_\alpha \mid \alpha \in A\}$ of open subsets of X such that $\bigcup_{\alpha \in A} G_\alpha = X$, there is a finite subset F of A such that $\bigcup_{\alpha \in F} G_\alpha = X$.

Example 2.18. $\mathbb{R}$ is not compact since the cover of $\mathbb{R}$ by the open sets $(-n, n)$ for $n \in \mathbb{N}$, has no finite subcover.

Remark 2.19. $\mathbb{R}$ is complete but not compact.

Remark 2.20. Every compact space (X, d) is totally bounded. But the converse is not true.

Theorem 2.21. *([4], p.298) A metric space (X, d) is compact if and only if it is both complete and totally bounded.*

CHAPTER III

METRIC-PRESERVING FUNCTIONS

In this chapter, we will consider some basic properties of metric-preserving functions and strongly metric-preserving functions. Some of the results will play a key role in the proof of our main theorems in the next chapter.

Definition 3.1. A function $f : [0, \infty) \to [0, \infty)$ is said to be **metric-preserving** if for all metric spaces (X, d), $f \circ d$ is a metric on X.

Definition 3.2. A function $f : [0, \infty) \to [0, \infty)$ is said to be **strongly metric-preserving** if for all metric spaces (X, d), $f \circ d$ is a metric on X that is topologically equivalent to d.

From now on, we denote by $\mathcal{M}$ the set of all metric-preserving functions and by $\mathcal{SM}$ the set of all strongly metric-preserving functions.

Definition 3.3. Let $f : [0, \infty) \to [0, \infty)$. Then f is said to be **amenable** if for any $x \in [0, \infty)$ we have $f(x) = 0$ if and only if $x = 0$.

The next proposition identifies a basic property of all metric-preserving functions:

Proposition 3.4. *([3]) Every metric-preserving function is subadditive.*

Proof. Let f be a metric-preserving function, $a, b \in [0, \infty)$ and d the Euclidean metric on $\mathbb{R}$. Then

$$f(a) + f(b) = (f \circ d)(0, a) + (f \circ d)(a, a + b) \geq (f \circ d)(0, a + b) = f(a + b). \qquad \square$$

Corollary 3.5. *([3]) Given $f : [0,\infty) \to [0,\infty)$, suppose that f is strictly convex on $[0,c]$ for some $c > 0$ and $f(0) = 0$. Then f is not metric-preserving.*

Proof. Let c be a positive number for which f is strictly convex on $[0,c]$. Then $f(c/2) < f(c)/2$, and therefore $f(c/2) + f(c/2) < f(c)$, which violates subadditivity. □

Borsík and Doboš ([1]) extend the result in Corollary 3.5; we state their result in Theorem 3.37. The proof makes use of the symmetry between subadditive and convex amenable functions, which is developed in the following remark:

Remark 3.6. ([1], [3])

(i) Suppose $f : [0,\infty) \to [0,\infty)$ is subadditive. Then for all positive integers $n, f(nx) \le nf(x)$ and $f(x/2^n) \ge f(x)/2^n$ whenever $x \ge 0$.

(ii) Suppose f is amenable and convex on $[0,c]$. Then for all positive integers $n,\ f(x/2^n) \le f(x)/2^n$ whenever $0 \le x \le c$.

Proof. (i) We will show that for all $n \in \mathbb{N},\ f(nx) \le nf(x)$, by induction.

Basic step. For $n = 1,\ f(x) \le f(x)$ is true.

For $n = 2,\ f(2x) = f(x + x) \le f(x) + f(x) = 2f(x)$.

Induction step. Let $k \ge 2$. Assume that $f(kx) \le kf(x)$.

Thus $f((k+1)x) = f(kx + x) \le f(kx) + f(x) \le kf(x) + f(x) = (k+1)f(x)$.

Next, we will show that for all $n \in \mathbb{N},\ f(x/2^n) \ge f(x)/2^n$.

Basic step. For $n = 1,\ f(x) = f(x/2 + x/2) \le f(x/2) + f(x/2) = 2f(x/2)$.

So $f(x/2) \ge f(x)/2$.

Induction step. Let $k \in \mathbb{N}$. Assume that $f(x/2^k) \ge f(x)/2^k$.

So $f(\frac{x}{2^{k+1}}) = f(\frac{1}{2} \cdot \frac{x}{2^k}) \ge \frac{1}{2} f(\frac{x}{2^k}) \ge \frac{1}{2} \frac{f(x)}{2^k} = \frac{f(x)}{2^{k+1}}$.

(ii) Suppose f is amenable and convex on $[0,c]$.

We will show that for all $n \in \mathbb{N},\ f(x/2^n) \le f(x)/2^n$ when $0 \le x \le c$.

Basic step. For $n = 1$, $f(\frac{x}{2}) = f(\frac{x+0}{2}) \le \frac{f(x)+f(0)}{2} = \frac{f(x)}{2}$.

Induction step. Let $k \in \mathbb{N}$. Assume that $f(x/2^k) \le f(x)/2^k$.

So $f(\frac{x}{2^{k+1}}) = f(\frac{x}{2^k \cdot 2}) \le \frac{f(x/2^k)}{2} \le \frac{f(x)}{2^k \cdot 2} = \frac{f(x)}{2^{k+1}}$. □

While subadditivity is an important necessary condition for f to be metric-preserving, the function

$$f(x) = \begin{cases} x & \text{if } 0 \le x \le 1, \\ \frac{1}{2} & \text{otherwise} \end{cases}$$

shows that subadditivity is not sufficient for an amenable function to be metric-preserving ([7]). However, adding "nondecreasing" to subadditivity does yield a sufficient condition:

Proposition 3.7. *([3]) If f is amenable, subadditive, and nondecreasing, then f is metric-preserving.*

Proof. Let (X, d) be a metric space. We show that $f \circ d$ is a metric on X. Properties (M1) and (M2) are easy to check. For (M3), let $x, y, z \in X$, and let $a = d(x, y)$, $b = d(y, z)$, and $c = d(x, z)$. It suffices to show that $f(a)+f(b) \ge f(c)$. But

$$\begin{aligned} f(a) + f(b) &\ge f(a+b) && \text{(subadditive)} \\ &\ge f(c) && \text{(nondecreasing)}, \end{aligned}$$

as required. □

The next proposition shows that concave amenable functions must be subadditive and nondecreasing, we can use Proposition 3.7 to conclude that they are metric-preserving.

Proposition 3.8. *([2]) If $f : [0,\infty) \to [0,\infty)$ is a concave amenable function, then f is metric-preserving.*

Proof. Since f is amenable, for any $x \in [0,\infty)$ we have $f(x) = 0$ if and only if $x = 0$.

Since f is concave on $[0,\infty)$, for all $\alpha \in (0,1)$ and for all $a, b \in [0,\infty)$ we have

$$f(\alpha a + (1-\alpha)b) \geq \alpha f(a) + (1-\alpha) f(b). \tag{3.1}$$

Let $x, y \in [0,\infty)$ be such that $x < y$.

From (3.1) when we choose $1-\alpha = \frac{x}{y}$, $a = 0$ and $b = y$ we have $\alpha = 1 - \frac{x}{y} = \frac{y-x}{y}$.

So $f(x) \geq \frac{x}{y} \cdot f(y)$. Thus $yf(x) - xf(y) \geq 0$.

When we put $\alpha = \frac{x}{y}$, $a = x$ and $b = x+y$ we obtain $1-\alpha = 1 - \frac{x}{y} = \frac{y-x}{y}$.

So

$$f(y) = f(\frac{x^2+y^2-x^2}{y}) = f(\frac{x^2}{y} + \frac{(y-x)(x+y)}{y}) \geq \frac{x}{y} f(x) + (\frac{y-x}{y}) f(x+y).$$

Thus $\frac{yf(y)-xf(x)}{y-x} \geq f(x+y)$.

But

$$\begin{aligned}\frac{yf(y)-xf(x)}{y-x} &= \frac{yf(x)+yf(y)-xf(x)-xf(y)-yf(x)+xf(y)}{y-x} \\ &= \frac{(y-x)(f(x)+f(y))}{y-x} - \frac{yf(x)-xf(y)}{y-x} \\ &= f(x)+f(y) - \frac{yf(x)-xf(y)}{y-x}.\end{aligned}$$

Therefore $f(x)+f(y) \geq f(x)+f(y) - \frac{yf(x)-xf(y)}{y-x} \geq f(x+y)$.

By the assumption when we put $\alpha = \frac{1}{2}$, $a = 0$ and $b = 2x$ we have $f(x) \geq f(2x)/2$.

