อิเล็กตรอนในศักย์รูปอานม้าและศักย์แบบสุ่มของระบบควอนตัมฮอลล์

นาย สรไกร ศรีศุภผล

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ELECTRON IN SADDLE POINT AND RANDOM POTENTIALS OF QUANTUM HALL SYSTEM

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หลังจาก เฟอร์ทิก และ ฮาล์เปอร์ริน ได้ทำการศึกษา การไม่ประจำที่ และการทันเนลของอิเล็กตรอนในปัญหาควอนตัมฮอลล์ โดยใช้แบบจำลองของอิเล็กตรอนซึ่งเคลื่อนที่ในสองมิติภายใต้อิทธิพลของสนามแม่เหล็กตั้งฉาก และศักย์รูปอานม้าแบบแอนท์ไอโซโทรปิก $V_{sp}(x, y) = V_0 + (m/2)(\Omega_x^2 x^2 - \Omega_y^2 y^2)$ คากาลอร์ฟสกีย์ ได้ใช้วิธีการอินทิเกรตตามวิถีของฟายน์แมน และเมทริกซ์ของปาปาโดเปาลอส ในการพิจารณาตัวแผ่กระจายของอิเล็กตรอนในกรณีที่ศักย์อานม้าเป็นแบบไอโซโทรปิกหรือ $\Omega_x^2 = \Omega_y^2$ ในวิทยานิพนธ์นี้ได้คำนวณตัวแพร่กระจายของอิเล็กตรอนในศักย์อานม้าแบบแอนไอโซโทรปิกโดยใช้วิธีการอินทิเกรตตามวิถีของฟายน์แมน และประยุกต์ผลลัพธ์ที่ได้ไปคำนวณความหนาแน่นของสถานะ ซึ่งแสดงให้เห็นได้ว่าวิธีการอินทิเกรตตามวิถีของฟายน์แมนสามารถใช้คำนวณหาค่าพลังงานของระดับแลนดาว และพิจารณาการไม่ประจำที่ของอิเล็กตรอนในแบบจำลองของเฟอร์ทิก และฮาล์เปอร์รินได้ และเมื่อนำไปพิจารณาในกรณีตัวกวัดแกว่งแบบไอโซโทรปิกแอนฮาร์โมนิกส์ ก็ได้ผลลัพธ์เช่นเดียวกับของคากาลอร์ฟสกีย์

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An electron moving in two dimensions under the influence of a transverse magnetic field and an anisotropic saddle point potential $V_{sp}(x,y) = V_0 + (m/2)(\Omega_x^2 x^2 - \Omega_y^2 y^2)$ was proposed by Fertig and Halperin for studying the tunneling and delocalization of electrons in the quantum Hall problem. The exact propagator for an electron in the isotropic case which $\Omega_x^2 = \Omega_y^2$ was considered by Kagalovsky by using the Feynman path integral, and the matrix method introduced by Papadopoulos. In this thesis, we derive the exact propagator for an electron in the anisotropic case by using the Feynman path integral. Our result is applied to calculate the density of states. It is shown that the energy of the Landau level of Fertig and Halperin can be obtained from the path integral and the delocalization of electrons is discussed. For the isotropic anharmonic oscillator we obtain the result of Kagalovsky.

Department Physics Student's signature..

Field of study Physics Advisor's signature..

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CHAPTER I

INTRODUCTION

For a system of non-interacting electrons confined in two-dimensions under the influence of a transverse magnetic field, an electron occupies the discrete level known as the Landau level [1]. In real systems, in which impurities are present, each Landau level is broadened into a band, which is called " Landau band" [2]. The state of electron in Landau band consists of extended (delocalized) state and localized state [3]. This gives rise to the quantum Hall effect [4].

If the magnetic field is very strong, the localization in quantum Hall effect can be discussed in terms of semiclassical approximation and percolation [5]. In semiclassical limit, the motion of electron is decomposed into a rapid cyclotron orbiting and a slow drift of orbit guiding centers move along equipotential line of impurity potential. Therefore, in the region where the potential forms hills or valleys the equipotential line is closed corresponding to localization. At the saddle point in potential, the equipotential line percolates and the corresponding state is extended.

From a semiclassical picture, Fertig and Halperin [6] were the pioneers in studying the motion of electron in a saddle point potential under the influence of a transverse magnetic field. They proposed that in vicinity of the saddle point potential, tunneling of electron can occur. In order to study tunneling of electron in this region they were assumed an electron mass m moving in xy-plane and the saddle point potential is in the form $V_{sp}(x, y) = V_0 + (m/2)(\Omega_x^2 x^2 - \Omega_y^2 y^2)$ where Ω_x and Ω_y are parameters representing the saddle point potential and V_0 is the height of the saddle point potential. They calculated the exact transmission coefficient of an electron in this potential and an arbitrary uniform perpendicular magnetic field.

Recently, the model of Fertig and Halperin has been widely used to study the levitation of extended state in quantum Hall effect [7-10]. Therefore, the propagator and density of states of electron are necessary elements of this model. In this thesis, we use the Feynman path integral to evaluate the propagator and density of states of electrons in this model.

This thesis is organized as follows. In Chapter 2, we review the quantum Hall effect, semiclassical approximation and the work of Fertig and Halperin. In Chapter 3, we review the Feynman path integral approach to calculate the propagator of electron and density of state. In Chapter 4, we present an exact evaluation of a propagator for electron moving in the a two-dimensional saddle point potential $V_{sp}(x, y)$ under the influence of a transverse magnetic field and apply our result to calculate density of states. Discussion and conclusion are present in Chapter 5.

CHAPTER II

QUANTUM HALL EFFECT

As mentioned in the previous chapter, we are interested in the problem of electron in a saddle point potential with a transverse magnetic field, which plays an important role in delocalization in a quantum Hall system. In this chapter, we will review the topic that led to this problem. The contents in this chapter are as follows: Section 2.1, we present an exact evaluation of Landau quantization by solving the Schrodinger equation for an electron moving in two-dimensions under the influence of a transverse magnetic field. Section 2.2, we review the quantum Hall effect that has been a direct consequence of Landau quantization and impurity effect. Section 2.3, we discuss the behavior of electron in quantum Hall effect using semiclassical approximation. Section 2.4, we present the work of Fertig and Halperin.

2.1 The Landau Level

The knowledge of the quantum mechanics of a free electron moving in a two-dimensional plane, subject to a magnetic field directed perpendicular to the plane, is central to the understanding of quantum Hall effect [11]. In this section we will describe the quantum mechanics of a two-dimensional electron in a magnetic field using theory developed by Landau [12].

Let us assume that a two-dimensional free electron gas, with effective mass m^*, is moving in a time-independent and uniform magnetic field $\vec{B}$ applied along the z direction. The magnetic field affects both the orbital motion and the spin dynamics of the electrons. For simplicity we will neglect the interaction between electron spin and the magnetic field. The Hamiltonian, H_0, for an electron moving under the uniform magnetic field is given by

$$H_0 = \frac{1}{2m^*}\left(\vec{p} + \frac{e}{c}\vec{A}\right)^2 , \qquad (2.1)$$

where e is the magnitude of the electronic charge, c is the speed of light, $\vec{p} = \frac{\hbar}{i}\vec{\nabla}$ is the electron momentum operator, $\hbar$ is the Planck's constant and $\vec{A}$ is the vector potential associated with the magnetic field.

Landau simplified (2.1) by introducing the gauge (known as Landau gauge) in which

$$\vec{A} = (0, Bx, 0)\ . \tag{2.2}$$

One can show that this gauge satisfies the requirement that $\vec{B} = \vec{\nabla} x \vec{A}$. Using Eq. (2.2) the Schrodinger equation corresponding to the Hamiltonian H becomes

$$\left\{\frac{\partial^2}{\partial x^2} + \left(\frac{\partial}{\partial y} + \frac{ieB}{\hbar c}x\right)^2 + \frac{2m^* E}{\hbar^2}\right\}\psi(x, y) = 0\ , \tag{2.3}$$

which can be solved by writing the wave function $\psi(x, y)$ in the form

$$\psi(x, y) = U(x)\exp(ik_y y)\ . \tag{2.4}$$

Substituting Eq. (2.4) into Eq. (2.3), the wave equation for U(x) can be expressed as [1]

$$\frac{-\hbar^2}{2m^*}\frac{\partial^2 U(x)}{\partial x^2} + \frac{m^*}{2}\left(\frac{eB}{m^* c}x - \frac{\hbar k_y}{m^*}\right)^2 U(x) = E\,U(x)\ . \tag{2.5}$$

Eq. (2.5) is equivalent to the Schrodinger equation for a one-dimensional simple harmonic oscillator with cyclotron frequency ω and equilibrium position

$$x_0 = \frac{\hbar k_y}{m^* \omega}\ . \tag{2.6}$$

The eigen value E of Eq. (2.5) is given by the well-known expression for simple harmonic oscillators

$$E = (n + \frac{1}{2})\hbar\omega, \quad n=0,1,2,\ldots . \tag{2.7}$$

These quantized energy levels are known as Landau levels. If we put the system in the plane of size $L_x L_y$ where L_x and L_y are the dimensions of the system in the x and y direction, respectively. From Eq. (2.6), the x dimension of the system is confined to 0< x <L_x we see that 0 < k_y < $m^*\omega L_x / \hbar$. Let us impose the periodic boundary conditions $\psi(x, y) = \psi(x, y + L_y)$. Then $k_y = 2\pi p / L_y$ with p an integer. Together with the condition on the range k_y this implies that the total number of states or degeneracy N in a Landau level is $\frac{1}{2\pi\ell^2} L_x L_y$, or the number of states per unit area of full Landau level is

$$N_B = \frac{1}{2\pi\ell^2} = \frac{eB}{2\pi\hbar c}, \tag{2.8}$$

where $\ell = \sqrt{c\hbar / eB}$ is called magnetic length. From Eqs. (2.7) and (2.8) the corresponding density of states is in the form of delta functions at energies E_n as shown in the Fig. 1. In a real sample, the impurity potential lifts the Landau levels degeneracy, and leads to the localization and delocalization of an electron that we will discuss it's mechanism in Section 2.3.

The filling factor is defined as the ratio between the density of the electrons, N and N_B

$$\upsilon = \frac{N}{N_B}. \tag{2.9}$$

It is an integer, υ=n+1, if the states of the lowest n+1 Landau level are completely filled with electrons and the other levels are empty.

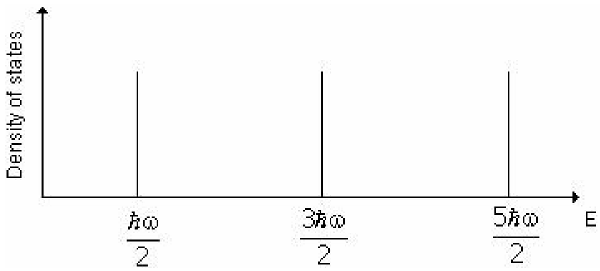

Fig. 1. The corresponding density of states for the free electron confined to move in a plane perpendicular to the magnetic field.

2.2 Quantum Hall Effect

The quantum Hall effect was discovered by Klaus von Klitzing in 1980 [13]. He investigated the electrical transport properties of silicon MOSFET (metal oxide semiconductor field effect transistor) subject to a magnetic field of about 18 T at a temperature of about 1.5 K. The geometry of the sample used by von Klitzing et.al. is shown in Fig. 2. The cross section of the sample is shown schematically in Fig. 3. The two-dimensional electron gas, which is central to the experiment, is confined in an inversion layer at interface between the silicon dioxide (SiO_2) and the p-type Si substrate. The band bending at this interface, when the substrate is biased to produce an inversion layer, is shown in Fig. 4. The substrate, SiO_2, layer and top metal electrode (known as the gate) form a parallel plate capacitor. The total amount of charge on these electrodes is proportional to the gate voltage V_g. As a result, the electron density N can be varied continuously by changing V_g. In the experiment a constant magnetic field was applied perpendicular to the sample along the z direction. A constant current was maintained in the sample in the x direction via an applied potential while the voltage drops across the sample in the x and y directions (denote by U and U_H, respectively) were measured.

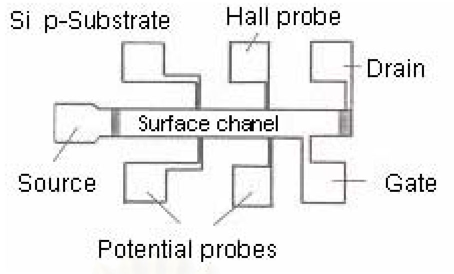

Fig. 2. Top view of the MOSFET Hall "bar" used in experiment of von Klitzing et.al.

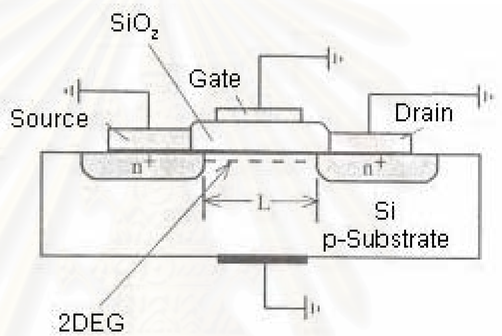

Fig. 3. Cross-sectional view of the sample in MOSFET along the surface channel showing the two-dimensional electron gas (2DEG) under the gate.

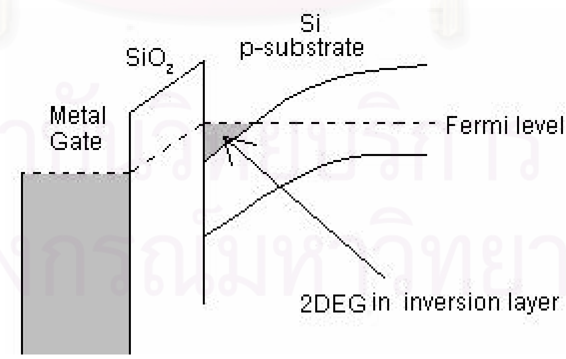

Fig. 4. The spatial variation in electron energy across the MOSFET when the gate voltage is biased such that an inversion layer is formed at the Si substrate and the oxide interface.

From the experimental result, they found that the transverse resistance or Hall resistance R_H exhibits plateaus that were given by integer fractions of h/e^2 when gate voltages are varied (Fig. 5)

$$R_H = \frac{1}{j}\frac{h}{e^2} \quad (j = 1,2,3...) . \tag{2.10}$$

At the same gate voltages they observed the longitudinal resistance was extremely small. Since the Hall conductance, the inverse of the Hall resistance, was quantized in

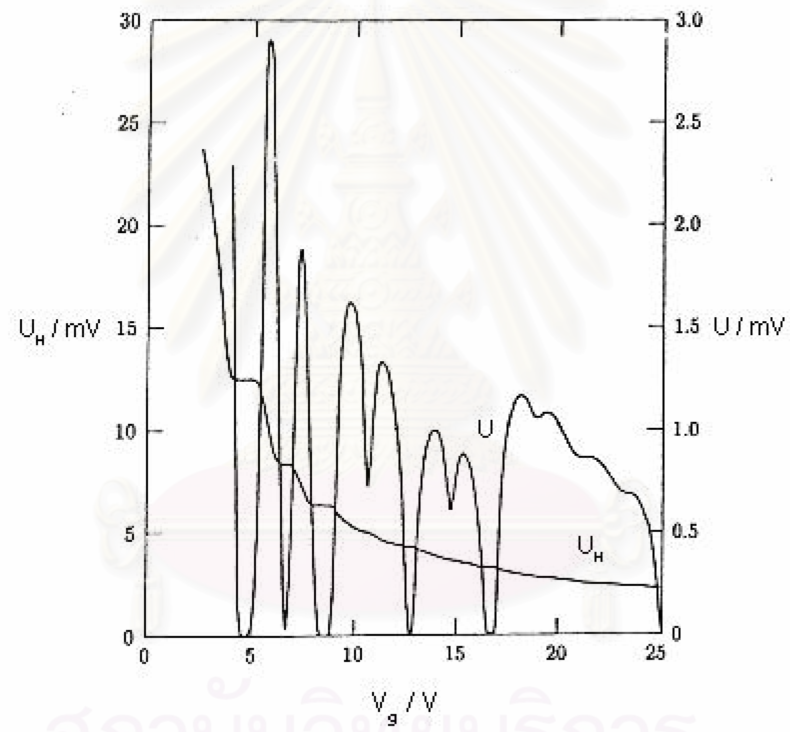

Fig. 5. The integer quantum Hall effect as discovered in 1980 by K. von Klitzing using silicon MOSFET. The Hall voltage U_H and the voltage V at two contacts parallel to the source-drain current ($\mathbf{I}$=1 μA), U are shown as a function of the voltage V_g at the gate of the MOSFET at the temperature of T= 1.5 K and magnetic flux of B =18 T.

integer units of e^2/h, this was called later on the integer quantum Hall effect. The quantization of Hall resistance can be as precise as a few parts in 10^8. Explicitly

$$R_H = (25{,}812.807)\,\Omega / j. \tag{2.11}$$

This precise quantization leads to a very accurate determination of the fine structure constant α, α^{-1} =137.035, in accordance with

$$\alpha = \frac{\mu_0 c}{2}\frac{e^2}{h}\ , \tag{2.12}$$

where $\mu_0 = 4\pi \times 10^{-7}$ H/m is the permeability of vacuum and c is the speed of light 299,792,458 m/s. Since the speed of light c is known very precisely, the determination of Hall plateaus provides a very accurate method of determining α. At the same time the quantized Hall resistance can be used as an absolute resistance standard.

