การอธิบายการพองตัวและพลังงานมืดของจักรวาลโดยใช้แบบจำลองทรีฟอร์ม

นายไตรทศ งามปิติพันธ์

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรมหาบัณฑิต

สาขาวิชาฟิสิกส์ ภาควิชาฟิสิกส์

คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย

ปีการศึกษา 2553

EXPLAINING INFLATION AND DARK ENERGY OF UNIVERSE USING THREE-FORM MODELS

Mr. Tritos Ngampitipan

A Thesis Submitted in Partial Fulfillment of the Requirements
for the Degree of Master of Science Program in Physics
Department of Physics
Faculty of Science
Chulalongkorn University
Academic Year 2010

Thesis Title	EXPLAINING INFLATION AND DARK ENERGY OF UNIVERSE USING THREE-FORM MODELS
By	Mr. Tritos Ngampitipan
Field of Study	Physics
Thesis Advisor	Assistant Professor Auttakit Chatrabhuti, Ph.D.

Accepted by the Faculty of Science, Chulalongkorn University in Partial Fulfillment of the Requirements for the Master's Degree

.... Dean of the Faculty of Science
(Professor Supot Hannongbua, Dr.rer.nat.)

THESIS COMMITTEE

.... Chairman
(Ahpisit Ungkitchanukit, Ph.D.)

.... Thesis Advisor
(Assistant Professor Auttakit Chatrabhuti, Ph.D.)

.... Examiner
(Varagorn Hengpunya, Ph.D.)

.... External Examiner
(Khamphee Karwan, Ph.D.)

ไตรทศ งามปิติพันธ์: การอธิบายการพองตัวและพลังงานมืดของจักรวาลโดยใช้แบบจำลองทรีฟอร์ม. (EXPLAINING INFLATION AND DARK ENERGY OF UNIVERSE USING THREE-FORM MODELS) อ. ที่ปรึกษาวิทยานิพนธ์หลัก: ผศ. ดร. อรรถกฤต ฉัตรภูติ, 66 หน้า.

สนามทรีฟอร์มสามารถให้คำอธิบายเรื่องราวของพลังงานมืดและการพองตัวได้ เพื่อที่จะอธิบายการขยายตัวด้วยความเร่งของเอกภพในปัจจุบันนี้และปัญหาความบังเอิญไปพร้อม ๆ กัน สนามทรีฟอร์มจำเป็นต้องคู่ควบกับสสารมืด ในวิทยานิพนธ์ฉบับนี้ เราศึกษาการคู่ควบ 4 ชนิด เราพบว่า การคู่ควบ 3 อันแรก ซึ่งขึ้นอยู่กับความหนาแน่นพลังงานของสสารมืด ρ_c สามารถทำให้เอกภพขยายตัวด้วยความเร่งได้ อย่างไรก็ตาม การคู่ควบเหล่านั้นไม่สามารถแก้ปัญหาความบังเอิญได้ สำหรับการคู่ควบอันที่ 4 ซึ่งไม่ขึ้นอยู่กับ ρ_c นั้นสามารถให้เอกภพที่ขยายตัวด้วยความเร่งและแก้ปัญหาความบังเอิญได้

ภาควิชา.............ฟิสิกส์.......................... ลายมือชื่อนิสิต...ไตรทศ งามปิติพันธ์..........
สาขาวิชา.............ฟิสิกส์.......................... ลายมือชื่อ อ.ที่ปรึกษาวิทยานิพนธ์หลัก อรรถกฤต ฉัตรภูติ
ปีการศึกษา.............2553........................

5172296823 : MAJOR PHYSICS
KEYWORDS: DARK ENERGY/ A THREE-FORM FIELD/ AN AUTONOMOUS SYSTEM

TRITOS NGAMPITIPAN : EXPLAINING INFLATION AND DARK ENERGY OF UNIVERSE USING THREE-FORM MODELS. THESIS ADVISOR : ASST. PROF. AUTTAKIT CHATRABHUTI, Ph.D., 66 pp.

Three-form fields can give rise to viable cosmological scenarios of dark energy and inflation. To simultaneously explain the present accelerated expansion of the universe and the coincidence problem, three-form field needs to couple to dark matter. In this dissertation we study four types of dark matter couplings. It has been found that the first three couplings which depend on dark matter energy density ρ_c can give the accelerated expansion of the universe; however, they cannot solve the coincidence problem. For the fourth coupling which does not depend on ρ_c, it is capable of both providing the accelerated expanding universe and solving the coincidence problem.

Department:.........Physics.........Student's Signature ..ไตรทศ..งามปิติพันธ์

Field of Study:........Physics........Advisor's Signature ..อรรถกฤต..ฉัตรภูติ

Academic Year:.........2010.........

ACKNOWLEDGEMENTS

I would like to thank Dr.Auttakit Chatrabhuti, my advisor, for his great deal of assistance to me and for valuable discussions. I also would like to thank Pitayuth Wongjun playing a vital role in this thesis.

I would like to thank Dr.Varagorn Hengpunya, Dr.Ahpisit Ungkitchanukit and Dr.Khamphee Karwan, committees, for examining my thesis.

Finally, I would like to thank all of my friends to give me assistance. I also give the special thanks to Picththida Jittamaro for encouraging me to write my thesis.

CONTENTS

Chapter **page**

LIST OF TABLES

Table **page**

LIST OF FIGURES

Figure **page**

Chapter I

Introduction

The accelerated expansion of the universe has been discovered for more than a decade [1, 2]. What is responsible for this acceleration is called dark energy. A number of explanations of dark energy have therefore been made. The simplest description concerns with introducing a cosmological constant in the Einstein field equations. It is in agreement with most of the cosmological observations. As a phenomenological model, it is successful. However, the value of the cosmological constant is so small. Its value is more than 30 orders of magnitude smaller than the Planck scale. This considerable difference causes a theory not to be natural. This is the naturalness problem or the cosmological constant problem. Moreover, there is also the coincidence problem of why the dark energy and the matter have comparable contributions to the energy density at the present time. Since the energy density of the cosmological constant remains constant throughout the history of the universe while the energy density of the matter decreases as the universe expands, the coincidence problem is therefore too difficult to be understood if the dark energy is the true cosmological constant.

Apart from cosmological constant there are other alternative models of dark energy which are dynamical. The most popular model is the quintessence model in which a scalar field plays the role of the dark energy [3, 4]. Due to their homogeneity and isotropy, quintessence and other scalar field models agree with the cosmological principle. Although scalar field models do not conflict with the observational data but so far no one has discovered the fundamental scalar particle. Moreover, at a more fundamental level there is no reason to exclude the possibility of some other higher form field to be the dark energy. These higher form fields can exist in some theories such as the string theory. The presence of them does not necessarily violate the cosmological principle. For these reasons an effort has been made in using a vector field, a one-form field, to play a role of dark energy [5, 6, 7, 8]. However, most of the vector field models encounter instabilities [9]. Generalizations to higher form fields have also been proposed [10, 11]. Two-form field models also have the same problem as vector field models. On the other hand three-form field models have no such problem: they are stable [11]. Therefore, it is of interest to consider a three-form field as a candidate for dark energy. Moreover, there is also the accelerated expansion in the very early universe called inflation. What drives inflation is called inflaton. Similar to dark energy, a three-form field is proposed to be the candidate for inflaton [11, 12]. The possibility for a three-form field to be dark energy and inflaton and to solve the coincidence problem is the aim of this thesis.

This thesis is organized as follows. In Chapter II, we give the basic cosmology necessary for this thesis. n-form field models of dark energy and inflation are discussed in Chapter III. We focus in detail on scalar field and three-form field models only. In Chapter IV, we introduce the autonomous system and apply it to the scalar field and three-form field models. The Chapter V is devoted to the discussion of coupling of the three-form field to the dark matter. In this thesis we study four types of couplings and analyze stability of the models. The conclusions are in the final chapter.

Chapter II

Basic cosmology

2.1 The Einstein field equations in FRW universe

To study the universe, we make use of the basic important assumption called the cosmological principle stating that the universe is homogeneous and isotropic at large scales. The metric satisfying this principle is the Friedmann-Robertson-Walker (FRW) metric with line-element,

$$ds^2 = -dt^2 + a^2(t)d\mathbf{x}^2, \tag{2.1}$$

where $a(t)$ is the scale factor. The dynamics of the universe can be described by the Einstein field equations

$$G^\mu_\nu = 8\pi G T^\mu_\nu, \tag{2.2}$$

where $G_{\mu\nu} \equiv R_{\mu\nu} - (1/2)Rg_{\mu\nu}$ is the Einstein tensor, $R_{\mu\nu}$ is the Ricci tensor, R is the Ricci scalar, and $T_{\mu\nu}$ is the energy momentum tensor.

The Ricci tensor can be expressed as

$$R_{\mu\nu} = \partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\nu \Gamma^\alpha_{\mu\alpha} + \Gamma^\alpha_{\beta\alpha}\Gamma^\beta_{\mu\nu} - \Gamma^\alpha_{\beta\nu}\Gamma^\beta_{\mu\alpha}, \tag{2.3}$$

where $\Gamma^\alpha_{\mu\nu}$ is the Christoffel symbol given by

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\beta}(\partial_\mu g_{\nu\beta} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu}). \tag{2.4}$$

The metric tensor and its inverse in the FRW metric (2.1) take the form

$$g_{\mu\nu} = diag(-1, a^2(t), a^2(t), a^2(t)), \tag{2.5}$$

$$g^{\mu\nu} = diag(-1, 1/a^2(t), 1/a^2(t), 1/a^2(t)). \tag{2.6}$$

Using the FRW metric (2.1), we can obtain all components of the Christoffel symbol,

$$\Gamma^0_{ij} = a\dot{a}\delta_{ij}, \tag{2.7}$$

$$\Gamma^i_{j0} = \frac{\dot{a}}{a}\delta^i_j, \tag{2.8}$$

where the dot denotes differentiation with respect to time t and the other components are all zero. Then we can derive the nonvanishing components of the Ricci tensors

$$\begin{aligned} R_{00} &= \partial_\alpha \Gamma^\alpha_{00} - \partial_0 \Gamma^\alpha_{0\alpha} + \Gamma^\alpha_{\beta\alpha}\Gamma^\beta_{00} - \Gamma^\alpha_{\beta 0}\Gamma^\beta_{0\alpha} \\ &= -3\frac{\ddot{a}}{a}, \end{aligned} \tag{2.9}$$

$$\begin{aligned} R_{ij} &= \partial_\alpha \Gamma^\alpha_{ij} - \partial_j \Gamma^\alpha_{i\alpha} + \Gamma^\alpha_{\beta\alpha}\Gamma^\beta_{ij} - \Gamma^\alpha_{\beta j}\Gamma^\beta_{i\alpha} \\ &= (a\ddot{a} + 2\dot{a}^2)\delta_{ij}. \end{aligned} \tag{2.10}$$