Thus $f(x)+f(x) = 2f(x) \geq f(2x) = f(x+x)$.

Therefore for all $x, y \in [0,\infty)$ we have $f(x+y) \leq f(x)+f(y)$, that is f is subadditive.

Now suppose that there are $x, y \in [0,\infty)$ such that $x < y$ and $f(x) > f(y)$.

Put $z = \frac{yf(x)-xf(y)}{f(x)-f(y)}$. So $z \geq 0$.

Put $\alpha = \frac{f(y)}{f(x)}$, $a = x$ and $b = z$. Then $1-\alpha = \frac{f(x)-f(y)}{f(x)}$. By (3.1) we have

$$f(\frac{f(y)}{f(x)} \cdot x + (\frac{f(x)-f(y)}{f(x)})(\frac{yf(x)-xf(y)}{f(x)-f(y)})) \geq \frac{f(y)}{f(x)} \cdot f(x) + (\frac{f(x)-f(y)}{f(x)}) \cdot f(z).$$

So $f(y) \geq f(y) + (1-\frac{f(y)}{f(x)}) \cdot f(z)$. Thus $(1-\frac{f(y)}{f(x)}) \cdot f(z) \leq 0$. Then $f(z) \leq 0$, which contradicts to the assumption. Therefore for all $x, y \in [0,\infty)$, $x \leq y$ implies $f(x) \leq f(y)$, that is f is nondecreasing. By Proposition 3.7, $f \in \mathcal{M}$. □

Example 3.9. ([3]) The function f with $f(x) = \log_a(1+x)$, for $a > 1$, and the function g with $g(x) = x^r$, where $0 < r \leq 1$, are metric-preserving.

It is known that f is amenable and nondecreasing, so it is enough to show that f is subadditive. Let $x, y \in [0,\infty)$ be such that $x \leq y$.

So $1 + (x+y) \leq 1 + x + y + xy = (1+x)(1+y)$.

Thus

$$\begin{aligned}\log_a(1+(x+y)) &\leq \log_a(1+x)(1+y) \quad \text{(nondecreasing)}\\ &= \log_a(1+x) + \log_a(1+y),\end{aligned}$$

that is $f(x+y) \leq f(x) + f(y)$. Therefore f is subadditive. By Proposition 3.7, we have $f \in \mathcal{M}$.

The function $g : [0,\infty) \to [0,\infty)$ defined by $g(x) = x^r$, with $0 < r \leq 1$ is metric-preserving, since it is concave and amenable.

Our examples so far have been both nondecreasing and continuous. A simple example of a discontinuous nondecreasing metric-preserving function is

$$f(x) = \begin{cases} 0 & \text{if } x = 0, \\ c & \text{otherwise,} \end{cases}$$

where c is a positive constant. Proposition 3.7 ensures that this function is metric-preserving. It is also possible to obtain a continuous, metric-preserving function that fails to be nondecreasing. In order to construct this and other related examples, we need the notion first a **triangle triplet**, which is used to characterize metric-preserving functions. This notion first appeared in Sreenivasan's early paper ([6]).

Definition 3.10. A **triangle triplet** is a triple (a, b, c) of nonnegative reals for which $a \leq b + c,\ b \leq a + c$, and $c \leq a + b$; equivalently, $|a - b| \leq c \leq a + b$.

We denote by $\triangle$ the set of all triangle triplets.

Remark 3.11.

(i) For all $a \in [1, \infty)$, $(a, a+1, a+2) \in \triangle$.

(ii) For all $x \in \triangle$, for all $k > 0$, $kx \in \triangle$.

(iii) For all $x \in \triangle$, for all permutations P on x, $Px \in \triangle$.

Proof. (i) Let $a \in [1, \infty)$. Since

$$a \leq a + 1 \leq (a+1) + (a+2),$$

$$a + 1 \leq a + 2 \leq (a+2) + a,$$

$$\text{and } \ a + 2 = a + 1 + 1 \leq (a+1) + a,$$

we have $(a, a+1, a+2) \in \triangle$.

Next, we will show that for all $a \in [0, 1)$, $(a, a+1, a+2) \in \triangle$ is not true. Choose $a = \frac{1}{2}$. Since $\frac{5}{2} \nleq \frac{4}{2} = \frac{1}{2} + \frac{3}{2}$, we have $(\frac{1}{2}, \frac{3}{2}, \frac{5}{2}) \notin \triangle$.

Now we show (ii). Let $(a, b, c) \in \triangle$ and let $k > 0$. Claim that $(ka, kb, kc) \in \triangle$. Since $a \leq b + c$ and $k > 0$, we have $ka \leq kb + kc$. Similarly we have $kb \leq ka + kc$ and $kc \leq ka + kb$. So we have the claim.

Next we show (iii). Let $(a_1, a_2, a_3) \in \triangle$. Let $\mathbf{S}$ be the set of all permutations on x. Then

$\mathbf{S} = \{(a_1, a_2, a_3), (a_1, a_3, a_2), (a_3, a_2, a_1), (a_2, a_1, a_3), (a_3, a_1, a_2), (a_2, a_3, a_1)\}$.

It is clear that for each $b \in \mathbf{S},\ b \in \triangle$. $\square$

Triangle triplets are precisely those triples of nonnegative reals that are of the form $(d(x,y), d(y,z), d(x,z))$ for some metric space (X,d) and some $x, y, z \in X$. This observation follows from Proposition 3.12 and the proof of Proposition 3.14.

Proposition 3.12. *([3]) If (X,d) is a metric space and $x, y, z \in X$, then $(d(x,y), d(y,z), d(x,z)) \in \triangle$.*

Proof. This is immediate from the triangle inequality. $\square$

Lemma 3.13. *([2]) Every metric-preserving function is amenable.*

Proof. Consider $\mathbb{R}$ and $d(x,y) = |x-y|$ for each $x, y \in \mathbb{R}$. Let f be a metric-preserving function. Then $(\mathbb{R}, f \circ d)$ is a metric space and for each $a \in [0,\infty)$ we have $d(a,0) = a$. Let $a \in [0,\infty)$. Then $0 = f(a) = (f \circ d)(a,0)$ if and only if $a = 0$. $\square$

Proposition 3.14. *([2], [3]) Let $f : [0,\infty) \to [0,\infty)$. Then f is metric-preserving if and only if f is amenable and for each $(a,b,c) \in \triangle$, we have $(f(a), f(b), f(c)) \in \triangle$.*

Proof. $(\Rightarrow)$. By Lemma 3.13 we have f is amenable. Next, given $(a,b,c) \in \triangle$, let d be the Euclidean metric on $\mathbb{R}^2$. Choose $u = (0,0),\ v = (0,a)$ and $w = (x,y)$ where $x = \frac{c^2-b^2+a^2}{2a}$, $y = \sqrt{c^2 - x^2}$. Thus there are $u, v, w \in \mathbb{R}^2$ such that

$d(u,v) = a,\ d(v,w) = b$, and $d(u,w) = c$.

Since $f \circ d$ is a metric, by Proposition 3.12 we have

$(f(a), f(b), f(c)) = (f \circ d(u,v), f \circ d(v,w), f \circ d(u,w)) \in \triangle$.

($\Leftarrow$). Given (X, d), we verify that $f \circ d$ is a metric. Properties (M1) and (M2) are immediate. For (M3), by Proposition 3.12 we obtain $(d(x,y), d(y,z), d(x,z)) \in \triangle$. By assumption we have $(f \circ d(x,y), f \circ d(y,z), f \circ d(x,z)) \in \triangle$.

So $f \circ d(x,y) \leq f \circ d(x,z) + f \circ d(z,y)$. □

Corollary 3.15. *([3]) For every metric-preserving function f, $|f(a) - f(b)| \leq f(|a - b|)$ for every $a, b \in [0, \infty)$.*

Remark 3.16. ([3]) Let $f : [0, \infty) \to [0, \infty)$. Then $f \in \mathcal{M}$ if and only if f is amenable and for each $(a, b, c) \in \triangle$, we have $f(a) \leq f(b) + f(c)$.

Proposition 3.17. *([3]) An amenable function f is metric-preserving if and only if for each metric d on $\mathbb{R}^2$, $f \circ d$ is a metric on $\mathbb{R}^2$.*

Proof. ($\Rightarrow$). Clear, by the definition of metric-preserving function.