The experiments show that between two adjacent Landau levels, the Hall resistance has fixed values and the longitudinal resistance vanishes, which means that the electrons are localized in this region. Localization is a key point to interpret the integer quantum Hall effect.

Due to impurities, the density of states will evolve from sharp Landau levels to a broader spectrum of levels (Fig. 6). There are two kinds of levels, localized and extended, in the new spectrum, and it is expected that the extended (delocalized) states occupy a core near the original Landau level energy while the localized states are more spread out in energy. Only the extended states can carry current. Therefore, if the occupation of the extended states does not change, the current will not change. An argument due to Laughlin [14] and Halperin [15] showed that extended states indeed exist at the cores of the Landau levels and if these states are full, (i.e., the Fermi level is not in the core of extended states) then they carry exactly the right current to give (2.10).

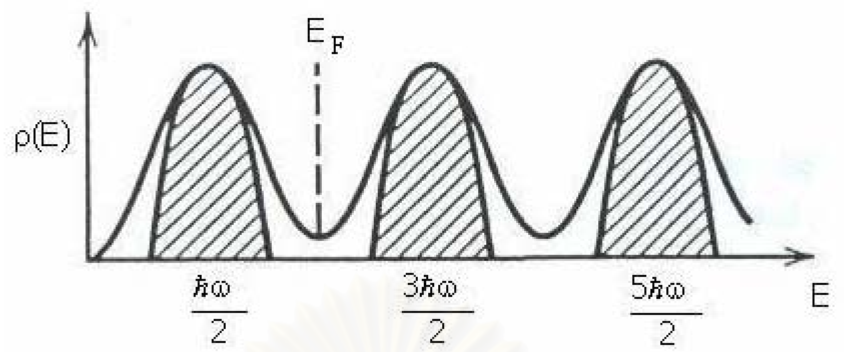

Fig. 6. In the presence of impurities, the Landau levels broaden to a band. Regions of extended state are shaded.

The existence of the localized states can explain the appearance of plateaus. As the density of electrons is increased the localized states gradually fill up without any change in occupation of the extended states, thus without any change in the Hall resistance. For these densities the Hall resistance is on a step in Fig. 5 and the longitudinal resistance vanishes. It is only as the Fermi level passes through the core of extended states that the longitudinal resistance becomes appreciable and the Hall resistance makes its transition from one plateau step to the next.

2.3 Semiclassical and Percolation Picture

In Section 2.2, we have shown that existence of localized state is centrally important to understand integer quantum Hall effects. In this section, we will show that for the disorder potential, smooth on the scale of the magnetic length, the localization in integer quantum Hall effect can be discussed in terms of semiclassical approximation.

From Hamiltonian H_0 in Eq. (2.1), the semiclassical approximation is most conveniently derived by replacing the coordinate (x,y) by guiding center coordinates (X,Y) and relative coordinate (ξ,η) given by

$$x = X + \xi\,, \qquad (2.13a)$$

$$y = Y + \eta \,, \tag{2.13b}$$

where

$$\xi = -\frac{c}{eB}(p_y - \frac{e}{c}A_y), \tag{2.14a}$$

$$\eta = \frac{c}{eB}(p_x - \frac{e}{c}A_x). \tag{2.14b}$$

From the Heisenberg equation of motion, we get

$$\dot{\xi} = \frac{i}{\hbar}[H_o, \xi] = \omega\eta \,, \tag{2.15a}$$

$$\dot{\eta} = \frac{i}{\hbar}[H_o, \eta] = -\omega\xi \,. \tag{2.16b}$$

We see that (ξ,η) indeed rotate with angular frequency ω around guiding center (cyclotron motion). Due to the commutation relation of $\vec{p}$ and $\vec{r}$, both guiding center coordinate and relative coordinate obey canonical commutation relations,

$$[\xi, \eta] = i\ell^2 \,, \tag{2.17a}$$

$$[X, Y] = -i\ell^2 \,, \tag{2.17b}$$

Using Eq. (2.14), the Hamiltonian H_o can be written in terms of ξ and η,

$$H_o = \frac{\hbar\omega}{2\ell^2}\left(\xi^2 + \eta^2\right), \tag{2.18}$$

which expresses the degeneracy of Landau levels as H_o does not depend on X and Y.

In the presence of a disorder potential V(x,y), the degeneracy is lifted. The equations of motion for the center coordinates are

$$\dot{X} = \frac{i}{\hbar}[H_0 + V(x,y), X] = \frac{-\ell^2}{\hbar}\frac{\partial V}{\partial y}, \tag{2.19a}$$

$$\dot{Y} = \frac{i}{\hbar}[H_0 + V(x,y), Y] = \frac{\ell^2}{\hbar}\frac{\partial V}{\partial x}. \tag{2.19b}$$

Due to impurity ions which are randomly located in sample, creating fluctuations in V(x,y). If the sample is penetrated by a strong magnetic field, the cyclotron radius is much smaller than the potential fluctuations in the sense that $\left|\vec{\nabla} V(x,y)\right| << \hbar\omega/\ell$. Consequently, the potential V(x,y) is smooth on the scale of magnetic length ℓ , we can replace V(x,y) by V(X,Y) and obtain a drift of the guiding center along equipotential,

$$\dot{X} = -\frac{\ell^2}{\hbar}\frac{\partial V}{\partial Y}, \tag{2.20a}$$

$$\dot{Y} = \frac{\ell^2}{\hbar}\frac{\partial V}{\partial X}. \tag{2.20b}$$

In this limit, the eigen energies are

$$E = (n + \frac{1}{2})\hbar\omega + V(X,Y). \tag{2.21}$$

This discussion shows that in the semiclassical limit the motion of electrons can be decomposed into guiding center motion along equipotential line of disorder potential and cyclotron motion with frequency ω around guiding center. It then seems very reasonable that in the presence of smooth disorder potential the eigenfunction will live on contour lines of constant energy on the random energy surface. Thus low energy states will be found lying along contours in valleys in the potential energy landscape while high-energy state will be found encircling hills in the landscape. In both cases the corresponding states are localized.

For the delocalization in the smooth disorder potential landscape, Trugman [16] pointed out that in this limit the delocalization of electronic states is associated with the percolation of equipotential lines which percolate at the saddle points in potential landscape. The energy of extended state is thus clearly determined by saddle point potential in this limit. For a disorder potential symmetrically distributed about zero, then all saddle point in potential landscape are identified with zero energy.

In order to understand the electron transport from the above picture, it is useful to imaging gradually filling a random landscape with water, in this analogy sea level represents the Fermi energy for electron. When only a small amount of water has been added, the water will fill in the valleys to form small lakes. As the sea level is increased, at a "height" of a saddle point in potential energy landscape (zero energy) their shorelines will be percolate from one side of the system to the other (Fig. 7). As the sea level is raised still further, the ocean will cover the majority of the landscape and only the hill will stick out above the water. The shoreline will no longer percolate but only surround the hill.

2.4 Fertig and Halperin Model

In the previous section we presented that behavior of an electron moving in two-dimensional disordered system with strong magnetic field can be understood using a semiclassical picture. However, semiclassical picture dose not include the effect of quantum tunneling which can occur in the vicinity of saddle point when two orbits on equipotential line at the same energy approach each other on the distance less than the cyclotron radius (Fig. 8). In this section, we present the work of Fertig and Halperin who studied tunneling in this region by assuming the saddle point potential in the form $V_{sp}(x, y) = (m/2)(\Omega_x^2 x^2 - \Omega_y^2 y^2) + V_0$. The parameters Ω_x and Ω_y are representing the harmonic and inverted harmonic of saddle point potential in the x and y directions, respectively, and V_0 is the top of the saddle point potential. The Hamiltonian for this system is

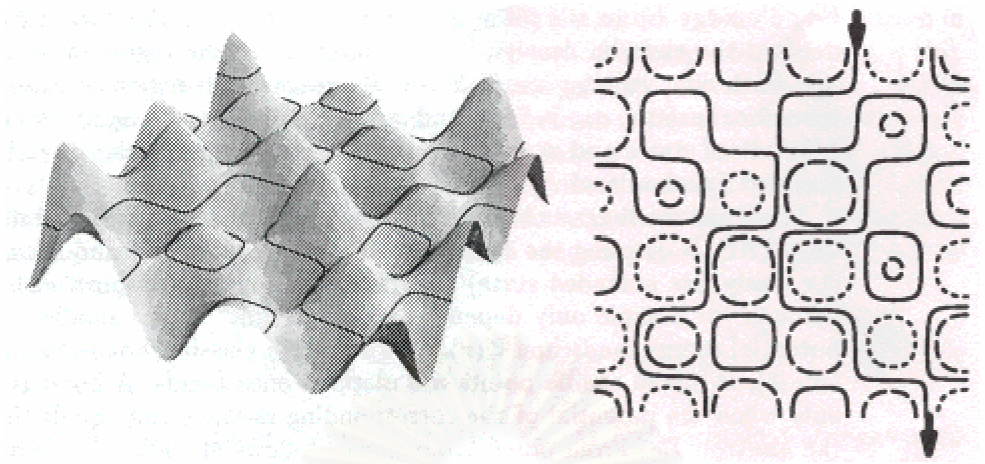

Fig. 7. Left: Schematic plot of smooth random potential V(x,y) with equipotential lines at E equal to the “height” of the saddle points in the potential energy landscape. Right: Equipotential lines of the same potential for E equal to valley, zero energy (“height” of saddle point), and hill corresponding to long dashed, solid and short dashed lines respectively. Note the solid line percolating the system from top to bottom as indicated by the arrows.

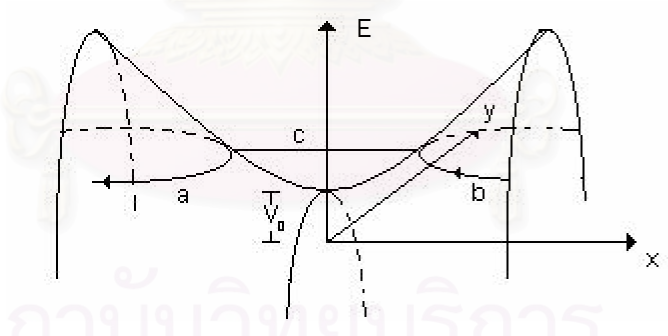

Fig. 8. Sketch of the saddle point potential corresponding to the model of Fertig and Halperin. The electron drifts along equipotential lines (a and b) and can tunnel between different contours where they get close in distance less than cyclotron radius (c< ℓ).

$$H = \frac{1}{2m}\left(\vec{p} + \frac{e}{c}\vec{A}\right)^2 + (\frac{m}{2})(\Omega_x^2 x^2 - \Omega_y^2 y^2) + V_0, \tag{2.22}$$

By using symmetric gauge $\vec{A} = \frac{B}{2}(-y,x,0)$ there is coupling between x and y in the Hamiltonian. In order to decouple this Hamiltonian, they express Eq. (2.22) in the form

$$H = \tilde{\omega}(\hat{a}_1^+\hat{a}_1 + \hat{a}_2^+\hat{a}_2 + 1) + \frac{\omega}{2i}(\hat{a}_1^+\hat{a}_2 - \hat{a}_2^+\hat{a}_1) + \gamma[(\hat{a}_2^+ + \hat{a}_2)^2 - (\hat{a}_1^+ + \hat{a}_1)^2] + V_0, \tag{2.23}$$

where $\tilde{\omega} = \sqrt{\frac{\omega^2}{4} + \tilde{\Omega}^2}$, $\tilde{\Omega}^2 = \frac{\Omega_x^2 - \Omega_y^2}{2}$, $\gamma = \frac{\Omega^2}{\tilde{\omega}}$, $\Omega^2 = \frac{\Omega_x^2 + \Omega_y^2}{2}$, and operators $\hat{a}_1$ and $\hat{a}_2$ are given by

$$\hat{a}_1 = \frac{1}{\sqrt{2}}\left\{\sqrt{m\tilde{\omega}}x + \frac{1}{\sqrt{m\tilde{\omega}}}\frac{\partial}{\partial x}\right\}, \tag{2.24a}$$

$$\hat{a}_2 = \frac{1}{\sqrt{2}}\left\{\sqrt{m\tilde{\omega}}y + \frac{1}{\sqrt{m\tilde{\omega}}}\frac{\partial}{\partial y}\right\}, \tag{2.24b}$$

so that $[\hat{a}_1, \hat{a}_1^+]$=$[\hat{a}_2, \hat{a}_2^+]$=1 and $[\hat{a}_1, \hat{a}_2]$=$[\hat{a}_1, \hat{a}_2^+]$=0 (omitted $\hbar$ in momentum operator).

Hamiltonian Eq. (2.23) can decouple into a sum of two commuting Hamiltonian by introduce a Bogoliubov transformation

$$\begin{pmatrix}\hat{a}_1\\ \hat{a}_2\end{pmatrix} = \begin{pmatrix} i\cos\phi & \sin\phi \\ -\sin\phi & -i\cos\phi\end{pmatrix}\begin{pmatrix}\hat{b}_1\\ \hat{b}_2\end{pmatrix}, \tag{2.25}$$

with $\tan(2\phi) = \frac{-\omega}{4\gamma}$, Eq. (2.23) becomes H_1+H_2 , where

$$H_1 = \begin{pmatrix}\hat{b}_1^+ & \hat{b}_1\end{pmatrix}\begin{pmatrix}\frac{\tilde{\omega}}{2} - \sqrt{\gamma^2 + (\frac{\omega}{4})^2} & \gamma \\ \gamma & \frac{\tilde{\omega}}{2} - \sqrt{\gamma^2 + (\frac{\omega}{4})^2}\end{pmatrix}\begin{pmatrix}\hat{b}_1\\ \hat{b}_1^+\end{pmatrix}, \tag{2.26a}$$

$$H_2 = \begin{pmatrix} \hat{b}_2^+ & \hat{b}_2 \end{pmatrix} \begin{pmatrix} \frac{\tilde{\omega}}{2} + \sqrt{\gamma^2 + (\frac{\omega}{4})^2} & -\gamma \\ -\gamma & \frac{\tilde{\omega}}{2} + \sqrt{\gamma^2 + (\frac{\omega}{4})^2} \end{pmatrix} \begin{pmatrix} \hat{b}_2 \\ \hat{b}_2^+ \end{pmatrix} + V_0 . \quad (2.26b)$$

Operators $\hat{b}_1$ and $\hat{b}_2$ satisfies commutation relation $[\hat{b}_1, \hat{b}_2]=[\hat{b}_1, \hat{b}_2^+]=0$ and $[\hat{b}_1, \hat{b}_1^+]=[\hat{b}_2, \hat{b}_2^+]=1$. From Eq. (2.26) H_2 can be diagonalized with a second Bogoliubov transformation of the form

$$\begin{pmatrix} \hat{b}_2 \\ \hat{b}_2^+ \end{pmatrix} = \begin{pmatrix} \cosh\theta_2 & \sinh\theta_2 \\ \sinh\theta_2 & \cosh\theta_2 \end{pmatrix} \begin{pmatrix} \hat{c}_2 \\ \hat{c}_2^+ \end{pmatrix}, \quad (2.27)$$

with

$$\tan(2\theta_2) = \frac{\gamma}{\frac{\tilde{\omega}}{2} + \sqrt{\gamma^2 + (\frac{\omega}{4})^2}} .$$

For this transformation, $[\hat{c}_2, \hat{c}_2^+]=1$, Eq. (2.26b) becomes

$$H_2 = \frac{1}{2} E_2 (\hat{c}_2^+ \hat{c}_2 + \frac{1}{2}) + V_0 , \quad (2.28)$$

where

$$E_2 = \frac{1}{\sqrt{2}} ((\omega^4 + 4\omega^2\tilde{\Omega}^2 + 4\Omega^4)^{1/2} + \omega^2 + 2\tilde{\Omega}^2)^{1/2} . \quad (2.29)$$

Equation Eq. (2.28) shows that H_2 has the form of harmonic oscillator Hamiltonian. Unlike H_2, H_1 cannot be written in the harmonic oscillator form. However, H_1 can be written in a convenient form, which preserve the commutation relation $[\hat{c}_1, \hat{c}_1^+]=1$,

$$H_1 = E_1 (\hat{c}_1^2 + \hat{c}_1^{+2}) , \quad (2.30)$$

when

$$\begin{pmatrix} \hat{b}_1 \\ \hat{b}_1^+ \end{pmatrix} = \begin{pmatrix} \cosh\,\theta_1 & \sinh\,\theta_1 \\ \sinh\,\theta_1 & \cosh\,\theta_1 \end{pmatrix} \begin{pmatrix} \hat{c}_1 \\ \hat{c}_1^+ \end{pmatrix}, \tag{2.31}$$

$$\tan(\,2\theta_1\,) = \frac{\dfrac{\tilde{\omega}}{2} - \sqrt{\gamma^2 + (\dfrac{\omega}{4})^2}}{-\gamma}\,, \tag{2.32}$$

and

$$E_1 = \frac{1}{2\sqrt{2}}((\omega^4 + 4\omega^2\tilde{\Omega}^2 + 4\Omega^4)^{1/2} - \omega^2 - 2\tilde{\Omega}^2)^{1/2}\,. \tag{2.33}$$

Using Eqs. (2.28) and (2.30), full Hamiltonian may be written as

$$H = E_1\left(\,\hat{c}_1^2 + \hat{c}_1^{+2}\right) + \frac{1}{2}E_2(\hat{c}_2^+\hat{c}_2 + \frac{1}{2}) + V_0\,, \tag{2.34}$$

with $[\hat{c}_1, \hat{c}_2]$=$[\hat{c}_1, \hat{c}_2^+]$=0, and $[\hat{c}_1, \hat{c}_1^+]$=$[\hat{c}_2, \hat{c}_2^+]$=1.