Now, we can find the Ricci scalar by contracting the Ricci tensors,

$$\begin{aligned} R &\equiv g^{\mu\nu} R_{\mu\nu} \\ &= 6(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}). \end{aligned} \tag{2.11}$$

The nonvanishing components of the Einstein tensors can be obtained.

$$\begin{aligned} G_{00} &= R_{00} - \frac{1}{2} R g_{00} \\ &= 3\frac{\dot{a}^2}{a^2}, \end{aligned} \tag{2.12}$$

$$\begin{aligned} G_{ij} &= R_{ij} - \frac{1}{2} R g_{ij} \\ &= (-2a\ddot{a} - \dot{a}^2)\delta_{ij}. \end{aligned} \tag{2.13}$$

We can raise a lower index of the Einstein tensors by using the inverse metric tensor

$$\begin{aligned} G^0_0 &= g^{0\mu} G_{\mu 0} \\ &= -3\frac{\dot{a}^2}{a^2}, \end{aligned} \tag{2.14}$$

$$\begin{aligned} G^i_j &= g^{i\mu} G_{\mu j} \\ &= (-\frac{2\ddot{a}}{a} - \frac{\dot{a}^2}{a^2})\delta^i_j. \end{aligned} \tag{2.15}$$

Then, consider the right hand side of the Einstein field equations (2.2). We assume that the universe consists of matters (both baryons and cold dark matters) and radiations. We describe each of these species as a fluid. In this thesis the fluid is assumed to be perfect, the fluid with no viscosity and momentum density. For the perfect fluid the energy momentum tensor takes the form

$$T^\mu_\nu = Diag(-\rho, p, p, p), \tag{2.16}$$

where ρ is the energy density of the fluid and p is the pressure density. Substituting (2.14) - (2.16) into the Einstein equations, the $(0,0)$ component gives

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho, \tag{2.17}$$

while the (i,j) component gives

$$\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{2a^2} = -4\pi G p. \tag{2.18}$$

The Hubble parameter is defined as

$$H \equiv \frac{\dot{a}}{a}. \tag{2.19}$$

Its derivative is given by

$$\dot{H} = \frac{\ddot{a}}{a} - H^2. \tag{2.20}$$

The $(0,0)$ component (2.17) becomes

$$H^2 = \frac{8\pi G}{3}\rho. \tag{2.21}$$

This is the Friedmann equation describing the evolution of the universe. The (i,j) component (2.18) becomes

$$\dot{H} = -4\pi G(\rho + p). \tag{2.22}$$

This is the acceleration equation accounting for the accelerated expansion of the universe. The conservation of the energy momentum tensor leads to the continuity equation

$$\nabla_\mu T^\mu_\nu \equiv \partial_\mu T^\mu_\nu + \Gamma^\mu_{\alpha\mu} T^\alpha_\nu - \Gamma^\alpha_{\nu\mu} T^\mu_\alpha = 0. \tag{2.23}$$

For the time component we obtain

$$\dot{\rho} + 3H(\rho + p) = 0. \tag{2.24}$$

2.2 Evolution of the universe

We can study the evolution of the universe by using the Friedmann equation (2.21). We will consider the radiation and matter dominated universe. The pressure for radiation can be written as $p = \rho/3$, while the pressure for matter vanishes. We define the equation of state parameter as

$$w \equiv \frac{p}{\rho}. \tag{2.25}$$

Therefore the equation of state parameter for radiation is $w_r = 1/3$ and for matter we have $w_m = 0$. The continuity equation (2.24) can be rewritten as

$$\dot{\rho} + 3H\rho(1 + w) = 0.$$

It can be solved by straightforward integration and the solution is

$$\rho = Ca^{-3(1+w)}, \tag{2.26}$$

where C is constant. Hence, for radiation, the energy density ρ_r is proportional to a^{-4}:

$$\rho_r = Ca^{-4}, \tag{2.27}$$

and for the matter

$$\rho_m = Ca^{-3}. \tag{2.28}$$

Putting the energy density (2.26) into the Friedmann equation (2.17) gives

$$a = A(t + c_1)^{2/3(1+w)}, \tag{2.29}$$

where A and c_1 are constant. If we choose the initial condition that $a = 0$ at $t = 0$, then $c_1 = 0$ and hence

$$a = At^{2/3(1+w)}. \tag{2.30}$$

Therefore the evolution of the scale factor of the radiation dominated universe is

$$a = At^{1/2}, \tag{2.31}$$

and the evolution of the scale factor of the matter dominated universe is

$$a = At^{2/3}. \tag{2.32}$$

We have seen that the radiation and matter dominated universe has a decelerated expansion.

2.3 Differential forms

A differential n-form is a $(0, n)$ tensor which is totally antisymmetric. Therefore, a scalar is a zero-form and a dual vector is a one-form. In d-dimensional spacetime, there are no n-forms with $n > d$ due to their antisymmetric property.

For n-form A and m-form B, we can build $(n + m)$-form $A \wedge B$ by wedge product

$$(A \wedge B)_{M_1 \ldots M_{n+m}} \equiv \frac{(n+m)!}{n!m!} A_{[M_1 \ldots M_n} B_{M_{n+1} \ldots M_{n+m}]}, \tag{2.33}$$

with $M_1, ..., M_{n+1} = 1, ..., n$. The square bracket denotes antisymmetrization:

$$T_{[M_1 \ldots M_n]} = \frac{1}{n!}(T_{M_1 \ldots M_n} + \text{alternating sum over permutations of indices } M_1 \ldots M_n). \tag{2.34}$$

Alternating sum means that odd permutations give a minus sign, for example,

$$T_{[M_1M_2M_3]} = \frac{1}{3!}(T_{M_1M_2M_3} - T_{M_1M_3M_2} + T_{M_3M_1M_2} - T_{M_3M_2M_1} + T_{M_2M_3M_1} - T_{M_2M_1M_3}). \tag{2.35}$$

The wedge product of two 1-forms is

$$(A \wedge B)_{MN} = 2A_{[M}B_{N]} = A_M B_N - A_N B_M. \tag{2.36}$$

By the definition we find that

$$A \wedge B = (-1)^{nm} B \wedge A. \tag{2.37}$$

We can form the $(n+1)$-form from an n-form by the **exterior derivative** d defined as

$$(\mathrm{d}A)_{M_1 \ldots M_{n+1}} = (n+1)\partial_{[M_1} A_{M_2 \ldots M_{n+1}]}. \tag{2.38}$$

The exterior derivative of a 0-form, for example, is simply the gradient

$$(\mathrm{d}\phi)_M = \partial_M \phi. \tag{2.39}$$

Exterior derivatives obey a modified version of the Leibniz rule

$$\mathrm{d}(A \wedge B) = (\mathrm{d}A) \wedge B + (-1)^n A \wedge (\mathrm{d}B). \tag{2.40}$$

Another interest of the exterior derivative is that

$$\mathrm{d}(\mathrm{d}A) = 0, \tag{2.41}$$

which is written as $\mathrm{d}^2 = 0$. This results from the definition of d and partial derivatives commute. We define an n-form A to be **closed** if $\mathrm{d}A = 0$ and **exact** if $A = \mathrm{d}C$ for some $(n-1)$-form C. All exact forms are closed, but the converse is not necessarily true.

2.4 Inflation

Inflation is the period of the very early universe with an accelerated expansion. It has been introduced to solve some problems in which the Big Bang model cannot handle such as the flatness problem, the horizon problem and the origin of structure problem. It does not replace the Big Bang idea. Instead it adds some ideas on the Big Bang model. Inflation in a more abstract form will be mentioned in the next chapter.

2.5 The acceleration of the universe

Observations tell us that the universe nowadays has the accelerated expansion. Moreover, there is also the period of the universe with an accelerated expansion in the very early time called inflation. To explain these accelerations, consider the acceleration equation by substituting (2.17) into (2.18),

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p). \tag{2.42}$$

Therefore, the accelerated expansion of the universe requires that

$$\rho + 3p < 0, \tag{2.43}$$

giving the equation of state parameter

$$w = \frac{p}{\rho} < -\frac{1}{3}. \tag{2.44}$$

Therefore, both radiations and matters cannot give rise to the acceleration of the universe. In order to explain this acceleration, we need to introduce new species with negative pressure. In the very early universe we call the species driving inflation **inflaton** and in the present we call the species producing the accelerated expansion of the universe **dark energy**. In the next chapter, we will study models of dark energy.

Chapter III

Dark energy and inflation from an n-form field

Thanks to its characteristic, the isotropy, the most popular candidate for dark energy and inflaton is a scalar field. However, there are no reasons to exclude the possibility of a higher form field to be dark energy and inflaton. Vector inflation has first been proposed [13]. However, vector inflation has difficulty since it induces anisotropy and faces the problem of slow-roll. In [14] Golovnev, Mukhanov, and Vanchurin have shown that these problems can be overcome. To reduce the degree of anisotropy, vector fields have to form either a triplet of mutually orthogonal vector fields or a large number of randomly directed vector fields. We obtain the isotropic universe in the first case and the slightly anisotropic universe in the other case with the anisotropy of order $1/\sqrt{N}$ for N vector fields. To handle the slow-roll, a vector field needs to non-minimally couple to gravity. Therefore, a non-minimally coupled vector field behaves as a minimally coupled scalar field. However, the models of vector field have instabilities [9]. That is the perturbations around the background diverge. The models of two-form field also face the same problem [11]. On the other hand, the three-form field models are stable. In this chapter, we will review models of dark energy and inflation from an n-form field, concentrating only on a scalar field and a three-form field.