($\Leftarrow$). Assume that for each metric d on $\mathbb{R}^2$, $f \circ d$ is a metric on $\mathbb{R}^2$. Let $(a, b, c) \in \triangle$ and let d be the Euclidean metric on $\mathbb{R}^2$. Choose $u, v, w \in \mathbb{R}^2$ as in the proof of Proposition 3.14. Then $d(u,v) = a$, $d(v,w) = b$ and $d(u,w) = c$. Since $f \circ d$ is a metric on $\mathbb{R}^2$, $(\mathbb{R}^2, f \circ d)$ is a metric space. By Proposition 3.12, we have $(f \circ d(u,v), f \circ d(v,w), f \circ d(u,w)) \in \triangle$, that is $(f(a), f(b), f(c)) \in \triangle$. By Proposition 3.14, we have $f \in \mathcal{M}$. □

Lemma 3.18. *([2]) If $f \in \mathcal{M}$, then for each $a, b \in [0, \infty)$, $a \leq 2b$ implies $f(a) \leq 2f(b)$.*

Proof. Let $a, b \in [0, \infty)$ be such that $a \leq 2b$. Since $|a - b| \leq b \leq a + b$, by Remark 3.16, we have $f(a) \leq f(b) + f(b) = 2f(b)$. □

Theorem 3.19. *([2]) Let f be metric-preserving. Then the following assertions are equivalent:*

(i) f is continuous on $[0, \infty)$,

(ii) f is continuous at 0,

(iii) for each $\epsilon > 0$, there is an $x > 0$ such that $f(x) < \epsilon$.

Proof. (i) $\Rightarrow$ (ii). It is clear.

(ii) $\Rightarrow$ (i). Assume that f is continuous at 0. To show that f is continuous on $[0, \infty)$, let $a > 0$ and $\epsilon > 0$. Then there is a $\gamma > 0$ such that for all $x \in [0, \infty)$ with $x < \gamma$ we have $f(x) < \epsilon$. Put $\delta = min\{\frac{\gamma}{2}, \frac{a}{2}\}$. Since $\delta < \gamma$, we have $f(x) < \epsilon$. Let $x \in [0, \infty)$ with $|x - a| < \delta$. Since $|x - a| < \delta \leq x + a$, by Proposition 3.14, we have $f(x) \leq f(a) + f(\delta)$ and $f(a) \leq f(x) + f(\delta)$. Thus $|f(x) - f(a)| \leq f(\delta) < \epsilon$. Hence f is continuous at a.

(ii) $\Rightarrow$ (iii). Assume that f is continuous at 0. Let $\epsilon > 0$. Then there is a $\delta > 0$ such that for all $x \in [0, \infty)$, if $x < \delta$, then $f(x) < \epsilon$. So $f(\frac{\delta}{2}) < \epsilon$.

(iii)$\Rightarrow$ (ii). Let $\epsilon > 0$. Then there is an $a > 0$ such that $f(a) < \frac{\epsilon}{2}$. By Lemma 3.18, we obtain for each $x \in [0, \infty)$ if $x \leq 2a$, then $f(x) \leq 2f(a) < \epsilon$. Put $\delta = 2a$. Then for each $\epsilon > 0$ there is a $\delta > 0$ such that for all $x \in [0, \infty)$ with $x < \delta$ implies $f(x) < \epsilon$. Hence f is continuous at 0. □

Proposition 3.20. *([3]) Suppose that f is metric-preserving.*

(i) For each $x_0 > 0$, there is an $\epsilon > 0$ such that $f(x) \geq \epsilon$ for each $x \geq x_0$.

(ii) If f is discontinuous at 0, then there is some $\epsilon > 0$ such that $f(x) > \epsilon$ for all $x > 0$.

Proof. (i) Suppose that the assertion is false. Then there is $x_0 > 0$ such that for all $\epsilon > 0$ there exists $x \geq x_0$ such that $f(x) < \epsilon$. So for all $n \in \mathbb{N}$ there exists $x_n \geq x_0$ such that $f(x_n) < \frac{1}{n}$. Thus there is a sequence (x_n) such that $x_n \geq x_0$

for all $n \in \mathbb{N}$, and $\lim_{n\to\infty} f(x_n) = 0$. Let $k \in \mathbb{N}$ be such that $f(x_k) < f(x_0)/2$. Then $(x_k, x_k, x_0) \in \triangle$. Since $f(x_k) < f(x_0)/2$, $f(x_0) \nleq f(x_k) + f(x_k)$. Hence $(f(x_k), f(x_k), f(x_0)) \notin \triangle$. This contradicts to Proposition 3.14.

(ii) Immediately from Theorem 3.19. □

Proposition 3.20 shows that metric-preserving functions cannot have the x-axis as a horizontal asymptote; thus, the function $x/(1+x^2)$ is not metric-preserving.

Definition 3.21. An amenable function f is **tightly bounded** if there exists a $v > 0$ such that $f(x) \in [v, 2v]$ for all $x > 0$.

Proposition 3.22. *([2], [3]) If f is tightly bounded, then f is metric-preserving.*

Proof. Let $v > 0$ be such that $f(x) \in [v, 2v]$ for all $x > 0$, and let $(a, b, c) \in \triangle$. Since the cases in which $abc = 0$ are trivial, we assume $abc > 0$. Therefore $f(a) \leq 2v = v + v \leq f(b) + f(c)$, and by Remark 3.16, we have $f \in \mathcal{M}$. □

Now, we will see that a metric-preserving function can be strictly decreasing on an interval (a, ∞) where $a \geq 0$. For each $f : [0, \infty) \to [0, \infty)$ and each $r > 0$ we define

$$U_{f,r}(x) = \begin{cases} 0 & \text{if } x = 0, \\ f(x) + r & \text{if } x > 0. \end{cases}$$

Proposition 3.23. *([3]) Suppose $f : [0, \infty) \to [0, \infty)$ is bounded above. Then there is an $r_0 > 0$ such that $U_{f,r} \in \mathcal{M}$ for all $r \geq r_0$.*

Proof. Let r_0 be an upper bound for f. We see that $U_{f,r}$ is amenable for all $r \geq r_0$. Claim that $U_{f,r}$ is tightly bounded for all $r \geq r_0$. Let $r \geq r_0$. Let $x > 0$. Then $U_{f,r}(x) = f(x) + r$. Thus $r \leq f(x) + r \leq r_0 + r \leq r + r = 2r$. So there exists an $r > 0$ such that $U_{f,r}(x) \in [r, 2r]$ for all $x > 0$, that is $U_{f,r}$ is tightly bounded for all $r \geq r_0$. By Proposition 3.22, $U_{f,r} \in \mathcal{M}$ for all $r \geq r_0$. □

Example 3.24. ([3]) *There is a metric-preserving function which is strictly decreasing on* $(0, \infty)$. Define

$$g(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1 + \frac{1}{x+1} & \text{if } x > 0. \end{cases}$$

Now, $g = U_{f,1}$ where

$$f(x) = \begin{cases} 0 & \text{if } x = 0, \\ \frac{1}{x+1} & \text{if } x > 0. \end{cases}$$

Since there is $1 > 0$ such that $f(x) \leq 1$ for all $x \in [0, \infty)$ and there is $r_0 = 1$, by Proposition 3.23, we have $g \in \mathcal{M}$. Claim that g is strictly decreasing on $(0, \infty)$. Let $x, y \in (0, \infty)$ be such that $x < y$. Thus $x + 1 < y + 1$.
So $g(x) = 1 + \frac{1}{x+1} > 1 + \frac{1}{y+1} = g(y)$. Thus $g \in \mathcal{M}$ and g is strictly decreasing on $(0, \infty)$.

Lemma 3.25. *([2]) Let f be a real valued continuous function on $[a, b]$, where $a, b \in \mathbb{R}$, $a < b$. Let $f(a) = f(b)$. Then for all $\epsilon > 0$, there are $u, v \in [a, b]$ such that $0 < |u - v| < \epsilon$ and $f(u) = f(v)$.*

Proposition 3.26. *([2]) Let f be a real valued continuous function on $[a, b]$, where $a, b \in \mathbb{R}$, $a < b$. Then for all $\epsilon > 0$, there are $x, y \in [a, b]$ such that $0 < |x - y| < \epsilon$ and $\frac{f(x) - f(y)}{x - y} = \frac{f(a) - f(b)}{a - b}$.*

Proof. Define $g : [a, b] \to \mathbb{R}$ by

$$g(x) = f(x) + \frac{(f(a) - f(b)) \cdot (a - x)}{a - b} \quad \text{for each } x \in [a, b].$$

Then g is a real valued continuous function on $[a, b]$ such that $g(a) = g(b)$. Thus by Lemma 3.25 we have

for all $\epsilon > 0$ there are $x, y \in [a, b]$ such that $0 < |x - y| < \epsilon$ and $g(x) = g(y)$.