In order to solve the Schrodinger equation from Eq. (2.34), it is convenient to define the following operator:

$$X = \frac{1}{\sqrt{2}\,i}(\hat{c}_1^+ - \hat{c}_1)\,, \tag{2.35a}$$

$$P = \frac{1}{\sqrt{2}}(\hat{c}_1^+ + \hat{c}_1)\,, \tag{2.35b}$$

$$s = \frac{1}{\sqrt{2}}(\hat{c}_2^+ + \hat{c}_2)\,, \tag{2.35c}$$

$$p = \frac{1}{\sqrt{2}\,i}(\hat{c}_2 - \hat{c}_2^+)\,. \tag{2.35d}$$

Using Eq. (2.35), the Hamiltonian H_1 and H_2 may be written as

$$H_1 = E_1\left(P^2 - X^2\right), \tag{2.36a}$$

$$H_2 = \frac{1}{2}E_2\left(p^2 + s^2\right) + V_0 , \qquad (2.36b)$$

with $[X, P]=i$, $[s,p]=i$, and $[s, X]=[s, P]=[p, X]=[p, P]=0$.

The full Hamiltonian is a sum of two commuting Hamiltonians, the first (H_1) being equivalent to that of a one-dimensional inverted harmonic oscillator, and the other (H_2) representing the one-dimensional particle in a confining harmonic potential. Physically, the coordinate s is associated with the cyclotron motion of electron, and the coordinate X with the guiding center motion.

Thus, the wave function of this system can be written in the form $\Psi(X, s)$ where X and s are real numbers. Since $H_2 - V_0$ is Hamiltonian of one-dimensional harmonic oscillator, $\Psi(X, s)$ can be chosen to be $\phi(X)\,\varphi_n(s)$ where $H_2\,\varphi_n(s) = \left\{(n + \frac{1}{2})E_2 + V_0\right\}\varphi_n(s)$, and $\varphi_n(s)$ to be normalized to unity ($\int_{-\infty}^{\infty} ds\,|\,\varphi_n(s)\,|^2 = 1$). This implies that energy spectrum is unlike discrete Landau levels, the saddle-point potential allows for a continuous energy E for each discrete state n. For $\Omega_x^2, \Omega_y^2 \to 0$ the energy E becomes discrete Landau level [8]. Consequently, Schrodinger's equation for wave function of this form may be written as

$$H_1\phi(X) = E_1\left(P^2 - X^2\right)\phi(X) = (E - (n + \frac{1}{2})E_2 - V_0)\phi(X) , \qquad (2.37)$$

where $E - (n + \frac{1}{2})E_2$ is the guiding center energy of electron. Eq. (2.25) implies that the probability that the electron in this system will go through the saddle point is equivalent to the probability that the one-dimensional electron will be transmitted through the inverted harmonic oscillator potential.

From the work of Fertig and Halperin, in Chapter 4, we will use the Feynman path integral to derive the exact propagator and evaluate the density of states of electron in this model.

CHAPTER III

THE PROPAGATOR AND FEYNMAN PATH INTEGRAL

Motivated by the work of Fertig and Halperin, who treat the problem of two-dimensional electron in a saddle point potential with a transverse magnetic field by solving Schrodinger equation, we will study the electron energy spectrum in this system by using Feynman path integral approach. Before we present our calculation in next chapter, in this chapter we will review the Feynman path integral and some applications that can be applied to our work.

3.1 The Propagator and Feynman Path Integrals

In quantum mechanics, the dynamical information of a quantum mechanical system is contained in the wave function. It is a function, sometimes called the probability amplitude that determines the wave associated with a particle. In practice, we can obtain their wave function by solving the Schrodinger's equation.

In Schrodinger's picture [17], there exists the state vector $|\Psi(t)\rangle$ that evolves as

$$|\Psi(t)\rangle = U(t,t')\,|\Psi(t')\rangle \tag{3.1}$$

where U(t,t') is the time evolution operator satisfying the following properties,

i) $i\hbar\frac{\partial}{\partial t}U(t,t') = HU(t,t')$

ii) $U(t',t') = 1$ {initial condition}

iii) $U(t'',t)U(t,t') = U(t'',t')$ {composition law}

iv) $U^{+}(t'',t') = U^{-1}(t'',t') = U(t',t'')$

and H is the Hamiltonian. If the Hamiltonian is not an explicit function of time then the evolution operator is of the form

$$U(t'',t') = \exp\left\{-\frac{i}{\hbar}H(t'',t')\right\} \ . \tag{3.2}$$

In the configuration representation Eq. (3.1) becomes

$$\left\langle \vec{x}'' \middle| \Psi(t'') \right\rangle = \int_{-\infty}^{\infty} d^3x' \left\langle \vec{x}'' \middle| U(t'', t') \middle| \vec{x}' \right\rangle \left\langle \vec{x}' \middle| \Psi(t') \right\rangle , \tag{3.3}$$

where the complete set

$$\int_{-\infty}^{\infty} d^3x' \left| \vec{x}' \right\rangle \left\langle \vec{x}' \right| = 1 . \tag{3.4}$$

We can rewrite Eq. (3.3) as

$$\left\langle \vec{x}'' \middle| \Psi(t'') \right\rangle = \int_{-\infty}^{\infty} d^3x' \, K(\vec{x}'', t''; \vec{x}', t') \left\langle \vec{x}' \middle| \Psi(t') \right\rangle , \tag{3.5}$$

where

$$K(\vec{x}'', t''; \vec{x}', t') = \left\langle \vec{x}'' \middle| U(t'', t') \middle| \vec{x}' \right\rangle . \tag{3.6}$$

$K(\vec{x}'', t''; \vec{x}', t')$ is called the "propagator" as the probability amplitude of a particle to go from $\vec{x}'$ at time t' to $\vec{x}''$ at time t''.

According to Feynman's idea [18], there are infinitely many paths for a particle to go from the initial point to the final point under the restrictive condition that $\vec{x}(t') = \vec{x}'$, $\vec{x}(t'') = \vec{x}''$. Each trajectory contributes to the total amplitude, to go from $\vec{x}'$ to $\vec{x}''$. They contribute equal amounts to the total amplitude, but at different phases. The phase of the contribution from a given path is the action S for that path in unit of action $\hbar$. That is, to summarize, the probability $P(\vec{x}'', \vec{x}')$ to go from the point $\vec{x}'$ at t' to the point $\vec{x}''$ at t'' is the absolute square of an amplitude $K(\vec{x}'', t''; \vec{x}', t')$, $P(\vec{x}'', \vec{x}') = \left| K(\vec{x}'', t''; \vec{x}', t') \right|^2$ to go from $\vec{x}'$ to $\vec{x}''$. This amplitude is the sum of all amplitude contributions $\Phi[\vec{x}(t)]$ from each path, that is

$$K(\vec{x}'', t''; \vec{x}', t') = \sum_{\substack{\text{over all path} \\ \text{from } \vec{x}' \text{ to } \vec{x}''}} \Phi[\vec{x}(t)] . \tag{3.7}$$

The contribution of a path has a phase proportional to the action S,

$$\Phi[\vec{x}(t)] = [\text{constant}]\exp\left\{\frac{i}{\hbar}S[\vec{x}(t)]\right\}, \tag{3.8}$$

and
$$S = \int_{t'}^{t''} dt\ L(\vec{x}, \dot{\vec{x}}), \tag{3.9}$$

with the Lagrangian
$$L(\vec{x}, \dot{\vec{x}}) = \frac{1}{2}m\dot{\vec{x}}^2 - V(\vec{x}). \tag{3.10}$$

Actually, we can not evaluate $K(\vec{x}'', t''; \vec{x}', t')$ from Eq. (3.7) directly because of the infinitely many paths contributing. Feynman [18] proposed another way to perform a new formalism of $K(\vec{x}'', t''; \vec{x}', t')$. By dividing the time variable into steps of width $\varepsilon \to 0$, this gives us a set of value t_i spaced at a distance ε apart between the values t' and t'' .At each time t_i we select some special $\vec{x}_i$ and construct a path by connecting all points. It is possible to define a sum over all paths in this manner by taking a multiple integral over all values of $\vec{x}_i$ for i between 1 and N-1, where

$$N\varepsilon = t'' - t'$$
$$\varepsilon = t_i - t_{i-1}$$
$$t_o = t'$$
$$t_N = t''$$
$$\vec{x}_o = \vec{x}'; \ \vec{x}_N = \vec{x}''$$

The resulting equation is

$$K(\vec{x}'', t''; \vec{x}', t') = \lim_{N\to\infty}\frac{1}{A}\int\int\cdots\int\frac{d^3x_1}{A}\frac{d^3x_2}{A}\cdots\frac{d^3x_{N-1}}{A}\exp\left\{\frac{i}{\hbar}S[\vec{x}(t)]\right\}, \tag{3.11}$$

where $S = \int_{t'}^{t''} dt\ L(\vec{x}, \dot{\vec{x}})$ and the normalizing factor $A = (\frac{2\pi i\hbar\varepsilon}{m})^{3/2}$.

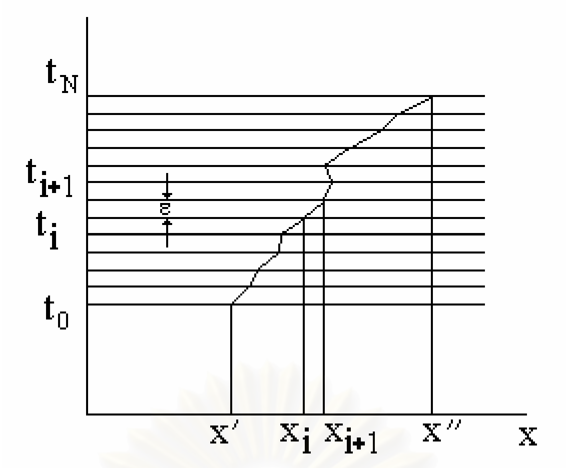

Fig. 9. The sum over paths is defined as a limit, in which at first is specified by giving only its coordinate x at a large number of specified time separated by very small interval ε. The path sum is then an integral over all these specified coordinates. Then to achieve the correct measure, the limit is taken as ε approaches zero.

For small time slices,

$$S(t_i, t_{i-1}) = \int_{t_{i-1}}^{t_i} dt\ L(\vec{x}, \dot{\vec{x}}) = \frac{m}{2\varepsilon}\left(\vec{x}_i - \vec{x}_{i-1}\right)^2 - \varepsilon V\left(\vec{x}_i\right), \tag{3.12}$$

so that Eq. (3.11) can be written as

$$K(\vec{x}'', t''; \vec{x}', t') = \lim_{N\to\infty}\left(\frac{m}{2\pi i\hbar\varepsilon}\right)^{\frac{3N}{2}} \int\int\ldots\int d^3x_1 d^3x_2 \ldots d^3x_{N-1} \exp\left\{\frac{i}{\hbar}\sum_{i=1}^{N}\frac{m}{2\varepsilon}\left(\vec{x}_i - \vec{x}_{i-1}\right)^2 - \varepsilon V\left(\vec{x}_i\right)\right\}. \tag{3.13}$$

Feynman wrote this sum over all paths in a less restrictive notation as

$$K(\vec{x}'', t''; \vec{x}', t') = \int D[\vec{x}(t)] \exp\left\{\frac{i}{\hbar} S[\vec{x}'', \vec{x}']\right\}, \tag{3.14}$$

which is called the Feynman path integral. Here the symbol $\int D[\vec{x}(t)]$ is defined by Eq. (3.13) and represents an integration over all possible paths connecting the point $(\vec{x}', t')$ and $(\vec{x}'', t'')$.

3.2 Path Integral of a Free Particle

From Eq. (3.13) we can compute the propagator of a free particle. The Lagrangian for a free particle is

$$L(\vec{x}, \dot{\vec{x}}) = \frac{m}{2}\dot{\vec{x}}^2 . \tag{3.15}$$

The three dimensional propagator is simply the product of three one-dimensional propagators, so that these is no point in cluttering our equation with vector. We wish to evaluate

$$K(\vec{x}'', t''; \vec{x}', t') = \lim_{N\to\infty}\left(\frac{m}{2\pi i\hbar\varepsilon}\right)^{\frac{N}{2}} \int\int\dots\int dx_1 dx_2 \dots dx_{N-1} \exp\left\{\frac{i}{\hbar}\sum_{i=1}^{N}\frac{m}{2\varepsilon}\left(\vec{x}_i - \vec{x}_{i-1}\right)^2\right\}. \tag{3.16}$$

This is an integral of the form $\int_{-\infty}^{\infty} dx \exp[-ax^2 + bx]$, which is called a gaussian integral. Since the integral of a gaussian is again gaussian, we may carry out the integration on one variable after the other with the help of the formula.

$$\int_{-\infty}^{\infty} dx_1 \left(\frac{m}{2\pi i\hbar\varepsilon}\right)^{-2/2} \exp\left\{\left(\frac{m}{2\pi i\hbar\varepsilon}\right)\left[(x_2 - x_1)^2 - (x_1 - x_0)^2\right]\right\}$$
$$= \left(\frac{m}{2\pi i\hbar(2\varepsilon)}\right)^{1/2} \exp\left\{\left(\frac{m}{2\pi i\hbar(2\varepsilon)}\right)(x_2 - x_0)^2\right\}. \tag{3.17}$$

After the integration is completed, the limit may be taken. The result is

$$K(\vec{x}'', t''; \vec{x}', t') = \left(\frac{m}{2\pi i\hbar(t'' - t')}\right)^{\frac{1}{2}} \exp\left\{\left(\frac{im}{2\hbar(t'' - t')}\right)(\vec{x}'' - \vec{x}')^2\right\} . \tag{3.18}$$

3.3 The Quadratic Lagrangian

In principle, if the path integral is still in a gaussian form, it is possible to carry out the integral over all paths in the way described in the previous section. But in real practice it is too complicated to perform, for example, the harmonic oscillator problem. We now introduce some additional mathematical techniques, which help us to sum over paths in some certain situations. The simplest example to be studied is a quadratic Lagrangian, this corresponds to a case in which the action S contains the path x(t) up to the second power.

To illustrate how the method works in such case, consider a particle whose Lagrangian has the form

$$L(x, \dot{x}, t) = a(t)\dot{x}^2 + b(t)x\dot{x} + c(t)x^2 + d(t)\dot{x} + e(t)x + f(t) . \qquad (3.19)$$

The action is the integral of this function with respect to time between two fixed end points. We wish to determine

$$K(x'', t''; x', t') = \int D[x(t)] \exp\left\{ \frac{i}{\hbar} \int_{t'}^{t''} dt\, L(x, \dot{x}, t) \right\} , \qquad (3.20)$$

the integral over all paths which go from (x', t') to (x'', t''). Of course, it is possible to carry out this integral over all paths in the way which was first described by dividing the region into short time elements, and so on. But we shall not go through this tedious calculation, since we can determine the most important characteristics of the propagator in the following way.