3.1 n-form field models of dark energy and inflation

In this section, we review [11] in which we consider an n-form field A in 4-dimensional spacetime with the action

$$S_A = \int d^4x\sqrt{-g}\left[\frac{R}{16\pi G} - \frac{1}{2(n+1)!}F^2 - V(A^2) - \frac{1}{2n!}\xi A^2 R\right], \tag{3.1}$$

where $F_{\mu_1\ldots\mu_{n+1}} \equiv (n+1)\nabla_{[\mu_1}A_{\mu_2\ldots\mu_{n+1}]}$ and $A^2 \equiv A^{\mu_1\cdots\mu_n}A_{\mu_1\ldots\mu_n}$. The equation of motion of the n-form field obtained by varying the action (3.1) with respect to the field is

$$\nabla^{\mu} F_{\mu\mu_1\ldots\mu_n} = (2n!V' + \xi R)\, A_{\mu_1\ldots\mu_n}. \tag{3.2}$$

The energy momentum tensor of the n-form field is derived by

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_A}{\delta g^{\mu\nu}}, \tag{3.3}$$

which can be written as

$$\begin{aligned} T_{\mu\nu} = {} & \frac{1}{n!} F_{\mu\mu_1\ldots\mu_n} F_{\nu}^{\mu_1\ldots\mu_n} + 2nV' A_{\mu\mu_1\ldots\mu_{n-1}} A_{\nu}^{\mu_1\ldots\mu_{n-1}} - g_{\mu\nu}\left(\frac{1}{2(n+1)!}F^2 + V(A^2)\right) \\ & + \frac{\xi}{n!}\left[nRA_{\mu\mu_1\ldots\mu_{n-1}} A_{\nu}^{\mu_1\ldots\mu_{n-1}} + \left(G_{\mu\nu} + g_{\mu\nu}\Box - \nabla_\mu \nabla_\nu\right) A^2\right]. \end{aligned} \tag{3.4}$$

Let us now consider examples of n-form dark energy for $n = 0$ [15] and $n = 3$ [12].

3.2 Scalar field models of dark energy and inflation

In this section we follow [15]. We will start with a zero-form field or a scalar field in 4-dimensional spacetime. The action of the scalar field is given by

$$S = \int d^4x \sqrt{-g}\left[\frac{R}{16\pi G} - \frac{1}{2} g^{\mu\nu}\nabla_\mu \phi \nabla_\nu \phi - V(\phi)\right], \tag{3.5}$$

where $V(\phi)$ is the potential of the scalar field. The equation of motion of the scalar field obtained by varying the action (3.5) with respect to ϕ is

$$\Box\phi - \frac{\mathrm{d}V}{\mathrm{d}\phi} = 0. \tag{3.6}$$

For the FRW metric (2.1), the equation of motion (3.6) takes the form

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\mathrm{d}V}{\mathrm{d}\phi} = 0. \tag{3.7}$$

The energy momentum tensor of the scalar field is derived by

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}}, \tag{3.8}$$

which can be expressed as

$$T_{\mu\nu} = \nabla_\mu \phi \nabla_\nu \phi - g_{\mu\nu}\left[\frac{1}{2} g^{\alpha\beta}\nabla_\alpha \phi \nabla_\beta \phi + V(\phi)\right]. \tag{3.9}$$

The energy density and pressure of the scalar field are

$$\begin{aligned} \rho_\phi &= -T^0_0 \\ &= \frac{1}{2}\dot{\phi}^2 + V(\phi) \qquad (3.10) \\ p_\phi &= T^i_i \\ &= \frac{1}{2}\dot{\phi}^2 - V(\phi). \qquad (3.11) \end{aligned}$$

From (2.21) and (3.10) we obtain

$$H^2 = \frac{8\pi G}{3}\left[\frac{1}{2}\dot{\phi}^2 + V(\phi)\right]. \qquad (3.12)$$

The equation of state parameter for the scalar field is

$$\begin{aligned} w_\phi &= \frac{p_\phi}{\rho_\phi} \\ &= \frac{\dot{\phi}^2 - 2V(\phi)}{\dot{\phi}^2 + 2V(\phi)}. \qquad (3.13) \end{aligned}$$

It ranges in the region $-1 \leq w_\phi \leq 1$. From the condition of the accelerated expansion (2.44), we obtain

$$\dot{\phi}^2 < V(\phi).$$

That is the universe has the accelerated expansion when the kinetic term of a scalar field is less than the potential term.

In the context of inflation we need the additional condition. To explain inflation, we introduce the standard approximation technique called the **slow-roll approximation**. This approximation eliminates the first term of (3.7) and the first term of (3.12)

$$3H\dot{\phi} \simeq -\frac{\mathrm{d}V}{\mathrm{d}\phi}, \qquad (3.14)$$

$$H^2 \simeq \frac{8\pi G}{3}V, \qquad (3.15)$$

where $\simeq$ means that quantities in the right hand side and the left hand side are equal in the slow-roll approximation. This approximation will be valid if two parameters satisfy

$$\epsilon(\phi) \ll 1 \quad \text{and} \quad |\eta(\phi)| \ll 1, \qquad (3.16)$$

where the slow-roll parameters ϵ and η are defined as

$$\begin{aligned} \epsilon(\phi) &= \frac{1}{16\pi G}\left(\frac{1}{V}\frac{\mathrm{d}V}{\mathrm{d}\phi}\right)^2, \\ \eta(\phi) &= \frac{1}{8\pi G}\frac{1}{V}\frac{\mathrm{d}^2V}{\mathrm{d}\phi^2}. \end{aligned}$$

These conditions are necessary for the slow-roll approximation to be valid. However, they are not sufficient conditions because they are only related to the form of the potential. We can freely choose the value of $\dot{\phi}$ because of the second order of the scalar field equation of motion (3.7) and actually we can select $\dot{\phi}$ to violate the slow-roll approximation. Therefore, it is necessary to introduce the additional condition to make the slow-roll approximation valid. Under such condition, $\dot{\phi}$ satisfies (3.14). (3.14) and (3.16) are referred to as the **slow-roll conditions**. We see that (3.15) is a consequence of (3.14) and $\epsilon(\phi) \ll 1$. Inflation comes to an end when the slow-roll conditions are violated.

3.3 Three-form field models of dark energy and inflation

For $n = 3$ case, we review [12]. The action for a three-form field $A_{\mu\nu\rho}$ minimally coupled to gravity can be written as

$$S = \int d^4x\sqrt{-g}\left[\frac{R}{16\pi G} - \frac{1}{48}F^2 - V(A^2)\right]. \tag{3.17}$$

Here, we define

$$F_{\mu\nu\rho\sigma} \equiv 4\nabla_{[\mu}A_{\nu\rho\sigma]} \text{ and } A^2 \equiv A^{\mu\nu\rho}A_{\mu\nu\rho}.$$

From (3.2), we get the equation of motion of the three-form field

$$\nabla^\alpha F_{\alpha\mu\nu\rho} = 12V'(A^2)A_{\mu\nu\rho}. \tag{3.18}$$

The energy momentum tensor of the three-form field is given by (3.4)

$$T_{\mu\nu} = \frac{1}{6}F_{\mu\alpha\beta\rho}F_\nu^{\alpha\beta\rho} + 6V'A_{\mu\alpha\beta}A_\nu^{\alpha\beta} - g_{\mu\nu}\left(\frac{1}{48}F^2 + V(A^2)\right). \tag{3.19}$$

For the FRW metric (2.1), the timelike component of the three-form field is non-dynamical because its equation of motion (3.18) reads

$$12V'(A^2)A_{ij0} = 0, \tag{3.20}$$

which is an algebraic constraint. This implies $A_{ij0} = 0$. Therefore, we can only focus on the spacelike components which take the form

$$A_{ijk} = a^3(t)\epsilon_{ijk}X(t), \tag{3.21}$$

where we will study the scalar field X rather than the field A. In 4-dimensional spacetime a three-form field is dual to a scalar field. The scalar field X are related to the field $A_{\mu\nu\rho}$ via

$$\begin{aligned} A^2 &= A^{\mu\nu\rho}A_{\mu\nu\rho} \\ &= 6X^2. \end{aligned}$$

From (3.18) we obtain the equation of motion of the field X

$$\ddot{X} = -3H\dot{X} - 3\dot{H}X - 12V'(A^2)X.$$

Since

$$V'(A^2) = \frac{1}{12X}V_{,X}\,, \tag{3.22}$$

where $V_{,X} \equiv dV/dX$, we get the equation of motion of X as

$$\ddot{X} = -3H\dot{X} - 3\dot{H}X - V_{,X}\,. \tag{3.23}$$

From (3.4), each component of the energy momentum tensor of the three-form field is

$$T_{00} = \frac{1}{2}\left(\dot{X} + 3HX\right)^2 + V(A^2), \tag{3.24}$$

$$T_{ij} = a^2\left[-\frac{1}{2}\left(\dot{X} + 3HX\right)^2 + V_{,X}\,X - V(A^2)\right]\delta_{ij}. \tag{3.25}$$

The energy density and pressure of the three-form field are given by

$$\begin{aligned} \rho_X &= -T^0_0 \\ &= \frac{1}{2}\left(\dot{X} + 3HX\right)^2 + V(A^2), \end{aligned} \tag{3.26}$$

$$\begin{aligned} p_X &= T^i_i \\ &= -\frac{1}{2}\left(\dot{X} + 3HX\right)^2 + V_{,X}\,X - V(A^2). \end{aligned} \tag{3.27}$$

The equation of state parameter of the three-form field is

$$\begin{aligned} w_X &= \frac{p_X}{\rho_X} \\ &= -1 + \frac{V_{,X}\,X}{\rho_X}. \end{aligned} \tag{3.28}$$

Note that the value of w_X depends on the slope of the potential and the properties of the field. From the condition of the accelerated expansion (2.44), we obtain

$$\frac{V_{,X}\,X}{\rho_X} < \frac{2}{3}.$$

We can choose the suitable potential for the three-form field to produce an accelerated expansion. In the context of inflation, the slow-roll conditions are no longer required for the three-form field. In other words, inflation can occur even if the three-form field is not slowly rolling [11].

In the next chapter we will introduce you to an autonomous system playing an important role in cosmology.