Since $g(x) = g(y)$, $f(x) + \frac{(f(a)-f(b))\cdot(a-x)}{a-b} = f(y) + \frac{(f(a)-f(b))\cdot(a-y)}{a-b}$.
Thus $f(x) - f(y) = \frac{(f(a)-f(b))}{a-b}(a - y - a + x)$. Hence $\frac{f(x)-f(y)}{x-y} = \frac{f(a)-f(b)}{a-b}$. $\square$

Proposition 3.27. *([2]) Suppose that f is metric-preserving and $d, k > 0$. Define $T_{f,d,k} : [0, \infty) \to [0, \infty)$ by*

$$T_{f,d,k}(x) = g(x) = \begin{cases} kx & \text{if } x \in [0, d), \\ f(x) & \text{otherwise.} \end{cases}$$

Then g is metric-preserving if and only if $f(d) = kd$ and $|f(x) - f(y)| \leq k|x - y|$ for all $x, y \in [d, \infty)$.

Proof. ($\Rightarrow$). Let g be metric-preserving. Since g is continuous at 0, by Theorem 3.19 we obtain that g is continuous on $[0, \infty)$. Thus

$$kd = \lim_{x \to d^-} g(x) = \lim_{x \to d} g(x) = g(d) = f(d).$$

Suppose that there are $x, y \in [d, \infty)$ such that $|f(x) - f(y)| > k|x - y|$. Let $x < y$. Since $f_{|_{[d,\infty)}} = g_{|_{[d,\infty)}}$ is continuous, f is continuous on $[x, y]$. Then by Proposition 3.26 we have there are $u, v \in [x, y]$ such that $0 < |u - v| < d$ and $\frac{f(u)-f(v)}{u-v} = \frac{f(x)-f(y)}{x-y}$. Hence

$$|f(u) - f(v)| = |u - v| \cdot \frac{|f(x) - f(y)|}{|x - y|} > |u - v| \cdot \frac{k|x - y|}{|x - y|} = k|u - v|.$$

Put $a = u$, $b = v$ and $c = |u - v|$. Thus

$$|f(b) - f(a)| > kc. \tag{3.2}$$

Since $|a - b| \leq c \leq a + b$ and $g \in \mathcal{M}$,

$$|g(a) - g(b)| \leq g(c) = kc. \tag{3.3}$$

By (3.2), $|g(a) - g)(b)| > kc$, which contradicts to (3.3).

($\Leftarrow$). Let $f(d) = kd$ and $|f(x) - f(y)| \leq k|x - y|$ for all $x, y \in [d, \infty)$.

Let $a, b, c \in [0, \infty)$ be such that $|a - b| \leq c \leq a + b$.

1) Suppose that $a, b \in [0, d)$. Then $c \in [0, 2d)$.

If $c \in [0, d)$, then $g(a) = ka \leq kb + kc = g(b) + g(c)$.

If $c \in [d, 2d)$, then $kd - f(c) = f(d) - f(c) \leq |f(c) - f(d)| \leq k|c - d| = k(c - d)$, which yields $-f(c) \leq k(c - 2d)$. Then $ka - f(c) \leq k(a + c - 2d)$.

Hence

$$\begin{aligned} g(a) = ka &\leq f(c) + k(a + c - 2d) \\ &\leq k(a + (a + b) - 2d) + f(c) \\ &\leq k(d + (d + b) - 2d) + f(c) \\ &= g(b) + g(c). \end{aligned}$$

2) Suppose that $a \in [0, d)$, $b \in [d, \infty)$. Then $c \in [0, \infty)$.

If $c \in [0, d)$, then $kd - f(b) = f(d) - f(b) \leq |f(b) - f(d)| \leq k|b - d| = k(b - d)$, which yields $-f(b) \leq k(b - 2d)$. Then $ka - f(b) \leq k(a + b - 2d)$.

Hence

$$\begin{aligned} g(a) = ka &\leq f(b) + k(a + b - 2d) \\ &\leq f(b) + k(a + (a + c) - 2d) \\ &\leq f(b) + k(d + (d + c) - 2d) \\ &= f(b) + kc = g(b) + g(c). \end{aligned}$$

If $c \in [d, \infty)$, then by Lemma 3.18 we obtain

$$\text{for all } x \in [0, \infty) \text{ if } d \leq 2x, \text{ then } f(d) \leq 2f(x).$$

Hence for all $x \in [0, \infty)$ if $x \geq \frac{d}{2}$, then $f(x) \geq \frac{f(d)}{2} = \frac{kd}{2}$.

Then $g(a) = ka < kd = k\frac{d}{2} + k\frac{d}{2} \leq f(b) + f(c) = g(b) + g(c)$.

3) Suppose that $a \in [d, \infty)$, $b \in [0, d)$. Then $c \in [0, \infty)$.

If $c \in [0, d)$, then $f(a) - kd = f(a) - f(d) \le |f(a) - f(d)| \le k|a - d| = ka - kd$, which yields $f(a) \le ka$. Then $g(a) = f(a) \le ka \le kb + kc = g(b) + g(c)$.

If $c \in [d, \infty)$, then $f(a) - f(c) \le |f(a) - f(c)| \le k|a - c| \le kb$.

Thus $g(a) = f(a) \le kb + f(c) = g(b) + g(c)$.

4) Suppose that $a, b \in [d, \infty)$. Then $c \in [0, \infty)$.

If $c \in [0, d)$, then $f(a) - f(b) \le |f(a) - f(b)| \le k|a - b| \le kc$, which yields $g(a) = f(a) \le f(b) + kc = g(b) + g(c)$.

If $c \in [d, \infty)$, then $g(a) = f(a) \le f(b) + f(c) = g(b) + g(c)$.

Thus for all $a, b, c \in [0, \infty)$ such that $|a - b| \le c \le a + b$ implies $g(a) \le g(b) + g(c)$. By Remark 3.16 we obtain $g \in \mathcal{M}$. $\square$

Example 3.28. ([3]) *There is a metric-preserving, continuous function that is strictly decreasing on* $(1, \infty)$. Let g be as in Example 3.24. Define

$$T(x) = \begin{cases} \frac{3}{2}x & \text{if } x \in [0, 1], \\ g(x) & \text{otherwise.} \end{cases}$$

Clearly, T is continuous and strictly decreasing on $(1, \infty)$. Since $T = T_{g,1,\frac{3}{2}}$, Proposition 3.27 ensures that $T \in \mathcal{M}$. This example shows that continuous metric-preserving functions need not be nondecreasing.

Example 3.29. ([3]) *There is a continuous, nondecreasing, metric-preserving function that is not concave.* Define

$$f(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } 0 < x < 2, \\ x - 1 & \text{if } 2 \le x < 3, \\ 2 & \text{otherwise.} \end{cases}$$

Since f is tightly bounded, by Proposition 3.22, we obtain $f \in \mathcal{M}$.

Define

$$T(x) = \begin{cases} x & \text{if } x \in [0,1], \\ f(x) & \text{otherwise.} \end{cases}$$

Clearly, T is continuous and nondecreasing. T is not concave since

$$T(\frac{a+b}{2}) < \frac{T(a)+T(b)}{2}$$

when $a = 1$ and $b = 3$. Since $T = T_{f,1,1}$, Proposition 3.27 can be applied to show $T \in \mathcal{M}$.

Example 3.30. ([3]) The Jůza's function T is in $\mathcal{M}$, where

$$T(x) = \begin{cases} x & \text{if } x \le 2, \\ 1 + \frac{1}{x-1} & \text{if } x > 2. \end{cases}$$

Proof. Consider

$$g(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } 0 < x < 2, \\ \frac{1}{x-1} & \text{if } x \ge 2. \end{cases}$$

So there is $M = 1 > 0$ such that $g(x) \le 1$ for all $x \ge 0$. Thus g is bounded above.

Define

$$f(x) = U_{g,1}(x) = \begin{cases} 0 & \text{if } x = 0, \\ g(x) + 1 & \text{if } x > 0. \end{cases}$$

By Proposition 3.23, we have $f = U_{g,1} \in \mathcal{M}$. Consider

$$T(x) = T_{f,2,1}(x) = \begin{cases} x & \text{if } 0 \le x < 2, \\ f(x) & \text{if } x \ge 2. \end{cases}$$

Claim that $T \in \mathcal{M}$. We have $f(2) = \frac{1}{2-1} + 1 = 1 + 1 = 2 = (1)(2)$. Let $x, y \geq 2$. Thus

$$\begin{aligned}|f(x) - f(y)| &= |\frac{1}{x-1} + 1 - \frac{1}{y-1} - 1| \\ &= |\frac{1}{x-1} - \frac{1}{y-1}| \\ &= |\frac{y-1-x+1}{(x-1)(y-1)}| \\ &= |y-x| \cdot |\frac{1}{x-1}| \cdot |\frac{1}{y-1}| \\ &\leq |x-y|\end{aligned}$$

By Proposition 3.27, we have $T \in \mathcal{M}$. □

Theorem 3.31. *([1], [2], [3])*

(i) If $f, g \in \mathcal{M}$ and $m > 0$, then $f \circ g$, $f + g$, mf and $\max\{f, g\} \in \mathcal{M}$.