Let $\bar{x}(t)$ be the classical path between the specified end points. This is the path, which has an extremum for the action S. For this notation we will use

$$S_{cl}[x'', x'] = S[\bar{x}(t)] . \qquad (3.21)$$

We can represent x in term of $\bar{x}$ and y

$$x = \bar{x} + y \,. \tag{3.22}$$

That is to say, instead of defining a point on the path by its distance $x(t)$ from an arbitrary coordinate axis, we measure instead the deviation y(t) from the classical path, as shown in Fig. 10.

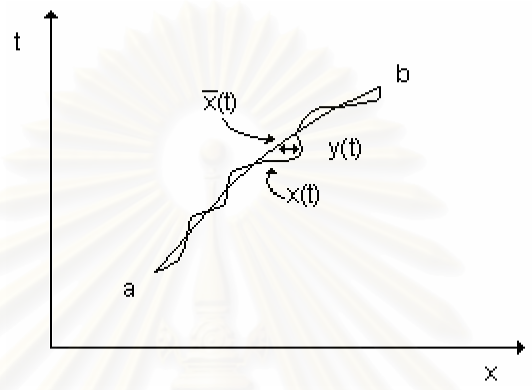

Fig. 10. The difference between the classical path $\bar{x}$(t) and some possible alternative path. The end point y(t'')= y(t')=0.

At each t the variable x and y differ by the constant $\bar{x}$. Therefore, clearly, dx_i = dy_i for each specific point t_i in the subdivision of time. In general, we may say D[x(t)] = $D[y(t)]$. The integral for the action can be written as

$$S[x(t)] = S[\bar{x} + y(t)] = \int_{t'}^{t''} dt\,[a(t)(\dot{\bar{x}}^2 + 2\dot{\bar{x}}\dot{y} + \dot{y}^2) + ..] \,. \tag{3.23}$$

If all the terms, which does not involve y are collected, the resulting integral is just $S[\bar{x}(t)] = S_{cl}$. If all the terms, which contains y as a linear factor, are collected, the resulting integral vanishes. This could be proved by actually carrying out the integration, however, such a calculation is unnecessary, since we already know the result is true. The function $\bar{x}(t)$ is determined by this very requirement Eq. (3.21). That is, $\bar{x}$ is so chosen that there is no change in S, to first order, for variation of path around $\bar{x}$. All that remains are the second order term in y. These can be easily picked out, so that we can write

$$S[x(t)] = S_{cl}[x'', x'] + \int_{t'}^{t''} dt[a(t)\dot{y}^2 + b(t)y\dot{y} + c(t)y^2] . \tag{3.24}$$

The integral over paths dose not depend upon the classical path, so the propagator can be written

$$K(x'', t''; x', t') = \exp\left\{\frac{i}{\hbar} S_{cl}[x'', x']\right\} \int D[y(t)] \exp\left\{\frac{i}{\hbar}\int_{t'}^{t''} dt\, [a(t)\dot{y}^2 + b(t)y\dot{y} + c(t)y^2]\right\}. \tag{3.25}$$

Since all path y(t) start from and return to the point y=0, the integral over paths can be a function only of times at the end points. This means that the kernel can be written as

$$K(x'', t''; x', t') = F(t'', t') \exp\left\{\frac{i}{\hbar} S_{cl}[x'', x']\right\} . \tag{3.26}$$

So the propagator is determined except for a multiplying factor $F(t'', t')$, which may be determined by some other known properties of the solution. However, for a quadratic Lagrangian, van-Vleck [19] and Pauli [20] had verified that the pre-factor $F(t'', t')$ can be evaluated exactly by using the formula

$$F(t'', t') = \sqrt{\det\left\{\frac{i}{2\pi\hbar}\frac{\partial^2}{\partial x''\partial x'} S_{cl}[x'', x']\right\}} \ , \tag{3.27}$$

so that Eq. (3.26) becomes

$$K(x'', t''; x', t') = \sqrt{\det\left\{\frac{i}{2\pi\hbar}\frac{\partial^2}{\partial x''\partial x'} S_{cl}[x'', x']\right\}} \ \exp\left\{\frac{i}{\hbar} S_{cl}[x'', x']\right\}. \tag{3.28}$$

It is interesting to note that the expression $K \sim \exp[\frac{i}{\hbar} S_{cl}]$ is exact for the case that S is a quadratic form.

3.4 Path Integral of a Harmonic Oscillator

From Eq. (3.28) we can compute the propagator of a harmonic oscillator. For a harmonic oscillator the Lagrangian is

$$L(x, \dot{x}) = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} m \omega^2 x^2 . \quad (3.29)$$

This Lagrangian leads to equation of motion [21]

$$\ddot{x} + \omega^2 x^2 = 0 , \quad (3.30)$$

Subject to the boundary condition

$$x(T) = x_T , \; x(0) = x_o , \quad (3.31)$$

and has the solution

$$x(t) = A \cos \omega t + B \sin \omega t . \quad (3.32)$$

After applying the boundary conditions, Eq. (3.32) becomes

$$x(t) = \frac{1}{\sin \omega T} \{ x_T \sin \omega t + x_o \sin \omega (T - t) \}. \quad (3.33)$$

Using this result, the action for the classical trajectory Eq. (3.33) is given by

$$S_{cl} = \frac{m}{2} \int_0^T dt \left\{ \left(\frac{dx}{dt} \right)^2 - \omega^2 x^2 \right\} = \frac{m\omega}{2 \sin \omega T} \left\{ \left(x_o^2 + x_T^2 \right) \cos \omega T - 2 x_o x_T \right\}. \quad (3.34)$$

The pre-factor associated with the propagator can be evaluated exactly. It is found that

$$F(T) = \sqrt{\frac{m\omega}{2\pi i\hbar \sin \omega T}}\,. \tag{3.35}$$

From Eqs. (3.26), (3.34) and (3.35) we obtain the propagator

$$K(x_T, T; x_o, 0) = \sqrt{\frac{m\omega}{2\pi i\hbar \sin \omega T}}\,\exp\left\{\frac{im\omega}{2\hbar \sin \omega T}\left[\left(x_o^2 + x_T^2\right)\cos \omega T - 2x_o x_T\right]\right\}. \tag{3.36}$$

3.5 Density of States and Energy Spectrum

The fundamental quantity that calculated from a path integral is the time evolution amplitude or propagator of a system $K(\vec{x}'', t''; \vec{x}', t')$. In this section we give a general method of evaluation the density of states and energy spectrum from the propagator [22].

For a system with a time-independent Hamiltonian, the propagator can be written as matrix element of the time evolution operator $U(t'', t') = \exp\left\{-\frac{i}{\hbar}H(t'' - t')\right\}$:

$$K(\vec{x}'', t''; \vec{x}', t') = \left\langle \vec{x}'' \middle| U(t'', t') \middle| \vec{x}' \right\rangle = \left(\left\langle \vec{x}'' \middle| \exp\left\{-\frac{i}{\hbar}Ht''\right\}\right)\left(\exp\left\{\frac{i}{\hbar}Ht'\right\}\middle| \vec{x}' \right\rangle\right),$$

which implies

$$\left\langle \vec{x}'', t'' \right| = \left\langle \vec{x}'' \right| \exp\left\{-\frac{i}{\hbar}Ht''\right\}, \left| \vec{x}', t' \right\rangle = \exp\left\{\frac{i}{\hbar}Ht'\right\}\left| \vec{x}' \right\rangle. \tag{3.37}$$

We now want to use the explicit form of the propagator in order to calculate the energy spectrum of a particle in the potential. To this end we consider the trace of the time-evolution operator:

$$
\begin{aligned}
G(t''-t') &= \mathrm{Tr}\left[\exp\left\{-\frac{i}{\hbar}H(t''-t')\right\}\right] \\
&= \int_{-\infty}^{\infty} d^3x_0 \left\langle \vec{x}_0 \left| \exp\left\{-\frac{i}{\hbar}H(t''-t')\right\} \right| \vec{x}_0 \right\rangle
\end{aligned}
$$

4

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} d^3x_0 \; K(\vec{x}_0, t''; \vec{x}_0, t') \\
&= \int_{-\infty}^{\infty} d^3x_0 \sum_{n=0}^{\infty} \left| \varphi_n(\vec{x}_0) \right|^2 \exp\left\{-\frac{i}{\hbar}E_n(t''-t')\right\} \\
G(T) &= \sum_{n=0}^{\infty} \exp\left\{-\frac{i}{\hbar}E_n T\right\},
\end{aligned}
$$

i.e., where $\varphi_n(\vec{x}_0) = \langle \vec{x}_0 | n \rangle$ is the eigenfunction with eigenvalue n and $T = t'' - t'$, we have found

$$
G(T) = \mathrm{Tr}\left[\exp\left\{-\frac{i}{\hbar}HT\right\}\right] = \sum_{n=0}^{\infty} \exp\left\{-\frac{i}{\hbar}E_n T\right\}. \tag{3.38}
$$

With the Fourier transform

$$
G(E) = \int_{0}^{\infty} dT \exp\left\{\frac{iET}{\hbar}\right\} G(T), \tag{3.39}
$$

we immediately obtain

$$
\begin{aligned}
G(E) &= \sum_{n=0}^{\infty} \int_{0}^{\infty} dT \exp\left\{\frac{i}{\hbar}(E-E_n)T\right\} \\
&= \sum_{n=0}^{\infty} \frac{i\hbar}{E-E_n}.
\end{aligned} \tag{3.40}
$$

As an example of the foregoing formalism, we consider the harmonic oscillator. It is easy to deduce the energy levels E_n by forming the trace

$$G(T) = \int_{-\infty}^{\infty} dx\ K(x, T; x,0)$$
$$= \sqrt{\frac{m\omega}{2\pi i\hbar \sin \omega T}} \int_{-\infty}^{\infty} dx \ \ \exp\left\{\frac{i}{\hbar} \frac{m\omega x^2}{\sin \omega T}(\cos \omega T - 1)\right\}$$
$$= \frac{1}{2i \sin \frac{\omega T}{2}}$$
$$= \sum_{n=0}^{\infty} \exp\left\{-i\left(n + \frac{1}{2}\right)\omega T\right\}.$$

(3.41)

The Fourier transform is

$$G(E) = \sum_{n=0}^{\infty} \frac{i\hbar}{E - \left(n + \frac{1}{2}\right)\hbar\omega}, \tag{3.42}$$

which gives the energy level

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega , \ n = 0,1,2,... \quad . \tag{3.43}$$

From Eq. (3.38), G(T) can be rewritten as an integral

$$G(T) = \int_{0}^{\infty} dE\ \rho(E) \exp\left\{\frac{-iET}{\hbar}\right\}, \tag{3.44}$$

with

$$\rho(E) = \sum_{n} \delta(E - E_n) , \tag{3.45}$$

being the density of states available to the system in an energy interval (E+dE). The density of states may also be written formally as

$$\rho(E) = \frac{1}{\pi\hbar} \mathrm{Re}\left[\int_{0}^{\infty} dT\ \mathrm{Tr}\left[K(\vec{r}_T, T; \vec{r}_0, 0)\right] \exp\left\{\frac{iET}{\hbar}\right\}\right]. \tag{3.46}$$

3.6 Path Integral of an Electron Confined in Two Dimensions with Perpendicular Magnetic Field

We now return to the system of an electron confined in two dimensions under the influence of a transverse magnetic field B, in z direction. Using the symmetric gauge $\vec{A} = B(-y/2, x/2, 0)$, the Lagrangian of the corresponding classical system is

$$L = \frac{m}{2}\left\{\dot{x}^2 + \dot{y}^2 + \omega\left(x\dot{y} - y\dot{x}\right)\right\}, \tag{3.47}$$

where $\omega = eB/mc$ is the cyclotron frequency. Eq. (3.47) leads to the equation of motion [21]

$$\ddot{x} - \omega\dot{y} = 0\,, \tag{3.48}$$

$$\ddot{y} + \omega\dot{x} = 0\,. \tag{3.49}$$

Subject to the boundary condition

$$x(0) = x_0\,,\, x(T) = x_T\,,\, y(0) = y_0\,,\, y(T) = y_T\,. \tag{3.50}$$

In order to solve the classical motion, we substitute Eq. (3.49) into Eq. (3.48), then we have

$$\frac{d^3x}{dt^3} + \omega^2\frac{dx}{dt} = 0\,,$$

which has the solution

$$x(t) = A\sin\omega t + B\cos\omega t + C\,. \tag{3.51}$$

Using Eqs. (3.51) and (3.48), we obtain

$$y(t) = A\cos\omega t - B\sin\omega t + D \tag{3.52}$$

where A,B,C and D are arbitrary constants. After applying boundary condition, Eqs. (3.51) and (3.52) becomes

$$\begin{aligned} x(t) = x_0 &+ \frac{(x_T - x_0)}{4\sin^2\frac{\omega T}{2}}\left[\sin\omega T\sin\omega t + (\cos\omega T - 1)(\cos\omega t - 1)\right] \\ &+ \frac{(y_T - y_0)}{4\sin^2\frac{\omega T}{2}}\left[(\cos\omega T - 1)\sin\omega t + \sin\omega T(\cos\omega t - 1)\right], \end{aligned} \tag{3.53}$$

$$\begin{aligned} y(t) = y_0 &+ \frac{(x_T - x_0)}{4\sin^2\frac{\omega T}{2}}\left[\sin\omega T(\cos\omega t - 1) - \sin\omega t(\cos\omega T - 1)\right] \\ &+ \frac{(y_T - y_0)}{4\sin^2\frac{\omega T}{2}}\left[(\cos\omega T - 1)(\cos\omega t - 1) + \sin\omega T\sin\omega t\right]. \end{aligned} \tag{3.54}$$

Using Eq. (3.28) we obtain the propagator

$$\begin{aligned} K(\vec{r}_T, T; \vec{r}_0, 0) = &\frac{m}{2\pi i\hbar T}\left(\frac{\omega T/2}{\sin(\omega T/2)}\right) \\ &\exp\left\{ \frac{im}{2\hbar}\left(\frac{\omega}{2}\cot\frac{\omega T}{2}((x_T - x_0)^2 + (y_T - y_0)^2 + \omega(x_0 y_T - x_T y_0) \right) \right\}. \end{aligned} \tag{3.55}$$

Because $K(\vec{r}_T, T; \vec{r}_0, 0)$ obtained in Eq. (3.55) is translation invariant, having the property

$$K(\vec{r}_T, T; \vec{r}_0, 0) = K(\vec{r}_T - \vec{r}_0, T; 0, 0), \tag{3.56}$$

so that for finding the density of states, the end point $\vec{r}_T$ and initial point $\vec{r}_0$ must be the same. It therefore follows that [23]

$$\rho(E) = \frac{A}{\pi\hbar}\mathrm{Re}\left[\int_0^T dT\, K(0,0,T)\exp\left\{\frac{iET}{\hbar}\right\}\right], \tag{3.57}$$

where A is area of the system.

Now applying Eq. (3.57) to Eq. (3.55), one finds the density of states of an electron,

$$\rho(E) = \frac{m\omega A}{2\pi\hbar}\sum_{n=0}^{\infty}\delta(E-(n+\frac{1}{2})\hbar\omega). \tag{3.58}$$

Eq. (3.58) consists of discrete energy level at each Landau levels $(n+\frac{1}{2})\hbar\omega$ separated by cyclotron energy $\hbar\omega$, and degeneracy per Landau level per unit area is $\frac{m\omega}{2\pi\hbar}$ which is equivalent to Eqs. (2.7) and (2.8).