Chapter IV

Autonomous system of dark energy models

4.1 Autonomous system

In this chapter we review [15]. The dynamics of the universe can be described in terms of autonomous systems. Let us first present some basic definitions associated with dynamical systems by considering the following coupled differential equations for two variables $x(t)$ and $y(t)$

$$\dot{x} = f(x, y, t), \dot{y} = g(x, y, t), \tag{4.1}$$

where f and g are the functions of x, y, and t. The system (4.1) is autonomous if f and g are explicitly independent of time. A point (x_c, y_c) is a fixed point or a critical point of the autonomous system if

$$(f, g)|_{(x_c, y_c)} = (0, 0). \tag{4.2}$$

A fixed point (x_c, y_c) is called an attractor when it satisfies the condition

$$(x(t), y(t)) \to (x_c, y_c) \quad \text{for} \quad t \to \infty. \tag{4.3}$$

We can find an attractor by analyzing the stability of the fixed points. Let us consider small perturbations δx and δy around the fixed point (x_c, y_c),

$$x = x_c + \delta x, y = y_c + \delta y. \tag{4.4}$$

Substituting into (4.1) gives the first-order differential equations

$$\frac{d}{dN}\begin{pmatrix}\delta x \\ \delta y\end{pmatrix} = M \begin{pmatrix}\delta x \\ \delta y\end{pmatrix}, \tag{4.5}$$

where $N = \ln(a)$ is the e-foldings number which is a convenient parameter to use for the dynamics of dark energy. The relation between N and t is

$$\frac{\mathrm{d}}{\mathrm{d}N} = \frac{1}{H}\frac{\mathrm{d}}{\mathrm{d}t}. \tag{4.6}$$

The matrix M depends on x_c and y_c and is given by

$$M = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}_{(x=x_c, y=y_c)} . \tag{4.7}$$

This equation has two eigenvalues μ_1 and μ_2. Thus the stability around the fixed points depends on the characteristics of the eigenvalues, which is classified as follows.

(i) Stable node: $\mu_1 < 0$ and $\mu_2 < 0$.

(ii) Unstable node: $\mu_1 > 0$ and $\mu_2 > 0$.

(iii) Saddle point: $\mu_1 < 0$ and $\mu_2 > 0$ (or $\mu_1 > 0$ and $\mu_2 < 0$).

(iv) Stable spiral: The determinant of the matrix M is negative and the real parts of μ_1 and μ_2 are negative.

A fixed point is an attractor in the cases (i) and (iv), but it is not so in the cases (ii) and (iii).

In this chapter, we will find the dynamics of scalar field and three-form field dark energy in the presence of the background fluids (matters and radiations).

4.2 Autonomous system in scalar field dark energy models

Including the effects of the background fluids , from (2.21) and (2.22) we have

$$H^2 = \frac{\kappa^2}{3}\left[\frac{1}{2}\dot{\phi}^2 + V(\phi) + \rho_m\right], \tag{4.8}$$

$$\dot{H} = -\frac{\kappa^2}{2}\left[\dot{\phi}^2 + (1+w_m)\rho_m\right], \tag{4.9}$$

where $\kappa^2 = 8\pi G$ and m denotes the background fluids. Let us introduce the following dimensionless quantities

$$x \equiv \frac{\kappa\dot{\phi}}{\sqrt{6}H}, y \equiv \frac{\kappa\sqrt{V}}{\sqrt{3}H}, \lambda \equiv -\frac{V_{,\phi}}{\kappa V}, \tag{4.10}$$

where $V_{,\phi} \equiv \mathrm{d}V/\mathrm{d}\phi$. Then

$$x' = -3x + \frac{\sqrt{6}}{2}\lambda y^2 + \frac{3}{2}x\left[(1-w_m)x^2 + (1+w_m)(1-y^2)\right], \tag{4.11}$$

$$y' = -\frac{\sqrt{6}}{2}\lambda xy + \frac{3}{2}y\left[(1-w_m)x^2 + (1+w_m)\left(1-y^2\right)\right], \tag{4.12}$$

Name	x	y	w_{tot}	Existence	Stability
(a)	0	0	w_m	all λ and γ	no
(b1)	1	0	1	all λ and γ	no
(b2)	-1	0	1	all λ and γ	no
(c)	$\lambda/\sqrt{6}$	$\sqrt{1-\lambda^2/6}$	$\lambda^2/3-1$	$\lambda^2 < 6$	$\lambda^2 < 3\gamma$
(d)	$3\gamma/\sqrt{6}\lambda$	$\sqrt{3(2-\gamma)\gamma/2\lambda^2}$	(4.37)	$\lambda^2 > 3\gamma$	$\lambda^2 > 3\gamma$

Table 4.1: All the fixed points in the quintessence model.

where $x' \equiv \mathrm{d}x/\mathrm{d}N$ and $y' \equiv \mathrm{d}y/\mathrm{d}N$. The Friedmann equation (4.8) becomes the constraint equation

$$x^2 + y^2 + \frac{\kappa^2 \rho_m}{3H^2} = 1. \tag{4.13}$$

From (3.13), the equation of state parameter of the scalar field ϕ in terms of these dimensionless variables is

$$w_\phi = \frac{x^2 - y^2}{x^2 + y^2}. \tag{4.14}$$

Using (3.10), the density parameter for the scalar field is

$$\begin{aligned} \Omega_\phi &\equiv \frac{\kappa^2 \rho_\phi}{3H^2} \\ &= x^2 + y^2. \end{aligned} \tag{4.15}$$

The total effective equation of state is given by

$$\begin{aligned} w_{tot} &= \frac{p_\phi + p_m}{\rho_\phi + \rho_m} \\ &= x^2 - y^2 + w_m \left(1 - x^2 - y^2\right) \qquad (4.16) \\ &= w_\phi \Omega_\phi + w_m \left(1 - \Omega_\phi\right). \qquad (4.17) \end{aligned}$$

From (4.10), we consider the case of constant λ. The potential giving such the λ is

$$V(\phi) = V_0 e^{-\kappa\lambda\phi}. \tag{4.18}$$

We obtain the fixed points by setting $\mathrm{d}x/\mathrm{d}N = 0$ and $\mathrm{d}y/\mathrm{d}N = 0$. We summarize all the fixed points in the Table 4.1

Next we analyze the properties of these fixed points. We define new variable $\gamma \equiv 1 + w_m$. We are interested in the fluid with $1 < \gamma < 2$. Let us consider the existence of each fixed point. The existence condition is that x and y are real. For the fixed point (a), $(x = 0, y = 0)$, it exists for all λ and γ. For the stability, we analyze by using the matrix M given by (4.7). For the fixed point (a), the matrix M becomes

$$M_{(a)} = \begin{pmatrix} -3 + \frac{3}{2}\gamma & 0 \\ 0 & \frac{3}{2}\gamma \end{pmatrix}. \tag{4.19}$$

The eigenvalues of the matrix M obtained by using the characteristic equation are

$$\mu_1 = -\frac{3}{2}(2-\gamma) \quad \text{and} \quad \mu_2 = \frac{3}{2}\gamma. \tag{4.20}$$

Therefore, the fixed point (a) is a saddle point since $\mu_1 < 0$ and $\mu_2 > 0$ for the range of γ. For the fixed point (b1), $(x = 1, y = 0)$, it exists for all λ and γ. For the stability, the matrix M is

$$M_{(b1)} = \begin{pmatrix} 6 - 3\gamma & 0 \\ 0 & -\frac{\sqrt{6}}{2}\lambda + 3 \end{pmatrix}. \tag{4.21}$$

Its eigenvalues are given by

$$\mu_1 = 3(2-\gamma) \quad \text{and} \quad \mu_2 = 3 - \frac{\sqrt{6}}{2}\lambda. \tag{4.22}$$

Therefore, the fixed point (b1) is unstable for $\lambda < \sqrt{6}$ and a saddle point for $\lambda > \sqrt{6}$. For the fixed point (b2), $(x = -1, y = 0)$, it exists for all λ and γ. For the stability, the matrix M is

$$M_{(b2)} = \begin{pmatrix} 6 - 3\gamma & 0 \\ 0 & \frac{\sqrt{6}}{2}\lambda + 3 \end{pmatrix}. \tag{4.23}$$

Its eigenvalues are given by

$$\mu_1 = 3(2-\gamma) \quad \text{and} \quad \mu_2 = 3 + \frac{\sqrt{6}}{2}\lambda. \tag{4.24}$$

Therefore, the fixed point (b2) is unstable for $\lambda > -\sqrt{6}$ and a saddle point for $\lambda < -\sqrt{6}$. For the fixed point (c), $\left(x = \lambda/\sqrt{6}, y = \sqrt{1 - \lambda^2/6}\right)$, it exists if

$$\lambda^2 < 6. \tag{4.25}$$

For the stability, the matrix M is

$$M_{(c)} = \begin{pmatrix} 6 - \frac{\gamma\lambda^2}{2} & \frac{\lambda}{2}(2-\gamma)\sqrt{6-\lambda^2} \\ \frac{\lambda}{2}(1-\gamma)\sqrt{6-\lambda^2} & \frac{\gamma\lambda^2}{2} - 3\gamma \end{pmatrix}. \tag{4.26}$$

Its eigenvalues are given by

$$\mu_1 = \frac{1}{2}\left(\lambda^2 - 6\right) \quad \text{and} \quad \mu_2 = \lambda^2 - 3\gamma. \tag{4.27}$$

Thus it is stable for $\lambda^2 < 3\gamma$ and a saddle point for $3\gamma < \lambda^2 < 6$. From (4.16), this fixed point gives the total equation of state

$$w_{tot} = \frac{\lambda^2}{3} - 1. \tag{4.28}$$

The accelerated expansion occurs when $w_{tot} < -1/3$, then

$$\lambda^2 < 2. \tag{4.29}$$

That is this fixed point can give the accelerated expansion of the universe. From (4.15), the density parameter for the scalar field is

$$\Omega_\phi = 1. \tag{4.30}$$

Therefore, this fixed point gives the scalar field dominated solution. For the fixed point (d), $\left(x = 3\gamma/\sqrt{6}\lambda, y = \sqrt{3(2-\gamma)\gamma/2\lambda^2}\right)$, we find that x and y are real for the range of γ. However, when calculating the density parameter for the scalar field from (4.15), we obtain

$$\Omega_\phi = \frac{3\gamma}{\lambda^2}. \tag{4.31}$$

From the constraint equation (4.13), using (4.15) we find that

$$\Omega_\phi \leq 1. \tag{4.32}$$

Thus

$$\lambda^2 > 3\gamma. \tag{4.33}$$

This is the existence condition for the fixed point (d). For the stability, the matrix M is

$$M_{(d)} = \begin{pmatrix} -\frac{3}{2}(2-\gamma)\left(1+\frac{3\gamma^2}{\lambda^2}\right) & 3(1-\frac{3\gamma^2}{2\lambda^2})\sqrt{(2-\gamma)\gamma} \\ \frac{3}{2}\left[\frac{3(2-\gamma)\gamma}{\lambda^2}-1\right]\sqrt{(2-\gamma)\gamma} & (\gamma-3)\left[\frac{9(2-\gamma)\gamma}{4\lambda^2}\right] \end{pmatrix}. \tag{4.34}$$

Its eigenvalues are given by

$$\mu_{1,2} = -\frac{3(2-\gamma)}{4}\left[1 \pm \sqrt{1-\frac{8\gamma\left(\lambda^2-3\gamma\right)}{\lambda^2(2-\gamma)}}\right]. \tag{4.35}$$

They are real when

$$\lambda^2 < \frac{24\gamma^2}{9\gamma-2}. \tag{4.36}$$

Therefore, the fixed point (d) is stable for $3\gamma < \lambda^2 < 24\gamma^2/(9\gamma-2)$ and stable spiral for $\lambda^2 > 24\gamma^2/(9\gamma-2)$. From (4.16), the total equation of state of this fixed point is

$$w_{tot} = \frac{3\gamma(\gamma-1)}{\lambda^2} + w_m\left(\frac{\lambda^2-3\gamma}{\lambda^2}\right). \tag{4.37}$$

From the existence condition (4.33) and the range of γ, we get $w_{tot} > 0$. Therefore, this fixed point cannot explain the current accelerated expansion of the universe.