(ii) If (h_n) is a sequence of metric-preserving functions that converges pointwise to a function h and $h(x) > 0$ for all $x > 0$, then $h \in \mathcal{M}$. Likewise, if $\sum_{i=1}^{\infty} h_i$ converges to a function h, where each function $h_i \in \mathcal{M}$, then $h \in \mathcal{M}$.

(iii) Let $\mathcal{S} \subseteq \mathcal{M}, \mathcal{S} \neq \phi$. Let for all $x > 0$ the set $\mathcal{S}_x = \{\ f(x) \mid f \in \mathcal{S}\ \}$ be bounded. Define the function $g : [0, \infty) \to [0, \infty)$ by
$g(x) = \sup\{\ f(x) \mid f \in \mathcal{S}\ \}$ for each $x \in [0, \infty)$. Then $g \in \mathcal{M}$.

Proof. (i) Suppose that $f, g \in \mathcal{M}$ and $m > 0$. Claim that $f \circ g \in \mathcal{M}$. Let (X, d) be a metric space. Since $g \in \mathcal{M}$, $g \circ d$ is a metric. Since $f \in \mathcal{M}$, $f \circ (g \circ d)$ is a metric. But $(f \circ g) \circ d = f \circ (g \circ d)$. So $(f \circ g) \circ d$ is a metric on X. We have $f \circ g \in \mathcal{M}$.

Claim that $f + g \in \mathcal{M}$. It is easy to prove that $f + g$ is amenable. Next, let $(a, b, c) \in \triangle$. Since $f, g \in \mathcal{M}$, by Remark 3.16, we have

$$\begin{aligned}(f+g)(a) = f(a) + g(a) &\leq f(b) + f(c) + g(b) + g(c) \\ &= (f+g)(b) + (f+g)(c).\end{aligned}$$

So by Remark 3.16 again we obtain $f+g \in \mathcal{M}$.

Claim that $mf \in \mathcal{M}$. It is easy to see that mf is amenable. Now, let $(a,b,c) \in \triangle$. Since $f \in \mathcal{M}$, by Remark 3.16, we have $f(a) \leq f(b)+f(c)$. So

$$\begin{aligned}(mf)(a) = m(f(a)) &\leq m(f(b)+f(c)) \\ &= m(f(b)) + m(f(c)) \\ &= (mf)(b) + (mf)(c).\end{aligned}$$

Thus $mf \in \mathcal{M}$.

Claim that $\max\{f,g\} \in \mathcal{M}$. It is easy to see that $\max\{f,g\}$ is amenable. Next, let $(a,b,c) \in \triangle$. By Remark 3.16, we have

$$\begin{aligned}f(a) \leq f(b)+f(c) &\leq \max\{f(b),g(b)\} + \max\{f(c),g(c)\}, \\ \text{and}\quad g(a) \leq g(b)+g(c) &\leq \max\{f(b),g(b)\} + \max\{f(c),g(c)\},\end{aligned}$$

which yields

$$\begin{aligned}(\max\{f,g\})(a) = \max\{f(a),g(a)\} &\leq \max\{f(b),g(b)\} + \max\{f(c),g(c)\} \\ &= (\max\{f,g\})(b) + (\max\{f,g\})(c).\end{aligned}$$

By Remark 3.16, we obtain $\max\{f,g\} \in \mathcal{M}$.

(ii) We will prove that if (h_n) is a sequence of metric-preserving functions that $\lim\limits_{n\to\infty} h_n(x) = h(x)$ and $h(x) > 0$ for all $x > 0$, then $h \in \mathcal{M}$. Let $(a,b,c) \in \triangle$. Since for all $n \in \mathbb{N}$, $h_n \in \mathcal{M}$, for all $n \in \mathbb{N}$, $h_n(a) \leq h_n(b) + h_n(c)$. Which yields

$$\begin{aligned}(\lim_{n\to\infty} h_n)(a) = \lim_{n\to\infty}(h_n(a)) &\leq \lim_{n\to\infty}(h_n(b)+h_n(c)) \\ &= \lim_{n\to\infty} h_n(b) + \lim_{n\to\infty} h_n(c) \\ &= (\lim_{n\to\infty} h_n)(b) + (\lim_{n\to\infty} h_n)(c).\end{aligned}$$

By Remark 3.16, we have $\lim\limits_{n\to\infty} h_n \in \mathcal{M}$.

Next, we will show that if $\sum_{i=1}^{\infty} h_i = h$ where for all $i \in \mathbb{N}$, $h_i \in \mathcal{M}$, then $h \in \mathcal{M}$. Let $n \in \mathbb{N}$. Put $s_n = \sum_{i=1}^{n} h_i$. By Theorem 3.31(i), we have for each $i \in \mathbb{N}$, $s_i \in \mathcal{M}$. Let $a > 0$. Then for each $i \in \mathbb{N}$, $h_i(a) > 0$, which yields for all $n \in \mathbb{N}$, $s_n(a) = \sum_{i=1}^{n} h_i(a) \geq h_1(a)$. Thus $h(a) = \lim_{n\to\infty} (s_n(a)) \geq h_1(a) > 0$. By the above proof, we have $h \in \mathcal{M}$.

(iii) Since for all $x > 0$ we have $\{ f(x) \mid f \in \mathcal{S} \} \subseteq (0, \infty)$, $g(x) > 0$. Then for all $x > 0$ we have $g(a) \neq 0$. Thus g is amenable. Next, let $(a, b, c) \in \triangle$. Then for all $f \in \mathcal{S}$ we have $f(a) \leq f(b) + f(c) \leq g(b) + g(c)$. Thus $g(a) \leq g(b) + g(c)$. By Remark 3.16, $g \in \mathcal{M}$. □

Remark 3.32. By Theorem 3.31(i), we have

(i) If $f, g \in \mathcal{M}$, then $\frac{f+g}{2} \in \mathcal{M}$.

(ii) If $f \in \mathcal{M}$, then $f^n \in \mathcal{M}$ where $f^n = f \circ f^{n-1}$ for all $n \in \mathbb{N}$.

Example 3.33. ([3]) *There is a discontinuous and metric-preserving function that is not tightly bounded.* Define

$$f(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1 + |x - 1| & \text{otherwise.} \end{cases}$$

The function f is discontinuous at 0 and not tightly bounded. Now, $f = \max\{g, h\}$, where $g(x) = x$ and

$$h(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1 + |x - 1| & \text{if } x \in (0, 2), \\ 2 & \text{otherwise.} \end{cases}$$

Since h is tightly bounded, by Proposition 3.22, we have $h \in \mathcal{M}$. Since g is amenable, subadditive and nondecreasing, $g \in \mathcal{M}$. Since h and $g \in \mathcal{M}$, by Theorem 3.31(i), we have $f \in \mathcal{M}$.

Lemma 3.34. *([3]) Suppose $f \in \mathcal{M}$. Then the following assertions are equivalent:*

(i) f is discontinuous at 0,

(ii) $f \circ d$ is a discrete metric, for every metric d.