3.7 Path Integral of an Electron Confined in Two-Dimensional Harmonic Potential with Perpendicular Magnetic Field

From Eq. (3.47), in the presence of an isotropic quadratic potential, $V_\lambda(x,y) = \frac{m\lambda^2}{2}(x^2+y^2)$, where λ is a parameter representing this potential. The Lagrangian in Eq. (3.47) becomes

$$L = \frac{m}{2}\left\{\dot{x}^2+\dot{y}^2+\omega\left(x\dot{y}-y\dot{x}\right)-\lambda^2(x^2+y^2)\right\} \tag{3.59}$$

the equations of motion are in the form

$$\ddot{x}-\omega\dot{y}+\lambda^2 x = 0\,, \tag{3.60}$$

$$\ddot{y}+\omega\dot{x}+\lambda^2 y = 0\,. \tag{3.61}$$

In order to solve these equations, Papadopoulos [24] introduced a 2x2 matrix

$$\bar{J} = \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix}. \tag{3.62}$$

Using this matrix, Eqs. (3.60) and (3.61) can be written as

$$\ddot{\bar{r}} + \omega\bar{J}\dot{\bar{r}} + \lambda^2\bar{r} = 0 \ , \tag{3.63}$$

where the matrix $\bar{r} = \begin{vmatrix} x \\ y \end{vmatrix}$. Writing the solution of Eq. (3.63) in the form $\vec{r} \sim \exp(\bar{\gamma}t)$, the auxiliary equation associated with Eq. (3.63) is

$$\bar{\gamma}^2 + \omega\bar{J}\bar{\gamma} + \lambda^2\bar{I} = 0 \ . \tag{3.64}$$

where the unit matrix $\bar{I} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$. This equation is satisfied by the matrices

$$\bar{\gamma}_1 = -\frac{\omega}{2}\bar{J} + \omega'\bar{J} \ , \tag{3.65}$$

$$\bar{\gamma}_2 = -\frac{\omega}{2}\bar{J} - \omega'\bar{J} \ , \tag{3.66}$$

where $\omega' = \sqrt{\frac{\omega^2}{4} + \lambda^2}$. Thus, the solution of Eq. (3.63) is

$$\bar{r}(t) = \exp\{\frac{-\omega\bar{J}t}{2}\}\left(\exp\{\omega'\bar{J}t\}\bar{A} + \exp\{-\omega'\bar{J}t\}\bar{B}\right), \tag{3.67}$$

where $\bar{A}$ and $\bar{B}$ are arbitrary constants. Using the property $\exp\{\pm\bar{J}\beta\} = \bar{I}\cos\beta \pm \bar{J}\sin\beta$, and applying boundary condition $\bar{r}(0) = \bar{r}_0 = \begin{vmatrix} x_0 \\ y_0 \end{vmatrix}$ and $\bar{r}(T) = \bar{r}_T = \begin{vmatrix} x_T \\ y_T \end{vmatrix}$ Eq. (3.67) becomes

$$\bar{r}(t) = \frac{\exp\{\frac{-\omega\bar{J}t}{2}\}}{\sin\omega'T}\left(\exp\{\frac{\omega'\bar{J}T}{2}\}\sin\omega't\,\bar{r}_T + \sin\omega'(T-t)\,\bar{r}_0\right), \tag{3.68}$$

Using this result, the classical action is given by

$$S_{cl} = \frac{m\omega'}{2\sin\omega' T}\Big((x_0^2 + x_T^2 + y_0^2 + y_T^2)\cos\omega' T - 2(x_0 x_T - y_0 y_T)\cos\frac{\omega T}{2} + 2(x_0 y_T - x_T y_0)\sin\frac{\omega T}{2} \Big). \tag{3.69}$$

Substituting Eq. (3.69) into Eq. (3.28), we obtain the exact propagator

$$K(\vec{r}_T, T; \vec{r}_0, 0) = \frac{m}{2\pi i\hbar \sin\omega' T} \exp\Big\{ \frac{im\omega'}{2\hbar\sin\omega' T}\Big((x_0^2 + x_T^2 + y_0^2 + y_T^2)\cos\omega' T - 2(x_0 x_T - y_0 y_T)\cos\frac{\omega T}{2} + 2(x_0 y_T - x_T y_0)\sin\frac{\omega T}{2} \Big)\Big\}. \tag{3.70}$$

After taking the trace Eq. (3.70), we obtain

$$G(T) = \left(\frac{1}{2i\sin(\omega' + \frac{\omega}{2})T}\right)\left(\frac{1}{2i\sin(\omega' - \frac{\omega}{2})T}\right) = \sum_{p=0}^{\infty}\exp\left\{-i(p+\frac{1}{2})(\omega'+\frac{\omega}{2})T\right\}\sum_{q=0}^{\infty}\exp\left\{-i(q+\frac{1}{2})(\omega'-\frac{\omega}{2})T\right\}. \tag{3.71}$$

In comparison with Eq. (3.41), it is found that this is a product of two harmonic oscillators with renormalized frequency $\omega' + \frac{\omega}{2}$ and $\omega' - \frac{\omega}{2}$. Substituting Eq. (3.71) into Eq. (3.46), then the density of state is given by

$$\rho(E) = \sum_{p,q=0}^{\infty}\delta(E - (p+\frac{1}{2})\hbar(\omega'+\frac{\omega}{2}) - (q-\frac{1}{2})\hbar(\omega'-\frac{\omega}{2})). \tag{3.72}$$

Eq. (3.72) indicates the discrete level at each index p and q.

The matrix method of Papadopoulos has been used by Kagalovsky [25] to evaluate the exact propagator of a two-dimensional electron system in the model of Fertig and Halperin. Since $V_{sp}(x, y)$ of Fertig and Halperin is anisotropic which matrix method is not applicable, Kagalovsky assumed the saddle point potential in the isotropic form $V_{\lambda}(x, y) = \frac{m\lambda^2}{2}(x^2 - y^2)$. By using symmetric gauge $\vec{A} = B(-y/2, x/2, 0)$ for magnetic field in the z direction, the Lagrangian of the corresponding classical system is presented as

$$L = \frac{m}{2}\left(\dot{\bar{r}}^2 - \lambda^2 \sigma_3 \bar{r}^2 + \omega \dot{\bar{r}} \bar{J} \bar{r}\right), \tag{3.73}$$

where $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is a Pauli matrix. The action is calculated along the trajectory evolving via the equation of motion

$$\ddot{\bar{r}} + \omega \bar{J} \dot{\bar{r}} - \lambda^2 \sigma_3 \bar{r} = 0 . \tag{3.74}$$

The solution of Eq. (3.74) may be written in the form $\bar{r} \sim \exp(\bar{R} t)$, where the matrix $\bar{R}$ has the structure

$$\bar{R} = \begin{pmatrix} 0 & \beta \\ \alpha & 0 \end{pmatrix}. \tag{3.75}$$

After this substitution, Eq. (3.74) becomes

$$\bar{R}^2 + \omega \bar{J} \bar{R} - \lambda^2 \sigma_3 = 0 , \tag{3.76}$$

which has two solutions, $\bar{R}_1 = \begin{pmatrix} 0 & -\omega_2 \\ \omega_1 & 0 \end{pmatrix}$ and $\bar{R}_2 = -\begin{pmatrix} 0 & \omega_4 \\ \omega_3 & 0 \end{pmatrix}$ where $\omega_{1,3} = \pm(\omega^2 - 2\lambda^2 \pm \sqrt{\omega^4 + 4\lambda^4})/2\omega$, $\omega_{2,4} = \pm 2\omega\lambda^2(\omega^2 + 2\lambda^2 \pm \sqrt{\omega^4 + 4\lambda^4})$. The exponentials can be presented as matrices

$$\exp(\bar{R}_1 t) = \begin{pmatrix} \cos\omega' t & \frac{\omega_1}{\omega'}\sin\omega' t \\ -\frac{\omega_2}{\omega'}\sin\omega' t & \cos\omega' t \end{pmatrix}, \tag{3.77}$$

$$\exp(\bar{R}_2 t) = \begin{pmatrix} \cosh\omega'' t & \frac{\omega_3}{\omega''}\sinh\omega'' t \\ -\frac{\omega_4}{\omega''}\sinh\omega'' t & \cosh\omega'' t \end{pmatrix}, \tag{3.78}$$

where $\omega' = \sqrt{\omega_1\omega_2}$ and $\omega'' = \sqrt{\omega_3\omega_4}$.

Therefore the general solution of Eq. (3.74) is $\bar{r}(t) = \exp(\bar{R}_1 t)\,\bar{a} + \exp(\bar{R}_2 t)\,\bar{b}$. Matrices $\bar{a}$ and $\bar{b}$ are determined from the boundary conditions $\bar{r}(T) = \bar{r}_T$, $\bar{r}(0) = \bar{r}_0$. Since the Lagrangian in this case is quadratic the classical action can be evaluated from

$$S_{cl} = \frac{m}{2}(\bar{r}_T\dot{\bar{r}}_T - \bar{r}_0\dot{\bar{r}}_0)\,. \tag{3.79}$$

After applying boundary condition to $\bar{r}(t)$ then substituting into Eq. (3.79), the classical action is given by the following formula:

$$\begin{aligned} S_{cl} = \frac{m}{2\Delta}\{&(x_T^2 + x_0^2)[(\frac{\omega_2\omega_3}{\omega'} + \omega')\sin\omega' T\cosh\omega'' T \\ &\quad -(\frac{\omega_1\omega_4}{\omega''} + \omega'')\cos\omega' T\sinh\omega'' T] \\ &+ (2x_T x_0)[-(\frac{\omega_2\omega_3}{\omega'} + \omega')\sin\omega' T + (\frac{\omega_1\omega_4}{\omega''} + \omega'')\sinh\omega'' T] \\ &+ (y_T^2 + y_0^2)[(\omega' - \frac{\omega_1\omega_4}{\omega'})\sin\omega' T\cosh\omega'' T \\ &\quad -(\frac{\omega_2\omega_3}{\omega''} - \omega'')\cos\omega' T\sinh\omega'' T] \\ &+ (2y_T y_0)[(\frac{\omega_1\omega_4}{\omega'} - \omega')\sin\omega' T + (\omega'' - \frac{\omega_2\omega_3}{\omega''})\sinh\omega'' T] \\ &+ (x_T y_0 - y_T x_0)(\omega_1 + \omega_2 + \omega_3 - \omega_4)(\cos\omega' T - \cosh\omega'' T) \\ &+ (x_T y_T - y_o x_0)[(\omega_1 - \omega_2 - \omega_3 - \omega_4)(1 - \cos\omega' T\cosh\omega'' T) \\ &+ \frac{(\omega_3 - \omega_4)}{\omega''}\omega'\sin\omega' T\sinh\omega'' T + \frac{(\omega_1 - \omega_2)}{\omega'}\omega'']\}\,, \end{aligned} \tag{3.80}$$

where $\Delta = 2 - 2\cos\omega' T\cosh\omega'' T + \dfrac{\omega_2\omega_3 - \omega_1\omega_4}{\omega'\omega''}\sin\omega' T\sinh\omega'' T$. Using Van Vleck-Pauli formula to evaluate the pre-factor, the exact propagator is given by

$$
\begin{aligned}
K(\bar{r}_T, T; \bar{r}_0, 0) = \frac{m}{2\pi i\hbar}\{&4\sin^2\omega' T[\omega_1\omega_4 - \omega_2\omega_3 + \omega''^2 - \omega'^2] \\
&+ 4\sin\omega' T\sinh\omega'' T[2\omega'\omega'' + (\omega_1\omega_4 - \omega_2\omega_3)(\frac{\omega''}{\omega'} - \frac{\omega'}{\omega''}) + \frac{(\omega_1^2\omega_4^2 + \omega_2^2\omega_3^2)}{\omega'\omega''}] \\
&+ 4\sinh^2\omega'' T[\omega''^2 + \omega_1\omega_4 - \omega_2\omega_3 - \omega'^2] \\
&+ (\cosh\omega'' T - \cos\omega' T)^2[\omega_1 + \omega_2 + \omega_3 - \omega_4]^2\}^{1/2}\exp\{\frac{i}{\hbar}S_{cl}\}
\end{aligned}
\tag{3.81}
$$

This is the exact propagator of Kagalovsky, which can be used to consider the behavior of a tunneling electron. Kagalovsky pointed out that the propagator might be transformed to new coordinates is product of two independent propagators, K_1 for the inverted harmonic and K_2 for the harmonic oscillator. The relation between the propagator in “old” coordinate Eq. (3.81) and in coordinates from the work of Fertig and Halperin is given by the following integral:

$$
\begin{aligned}
K(\bar{r}_T, T; \bar{r}_0, 0) = \int\int\int\int & dX' ds' dX'' ds''\, K_1(X'', T; X', 0) K_2(s'', T; s', 0) \\
& \delta(x_T - \sqrt{\frac{2}{m\omega}}(\alpha_1 X'' - \beta_2 s''))\,\delta(x_o - \sqrt{\frac{2}{m\omega}}(\alpha_1 X' - \beta_2 s')) \\
& \exp\{\frac{i}{\hbar}\sqrt{\frac{m\omega}{2}}(\beta_3 X'' + \alpha_4 s'')\, y_T\}\exp\{\frac{i}{\hbar}\sqrt{\frac{m\omega}{2}}(\beta_3 X' + \alpha_4 s')\, y_0\},
\end{aligned}
\tag{3.82}
$$

where α_i and β_i are coefficient dependent on m, ω, and λ. In this thesis, we will show that the exact propagator in the case of anisotropic saddle point potential $V_{sp}(x,y)$ can be obtained. Since the matrix method cannot use in the case of anisotropy, we will develop a new method to solve the classical motion. The detailed calculation is presented in the next chapter.

CHAPTER IV

EXACT PROPAGATOR FOR TWO-DIMENSIONAL ELECTRON IN AN ANISOTROPIC QUADRATIC SADDLE POINT POTENTIAL IN A TRANSVERSE MAGNETIC FIELD

In this chapter, we derive the exact propagator for a two-dimensional system in the presence of transverse magnetic field and anisotropic quadratic saddle point potential. As presented in Chapter 3, when the potential is in the form of an isotropic quadratic potential the propagator can be evaluated exactly by using two-dimensional matrix of Papadopoulos. However, the matrix method is not applicable in our case due to the anisotropy of saddle point potential. In stead we introduce the complex variable path to decouple the two classical equations of motion. The classical solutions are obtained and used to calculate the classical action. Since the action is quadratic one can obtain the pre-factor by employing the Van Vleck-Pauli method which leads to the exact propagator. The propagator can be used to obtain the density of states and energy spectrum.

4.1 Classical Action and Exact Propagator

We now consider the problem of an electron confined in a two-dimensional quadratic saddle point potential $V_{sp}(x,y)$ = V0 + (m/2) $(\Omega_x^2 x^2 - \Omega_y^2 y^2)$ with a magnetic field B has the direction along the z-axis. The Lagrangian of the corresponding classical system is presented as

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) + \frac{m\omega}{2}(x\dot{y} - y\dot{x}) - \frac{m}{2}(\Omega_x^2 x^2 - \Omega_y^2 y^2) - V_o\,, \qquad (4.1)$$

where ω=eB/mc is the cyclotron frequency, Ω_x and Ω_y are parameters representing the harmonic and inverted harmonic of saddle point potential in the x and y directions, respectively, and V_0 is the top of the saddle point potential. The required propagator can be written in the path integral form,

$$K(\vec{r}_T, T; \vec{r}_0, 0) = \int D[\vec{r}(t)] \exp\left\{\frac{i}{\hbar} S[\vec{r}_T, \vec{r}_0]\right\}, \tag{4.2}$$

where $\vec{r} = (x,y)$, $S[\vec{r}_T, \vec{r}_0] = \int_0^T dt\, L(\vec{r}, \dot{\vec{r}}, t)$ is the action and $D[\vec{r}(t)]$ denote measure of the path integral to be carried out with the boundary conditions $\vec{r}(0) = \vec{r}_0$ and $\vec{r}(T) = \vec{r}_T$. The propagator in Eq. (4.1) can be rewritten as

$$K(\vec{r}_T, T; \vec{r}_0, 0) = \iint D[x(t)]\, D[y(t)] \exp\Big\{ \frac{im}{2\hbar} \int_0^T dt \Big((\dot{x}^2 + \dot{y}^2) + \omega(x\dot{y} - y\dot{x}) - (\Omega_x^2 x^2 - \Omega_y^2 y^2) \Big) - V_0 T \Big\}. \tag{4.3}$$

Since the Lagrangian given by Eq. (4.1) is quadratic the path integral can be evaluated exactly as

$$K(\vec{r}_T, T; \vec{r}_0, 0) = F(T) \exp\left\{\frac{i}{\hbar} S_{cl}\right\}, \tag{4.4}$$

where F(T) is a pre-exponential factor which can be calculated by using Eq. (3.27).