Name	x	y	w	w_{tot}	Existence
(a)	0	0	± 1	-	all λ and γ
(b1)	$\sqrt{\frac{2}{3}}$	1	0	-1	all λ and γ
(b2)	$-\sqrt{\frac{2}{3}}$	-1	0	-1	all λ and γ
(c)	x_{ext}	$\sqrt{\frac{3}{2}}x_{ext}$	0	depending on the potential	depending on the potential

Table 4.2: All the fixed points in the three-form field model.

4.3 Autonomous system in three-form field dark energy models

In this section we review [12]. From (2.21) and (2.22), we have

$$H^2 = \frac{\kappa^2}{3}\left[\frac{1}{2}\left(\dot{X} + 3HX\right)^2 + V(A^2) + \rho_m\right], \tag{4.38}$$

$$\dot{H} = -\frac{\kappa^2}{2}\left(V_{,X}\, X + \gamma\rho_m\right). \tag{4.39}$$

Define the dimensionless variables

$$x \equiv \kappa X, y \equiv \frac{\kappa}{\sqrt{6}}(X' + 3X), z^2 \equiv \frac{\kappa^2 V}{3H^2}, w^2 \equiv \frac{\kappa^2\rho_m}{3H^2}, \lambda(x) \equiv -\frac{V_{,X}}{\kappa V}. \tag{4.40}$$

From the Friedmann equation (4.38), we have the constraint equation

$$y^2 + z^2 + w^2 = 1. \tag{4.41}$$

We can eliminate z from the above autonomous system by using this equation. Then we obtain the autonomous system for the three-form field

$$x' = 3\left(\sqrt{\frac{2}{3}}y - x\right), \tag{4.42}$$

$$y' = -\frac{3}{2}\lambda(x)(1 - y^2 - w^2)\left(xy - \frac{\sqrt{6}}{3}\right) + \frac{3}{2}\gamma y w^2, \tag{4.43}$$

$$w' = -\frac{3}{2}w\left[\gamma + \lambda(x)x(1 - y^2 - w^2) - \gamma w^2\right]. \tag{4.44}$$

To find the fixed points we set x', y' and w' are equal to zero. We summarize all the fixed points in the Table 4.2

We can find the density parameter of the three-form field by using (3.26), (4.6), (4.40), and (4.41)

$$\begin{aligned}\Omega_X &\equiv \frac{\kappa^2\rho_X}{3H^2} \\ &= 1 - w^2.\end{aligned} \tag{4.45}$$

Let us analyze the stability of the fixed points. The fixed point (a) corresponds to the matter dominated solution. The eigenvalues are $(-3, -3\gamma/2, 3\gamma)$. Therefore it is not stable.

The fixed point (b) has the eigenvalues $(-3, 0, -3\gamma/2)$. Since one of the eigenvalues is zero, we cannot say anything about the stability of the fixed point from the linear analysis. We have to consider specific potentials and go to the nonlinear order. The eigenvector corresponding to the vanishing eigenvalue is $(\sqrt{2/3}, 1, 0)$. We analyze the stability along the zero eigenvalue direction $\delta r = \sqrt{2/3}\delta x + \delta y$ for which we obtain

$$\delta r' = \mu^{(n)} \delta r^n, \tag{4.46}$$

where $n > 1$ and $\mu^{(n)}$ is the coefficient and we use $\delta x = \sqrt{6}\delta r/5$ and $\delta y = 3\delta r/5$ such that $\mu^{(1)} = 1$. The general solution to this equation is

$$\delta r = \delta r_0 \left(1 - \delta r_0^{n-1}(n-1)\mu^{(n)} N\right)^{1/1-n}. \tag{4.47}$$

For a negative initial perturbation ($\delta r_0 < 0$) we require $\mu^{(n)} > 0$ if n is even and $\mu^{(n)} < 0$ if n is odd for the stability. For a positive perturbation we require $\mu^{(n)} < 0$ independent of the value of n. For the fixed point (b1) we have a negative perturbation and the fixed point (b2) we have a positive perturbation. Therefore for the fixed point (b1) to be stable $\mu^{(n)} > 0$ if n is even and $\mu^{(n)} < 0$ if n is odd. For the fixed point (b2) to be stable $\mu^{(n)} < 0$. From (4.45) we obtain the density parameter of the three-form field for these fixed points

$$\Omega_X = 1. \tag{4.48}$$

The fixed point (c) corresponds to the value of x at the extrema of the potential. Its stability depends on the specific form of the potential. From (4.45) we obtain the density parameter of the three-form field for these fixed points

$$\Omega_X = 1. \tag{4.49}$$

We will consider the potentials as follows.

1. $V = \exp(-\eta X)$

For this potential the fixed points (b) have

$$\mu_b^{(2)} = \frac{18}{25}\sqrt{6}\eta. \tag{4.50}$$

Therefore the fixed point (b1) is stable for $\eta > 0$ and the fixed point (b2) is stable for $\eta < 0$.

2. $V = \exp(-\eta X^2)$

The value of μ for the fixed point (b) is

$$\mu_{b1}^{(2)} = \frac{72}{25}\eta \quad \text{and} \quad \mu_{b2}^{(2)} = -\frac{72}{25}\eta. \tag{4.51}$$

Therefore the fixed points (b) are both stable if $\eta > 0$.

The fixed point (c) is $(0, 0, 0)$. Its eigenvalues are $-(3/2)(1 \pm \sqrt{1 + 8\eta/3}$ and $-3\gamma/2$. This fixed point is stable when $\eta < 0$.

3. $V = X^2 + k$

We take k to be a positive constant. The fixed points (b) have

$$\mu_{b1}^{(2)} = -\frac{72}{24}\frac{1}{2/3 + k} \quad \text{and} \quad \mu_{b2}^{(2)} = \frac{72}{24}\frac{1}{2/3 + k}. \tag{4.52}$$

Therefore they are unstable. The fixed point (c) is $(0, 0, 0)$. Its eigenvalues are

$$-\frac{3}{2}\left(1 \mp \sqrt{1 - \frac{8}{3k}}\right) \quad \text{and} \quad -\frac{3\gamma}{2}. \tag{4.53}$$

Therefore this fixed point is stable.

4. $V = X^4 + k$

Again k is a positive constant. For the fixed points (b) we have

$$\mu_{b1}^{(2)} = -\frac{96}{25}\frac{1}{4/9 + k} \quad \text{and} \quad \mu_{b2}^{(2)} = \frac{96}{25}\frac{1}{4/9 + k}. \tag{4.54}$$

Thus again they are unstable. For the fixed point (c) $(0, 0, 0)$ the eigenvalues are $(-3, 0, -3\gamma/2)$. Because of the zero eigenvalue, we have to go to the second order. We find $\mu_c^{(2)} = 0$. Thus we move to the third order

$$\mu_c^{(3)} = -\frac{72}{125k}. \tag{4.55}$$

Therefore this fixed point is stable.

5. $V = (X^2 - C^2)^2 + k$

We take $C > 0$. For the fixed points (b) we obtain

$$\mu_{b1}^{(2)} = -\frac{144}{25}\frac{2/3 - C^2}{(2/3 - C^2)^2 + k} \quad \text{and} \quad \mu_{b2}^{(2)} = \frac{144}{25}\frac{2/3 - C^2}{(2/3 - C^2)^2 + k}. \tag{4.56}$$

$V(X)$	(a)	(b1)	(b2)	(c)
$\exp(-\eta X)$	no	$\eta > 0$	$\eta < 0$	yes
$\exp(-\eta X^2)$	no	$\eta > 0$	$\eta > 0$	$\eta < 0$

Table 4.3: The stability of all the fixed points in the three-form field model.

Therefore they are stable when $C > \sqrt{2/3}$. For this type of the potential the fixed points (c) have three fixed points

$$\text{c1:} \quad (\pm C, \pm\sqrt{\frac{3}{2}}C, 0) \quad \text{and} \quad \text{c2:} \quad (0,0,0). \tag{4.57}$$

For the fixed point (c1) the eigenvalues are

$$-\frac{3}{2}\left(1 \pm \sqrt{1 - \frac{24C^2}{k}\left(C^2 - \frac{2}{3}\right)^2}\right) \quad \text{and} \quad -\frac{3\gamma}{2}. \tag{4.58}$$

Therefore it is stable for $k > 24C^2\left(C^2 - 2/3\right)^2$. Moreover it exists for $C < 2/3$. The eigenvalues of the fixed point (c2) are

$$-\frac{3}{2}\left(1 \pm \sqrt{1 + \frac{16C^2}{3\left(C^4 + k\right)}}\right) \quad \text{and} \quad -\frac{3\gamma}{2}. \tag{4.59}$$

Therefore it is unstable. We summarize the stability of the fixed points in the Table 4.3 We also note that from (4.48) and (4.49) both of the fixed points (b) and (c) give the dark energy dominated solutions and hence the coincidence problem is not be solved.