Proof. $(i) \Rightarrow (ii)$. Assume that f is discontinuous at 0. Let (X, d) be a metric space. Since $f \in \mathcal{M}$ and f is discontinuous at 0, by Proposition 3.20(ii), there is an $\epsilon > 0$ such that $f(z) > \epsilon$ for all $z > 0$. Then $B_{f \circ d}(x, \epsilon) = \{x\}$ for each $x \in X$, as required.
$(ii) \Rightarrow (i)$. Let d be the Euclidean metric on $\mathbb{R}$ and let $\epsilon > 0$ be such that $B_{f \circ d}(0, \epsilon) = \{0\}$. Since the sequence $(\frac{1}{n})$ converges to 0 (relative to d) but $\epsilon \leq f(d(\frac{1}{n}, 0)) = f(\frac{1}{n})$ for all $n \in \mathbb{N}$, f is discontinuous at 0. □

Theorem 3.35. *([3]) A metric-preserving function is strongly metric-preserving if and only if it is continuous at 0.*

Proof. $(\Rightarrow)$. Assume that $f \in \mathcal{SM}$. Claim that f is continuous at 0. Suppose not, that is f is discontinuous at 0. Let d be the Euclidean metric on $\mathbb{R}$. By Lemma 3.34, we have $f \circ d$ is a discrete metric. Since $f \in \mathcal{SM}$, $f \circ d$ and d are topologically equivalent and then d is a discrete metric. This is impossible.
$(\Leftarrow)$. Suppose that f is continuous at 0. Let (X, d) be any metric space. We will show that $f \circ d$ and d are topologically equivalent. Let $x \in X$ and $\epsilon > 0$. By continuity of f at 0, there is a $\gamma > 0$ such that for all $z \in [0, \infty)$ with $0 < z < \gamma$ implies $f(z) < \epsilon$. Choose $\delta = \min\{\gamma, \frac{\epsilon}{2}\}$. So there exists a $\delta \leq \epsilon$ such that $f(z) < \epsilon$ whenever $0 \leq z < \delta$. Claim that $B_d(x, \delta) \subseteq B_{f \circ d}(x, \epsilon)$. Let $y \in B_d(x, \delta)$. So $d(x, y) < \delta$. Thus $f \circ d(x, y) = f(d(x, y)) < \epsilon$. Then $y \in B_{f \circ d}(x, \epsilon)$. Therefore $B_d(x, \delta) \subseteq B_{f \circ d}(x, \epsilon)$.

Since $x \in X$ and $\epsilon > 0$, by Proposition 3.20(i), we obtain an $r > 0$ such that $f(z) \geq r$ for all $z \geq \epsilon$. Claim that $B_{f \circ d}(x, r) \subseteq B_d(x, \epsilon)$. Let $y \in B_{f \circ d}(x, r)$. So

$f(d(x,y)) = f \circ d(x,y) < r$. Thus $d(x,y) < \epsilon$. Then $y \in B_d(x,\epsilon)$. Therefore $B_{f\circ d}(x,r) \subseteq B_d(x,\epsilon)$. □

Theorem 3.36. *([2], [3]) Suppose $f \in \mathcal{M}$. Then the following assertions are equivalent:*

(i) $f \in \mathcal{SM}$,

(ii) f is continuous at 0,

(iii) f is continuous on $[0,\infty)$,

(iv) for each $\epsilon > 0$, there is an $x > 0$ such that $f(x) < \epsilon$.

Proof. By Theorem 3.19 and Theorem 3.35. □

Theorem 3.37. *([1], [3]) If $f \in \mathcal{M}$ and f is convex on $[0,c]$ where $c > 0$, then f is linear on $[0,c]$.*

Proof. Assume that $f \in \mathcal{M}$ and f is convex on $[0,c]$ where $c > 0$. Claim that for each $x \in [0,c]$ and each positive integer n, we have $f(x/2^n) = f(x)/2^n$. Since $f \in \mathcal{M}$, by Proposition 3.4, we have f is subadditive. By Remark 3.6(i), we have $f(x/2^n) \geq f(x)/2^n$ for each positive integer n. Since $f \in \mathcal{M}$, f is amenable. Since f is amenable and convex on $[0,c]$, by Remark 3.6(ii), we have $f(x/2^n) \leq f(x)/2^n$ for each positive integer n. So

$$f(x/2^n) = f(x)/2^n \quad \text{for each positive integer } n. \tag{3.4}$$

Next, from the convexity we obtain

$$\text{for all } \ a,b \in (0,\infty) \ \text{ with } \ 0 < a \leq b \leq c \ \text{ implies } \ \frac{f(a)}{a} \leq \frac{f(b)}{b}. \tag{3.5}$$

We will show that $f(x) = (f(c)/c)x$ for each $x \in [0,c]$. Since this relation is obvious for $x = 0$, let $x \in (0,c]$. Let $n \in \mathbb{N}$ be such that $\frac{c}{n} < x$. From $2^n > n$ we obtain $\frac{c}{2^n} < \frac{c}{n} < x$. So

$$\begin{aligned} \frac{f(c)}{c} = \frac{f(c)}{2^n} \cdot \frac{2^n}{c} = \frac{f(c/2^n)}{c/2^n} & \quad (by\ (3.4)) \\ \leq \frac{f(x)}{x} & \quad (by\ (3.5)) \\ \leq \frac{f(c)}{c} & \quad (by\ (3.5)). \end{aligned}$$

Then $f(x)/x = f(c)/c$. Hence $f(x) = (f(c)/c)x$. □

Note. ([3]) Let $f \in \mathcal{M}$ and (X, d) be a metric space. Then if f is continuous at 0, then by Theorem 3.35 $f \in \mathcal{SM}$, that is $f \circ d$ is topologically equivalent to d. If f is discontinuous at 0, then by Lemma 3.34 $f \circ d$ is a discrete metric.

CHAPTER IV

SOME CONSERVATIVE PROPERTIES OF METRIC-PRESERVING FUNCTIONS

In this chapter, we will prove our main results, stated in the Theorem 4.6 and Theorem 4.9. The theorems say that if $f \in \mathcal{SM}$, then $f \circ d$ has a certain property "P"if and only if d has.

It is noticed that if Q is a topological property (that is whenever a metric space has the property, so does every metric space homeomorphic to it), then if $f \in \mathcal{SM}$ then $f \circ d$ has the property Q if and only if d has.

We know that completeness and totally boundedness are not topological properties. However we will show that for any $f \in \mathcal{SM}$, $f \circ d$ is complete if and only if d is; and $f \circ d$ is totally bounded if and only if d is.

Theorem 4.1. *Let d be a metric on X. Let $f_n \in \mathcal{M}$ for all $n \in \mathbb{N}$ such that $f_n \circ d$ is numerically equivalent to d, that is for each $n \in \mathbb{N}$, there are $m_n, k_n > 0$ such that for all $(x, y) \in X \times X : m_n d(x, y) \leq f_n \circ d(x, y) \leq k_n d(x, y)$.*
Suppose that $\{m_n | n \in \mathbb{N}\}$ and $\{k_n | n \in \mathbb{N}\}$ are bounded and m_n keep away from zero for all $n \in \mathbb{N}$. That is there is a $\delta > 0$ such that $m_n > \delta$ for all $n \in \mathbb{N}$.
If $f_n \to f$ on $[0, \infty)$ and $f(x) > 0$ for all $x > 0$, then $f \in \mathcal{M}$ such that $f \circ d$ is numerically equivalent to d.

Proof. Assume that $f_n \to f$ on $[0, \infty)$. Since $f_n \in \mathcal{M}$ for all $n \in \mathbb{N}$ and $f(x) > 0$ for all $x > 0$, by Theorem 3.31(ii), we have $f \in \mathcal{M}$.
Next, we will show that $f \circ d$ is numerically equivalent to d.

We will see that for each $n \in \mathbb{N}$, there are $m_n, k_n > 0$ such that

$$for\ all\ (x,y) \in X \times X : m_n d(x,y) \leq f_n \circ d(x,y) \leq k_n d(x,y). \tag{4.1}$$

Choose $m = \inf \{ m_n \mid n \in \mathbb{N} \}$ and $k = \sup \{ k_n \mid n \in \mathbb{N} \}$. Then $0 < m \leq k$.

Let $(x,y) \in X \times X$.

We have $md(x,y) \leq m_n d(x,y) \leq f_n \circ d(x,y) = f_n(d(x,y))$ for each $n \in \mathbb{N}$.

Then

$$md(x,y) \leq \lim_{n\to\infty} f_n(d(x,y)) = f(d(x,y)) = f \circ d(x,y). \tag{4.2}$$

From (4.1), $f_n \circ d(x,y) \leq k_n d(x,y) \leq kd(x,y)$ for each $n \in \mathbb{N}$.

Thus $f(d(x,y)) = \lim_{n\to\infty} f_n(d(x,y)) \leq kd(x,y)$, that is

$$f \circ d(x,y) \leq kd(x,y). \tag{4.3}$$

From (4.2)and (4.3), we have $md(x,y) \leq f \circ d(x,y) \leq kd(x,y)$. So $f \circ d$ is numerically equivalent to d. □

Theorem 4.2. *Let $f_n \in \mathcal{SM}$ for all $n \in \mathbb{N}$. If $f_n \to f$ uniformly on $[0,\infty)$ and $f(x) > 0$ for all $x > 0$, then $f \in \mathcal{SM}$.*

Proof. Suppose that $f_n \to f$ uniformly on $[0,\infty)$.