We now wish to calculate the classical action S_{cl} corresponding to the Lagrangian in Eq. (4.1). The constituent equations of motion for this Lagrangian are

$$\ddot{x} - \omega\dot{y} + \Omega_x^2 x = 0\,, \tag{4.5}$$

$$\ddot{y} + \omega\dot{x} - \Omega_y^2 y = 0\,, \tag{4.6}$$

which are subject to the boundary condition

$$x(T) = x_T\,,\ x(0) = x_o\,,\ y(T) = y_T\,,\ y(0) = y_o\,. \tag{4.7}$$

We can express the equation of motion Eqs. (4.5) and (4.6) in the form

$$\ddot{\tilde{z}} + i\omega\dot{\tilde{z}} + \tilde{\Omega}^2\tilde{z} = -\Omega^2 z\,, \tag{4.8}$$

$$\ddot{z} - i\omega\dot{z} + \tilde{\Omega}^2 z = -\Omega^2\tilde{z}\,, \tag{4.9}$$

where $\Omega^2 = (\Omega_x^2 + \Omega_y^2)/2$, $\tilde{\Omega}^2 = (\Omega_x^2 - \Omega_y^2)/2$ and the variables z and $\tilde{z}$ are given by

$$z = x - iy\,, \tag{4.10}$$

$$\tilde{z} = x + iy\,. \tag{4.11}$$

Eq. (4.9) can be rewritten as

$$\tilde{z} = -\frac{1}{\Omega^2}\left(\ddot{z} - i\omega\dot{z} + \tilde{\Omega}^2 z\right). \tag{4.12}$$

Substituting Eq. (4.12) into Eq. (4.8), we obtain

$$\frac{d^4 z}{dt^4} + \left(\omega^2 + 2\tilde{\Omega}^2\right)\frac{d^2 z}{dt^2} + (\tilde{\Omega}^4 - \Omega^4)\,z = 0\ . \tag{4.13}$$

The auxiliary equation associated with Eq. (4.13) is

$$\beta^4 + \left(\omega^2 + 2\tilde{\Omega}^2\right)\beta^2 + (\tilde{\Omega}^4 - \Omega^4) = 0\,. \tag{4.14}$$

Solving Eq. (4.14), we obtain four roots

$$\begin{aligned} \beta_1 &= \omega_1, \\ \beta_2 &= -\omega_1, \\ \beta_3 &= i\omega_2, \\ \beta_4 &= -i\omega_2, \end{aligned} \tag{4.15}$$

where

$$\omega_1 = \frac{1}{\sqrt{2}}((\omega^4 + 4\omega^2\tilde{\Omega}^2 + 4\Omega^4)^{1/2} - \omega^2 - 2\tilde{\Omega}^2)^{1/2}\,, \tag{4.16}$$

$$\omega_2 = \frac{1}{\sqrt{2}}((\omega^4 + 4\omega^2\tilde{\Omega}^2 + 4\Omega^4)^{1/2} + \omega^2 + 2\tilde{\Omega}^2)^{1/2}\,. \qquad (4.17)$$

The solution of Eqs. (4.8) and (4.9) are

$$z(t) = C_1 \exp\{\omega_1 t\} + C_2 \exp\{-\omega_1 t\} + C_3 \exp\{i\omega_2 t\} + C_4 \exp\{-i\omega_2 t\}, \qquad (4.18)$$

$$\tilde{z}(t) = \Omega_1 C_1 \exp\{\omega_1 t\} + \frac{C_2}{\Omega_1}\exp\{-\omega_1 t\} + \Omega_2 C_3 \exp\{i\omega_2 t\} + \frac{C_4}{\Omega_2}\exp\{-i\omega_2 t\}, \qquad (4.19)$$

where $\Omega_1 = -\frac{1}{\Omega^2}(\omega_1(\omega_1 - i\omega) + \tilde{\Omega}^2)$, $\Omega_2 = -\frac{1}{\Omega^2}(\omega_2(\omega - \omega_2) + \tilde{\Omega}^2)$, and C_1 , C_2 , C_3 , C_4 are arbitrary constants.

To obtain the complete solutions, we assume $\Omega_1 = \exp\{i\theta\}$ and $\Omega_2 = \exp\{\gamma\}$, Eq. (4.19) becomes

$$\tilde{z}(t) = C_1 \exp\{\omega_1 t + i\theta\} + C_2 \exp\{-\omega_1 t - i\theta\} + C_3 \exp\{i\omega_2 t + \gamma\} + C_4 \exp\{-i\omega_2 t - \gamma\}. \qquad (4.20)$$

From Eqs. (4.18), (4.20), using relation $x(t) = \frac{\tilde{z} + z}{2}, y(t) = \frac{\tilde{z} - z}{2i}$ and arranging the arbitrary constant, $C_1 e^{\frac{i\theta}{2}} + C_2 e^{-\frac{i\theta}{2}} = A$, $C_1 e^{\frac{i\theta}{2}} - C_2 e^{-\frac{i\theta}{2}} = B$, $(C_3 e^{\frac{\gamma}{2}} + C_4 e^{-\frac{\gamma}{2}})/2\sqrt{\Omega_2} = C$, $i(C_3 e^{\frac{\gamma}{2}} - C_4 e^{-\frac{\gamma}{2}})/2\sqrt{\Omega_2} = D$, we get

$$x(t) = \cos\frac{\theta}{2}\left(A\cosh\,\omega_1 t + B\sinh\,\omega_1 t\right) + \left(\Omega_2 + 1\right)\left(C\cos\,\omega_2 t - D\sin\,\omega_2 t\right). \qquad (4.21)$$

Similarly,

$$y(t) = \sin\frac{\theta}{2}\left(A\cosh\,\omega_1 t + B\sinh\,\omega_1 t\right) + \left(\Omega_2 - 1\right)\left(C\cos\,\omega_2 t - D\sin\,\omega_2 t\right), \qquad (4.22)$$

where $\theta = \arctan(-\omega\omega_1/(\omega_1^2 + \tilde{\Omega}^2))$. After applying the boundary condition, Eqs. (4.21) and (4.22) becomes

$$x(t) = x_0 F_1(t) + y_0 F_2(t) + x_T F_3(t) + y_T F_4(t)\,, \qquad (4.23)$$

$$y(t) = x_0 F_5(t) + y_0 F_6(t) + x_T F_7(t) + y_T F_8(t)\,, \tag{4.24}$$

where

$$\begin{aligned}
F_1(t) = \frac{1}{\Delta}\Big\{ &\left(\Omega_2^2 - 1\right)[\cos\,\omega_2 T \cosh\,\omega_1 (T-t) + \cos\,\omega_2 (T-t) \cosh\,\omega_1 T \\
&- \cosh\,\omega_1 t - \cos\,\omega_2 t] \\
&+ \left(\Omega_2 + 1\right)^2 \tan\frac{\theta}{2} \sin\omega_2 (T-t) \sinh\omega_1 T \\
&- \left(\Omega_2 - 1\right)^2 \cot\frac{\theta}{2} \sin\omega_2 T \sinh\omega_1 (T-t) \Big\},
\end{aligned}$$

$$\begin{aligned}
F_2(t) = \frac{1}{\Delta}\Big\{ &\left(\Omega_2 + 1\right)^2 [\sin\,\omega_2 t - \sin\omega_2 T \cosh\,\omega_1 (T-t) \\
&+ \sin\omega_2 (T-t) \cosh\,\omega_1 T] \\
&+ \left(\Omega_2^2 - 1\right)\cot\frac{\theta}{2}[\cos\,\omega_2 (T-t) \sinh\,\omega_1 T - \sinh\,\omega_1 t \\
&- \cos\,\omega_2 T \sinh\,\omega_1 (T-t)] \Big\}
\end{aligned}$$

$$\begin{aligned}
F_3(t) = \frac{1}{\Delta}\Big\{ &\left(\Omega_2^2 - 1\right)[\cos\,\omega_2 t \cosh\,\omega_1 T + \cos\,\omega_2 T \cosh\,\omega_1 t \\
&- \cosh\,\omega_1 (T-t) - \cos\,\omega_2 (T-t)] \\
&+ \left(\Omega_2 + 1\right)^2 \tan\frac{\theta}{2} \sin\omega_2 t \sinh\omega_1 T \\
&- \left(\Omega_2 - 1\right)^2 \cot\frac{\theta}{2} \sin\omega_2 T \sinh\omega_1 t \Big\},
\end{aligned}$$

$$\begin{aligned}
F_4(t) = \frac{1}{\Delta}\Big\{ &\left(\Omega_2 + 1\right)^2 [\sin\,\omega_2 (T-t) + \sin\omega_2 t \cosh\,\omega_1 T \\
&- \sin\omega_2 T \cosh\,\omega_1 t] \\
&+ \left(\Omega_2^2 - 1\right)\cot\frac{\theta}{2}[\cos\,\omega_2 t \sinh\,\omega_1 T - \sinh\,\omega_1 (T-t) \\
&- \cos\,\omega_2 T \sinh\,\omega_1 t] \Big\},
\end{aligned}$$

$$\begin{aligned}
F_5(t) = \frac{1}{\Delta}\Big\{ &\left(\Omega_2^2 - 1\right)\tan\frac{\theta}{2}[\cos\,\omega_2 (T-t) \sinh\,\omega_1 t - \cos\,\omega_2 T \sinh\,\omega_1 (T-t) \\
&- \sinh\,\omega_1 t] \\
&+ \left(\Omega_2 - 1\right)^2 [\sin\omega_2 T \cosh\,\omega_1 (T-t) - \sin\omega_2 (T-t) \cosh\,\omega_1 T \\
&- \sin\omega_2 t] \Big\},
\end{aligned}$$

$$\begin{aligned}
F_6(t) = \frac{1}{\Delta}\Big\{ &\left(\Omega_2^2 - 1\right)[\cos\,\omega_2 T \cosh\,\omega_1 (T-t) + \cos\,\omega_2 (T-t) \cosh\,\omega_1 T \\
&- \cosh\,\omega_1 t - \cos\,\omega_2 t] \\
&+ \left(\Omega_2 + 1\right)^2 \tan\frac{\theta}{2} \sin\omega_2 T \sinh\omega_1 (T-t) \\
&- \left(\Omega_2 - 1\right)^2 \cot\frac{\theta}{2} \sin\omega_2 (T-t) \sinh\omega_1 T \Big\},
\end{aligned}$$

$$F_7(t) = \frac{1}{\Delta}\Big\{ (\Omega_2 - 1)^2 [\sin\omega_2(T-t) + \sin\omega_2 t \cosh\omega_1 T - \sin\omega_2 T \cosh\omega_1 t] + (\Omega_2^2 - 1)\tan\frac{\theta}{2}[\cos\omega_2 T \sinh\omega_1 t + \sinh\omega_1(T-t) - \cos\omega_2 t \sinh\omega_1 T] \Big\},$$

$$F_8(t) = \frac{1}{\Delta}\Big\{ (\Omega_2^2 - 1)[\cos\omega_2 T \cosh\omega_1 t + \cos\omega_2 t \cosh\omega_1 T - \cosh\omega_1(T-t) - \cos\omega_2(T-t)] + (\Omega_2 + 1)^2 \tan\frac{\theta}{2}\sin\omega_2 T \sinh\omega_1 t - (\Omega_2 - 1)^2 \cot\frac{\theta}{2}\sin\omega_2 t \sinh\omega_1 T \Big\},$$

and

$$\Delta = 2(1-\Omega_2^2)(1-\cos\omega_2 T\cosh\omega_1 T) - \frac{2(\cos\theta - 2\Omega_2 + \Omega_2^2\cos\theta)}{\sin\theta}\sin\omega_2 T \sinh\omega_1 T. \quad (4.25)$$

We now focus our attention on the classical action

$$S_{cl} = \frac{m}{2}\int_0^T dt \left((\dot{x}^2 + \dot{y}^2) + \omega(x\dot{y} - y\dot{x}) - (\Omega_x^2 x^2 - \Omega_y^2 y^2) \right) - V_0 T. \quad (4.26)$$

Integrating by parts the first term of Eq. (4.26) and applying the equation of motion Eqs. (4.5) and (4.6), we obtain

$$S_{cl} = \frac{m}{2}\left(x(T)\dot{x}(T) - x(0)\dot{x}(0) + y(T)\dot{y}(T) - y(0)\dot{y}(0)\right) - V_0 T. \quad (4.27)$$

The complete solution for the action is

$$S_{cl} = \frac{m}{2\Delta}\Big\{$$

$$[\omega_1(\Omega_2-1)^2\cot\frac{\theta}{2}+\omega_2(\Omega_2^2-1)]\sin\omega_2 T[(x_o^2+x_T^2)\cosh\omega_1 T-2x_o x_T]$$

$$-[\omega_1(\Omega_2^2-1)+\omega_2(\Omega_2+1)^2\tan\frac{\theta}{2}]\sinh\omega_1 T[(x_o^2+x_T^2)\cos\omega_2 T-2x_o x_T]$$

$$+[\omega_2(\Omega_2^2-1)-\omega_1(\Omega_2+1)^2\tan\frac{\theta}{2}]\sin\omega_2 T[(y_o^2+y_T^2)\cosh\omega_1 T-2y_o y_T]$$

$$+[\omega_2(\Omega_2-1)^2\cot\frac{\theta}{2}-\omega_1(\Omega_2^2-1)]\sinh\omega_1 T[(y_o^2+y_T^2)\cos\omega_2 T-2y_o y_T]$$

$$+\frac{2}{\sin\theta}[(\cos\omega_2 T-\cosh\omega_1 T)(\omega_1\cos\theta(\Omega_2^2-1)$$

$$+\omega_2\sin\theta(\Omega_2^2+1))](x_o y_T-x_T y_o)$$

$$+\frac{2}{\sin\theta}[(2\omega_2\Omega_2\sin\theta-\omega_1(\Omega_2^2-1))\cosh\omega_1 T\cos\omega_2 T$$

$$-(2\omega_1\Omega_2\sin\theta+\omega_2(\Omega_2^2-1))\sinh\omega_1 T\sin\omega_2 T$$

$$-(2\omega_2\Omega_2\sin\theta-\omega_1(\Omega_2^2-1))](x_o y_o-x_T y_T)\Big\}$$

$$-V_0 T. \qquad (4.28)$$

The pre-factor associated with the propagator can be evaluated exactly. It is found that (see appendix)

$$F(T)=\frac{m}{2\pi i\hbar\sqrt{\Delta\sin\theta}}\Big\{(\omega_1^2-\omega_2^2)(\Omega_2^2-1)\sin\theta-2\omega_1\omega_2(\cos\theta-2\Omega_2+\Omega_2^2\cos\theta)\Big\}^{\frac{1}{2}}. \qquad (4.29)$$

From Eqs. (4.4), (4.28) and (4.29) we obtain the propagator

$$K(\vec{r}_T,\vec{r}_o;T)=\frac{m}{2\pi i\hbar\sqrt{\Delta\sin\theta}}\Big\{(\omega_1^2-\omega_2^2)(\Omega_2^2-1)\sin\theta$$

$$-2\omega_1\omega_2(\cos\theta-2\Omega_2+\Omega_2^2\cos\theta)\Big\}^{\frac{1}{2}}\exp\left\{-\frac{iV_0 T}{\hbar}\right\}\exp\frac{im}{2\Delta\hbar}\Big\{$$

$$[\omega_1(\Omega_2-1)^2\cot\frac{\theta}{2}+\omega_2(\Omega_2^2-1)]\sin\omega_2 T[(x_0^2+x_T^2)\cosh\omega_1 T-2x_0 x_T]$$

$$-[\omega_1(\Omega_2^2-1)+\omega_2(\Omega_2+1)^2\tan\frac{\theta}{2}]\sinh\omega_1 T[(x_0^2+x_T^2)\cos\omega_2 T-2x_0 x_T]$$

$$
\begin{aligned}
&+[\omega_2(\Omega_2^2-1)-\omega_1(\Omega_2+1)^2\tan\frac{\theta}{2}]\sin\omega_2 T[(y_o^2+y_T^2)\cosh\omega_1 T-2y_o y_T]\\
&+[\omega_2(\Omega_2-1)^2\cot\frac{\theta}{2}-\omega_1(\Omega_2^2-1)]\sinh\omega_1 T[(y_o^2+y_T^2)\cos\omega_2 T-2y_o y_T]\\
&+\frac{2}{\sin\theta}[(\cos\ \omega_2 T-\cosh\ \omega_1 T)(\omega_1\cos\ \theta(\Omega_2^2-1)\\
&\qquad\qquad+\omega_2\sin\theta(\Omega_2^2+1))](x_o y_T-x_T y_o)\\
&+\frac{2}{\sin\theta}[(2\omega_2\Omega_2\sin\theta-\omega_1(\Omega_2^2-1))\cosh\ \omega_1 T\cos\ \omega_2 T\\
&-(2\omega_1\Omega_2\sin\theta+\omega_2(\Omega_2^2-1))\sinh\ \omega_1 T\sin\omega_2 T\\
&-(2\omega_2\Omega_2\sin\theta-\omega_1(\Omega_2^2-1))](x_o y_o-x_T y_T)\ \}.
\end{aligned}
\tag{4.30}
$$