In the next chapter, we will discuss the coincidence problem and the necessity for introducing the coupling between the three-form field and the dark matter in order to solve the coincidence problem.

Chapter V

Coupling three-form field dark energy with dark matter

According to observations, today the dark matter energy density is close in value to the energy density of dark energy. This leads to the so-called coincidence problem because their evolutions are considerably different throughout the universe history. Is it coincidence that their energy densities are of the same order? There have been many models proposed to explain the coincidence problem and it has been found that from the section 4.2 an uncoupled three-form field cannot solve the coincidence problem because it gives the three-form field dominated solution (4.48) and (4.49). If dark matter is capable of decaying into dark energy, the explanation of the similarity of their energy densities may be made. This introduces the coupling between three-form field dark energy and dark matter. In order to alleviate the coincidence problem we expect this coupling to lead to an accelerated scaling attractor solution

$$\frac{\Omega_{\text{dark energy}}}{\Omega_{\text{dark matter}}} = O(1) \quad \text{and} \quad \ddot{a} > 0. \tag{5.1}$$

The existence of the coupling can be represented by the modified continuity equations

$$\dot{\rho}_c = -3H\rho_c - Q, \tag{5.2}$$

$$\dot{\rho}_X = -3H(\rho_X + p_X) + Q, \tag{5.3}$$

where c stands for cold dark matter and Q, the coupling, is the energy transfer between dark energy and dark matter

$$Q > 0 \Rightarrow \text{dark matter} \rightarrow \text{dark energy},$$
$$Q < 0 \Rightarrow \text{dark energy} \rightarrow \text{dark matter},$$

while the background baryons and radiation still satisfy

$$\dot{\rho}_b = -3H\rho_b, \tag{5.4}$$

$$\dot{\rho}_r = -3H(\rho_r + p_r), \tag{5.5}$$

where b stands for baryon and r radiation. The explicit form of Einstein equations (2.21), (2.22) with baryons and radiation included become

$$H^2 = \frac{\kappa^2}{3}\left(\frac{1}{2}(\dot{X}+3HX)^2 + V(X) + \rho_b + \rho_r + \rho_c\right), \tag{5.6}$$

$$\dot{H} = -\frac{\kappa^2}{2}(V_{,X}X + \rho_b + \rho_r + p_r + \rho_c). \tag{5.7}$$

From equation (5.3), using (3.26) and (3.27), we obtain

$$\ddot{X} + 3H\dot{X} + 3\dot{H}X + V_{,X} = \frac{Q}{\dot{X}+3HX}. \tag{5.8}$$

We define the dimensionless variables for baryons, radiations and cold dark matters additional from (4.40)

$$w^2 \equiv \frac{\kappa^2\rho_b}{3H^2}, u^2 \equiv \frac{\kappa^2\rho_r}{3H^2}, v^2 \equiv \frac{\kappa^2\rho_c}{3H^2}. \tag{5.9}$$

From equation (2.22), (2.24) and (3.23), we can construct the autonomous system

$$x' = 3\left(\sqrt{\frac{2}{3}}y - x\right), \tag{5.10}$$

$$y' = \overline{\gamma} - \frac{3}{2}\lambda(x)z^2\left(xy - \sqrt{\frac{2}{3}}\right) + \frac{3}{2}\left(w^2 + \frac{4}{3}u^2 + v^2\right)y, \tag{5.11}$$

$$v' = -\frac{\overline{\gamma}y}{v} - \frac{3}{2}v\left[1 + \lambda(x)xz^2 - w^2 - \frac{4}{3}u^2 - v^2\right], \tag{5.12}$$

$$w' = -\frac{3}{2}w\left[1 + \lambda(x)xz^2 - w^2 - \frac{4}{3}u^2 - v^2\right], \tag{5.13}$$

$$u' = -\frac{3}{2}u\left[\frac{4}{3} + \lambda(x)xz^2 - w^2 - \frac{4}{3}u^2 - v^2\right]. \tag{5.14}$$

where

$$\overline{\gamma} = \frac{\kappa Q}{\sqrt{6}(\dot{X}+3HX)H^2}. \tag{5.15}$$

From (5.6), we obtain the constraint equation

$$y^2 + z^2 + w^2 + u^2 + v^2 = 1. \tag{5.16}$$

Generalizing (4.45), we obtain the density parameter of the three-form field

$$\Omega_X = 1 - w^2 - u^2 - v^2. \tag{5.17}$$

From (3.28), we obtain the equation of state parameter

$$w_X = -1 - \sqrt{\frac{2}{3}}\lambda(x)y\frac{1 - y^2 - w^2 - u^2 - v^2}{1 - w^2 - u^2 - v^2}. \tag{5.18}$$

What should Q be of the form? A coupling model should be phenomenological. There have been various models of a coupling proposed. Some of them have simple functional solutions such as $Q \propto a^n$. However These models are incomplete. They cannot be thoroughly tested against observations.

A good model requires at least that Q should be expressed in terms of the energy densities and other covariant quantities. For the scalar field model in [16] they represent three forms of the coupling

$$\begin{aligned}
\text{(I)} \quad \mathrm{Q} &= \sqrt{\tfrac{2}{3}}\kappa\beta\rho_c\dot{\phi} \\
\text{(II)} \quad \mathrm{Q} &= \alpha H\rho_c \\
\text{(III)} \quad \mathrm{Q} &= \Gamma\rho_c,
\end{aligned}$$

where β and α are dimensionless constants.

The coupling model (I) was obtained via the scalar-tensor theory [17]. It gives us the accelerated scaling solutions. Although it has a clear physical motivation, it is contradictory with the observations [18]. The accelerated scaling attractor is not connected to a matter period where the structure grows in the standard way. Generalizations of (I) with $\beta = \beta(\phi)$ also face the same problem [18].

The coupling model (II) does not come from a physical model of dark sector couplings, but is just for mathematical simplicity. This model and its generalization [19], $Q = \alpha H(\rho_c + \rho_x)$ are designed to produce an accelerated scaling attractor. The model (II) and its generalization avoid the problems the model (I) has with a nonstandard matter epoch [20]. They are phenomenologically useful, but it is difficult for them to appear from a physical explanation of dark sector couplings.

To avoid this problem the model (II) is improved and the non-local transfer rate, αH, is replaced by the local rate Γ, giving the model (III) $Q = \Gamma\rho_c$.

In this thesis in the three-form field model, the forms of Q are similar to ones in the scalar field model. They are

$$\begin{aligned}
\text{(I)} \quad & Q^\mu = -\sqrt{\tfrac{2}{3}}\kappa\beta\rho_c\tfrac{1}{24a^3}\epsilon^{\nu\rho\sigma\gamma}F_{\nu\rho\sigma\gamma}u^\mu, \\
\text{(II)} \quad & Q^\mu = -\alpha H\rho_c u^\mu, \\
\text{(III)} \quad & Q^\mu = -\Gamma\rho_c u^\mu,
\end{aligned}$$

where α, β and Γ are constant and u^μ is a four-velocity.

In this thesis we use the exponential potential ($V = V_0 e^{-\eta x}$) and the Gaussian potential ($V = V_0 e^{-\eta x^2}$).

Point	y_*	v_*^2	w_*	u_*	w_{tot}	Existence	Stability
A	$-\frac{2}{3}\beta$	$1-\frac{4}{9}\beta^2$	0	0	$-\frac{4}{9}\beta^2$	$\lvert\beta\rvert \leqslant \frac{3}{2}$	$\lvert\beta\rvert > \frac{3}{2}$
B	± 1	0	0	0	-1	all β, λ	$\beta > -\frac{3}{2}$
C	$-\frac{3}{2}\frac{1}{\beta}$	$-\frac{\sqrt{3/2}(4\beta^2-9)\lambda}{2\beta^2(2\beta-\sqrt{6}\lambda)}$	0	0	-1	Shown in Figure 5.1(a)	Shown in Figure 5.1(a)

Table 5.1: The properties of the fixed points for the coupling model (I).

5.1 Coupling model (I): $Q^\mu = -\sqrt{\frac{2}{3}}\kappa\beta\rho_c\frac{1}{24a^3}\epsilon^{\nu\rho\sigma\gamma}F_{\nu\rho\sigma\gamma}u^\mu$

For this covariant form, the interaction term that satisfies equations (5.2) and (5.3) can be expressed as $Q = Q_0 = \sqrt{\frac{2}{3}}\kappa\beta\rho_c(\dot{X} + 3HX)$. Therefore the interaction variable becomes

$$\overline{\gamma} = \beta v^2.$$

The autonomous system (5.10) - (5.14) becomes

$$\begin{aligned}
x' &= 3\left(\sqrt{\frac{2}{3}}y - x\right), \\
y' &= \beta v^2 - \frac{3}{2}\lambda(x)z^2\left(xy - \sqrt{\frac{2}{3}}\right) + \frac{3}{2}\left(w^2 + \frac{4}{3}u^2 + v^2\right)y, \\
v' &= -\beta yv - \frac{3}{2}v\left[1 + \lambda(x)xz^2 - w^2 - \frac{4}{3}u^2 - v^2\right], \\
w' &= -\frac{3}{2}w\left[1 + \lambda(x)xz^2 - w^2 - \frac{4}{3}u^2 - v^2\right], \\
u' &= -\frac{3}{2}u\left[\frac{4}{3} + \lambda(x)xz^2 - w^2 - \frac{4}{3}u^2 - v^2\right].
\end{aligned}$$

We summarize all the fixed points in the Table 5.1.

Next we will analyze each fixed point.

- **Fixed points A**

These fixed points are $(x, y, v, w, u) = (\sqrt{2/3}y_*, y_*, v_*, 0, 0)$, where

$$y_* = -\frac{2}{3}\beta \quad \text{and} \quad v_*^2 = 1 - y_*^2 = 1 - \frac{4}{9}\beta^2.$$

The existence condition is that each dimensionless variable is real. Therefore these fixed points exist when

$$|\beta| \leq \frac{3}{2}.$$

The total equation of state is given by

$$w_{tot} = \frac{-y_*^2 - (1 - y_*^2 - v_*^2 - w_*^2 - u_*^2)\left[1 + \lambda(x)x\right]}{1 - w_*^2 - u_*^2}. \tag{5.19}$$

At these fixed points the total equation of state is

$$w_{tot} = -\frac{4}{9}\beta^2.$$

The universe has an accelerated expansion $w_{tot} < -1/3$ when

$$|\beta| > \frac{\sqrt{3}}{2}.$$

To find their stabilities we calculate the eigenvalues of the matrix M (4.7). They are $(-3, (9 - 4\beta^2)/6, (9 - 4\beta^2)/3)$. Thus they are stable when $|\beta| > 3/2$. So we conclude that they are not stable.