Since $f_n \in \mathcal{SM}$, by Theorem 3.36 f_n is continuous on $[0,\infty)$ for each $n \in \mathbb{N}$. Then f is continuous on $[0,\infty)$ and by Theorem 3.31 $f \in \mathcal{M}$. Since $f \in \mathcal{M}$ and f is continuous on $[0,\infty)$, by Theorem 3.36, $f \in \mathcal{SM}$. □

Theorem 4.3. *If $f \in \mathcal{M}$ and (X,d) is a metric space, then if a subset G of X is open in (X,d) then it is open in $(X, f \circ d)$.*

Proof. Assume that $f \in \mathcal{M}$ and (X,d) is a metric space. Let G be open in (X,d). Let $x \in G$. Then there is an $\epsilon > 0$ such that $B_d(x,\epsilon) \subseteq G$. By Proposition 3.20,

$$there\ is\ a\ \delta > 0\ such\ that\ f(z) \geq \delta\ for\ all\ z \geq \epsilon. \tag{4.4}$$

To show that $B_{f\circ d}(x,\delta) \subseteq B_d(x,\epsilon)$, let $z \in B_{f\circ d}(x,\delta)$.

Then $f(d(x,z)) = f \circ d(x,z) < \delta$. By (4.4) $d(x,z) < \epsilon$. So $z \in B_d(x,\epsilon)$.

Thus $B_{f\circ d}(x,\delta) \subseteq B_d(x,\epsilon) \subseteq G$. So G is open in $(X, f\circ d)$. □

Corollary 4.4. *Let $f \in \mathcal{SM}$. Then for any metric space (X,d), a subset G of X is open in (X,d) if and only if it is open in $(X, f\circ d)$.*

It is known that compactness is a topological property, that is if (X,d_1) and (Y,d_2) are homeomorphic, then (X,d_1) is compact if and only if (Y,d_2) is compact, but completeness is not a topological property as shown in the following example.

Example 4.5. ([5]) Let $\mathbb{N}$ be the set of all positive integers. Let d be the usual absolute value metric on $\mathbb{N}$, that is $d(m,n) = |m-n|$ for all $m,n \in \mathbb{N}$. Then $(\mathbb{N}, d)$ is a complete metric space, since the only Cauchy sequences in $\mathbb{N}$ are those sequences which are constant from some point on.

Now define a metric d' on $\mathbb{N}$ by

$$d'(m,n) = |1/m - 1/n|, \text{ for } m,n \in \mathbb{N}.$$

It can be easily verified that d' is a metric on $\mathbb{N}$, and $(\mathbb{N},d)$ is homeomorphic to $(\mathbb{N},d')$. However, $(\mathbb{N},d')$ is not a complete metric space, since the sequence (s_n), $n \in \mathbb{N}$, in $\mathbb{N}$ defined by $s_n = n$ for each $n \in \mathbb{N}$ is a Cauchy sequence, but does not converge.

Although the condition that (X,d_1) and (X,d_2) are homeomorphic is not enough to yield that the space is complete if another is, the condition that f belongs to the class $\mathcal{SM}$ guarantees that if d is complete, then $f\circ d$ is complete and conversely. A part of the result follows from Theorem 3.20(i) which stated that " If $f \in \mathcal{M}$, then for each $x_0 > 0$, there is an $\epsilon > 0$ such that $f(x) \geq \epsilon$ for each $x \geq x_0$ "

The result can be stated precisely as the following theorem.

Theorem 4.6. *Let $f \in \mathcal{SM}$. Then (X,d) is a complete metric space if and only if $(X, f\circ d)$ is a complete metric space.*

Proof. $(\Rightarrow)$. Assume that (X,d) is a complete metric space. Claim that $(X, f\circ d)$ is a complete metric space. Let (x_n) be a Cauchy sequence in $(X, f\circ d)$. By Proposition 3.20,

$$for\ any\ \epsilon > 0\ there\ is\ an\ \epsilon' > 0\ such\ that\ f(z) \geq \epsilon'\ for\ all\ z \geq \epsilon. \tag{4.5}$$

Let ϵ be any positive real number, and ϵ' obtained from ϵ by (4.5). Since (x_n) is a Cauchy sequence in $(X, f\circ d)$, there exists a $K \in \mathbb{N}$, such that for all $n,m \geq K$ we have $f\circ d(x_n,x_m) < \epsilon'$. By (4.5) we have $d(x_n,x_m) < \epsilon$. Then for all $\epsilon > 0$ there exists a $K \in \mathbb{N}$ such that for all $n,m \geq K$, we have $d(x_n,x_m) < \epsilon$. Thus (x_n) is a Cauchy sequence in (X,d). Thus there exists an $x \in X$ such that (x_n) converges to x in (X,d).

Let $\epsilon > 0$ be given. Since $f\circ d$ and d are topologically equivalent, there is a $\delta > 0$ such that for all $y \in X$ with $d(x,y) < \delta$ implies $f\circ d(x,y) < \epsilon$. Since (x_n) converges to x in (X,d), there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ we have $d(x_n,x) < \delta$. Then for all $n \geq N$, we have $f\circ d(x_n,x) < \epsilon$. Hence $\lim\limits_{n\to\infty} x_n = x$ in $(X, f\circ d)$.

$(\Leftarrow)$. Assume that $(X, f\circ d)$ is a complete metric space. To show that (X,d) is a complete metric space, let (x_n) be a Cauchy sequence in (X,d). Since $f \in \mathcal{SM}$, f is continuous at 0. So

$for\ any\ \epsilon > 0\ \ there\ is\ a\ \delta > 0\ such\ that$

$$for\ all\ z \in [0,\infty),\ with\ z < \delta,\ we\ have\ f(z) < \epsilon. \tag{4.6}$$

Let $\epsilon > 0$ be given, and δ be as in (4.6). Since (x_n) is a Cauchy sequence in (X,d), there exists a $K \in \mathbb{N}$ such that for all $n,m \geq K$ we have $d(x_n,x_m) < \delta$.

So by (4.6) we obtain $f \circ d(x_n, x_m) < \epsilon$. Thus for all $\epsilon > 0$, there exists a $K \in \mathbb{N}$ such that for all $n, m \geq K$, $f \circ d(x_n, x_m) < \epsilon$. Then (x_n) is a Cauchy sequence in $(X, f \circ d)$. So there is an $x \in X$ such that (x_n) converges to x in $(X, f \circ d)$.
To show that (x_n) converges to x in (X, d), let $\epsilon > 0$ be given. Since $f \circ d$ and d are topologically equivalent, there is a $\delta > 0$ such that for all $y \in X$ with $f \circ d(x, y) < \delta$ implies $d(x, y) < \epsilon$. Since (x_n) converges to x in $(X, f \circ d)$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, we have $f \circ d(x_n, x) < \delta$. Then for all $n \geq N$, we obtain $d(x_n, x) < \epsilon$. Hence $\lim\limits_{n \to \infty} x_n = x$ in (X, d). $\square$

It is also known that the totally boundedness is not a topological property. The following proposition shows that there exist homeomorphic metric spaces (X, d_1) and (X, d_2) such that (X, d_1) is totally bounded, but (X, d_2) is not.

Proposition 4.7. *([4]) In $\mathbb{R}$, the Euclidean metric d_e is not totally bounded but the metric $d_\varphi(x, y) = |\frac{x}{1+|x|} - \frac{y}{1+|y|}|$ is totally bounded.*

Proof. To show $d_e(x, y) = |x - y|$ is not totally bounded, suppose that d_e is totally bounded. So there exists a finite subset F of $\mathbb{R}$, say $F = \{y_1, y_2, ..., y_n\}$ such that $\mathbb{R} = \bigcup_{i=1}^{n} B_{d_e}(y_i, 1)$.
For $i \in \{2, 3, ..., n\}$, we obtain

$$\begin{aligned} d_e(y_1 - 1, y_i) &= d_e(y_1 - 1, y_1) + d_e(y_1, y_i) \\ &= 1 + |y_1 - y_i|, \end{aligned}$$

that is $d_e(y_1 - 1, y_i) \geq 1$ for all $i \in \{2, 3, ..., n\}$. Then $y_1 - 1 \notin B_{d_e}(y_i, 1)$ for any $i \in \{2, 3, ..., n\}$. So $y_1 - 1 \notin \bigcup_{i=1}^{n} B_{d_e}(y_i, 1) = \mathbb{R}$. This is a contradiction. So d_e is not totally bounded.

Now we will show that d_φ is totally bounded.