4.2 Two Limiting Cases

Eq. (4.30) is the exact propagator of an electron moving in two dimensions under the influence of transverse magnetic field and a saddle point potential. To check the validity of our result, we consider the two limiting cases:

a) When the saddle point potential approaches zero, the system of interest corresponds to the $V_0 \to 0$, $\Omega_x^2 \to 0$ and $\Omega_y^2 \to 0$ case. We first consider the pre-factor of Eq. (4.30). In this case

$$
\begin{aligned}
\lim_{\Omega_x^2,\Omega_y^1\to 0} F(T) &= \lim_{\Omega_x^2,\Omega_y^2\to 0}\frac{m}{2\pi i\hbar\sqrt{\Delta\sin\theta}}\{(\omega_1^2-\omega_2^2)(\Omega_2^2-1)\sin\theta\\
&\qquad -2\omega_1\omega_2(\cos\ \theta-2\Omega_2+\Omega_2^2\cos\ \theta)\}^{\frac{1}{2}}\\
&=\lim_{\substack{\omega_2\to\omega\\ \omega_1\to 0}}\frac{m}{2\pi i\hbar\sqrt{\Delta\sin\theta}}\{(\omega_1^2-\omega_2^2)(\Omega_2^2-1)\sin\theta\\
&\qquad -2\omega_1\omega_2(\cos\ \theta-2\Omega_2+\Omega_2^2\cos\ \theta)\}^{\frac{1}{2}}\\
&=\frac{m}{2\pi i\hbar}\sqrt{\frac{\omega^2}{2(1-\cos\ \omega T)}}\\
&=\frac{m}{2\pi i\hbar T}\left(\frac{\omega T}{2\sin\frac{\omega T}{2}}\right).
\end{aligned}
\tag{4.31}
$$

One can verify that when $\omega_1 \to 0$ and $\omega_2 \to \omega$ the exponential term of Eq. (4.30) reduces to

$$\exp\{\frac{im}{2\hbar}\frac{\omega}{4\sin^2\frac{\omega T}{2}}[\left(x_0^2 + x_T^2\right)\sin\omega T - 2x_0 x_T \sin\omega T$$
$$+ \left(y_0^2 + y_T^2\right)\sin\omega T - 2y_0 y_T \sin\omega T$$
$$+ 2\left(x_0 y_T - x_T y_0\right)\left(1 - \cos\omega T\right)]\}$$

$$= \exp\{\frac{im}{2\hbar}\Big(\frac{\omega}{2}\cot\frac{\omega T}{2}((x_T - x_0)^2 + (y_T - y_0)^2$$
$$+ \omega(x_0 y_T - x_T y_0)\Big)\}. \tag{4.32}$$

Thus, from Eqs. (4.31) and (4.32) we get, when $V_0 \to 0$, $\Omega_x^2 \to 0$ and $\Omega_y^2 \to 0$

$$K(\vec{r}_T, T; \vec{r}_0, 0) = \frac{m}{2\pi i\hbar T}\left(\frac{\omega T}{2\sin\omega T}\right)$$
$$\exp\Big\{\frac{im}{2\hbar}\Big(\frac{\omega}{2}\cot\frac{\omega T}{2}((x_T - x_0)^2 + (y_T - y_0)^2$$
$$\omega(x_0 y_T - x_T y_0)\Big)\Big\}.$$

This is the propagator of an electron confined in two dimensions in the presence of a transverse magnetic field.

b) When the magnetic field goes to zero, this limiting case corresponds to the case when $\omega \to 0$. Then the pre-factor is

$$\lim_{\omega\to 0} F(T) = \lim_{\omega\to 0} \frac{m}{2\pi i\hbar\sqrt{\Delta \sin\theta}} \{ (\omega_1^2 - \omega_2^2)(\Omega_2^2 - 1)\sin\theta$$
$$- 2\omega_1\omega_2 (\cos\theta - 2\Omega_2 + \Omega_2^2 \cos\theta) \}^{\frac{1}{2}}$$

$$= \lim_{\substack{\omega_2\to\Omega_x \\ \omega_1\to\Omega_y}} \frac{m}{2\pi i\hbar\sqrt{\Delta \sin\theta}} \{ (\omega_1^2 - \omega_2^2)(\Omega_2^2 - 1)\sin\theta$$
$$- 2\omega_1\omega_2 (\cos\theta - 2\Omega_2 + \Omega_2^2 \cos\theta) \}^{\frac{1}{2}}$$

$$= \frac{m}{2\pi i\hbar}\sqrt{\frac{\Omega_x\Omega_y}{\sin\Omega_x T \sinh\Omega_y T}} \quad , \tag{4.33}$$

and the exponential term of Eq. (4.30) when $\omega \to 0$ becomes

$$\exp\{ \frac{im}{2\pi i\hbar} \frac{\omega}{4\tan\frac{\theta}{2}\sin\Omega_x T \sinh\Omega_y T}$$
$$[4\Omega_x \tan\frac{\theta}{2}\sinh\Omega_y T\left(\left(x_0^2 + x_T^2\right)\cos\Omega_x T - 2x_0 x_T\right)$$
$$+ 4\Omega_y \tan\frac{\theta}{2}\sin\Omega_x T\left(\left(y_0^2 + y_T^2\right)\cosh\Omega_y T - 2y_0 y_T\right)]\}$$

$$= \exp\frac{i}{\hbar}\{ \frac{m\Omega_x}{2\sin\Omega_x T}\left(\left(x_0^2 + x_T^2\right)\cos\Omega_x T - 2x_0 x_T\right) \tag{4.34}$$
$$+ \frac{m\Omega_y}{2\sinh\Omega_y T}\left(\left(y_0^2 + y_T^2\right)\cosh\Omega_y T - 2y_0 y_T\right) \}.$$

From Eqs. (4.33) and (4.34), it follows that

$$K(\vec{r}_T, T; \vec{r}_0, 0) = \frac{m}{2\pi i\hbar}\sqrt{\frac{\Omega_x\Omega_y}{\sin\Omega_x T \sinh\Omega_y T}} \tag{4.35}$$
$$\exp\left\{\frac{i}{\hbar}\frac{m\Omega_x}{2\sin\Omega_x T}\left(\left(x_0^2 + x_T^2\right)\cos\Omega_x T - 2x_0 x_T\right)\right.$$
$$\left. + \frac{m\Omega_y}{2\sinh\Omega_y T}\left(\left(y_0^2 + y_T^2\right)\cosh\Omega_y T - 2y_0 y_T\right) \right\}.$$

This is the propagator of an electron in the saddle point potential, which is represented by a product of two independent one-dimensional propagators, one for the harmonic oscillator (along x-axis) and the other for the inverse parabola potential (along y-axis). One can also show that for the case of an isotropic saddle point potential, parameters ω_1 and ω_2 reduce to

$\omega_1 = \frac{1}{\sqrt{2}}((\omega^4 + 4\Omega^4)^{1/2} - \omega^2)^{1/2}$ and $\omega_2 = \frac{1}{\sqrt{2}}((\omega^4 + 4\Omega^4)^{1/2} + \omega^2)^{1/2}$ which was given in the work of Kagalovsky [25].

4.3 Density of States and Energy Spectrum of the Fertig and Halperin Model

Starting from the exact propagator given in Eq. (4.30), it is possible to obtain the density of states by using Eq. (3.46). In order to obtain the trace of Eq. (4.30), we set $x_T = x_0 = x$ and $y_T = y_0 = y$ that lead to varnishing of coupling terms, $x_T y_0 - x_0 y_T$ and $x_0 y_0 - x_T y_T$, in the propagator. Then Eq. (4.30) reduce to

$$
\begin{aligned}
K(x, y, T; x, y, 0) = F(T)\exp\{ & \frac{im}{2\Delta\hbar}(\\
& 2x^2[\sin\omega_2 T(\cosh\omega_1 T - 1)\left(\omega_1(\Omega_2 - 1)^2 \cot\frac{\theta}{2} + \omega_2(\Omega_2^2 - 1)\right) \\
& - \sinh\omega_1 T(\cos\omega_2 T - 1)\left(\omega_1(\Omega_2^2 - 1) + \omega_2(\Omega_2 + 1)^2 \tan\frac{\theta}{2}\right)] \\
& + 2y^2[\sin\omega_2 T(\cosh\omega_1 T - 1)\left(\omega_2(\Omega_2^2 - 1) - \omega_1(\Omega_2 + 1)^2 \tan\frac{\theta}{2}\right) \\
& + \sinh\omega_1 T(\cos\omega_2 T - 1)\left(\omega_2(\Omega_2 - 1)^2 \cot\frac{\theta}{2} - \omega_1(\Omega_2^2 - 1)\right)]) \\
& - \frac{iV_0 T}{\hbar}\}.
\end{aligned}
$$

This is an exponential of the form $\exp[iax^2]$. Thus the trace of Eq. (4.30) can be evaluated exactly by using Gaussian integral, which can be performed immediately,

$$
\begin{aligned}
Tr\left[K(\vec{r}_T, T; \vec{r}_0, 0)\right] &= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, K(x, y, T; x, y, 0), \\
&= F(T)\exp\{-\frac{iV_0 T}{\hbar}\}\, \frac{\pi}{2}\sqrt{\frac{2i\Delta\hbar}{m}} \\
& \{ [\sin\omega_2 T(\cosh\omega_1 T - 1)\left(\omega_1(\Omega_2 - 1)^2 \cot\frac{\theta}{2} + \omega_2(\Omega_2^2 - 1)\right) \\
& - \sinh\omega_1 T(\cos\omega_2 T - 1)\left(\omega_1(\Omega_2^2 - 1) + \omega_2(\Omega_2 + 1)^2 \tan\frac{\theta}{2}\right)]
\end{aligned}
$$

$$[\sin\omega_2 T(\cosh\omega_1 T - 1)\left(\omega_2(\Omega_2^2 - 1) - \omega_1(\Omega_2 + 1)^2 \tan\frac{\theta}{2}\right)$$

$$+ \sinh\omega_1 T(\cos\omega_2 T - 1)\left(\omega_2(\Omega_2 - 1)^2 \cot\frac{\theta}{2} - \omega_1(\Omega_2^2 - 1)\right)] \}^{-1/2}$$

$$= \left(\frac{1}{2\sinh\frac{\omega_1 T}{2}}\right)\left(\frac{\exp\left\{\frac{-iV_0 T}{\hbar}\right\}}{2i\sin\frac{\omega_2 T}{2}}\right). \qquad (4.36)$$

This is the product of two independent terms which consist of the first factor $(2\sinh\frac{\omega_1 T}{2})^{-1}$ representing the one-dimensional propagator in an inverted harmonic potential and the second $\exp\left\{\frac{-iV_0 T}{\hbar}\right\}$ $(2i\sin\frac{\omega_2 T}{2})^{-1}$ for the harmonic potential with renormalized frequency ω_1 and ω_2, respectively. For the case which magnetic field goes to zero, Eq. (4.36) reduces to the trace of propagator of a two-dimensional electron in the anisotropic saddle point potential.

In order to evaluate the density of states we use the following identity

$$\frac{1}{2i\sin\frac{\omega_2 T}{2}} = \sum_{n=0}^{\infty} \exp\left[-i\left(n + \frac{1}{2}\right)\omega_2 T\right],$$

and in the large T approximation we use

$$\frac{1}{2\sinh\frac{\omega_1 T}{2}} \cong \exp\left[\frac{-\omega_1 T}{2}\right].$$

Thus the density of states becomes

$$\rho(E) = \frac{1}{\pi}\sum_{n=0}^{\infty} \frac{\frac{\hbar\omega_1}{2}}{\left(E - (n + \frac{1}{2})\hbar\omega_2 - V_0\right)^2 + \left(\frac{\hbar\omega_1}{2}\right)^2}, \qquad (4.37)$$

Eq. (4.37) is in the form of a summation of Lorentzian functions that have peaks at $E = V_0 + (n + \frac{1}{2})\hbar\omega_2$ and broadening of each peak is proportional to the parameter $\hbar\omega_1 / 2$ (Fig. 11). Therefore, the first term in the denominator gives the renormalized energy spectrum

$$E_n = V_0 + (n + \frac{1}{2})\hbar\omega_2 , \ n = 0,1,2,.. . \qquad (4.38)$$

From Eqs. (4.26) and (4.27) one can show that the energy spectrum is unlike discrete Landau levels, the saddle-point potential allows for a continuous energy E for each discrete state n. Furthermore, for $\Omega_x^2, \Omega_y^2, V_0 \to 0$ then $\omega_2 \to \omega$, $\omega_1 \to 0$, energy $E_n \to (n + \frac{1}{2})\hbar\omega$ corresponds to the usual discrete Landau levels.

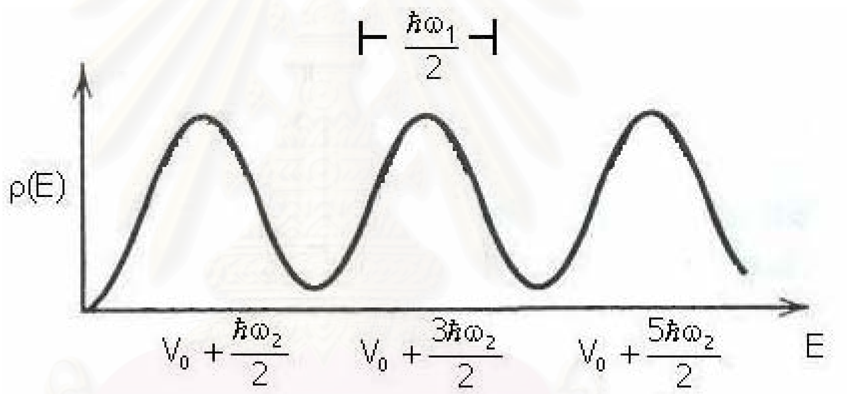

Fig. 11. Schematic diagram of the density of states which is given by Eq. (4.37).

CHAPTER V

CONCLUSIONS AND DISCUSSION

In this thesis, we have studied the system of non-interacting electron confined in two-dimensions under the influence of a transverse magnetic field and an anisotropic quadratic saddle point potential $V_{sp}(x, y) = V_0 + \frac{m}{2}(\Omega_x^2 x^2 - \Omega_y^2 y^2)$. This system was introduced by Fertig and Halperin to study tunneling and delocalization of electrons in the quantum Hall problem.

In the work of Fertig and Halperin, the symmetric gauge has been used for magnetic field in the z direction. The Hamiltonian of this system is given by Eq. (2.22),

$$H = \frac{1}{2m}\left\{(\frac{\hbar}{i}\frac{\partial}{\partial x} - \frac{eBy}{2c})^2 + (\frac{\hbar}{i}\frac{\partial}{\partial y} + \frac{eBx}{2c})^2\right\} + V_0 + \frac{m}{2}(\Omega_x^2 x^2 - \Omega_y^2 y^2)\,. \qquad (5.1)$$

By using Bogoliubov transformations, they were able to decouple the Hamiltonian into a sum of two commuting Hamiltonians in Eq. (2.36),

$$H = \left\{E_1\left(P^2 - X^2\right)\right\} + \left\{\frac{1}{2}E_2\left(p^2 + s^2\right) + V_0\right\}. \qquad (5.2)$$

where parameters are given by Eqs. (2.29) and (2.33),

$$E_1 = \frac{1}{2\sqrt{2}}((\omega^4 + 4\omega^2\tilde{\Omega}^2 + 4\Omega^4)^{1/2} - \omega^2 - 2\tilde{\Omega}^2)^{1/2}\,,$$

$$E_2 = \frac{1}{\sqrt{2}}((\omega^4 + 4\omega^2\tilde{\Omega}^2 + 4\Omega^4)^{1/2} + \omega^2 + 2\tilde{\Omega}^2)^{1/2}\,.$$

The first $E_1\left(P^2 - X^2\right)$ is equivalent to that of a one-dimensional particle in an inverted harmonic potential and the other $\frac{1}{2}E_2\left(p^2 + s^2\right) + V_0$ represents a one-dimensional particle confined by a harmonic potential. This implies that energy spectrum is unlike discrete Landau levels, the saddle-point potential allows for a continuous energy E.