- **Fixed points B**

These fixed points are $(x, y, v, w, u) = (\pm\sqrt{2/3}, \pm 1, 0, 0, 0)$. They correspond to the three-form dominated solution. They exist for all parameters λ, β. From (5.19) the total equation of state at these fixed points is

$$w_{tot} = -1. \tag{5.20}$$

therefore the three-form field at these fixed points acts as the cosmological constant and can give the accelerated expansion of the universe. From (5.17) we obtain $\Omega_X = 1$. Thus these fixed points cannot solve the coincidence problem because Ω_X/Ω_c is not of order of unity. The stability analysis gives the eigenvalues $(-3, 0, -3/2 - \beta)$. Because of the zero eigenvalue we have to consider the second order perturbation as in the case of an uncoupled three-form field in the previous chapter. However we can avoid this complication by analyzing their stability numerically.

In numerical method the condition of their stabilities is that each dimensionless variable converges. For the exponential potential we find that if $y_* = 1$, the potential parameter (η) needs to be positive and if $y_* = -1$, the potential parameter needs to be negative. For the Gaussian potential the stability requires that $\eta > 0$. Therefore they are stable when $\beta > -3/2$ and $y_*\eta > 0$ for the exponential potential and $\beta > -3/2$ and $\eta > 0$ for the Gaussian potential.

- **Fixed points C**

These fixed points are $(x, y, v, w, u) = (\sqrt{2/3}y_*, y_*, v_*, 0, 0)$, where

$$y_* = -\frac{3}{2\beta} \quad \text{and} \quad v_*^2 = -\frac{\sqrt{3/2}(4\beta^2 - 9)\lambda}{2\beta^2(2\beta - \sqrt{6}\lambda)}.$$

The existence condition and the stability of these fixed points are shown in the Figure 5.1 (a).

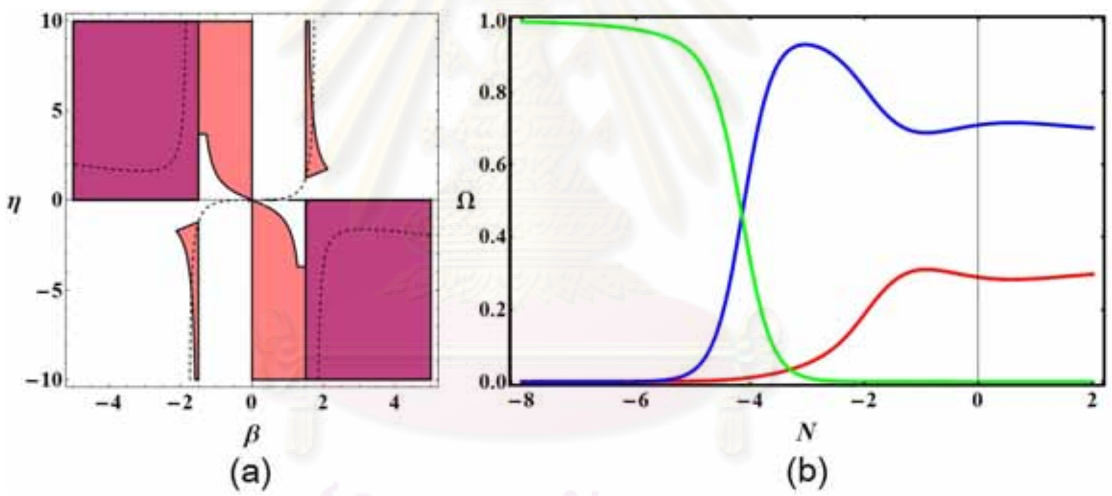

Figure 5.1: (a): This figure shows the region of stability (red, shaded) and existence (blue, shaded) in the (β, η) parameter space. The violet region indicates the compatible region of two conditions. Dashed line represents the point that the energy parameter ratio of dark energy to dark matter is 7:3.(b): This figure shows the evolution of the dynamical variables for the exponential potential $V = V_0 e^{-\eta x}$. We use $\beta = -2.13$ for this simulation. The red line represents the energy parameter of dark matter(Ω_c), The blue line represents the energy parameter of dark energy(Ω_X) and The green line represents the energy parameter of radiation(Ω_r).

From (5.18), we get the equation of state parameter for the three-form field

$$w_X = \frac{4\sqrt{6}\beta^2\lambda - 8\beta^3}{8\beta^3 - 9\sqrt{6}\lambda}.$$

From (5.19), the total equation of state at these fixed points is

$$w_{tot} = -1,$$

therefore, the three-form field at these fixed points acts as the cosmological constant like the fixed points B. From (5.17), we obtain

$$\Omega_X = \frac{8\beta^3 - 9\sqrt{6}\lambda}{4\beta^2(2\beta - \sqrt{6}\lambda)}.$$

We can find the density parameter for the cold dark matter

$$\Omega_c \equiv \frac{\kappa^2 \rho_c}{3H^2} = v^2. \tag{5.21}$$

At these fixed points it is

$$\Omega_c = -\frac{\sqrt{3/2}(4\beta^2 - 9)\lambda}{2\beta^2(2\beta - \sqrt{6}\lambda)}.$$

Therefore, the ratio of the dark energy to the dark matter is

$$\begin{aligned}\frac{\Omega_X}{\Omega_c} &= -\frac{8\beta^3 - 9\sqrt{6}\lambda}{\sqrt{6}(4\beta^2 - 9)\lambda} \\ &= -\frac{8\beta^3 - 9\sqrt{6}\lambda}{4\sqrt{6}\beta^2\lambda - 9\sqrt{6}\lambda}.\end{aligned}$$

For the exponential potential, from (4.40) we have $\lambda = \eta$. Thus

$$\frac{\Omega_X}{\Omega_c} = -\frac{8\beta^3 - 9\sqrt{6}\eta}{4\sqrt{6}\beta^2\eta - 9\sqrt{6}\eta}.$$

To solve the coincidence problem we require $\Omega_X/\Omega_c = 7/3$, that is

$$\eta = \frac{2\sqrt{6}\beta^3}{45 - 14\beta^2}.$$

This is represented by the dashed line in the Figure 5.1 (a). η diverges at $\beta = \pm\sqrt{45/14}$. This divergence also exists in the Gaussian potential. However these fixed points are contradictory with observations. From the Figure 5.1 (b), there is no the matter dominated period like in the case of the scalar field mentioned in the beginning of this chapter both for the exponential potential and for the Gaussian potential.

5.2 Coupling model (II): $Q^\mu = -\alpha H \rho_c u^\mu$

The interaction term that satisfies equations (5.2) and (5.3) of this coupling model can be expressed as $Q = Q_0 = \alpha H \rho_c$ and the interaction variable becomes

$$\overline{\gamma} = \frac{1}{2}\alpha \frac{v^2}{y}.$$

The autonomous system (5.10) - (5.14) becomes

$$\begin{aligned}
x' &= 3\left(\sqrt{\frac{2}{3}}y - x\right), \\
y' &= \frac{1}{2}\frac{\alpha v^2}{y} - \frac{3}{2}\lambda(x)z^2\left(xy - \sqrt{\frac{2}{3}}\right) + \frac{3}{2}\left(w^2 + \frac{4}{3}u^2 + v^2\right)y, \\
v' &= -\frac{1}{2}\alpha v - \frac{3}{2}v\left[1 + \lambda(x)xz^2 - w^2 - \frac{4}{3}u^2 - v^2\right], \\
w' &= -\frac{3}{2}w\left[1 + \lambda(x)xz^2 - w^2 - \frac{4}{3}u^2 - v^2\right], \\
u' &= -\frac{3}{2}u\left[\frac{4}{3} + \lambda(x)xz^2 - w^2 - \frac{4}{3}u^2 - v^2\right].
\end{aligned} \tag{5.22}$$

We summarize all the fixed points in the Table 5.2.

Point	y_*	v_*	w_*	u_*	w_{tot}	Existence	Stability
A	$\sqrt{-\frac{\alpha}{3}}$	$\pm\sqrt{1+\frac{\alpha}{3}}$	0	0	$\frac{\alpha}{3}$	$-3 \leqslant \alpha \leqslant 0$	$\alpha < -3$
B	± 1	0	0	0	-1	all α,λ	$-3 < \alpha$

Table 5.2: The properties of the fixed points for the coupling model (II). Note that, $x = \sqrt{\frac{2}{3}}y$ at the fixed points.

Next we will analyze each fixed point.

- **Fixed points A**

These fixed points are $(x, y, v, w, u) = (\sqrt{2/3}y_*, y_*, v_*, 0, 0)$, where

$$y_* = \sqrt{-\frac{\alpha}{3}} \quad \text{and} \quad v_*^2 = 1 - y_*^2 = 1 + \frac{\alpha}{3}.$$

These fixed points exist when

$$\alpha \leq 0,$$

and

$$\alpha \geq -3.$$

From (5.19), at these fixed points the total equation of state is

$$w_{tot} = \frac{\alpha}{3}.$$

The universe has an accelerated expansion $w_{tot} < -1/3$ when

$$\alpha < -1.$$

For their stabilities, the eigenvalues are $(-3, \alpha + 3, \alpha + 3)$. Thus they are stable for $\alpha < -3$. It follows that they are not stable.