Let $\epsilon > 0$ be given. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \frac{x}{1+|x|}$. So $\lim\limits_{x \to \infty} \dfrac{x}{1 + |x|} = 1$. Thus there exists an $N \in \mathbb{N}$ such that for all $x \geq N$, we have $|f(x) - 1| < \epsilon/2$.

Then for all $x, y \geq N$, we obtain $|f(x) - f(y)| < \epsilon$. That is there exists an $N \in \mathbb{N}$ such that for each $x \geq N, d_\varphi(x, N) < \epsilon$. Since $\lim_{x \to -\infty} f(x) = -1$, there exists a $K \in \mathbb{Z}^-$ such that for any $x \leq K$, we have $|f(x) - (-1)| < \epsilon/2$. Then for all $x, y \leq K$, we obtain $|f(x) - f(y)| < \epsilon$, that is there exists a $K \in \mathbb{Z}^-$ such that for each $x \leq K, d_\varphi(x, K) < \epsilon$. Choose $M = \max\{|K|, |N|\}$. Thus there is an $M \in \mathbb{N}$ such that

$$for\ any\ x \geq M,\ d_\varphi(x, M) < \epsilon \tag{4.7}$$

and

$$for\ any\ x \leq -M,\ d_\varphi(x, -M) < \epsilon. \tag{4.8}$$

Since f is uniformly continuous on $[-(M+1), M+1]$, there is a $\delta > 0$ with $0 < \delta < 1$ such that

$$for\ all\ y, z \in [-(M+1), M+1],\ |y - z| < \delta\ implies\ d_\varphi(y, z) < \epsilon. \tag{4.9}$$

Pick $x_i = \frac{i\delta}{2}$ for all $i \in \{-m, -(m-1), ..., -2, -1, 0, 1, 2, ..., m\}$, $x_{m+1} = M$ and $x_{-(m+1)} = -M$ where $m \geq (M+1)2/\delta$.

Claim that $\bigcup_{i=-(m+1)}^{m+1} B_{d_\varphi}(x_i, \epsilon) = \mathbb{R}$. Let $x \in \mathbb{R}$.

If $x \geq M$, then from(4.7), we obtain $d_\varphi(x, x_{m+1}) < \epsilon$.

If $-M < x < M$, then $x \in [-(M+1), M+1]$.

Thus there exists an $i \in \{-m, -(m-1), ..., -2, -1, 0, 1, 2, ..., m\}$ such that $|x - x_i| < \delta$. From (4.9), we obtain $d_\varphi(x, x_i) < \epsilon$.

If $x \leq -M$, then from (4.8), we have $d_\varphi(x, x_{-(m+1)}) < \epsilon$. So $x \in \bigcup_{i=-(m+1)}^{m+1} B_{d_\varphi}(x_i, \epsilon)$.

Hence $\bigcup_{i=-(m+1)}^{m+1} B_{d_\varphi}(x_i, \epsilon) = \mathbb{R}$. □

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By the Proposition 4.7 we notice that eventhough d_e and d_φ are topologically equivalent but d_e is not totally bounded while d_φ is. However, the next theorem will show that if $f \in \mathcal{SM}$ and if any one of the two spaces (X, d) or $(X, f \circ d)$ is totally bounded, then the other is also totally bounded.

Lemma 4.8. *Let $f \in \mathcal{M}$. If $(X, f \circ d)$ is totally bounded, then (X, d) is totally bounded.*

Proof. Assume that $(X, f \circ d)$ is totally bounded. We will show that (X, d) is totally bounded. Let $\epsilon > 0$ be given. By Proposition 3.20,

$$\textit{there exists a } \delta > 0 \textit{ such that } f(z) \geq \delta \textit{ for all } z \geq \epsilon. \tag{4.10}$$

Since $(X, f \circ d)$ is totally bounded, there is a finite subset F of X, say $F = \{a_1, a_2, ..., a_n\}$ such that $X = \bigcup_{i=1}^{n} B_{f \circ d}(a_i, \delta)$. Claim that $X \subseteq \bigcup_{i=1}^{n} B_d(a_i, \epsilon)$. Let $x \in X$. So $x \in \bigcup_{i=1}^{n} B_{f \circ d}(a_i, \delta)$, that is $x \in B_{f \circ d}(a_i, \delta)$ for some $i \in \{1, 2, ..., n\}$. Thus $f \circ d(a_i, x) < \delta$. By (4.10), we have $d(a_i, x) < \epsilon$. Then $x \in B_d(a_i, \epsilon)$. Therefore $x \in \bigcup_{i=1}^{n} B_d(a_i, \epsilon)$. So $X = \bigcup_{i=1}^{n} B_d(a_i, \epsilon)$. Hence (X, d) is totally bounded. □

Theorem 4.9. *Suppose that $f \in \mathcal{SM}$. Then (X, d) is totally bounded if and only if $(X, f \circ d)$ is totally bounded.*

Proof. $(\Rightarrow)$. Assume that (X, d) is totally bounded. We will show that $(X, f \circ d)$ is totally bounded. Let $\epsilon > 0$ be given. Since $f \in \mathcal{SM}$, f is continuous at 0. So there exists a $\delta > 0$ such that

$$\textit{for all } z \in [0, \infty) \textit{ with } 0 < z < \delta \textit{ implies } f(z) < \epsilon. \tag{4.11}$$

Since (X, d) is totally bounded, there is a finite subset F of X, say $F = \{a_1, a_2, ..., a_n\}$ such that $X = \bigcup_{i=1}^{n} B_d(a_i, \delta)$. To show that $X \subseteq \bigcup_{i=1}^{n} B_{f \circ d}(a_i, \epsilon)$, let $x \in X$. So $x \in \bigcup_{i=1}^{n} B_d(a_i, \delta)$, that is $x \in B_d(a_i, \delta)$ for some $i \in \{1, 2, ..., n\}$. Thus $d(a_i, x) < \delta$. By (4.11), we have $f \circ d(a_i, x) < \epsilon$. Then $x \in B_{f \circ d}(a_i, \epsilon)$. Therefore

$x \in \bigcup_{i=1}^n B_{f\circ d}(a_i, \epsilon)$. So $X = \bigcup_{i=1}^n B_{f\circ d}(a_i, \epsilon)$. Hence $(X, f \circ d)$ is totally bounded.

$(\Leftarrow)$. Follows from Lemma 4.8. $\square$

It is not difficult to show that if two metrics d and d' on a space X are numerically equivalent, then (X, d) is complete if and only if (X, d') is complete and (X, d) is totally bounded if and only if (X, d') is totally bounded.

Remark 4.10. $f : [0, \infty) \to [0, \infty)$ defined by $f(x) = \min\{1, x\}$ is strongly metric-preserving.

Proof. It is easy to show that f is amenable, subadditive and nondecreasing. Then by Proposition 3.7, $f \in \mathcal{M}$. Since f is continuous at 0, by Theorem 3.36 we have $f \in \mathcal{SM}$. $\square$

Corollary 4.11. *The space* $(\mathbb{R}^n, r)$, *where* $r(x, y) = \min\{1, d_e(x, y)\}$, *is a complete metric space. This metric is called the radar screen metric on* $\mathbb{R}^n$.

Proposition 4.12. *In* $\mathbb{R}^n$, *the radar screen metric* r *is topologically equivalent but not numerically equivalent to* d_e.

Proof. To show r is topologically equivalent to d_e, let $x \in X$ and $\epsilon > 0$.
Choose $\delta = \min\{1, \epsilon\}$. Let $y \in X$ be such that $d_e(x, y) < \delta$.
So $r(x, y) \leq d_e(x, y) < \delta \leq \epsilon$, that is $r(x, y) < \epsilon$.

Next, assume that $r(x, y) < \delta$. Then $d_e(x, y) = r(x, y) < \delta \leq \epsilon$. Thus r and d_e are topologically equivalent.

Now, we will show that r is not numerically equivalent to d_e. Suppose that r and d_e are numerically equivalent. Then there exist positive constants m, k such that for all $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$, we have $mr(x, y) \leq d_e(x, y) \leq kr(x, y) \leq k$. Thus $d_e(x, y) \leq k$. This is a contradiction. Hence r is not numerically equivalent to d_e. $\square$

Remark 4.13. The condition that $f \in \mathcal{SM}$ is weaker than that $f \circ d$ and d are numerically equivalent.

Conclusion. The Theorem 4.6 and 4.9 show that with the assumption that f is strongly metric-preserving, which is weaker than that $f \circ d$ and d are numerically equivalent, we still have the assertions that $f \circ d$ is complete if and only if d is, and $f \circ d$ is totally bounded if and only if d is.

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