Kagalovsky used the Feynman path integral to calculate the exact propagator of electrons in this system. Since the system is quadratic, the path integral can be carried out exactly by using Eq. (3.26),

$$K(\vec{r}_T, T; \vec{r}_0, 0) = F(T)\exp\left\{\frac{i}{\hbar}S_{cl}\right\}, \tag{5.3}$$

where S_{cl} is the classical action ,and pre-factor F(T) is given by Eq. (3.27),

$$F(T) = \sqrt{\det\left\{\frac{i}{2\pi\hbar}\frac{\partial^2}{\partial\vec{r}_T\partial\vec{r}_0}S_{cl}[\vec{r}_T, \vec{r}_0]\right\}} \quad . \tag{5.4}$$

The classical action is evaluated along the trajectory via the equations of motion Eqs. (4.5) and (4.6),

$$\ddot{x} - \omega\dot{y} + \Omega_x^2 x = 0\,, \tag{5.5}$$

$$\ddot{y} + \omega\dot{x} - \Omega_y^2 y = 0\,. \tag{5.6}$$

In order to evaluate classical action from Eqs. (5.5) and (5.6), Kagalovsky used matrix methods which are not applicable to a potential of anisotropic form. Therefore, he assumed $\Omega_x^2 = \Omega_y^2 = \lambda^2$ or $V_{sp}(x, y)$ in the isotropic form $\frac{m}{2}\lambda^2(x^2 - y^2)$. After solving the classical action and using formula Eq. (3.26), Kagalovsky obtained the exact propagator Eq. (3.81),

$$K(\bar{r}_T, T; \bar{r}_0, 0) = \frac{m}{2\pi i\hbar}\{4\sin^2\omega' T[\omega_1\omega_4 - \omega_2\omega_3 + \omega''^2 - \omega'^2]$$

$$+ 4\sin\omega' T\sinh\omega'' T[2\omega'\omega'' + (\omega_1\omega_4 - \omega_2\omega_3)(\frac{\omega''}{\omega'} - \frac{\omega'}{\omega''}) + \frac{(\omega_1^2\omega_4^2 + \omega_2^2\omega_3^2)}{\omega'\omega''}]$$

$$+ 4\sinh^2\omega'' T[\omega''^2 + \omega_1\omega_4 - \omega_2\omega_3 - \omega'^2]$$

$$+ (\cosh\omega'' T - \cos\omega' T)^2[\omega_1 + \omega_2 + \omega_3 - \omega_4]^2\}^{1/2}\exp\{\frac{i}{\hbar}S_{cl}\}\,,$$

where classical action S_{cl} in Eq. (3.81),

$$S_{cl} = \frac{m}{2\Delta}\{(x_T^2 + x_0^2)[(\frac{\omega_2\omega_3}{\omega'} + \omega')\sin\omega'T\cosh\omega''T$$

$$-(\frac{\omega_1\omega_4}{\omega''} + \omega'')\cos\omega'T\sinh\omega''T]$$

$$+(2x_T x_0)[-(\frac{\omega_2\omega_3}{\omega'} + \omega')\sin\omega'T + (\frac{\omega_1\omega_4}{\omega''} + \omega'')\sinh\omega''T]$$

$$+(y_T^2 + y_0^2)[(\omega' - \frac{\omega_1\omega_4}{\omega'})\sin\omega'T\cosh\omega''T$$

$$-(\frac{\omega_2\omega_3}{\omega''} - \omega'')\cos\omega'T\sinh\omega''T]$$

$$+(2y_T y_0)[(\frac{\omega_1\omega_4}{\omega'} - \omega')\sin\omega'T + (\omega'' - \frac{\omega_2\omega_3}{\omega''})\sinh\omega''T]$$

$$+(x_T y_0 - y_T x_0)(\omega_1 + \omega_2 + \omega_3 - \omega_4)(\cos\omega'T - \cosh\omega''T)$$

$$+(x_T y_T - y_o x_0)[(\omega_1 - \omega_2 - \omega_3 - \omega_4)(1 - \cos\omega'T\cosh\omega''T)$$

$$+\frac{(\omega_3 - \omega_4)}{\omega''}\omega'\sin\omega'T\sinh\omega''T + \frac{(\omega_1 - \omega_2)}{\omega'}\omega'']\},$$

, $\Delta = 2 - 2\cos\omega'T\cosh\omega''T + \frac{\omega_2\omega_3 - \omega_1\omega_4}{\omega'\omega''}\sin\omega'T\sinh\omega''T$ and parameters ω_i are given by $\omega_{1,3} = \pm(\omega^2 - 2\lambda^2 \pm \sqrt{\omega^4 + 4\lambda^4})/2\omega$, $\omega_{2,4} = \pm 2\omega\lambda^2(\omega^2 + 2\lambda^2 \pm \sqrt{\omega^4 + 4\lambda^4})$, $\omega' = \sqrt{\omega_1\omega_2}$, $\omega'' = \sqrt{\omega_3\omega_4}$. To relate the path integral approach to the work of Fertig and Halperin, Kagalovsky pointed out that the propagator may be transformed to a product of two independent propagators,

$$K(\vec{r}_T, T; \vec{r}_0, 0) \rightarrow K_1(X)K_2(s), \tag{5.9}$$

$K_1(X)$ for the inverted harmonic and the other $K_2(s)$ for the harmonic oscillator in new coordinate.

In Chapter 4, we presented the exact evaluation of the propagator for an electron in the model of Fertig and Halperin. By introducing the complex variable path $z = x - iy$ and $\tilde{z} = x + iy$ instead of the matrix method, the classical action for the case of anisotropy $\Omega_x^2 \neq \Omega_y^2$ can be obtained. By solving equation of motion from complex path and using Eq. (3.26) we obtain the exact propagator Eq. (4.30),

$$
\begin{aligned}
K(\vec{r}_T, \vec{r}_o; T) = \frac{m}{2\pi i\hbar\sqrt{\Delta \sin\theta}} \{ & (\omega_1^2 - \omega_2^2)(\Omega_2^2 - 1)\sin\theta \\
& - 2\omega_1\omega_2(\cos\theta - 2\Omega_2 + \Omega_2^2\cos\theta) \}^{\frac{1}{2}} \exp\left\{-\frac{iV_0 T}{\hbar}\right\} \exp\frac{im}{2\Delta\hbar}\{ \\
& [\omega_1(\Omega_2 - 1)^2\cot\frac{\theta}{2} + \omega_2(\Omega_2^2 - 1)]\sin\omega_2 T[(x_0^2 + x_T^2)\cosh\omega_1 T - 2x_0 x_T] \\
& - [\omega_1(\Omega_2^2 - 1) + \omega_2(\Omega_2 + 1)^2\tan\frac{\theta}{2}]\sinh\omega_1 T[(x_0^2 + x_T^2)\cos\omega_2 T - 2x_0 x_T] \\
& + [\omega_2(\Omega_2^2 - 1) - \omega_1(\Omega_2 + 1)^2\tan\frac{\theta}{2}]\sin\omega_2 T[(y_o^2 + y_T^2)\cosh\omega_1 T - 2y_o y_T] \\
& + [\omega_2(\Omega_2 - 1)^2\cot\frac{\theta}{2} - \omega_1(\Omega_2^2 - 1)]\sinh\omega_1 T[(y_o^2 + y_T^2)\cos\omega_2 T - 2y_o y_T] \\
& + \frac{2}{\sin\theta}[(\cos\omega_2 T - \cosh\omega_1 T)(\omega_1\cos\theta(\Omega_2^2 - 1) \\
& \qquad + \omega_2\sin\theta(\Omega_2^2 + 1))](x_o y_T - x_T y_o) \\
& \qquad + \frac{2}{\sin\theta}[(2\omega_2\Omega_2\sin\theta - \omega_1(\Omega_2^2 - 1))\cosh\omega_1 T\cos\omega_2 T \\
& \qquad - (2\omega_1\Omega_2\sin\theta + \omega_2(\Omega_2^2 - 1))\sinh\omega_1 T\sin\omega_2 T \\
& \qquad - (2\omega_2\Omega_2\sin\theta - \omega_1(\Omega_2^2 - 1))](x_o y_o - x_T y_T) \},
\end{aligned}
\tag{5.10}
$$

where

$\Delta = 2(1 - \Omega_2^2)(1 - \cos\omega_2 T\cosh\omega_1 T) - \dfrac{2(\cos\theta - 2\Omega_2 + \Omega_2^2\cos\theta)}{\sin\theta}\sin\omega_2 T\sinh\omega_1 T.$

$\theta = \arctan(-\omega\omega_1/(\omega_1^2 + \tilde{\Omega}^2))$, $\Omega_2 = -\dfrac{1}{\Omega^2}\left\{\omega_2(\omega - \omega_2) + \tilde{\Omega}^2\right\}$,

$\omega_1 = \dfrac{1}{\sqrt{2}}((\omega^4 + 4\omega^2\tilde{\Omega}^2 + 4\Omega^4)^{1/2} - \omega^2 - 2\tilde{\Omega}^2)^{1/2}$,

$\omega_2 = \dfrac{1}{\sqrt{2}}((\omega^4 + 4\omega^2\tilde{\Omega}^2 + 4\Omega^4)^{1/2} + \omega^2 + 2\tilde{\Omega}^2)^{1/2}$, $\Omega^2 = (\Omega_x^2 + \Omega_y^2)/2$ and

$\tilde{\Omega}^2 = (\Omega_x^2 - \Omega_y^2)/2$.

One can easily check that if the saddle point potential approaches zero, $\Omega_x^2 \to 0$ and $\Omega_y^2 \to 0$ then $\omega_1 \to 0$ and $\omega_2 \to \omega$. Then the propagator reduces to the well-known propagator for an electron in perpendicular magnetic field. In other case which the magnetic field goes to zero, this case corresponds to $\omega \to 0$, $\omega_1 \to \Omega_y$ and $\omega_2 \to \Omega_x$ then we obtain the propagator of electron in the anisotropic saddle point potential, which can be written as product of two independent one-dimensional contributions, one representing for the harmonic oscillator (along x-axis) and the other representing the inverse parabola potential (along y-axis).

From our propagator (5.10), one can show that for the isotropic case $\Omega_x^2 = \Omega_y^2 = \lambda^2$ parameter ω_1 and ω_2 are reduced to $\omega_1 = \frac{1}{\sqrt{2}}\sqrt{\sqrt{\omega^4 + 4\lambda^4} - \omega^2}$ and $\omega_2 = \frac{1}{\sqrt{2}}\sqrt{\sqrt{\omega^4 + 4\lambda^4} + \omega^2}$ equal to ω'' and ω', which given by Kagalovsky, respectively.

By taking the trace of the propagator, we can show that our result is consistent with the two commuting Hamiltonian of Fertig and Halperin, and the two independent propagators of Kagalovsky. The trace, $\mathrm{Tr}\left[K(\vec{r}_T, T; \vec{r}_0, 0)\right] = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, K(x, y, T; x, y, 0)$, can be evaluated exactly is Eq. (4.36)

$$\mathrm{Tr}\left[K(\vec{r}_T, T; \vec{r}_0, 0)\right] = \left(\frac{1}{2\sinh\frac{\omega_1 T}{2}}\right)\left(\frac{\exp\left\{\frac{-iV_0 T}{\hbar}\right\}}{2i\sin\frac{\omega_2 T}{2}}\right). \qquad (5.11)$$

This is in the form of a product of two independent terms. The first factor represents the one-dimensional particle in an inverted harmonic potential characterized by parameter ω_1 and the second for harmonic potential characterized by parameter ω_2. This trace corresponds to the Hamiltonian H_1, H_2 of Fertig and Halperin and propagator K_1, K_2 of Kagalovsky respectively.

For applications, we evaluated the density of states by using Eq. (3.46),

$$\rho(E) = \frac{1}{\pi\hbar}\mathrm{Re}\left[\int_0^{\infty} dT\ \mathrm{Tr}\left[K(\vec{r}_T, T; \vec{r}_0, 0)\right]\exp\left\{\frac{iET}{\hbar}\right\}\right]. \qquad (5.12)$$

In order to obtain the density of states, we expand $1/2i\sin(\omega_2 T/2) = \sum_{n=0}^{\infty}\exp\left[-i\left(n + \frac{1}{2}\right)\omega_2 T\right]$ and for large T approximation we approximate $1/2\sinh(\omega_1 T/2)$ by $\exp[-\omega_1 T/2]$. Thus the density of states can be expressed in Eq.(4.37),

$$\rho(E) = \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\frac{\hbar\omega_1}{2}}{\left(E-(n+\frac{1}{2})\hbar\omega_2 - V_0\right)^2 + \left(\frac{\hbar\omega_1}{2}\right)^2}, \tag{5.13}$$

This result is shown to be a series of a set of Lorentzian delta functions that have peaks at energy E_n in Eq.(4.38),

$$E_n = (n+\frac{1}{2})\hbar\omega_2 \text{, n=0,1,2,.. .}$$

The Lorentzian shape gives information about the renormalized Landau level and a continuous energy E for each discrete state n. The number of continuous states between E_n is proportional to the parameter ω_1. From Eq. (4.16), we find that parameter ω_1 goes to zero when inverse parabola term of $V_{sp}(x,y)$ approach to zero $\Omega_y^2 \to 0$. Therefore, the continuous state represents the unbound state of an electron in y direction reflecting delocalization which is a direct consequence of the inverse parabola term $(-\Omega_y^2 y^2)$ in the saddle point potential.

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APPENDIX

APPENDIX

SIMPLIFICATION OF THE PRE-FACTOR F(T)

In order to simplify the prefactor to the form given by Eq. (4.28), we use the Van Vleck-Pauli formula for a two-dimensional system,

$$F(T) = \sqrt{\det\left\{\frac{i}{2\pi\hbar}\frac{\partial^2 S_{cl}}{\partial \vec{r}_T \partial \vec{r}_0}\right\}} \;. \qquad (A.1)$$

After substituting the classical action from Eq. (4.27) we obtain

$$\begin{aligned} F(T) = \frac{m}{2\pi i\hbar\Delta\sin\theta}\{\{[(\omega_1^2-\omega_2^2)(\Omega_2^2-1)\sin\theta-2\omega_1\omega_2(\cos\theta-2\Omega_2+\Omega_2^2\cos\theta)] \\ x\,[(\Omega_2^2-1)\sin\theta(\sinh^2\omega_1 T-\sin^2\omega_2 T-\cosh^2\omega_1 T-\cos^2\omega_2 T \\ +2\cosh\omega_1 T\cos\omega_2 T) \\ -2\sinh\omega_1 T\sin\omega_2 T(\cos\theta-2\Omega_2+\Omega_2^2\cos\theta)]\} \\ +(\cos\omega_2 T-\cosh\omega_1 T)^2\,\Theta(\theta)\}^{1/2}, \end{aligned} \qquad (A.2)$$

where parameter Δ is given by Eq. (4.25) ,and

$$\begin{aligned} \Theta(\theta) = \omega_1^2(\Omega_2^2-1)^2\cos^2\theta+\omega_2^2(\Omega_2^2+1)^2\sin^2\theta+2\omega_1\omega_2(\Omega_2^4-1)\sin\theta\cos\theta \\ +(\Omega_2^2-1)\sin\theta[(\omega_1^2-\omega_2^2)(\Omega_2^2-1)\sin\theta-2\omega_1\omega_2(\cos\theta-2\Omega_2+\Omega_2^2\cos\theta)]. \end{aligned} \qquad (A.3)$$

From Eq. (A.3) we find that $\Theta(\theta)$ can be written as

$$\Theta(\theta) = (\omega_1(\Omega_2^2-1)+2\omega_2\Omega_2\sin\theta)^2 \,. \qquad (A.4)$$

Substituting parameter $\Omega_2 = -\frac{1}{\Omega^2}\left\{\omega_2(\omega-\omega_2)+\tilde{\Omega}^2\right\}$ and $\sin\theta = \frac{\omega\omega_1}{\Omega^2}$ from Eq. (4.19) into Eq. (A.4) we obtain

$$\Theta(\theta) = \frac{\omega_1^2}{\Omega^8}\{\omega_2^4-(\omega^2+2\tilde{\Omega}^2)\omega_2^2+\tilde{\Omega}^4-\Omega^4\}^2 \,. \qquad (A.5)$$

Substituting ω_2 from Eq. (4.17) we find that $\Theta(\theta) = 0$, thus the Eq. (A.2) becomes

$$F(T) = \frac{m}{2\pi i\hbar\Delta \sin\theta}\{\{[(\omega_1^2 - \omega_2^2)(\Omega_2^2 - 1)\sin\theta - 2\omega_1\omega_2(\cos\theta - 2\Omega_2 + \Omega_2^2\cos\theta)]$$
$$\text{x}\,[-(\Omega_2^2 - 1)\sin\theta(2 - 2\cosh\omega_1 T\cos\omega_2 T)$$
$$-2(\cos\theta - 2\Omega_2 + \Omega_2^2\cos\theta)\sinh\omega_1 T\sin\omega_2 T]\}\}^{1/2}.$$

When term

$-(\Omega_2^2 - 1)\sin\theta(2 - 2\cosh\omega_1 T\cos\omega_2 T) - 2(\cos\theta - 2\Omega_2 + \Omega_2^2\cos\theta)\sinh\omega_1 T\sin\omega_2 T =$ $\Delta\sin\theta$, thus

$$F(T) = \frac{m}{2\pi i\hbar\sqrt{\Delta\sin\theta}}\left\{(\omega_1^2 - \omega_2^2)(\Omega_2^2 - 1)\sin\theta - 2\omega_1\omega_2(\cos\theta - 2\Omega_2 + \Omega_2^2\cos\theta)\right\}^{\frac{1}{2}}.$$

VITAE

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