- **Fixed points B**

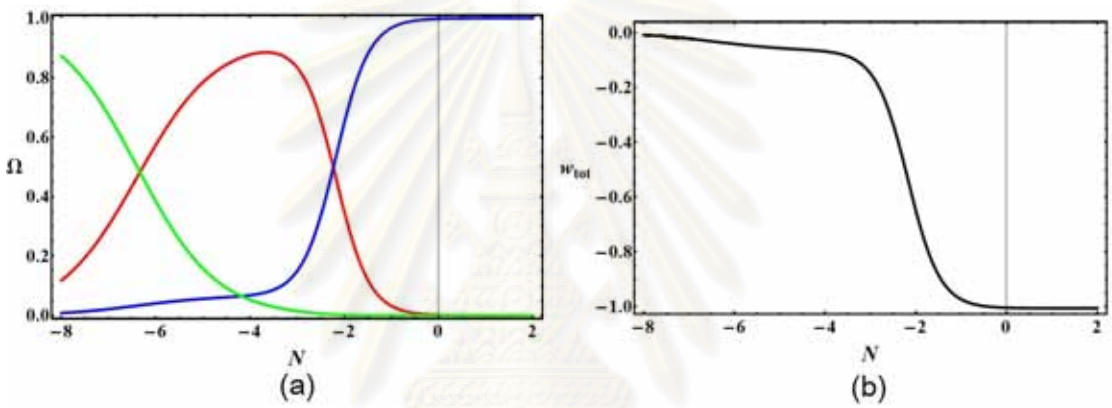

Figure 5.2: (a) shows the evolution of the dynamical variables for $y_* = +1$ fixed point with the exponential potential. We use $\alpha = -0.2$, $\eta = 1.0$. The red line represents the energy parameter of dark matter (Ω_c), the blue line represents the energy parameter of dark energy (Ω_X) and the green line represents the energy parameter of radiation(Ω_r). (b) shows the evolution behavior of the total equation of state parameter by using the same parameter with (a).

These fixed points are $(x, y, v, w, u) = (\pm\sqrt{2/3}, \pm 1, 0, 0, 0)$. They correspond to the three-form dominated solution. They exist for all parameters λ, α. From (5.19) at these fixed points the total equation of state is

$$w_{tot} = -1,$$

therefore the three-form field at these fixed points acts as the cosmological constant as in the model (I). From (5.17) we obtain $\Omega_X = 1$. Thus these fixed points cannot solve the coincidence problem. For their stabilities the eigenvalues are $(-3, 0, -3/2 - \alpha/2)$. We use the numerical method to find the stability condition as in the model (I). For $\alpha < 0$ we find that for the exponential potential if $y_* = 1$, the potential parameter $\eta > 0$ and if $y_* = -1$, the potential parameter $\eta < 0$. For the Gaussian potential the stability requires $\eta > 0$.

For $\alpha > 0$ with the exponential potential, we find that the evolution from numerical simulation encounters a singularity at the matter dominated era. This singularity can be seen in the interaction term of (5.22) when $v \neq 0$ and $y = 0$. However this singularity does not exist in the case of $\alpha < 0$ as shown in the Figure 5.2. This singularity also exists in the Gaussian potential when $\alpha > 0$. Therefore these fixed points are stable when $-3 < \alpha < 0$ and $y_*\eta > 0$ for the exponential potential and $-3 < \alpha < 0$ and $\eta > 0$ for the Gaussian potential.

5.3 Coupling model (III): $Q^\mu = -\Gamma\rho_c u^\mu$

The interaction term that satisfies equations (5.2) and (5.3) of this coupling model can be expressed as $Q = Q_0 = \Gamma\rho_c$. We cannot eliminate the Hubble parameter from the dynamical system. We have to define a new dimensionless variable for this model coupling

$$s \equiv \frac{H_0}{H}, \tag{5.23}$$

where H_0 is the Hubble parameter at the present time. Since from (2.21) H decreases as the time increases, the early universe corresponds to $s \to 0$ and the present time corresponds to $s = 1$. The interaction variable becomes

$$\overline{\gamma} = \frac{1}{2}\frac{\overline{\overline{\gamma}}sv^2}{y},$$

where $\overline{\overline{\gamma}} \equiv \Gamma/H_0$. The autonomous system (5.10) - (5.14) becomes

$$x' = 3\left(\sqrt{\frac{2}{3}}y - x\right),$$

$$y' = \frac{\overline{\overline{\gamma}}sv^2}{2y} - \frac{3}{2}\lambda(x)z^2\left(xy - \sqrt{\frac{2}{3}}\right) + \frac{3}{2}\left(w^2 + \frac{4}{3}u^2 + v^2\right)y, \tag{5.24}$$

$$v' = -\frac{\overline{\overline{\gamma}}sv}{2} - \frac{3}{2}v\left[\gamma_c + \lambda(x)xz^2 - \gamma_b w^2 - \gamma_r u^2 - \gamma_c v^2\right], \tag{5.25}$$

$$w' = -\frac{3}{2}w\left[1 + \lambda(x)xz^2 - w^2 - \frac{4}{3}u^2 - v^2\right],$$

$$u' = -\frac{3}{2}u\left[\frac{4}{3} + \lambda(x)xz^2 - w^2 - \frac{4}{3}u^2 - v^2\right],$$

and the dynamical equation of s is

$$s' = -\frac{3}{2}s[\lambda(x)xz^2 - \gamma_b w^2 - \gamma_r u^2 - \gamma_c v^2]. \tag{5.26}$$

Point	y_*	v_*	w_*	u_*	s_*	w_{tot}	Existence	Stability
A	$\sqrt{-\frac{s_*\bar{\bar{\gamma}}}{3}}$	$\pm\sqrt{1-y_*^2}$	0	0	any	$\frac{\bar{\bar{\gamma}}s}{3}$	$-3 \leqslant \bar{\bar{\gamma}}s \leqslant 0$	$\bar{\bar{\gamma}}s < -3$
B	±1	0	0	0	any	-1	all $\bar{\bar{\gamma}}$, λ	$\bar{\bar{\gamma}}s > -3$
C	y_*	$\pm\sqrt{\frac{2\lambda(x_*)y_*(1-y_*^2)}{2\lambda(x_*)y_*+\sqrt{6}}}$	0	0	$-\frac{3}{\bar{\bar{\gamma}}}$	-1	all $\bar{\bar{\gamma}}$, λ	-

Table 5.3: The properties of the fixed points for the coupling model (III). Note that, $x = \sqrt{\frac{2}{3}}y$ at the fixed points.

We summarize all the fixed points in the Table 5.3.

Next we will analyze each fixed point.

- **Fixed points A**

These fixed points are $(x, y, v, w, u, s) = (\sqrt{2/3}y_*, y_*, v_*, 0, 0, s_*)$, where

$$y_* = \pm\sqrt{-\frac{s_*\bar{\bar{\gamma}}}{3}}, v_* = \pm\sqrt{1-y_*^2} = \pm\sqrt{1+\frac{s_*\bar{\bar{\gamma}}}{3}},$$

and s_* can take any non-negative value. They exist when

$$s_*\bar{\bar{\gamma}} \leq 0,$$

and

$$s_*\bar{\bar{\gamma}} \geq -3.$$

From (5.19) at these fixed points the total equation of state is

$$w_{tot} = \frac{\bar{\bar{\gamma}}s}{3}.$$

The universe has an accelerated expansion $w_{tot} < -1/3$ when

$$\bar{\bar{\gamma}}s < -1.$$

For their stabilities the eigenvalues are $(-3, (s_*\bar{\bar{\gamma}} - 1)/2, s_*\bar{\bar{\gamma}} + 3, s_*\bar{\bar{\gamma}} + 3)$. They are stable when

$$s_*\bar{\bar{\gamma}} < -3.$$

Therefore they are not stable.

- **Fixed point B**

These fixed points are $(x, y, v, w, u, s) = (\pm\sqrt{2/3}, \pm 1, 0, 0, 0, s_*)$. They correspond to the three-form dominated solution. They exist for all parameters $\lambda, \bar{\bar{\gamma}}$. From (5.19), at these fixed points the total equation of state is

$$w_{tot} = -1,$$

therefore the three-form field at these fixed points acts as the cosmological constant as in the model (I) and (II). From (5.17), we obtain $\Omega_X = 1$. Thus these fixed points cannot solve the coincidence problem. For their stabilities the eigenvalues are $(-3, -2, 0, (-3 - s_*\bar{\bar{\gamma}})/2)$. We use the numerical method to find the stability condition as in the model (I). The result is that no dimensionless variables diverge. Therefore these fixed points are stable when $s_*\bar{\bar{\gamma}} > -3$.

The well behavior of the energy density evolution is allowed only for negative sign of interaction parameter, $\bar{\bar{\gamma}} < 0$. In other words, it is allowed energy transfer from dark matter to dark energy only. The physical interpretation of this phenomena is the same as mentioned in the model coupling (II). We do not show in detail here.

- **Fixed point C**

These fixed points are $(x, y, v, w, u, s) = (\sqrt{2/3}y_*, y_*, v_*, 0, 0, s_*)$, where y_* is any real value and

$$v_* = \pm\sqrt{\frac{2\lambda(x_*)y_*(1 - y_*^2)}{2\lambda(x_*)y_* + \sqrt{6}}}, \tag{5.27}$$

$$s_* = -\frac{3}{\bar{\bar{\gamma}}}. \tag{5.28}$$

They exist for all parameters $\lambda, \bar{\bar{\gamma}}$. From (5.19), the total equation of state at these fixed points is

$$w_{tot} = -1,$$

therefore the three-form field at these fixed points acts as the cosmological constant as in fixed points B. We find the key feature of these fixed points when considering the value of s_*. From (5.28), the definition of s (5.23), and the definition of $\overline{\overline{\gamma}}$ we have

$$\Gamma = -3H.$$

Then at these fixed points the coupling Q takes the form $Q = -3H\rho_c$ and therefore, from (5.2)

$$\dot{\rho}_c = 0.$$

Therefore, these fixed points are in contradiction with observations.

5.4 Extension to another coupling model

From the previous models, we have seen that there are no couplings models able to solve the coincidence problem. In order to solve it, we now have to introduce the new coupling model. This coupling model can be written in the covariant form as

$$Q^\mu = -\frac{\Gamma}{24a^3}\epsilon^{\nu\rho\sigma\gamma}F_{\nu\rho\sigma\gamma}u^\mu, \tag{5.29}$$

where Γ is constant. The interaction term that satisfies equations (5.2) and (5.3) can be expressed as $Q = Q_0 = \Gamma(\dot{X} + 3HX)$. The autonomous system (5.10) - (5.14) are

$$x' = 3\left(\sqrt{\frac{2}{3}}y - x\right),$$

$$y' = \overline{\gamma} - \frac{3}{2}\lambda(x)z^2\left(xy - \sqrt{\frac{2}{3}}\right) + \frac{3}{2}\left(w^2 + \frac{4}{3}u^2 + v^2\right)y, \tag{5.30}$$

$$v' = -\frac{\overline{\gamma}y}{v} - \frac{3}{2}v\left[1 + \lambda(x)xz^2 - w^2 - \frac{4}{3}u^2 - v^2\right], \tag{5.31}$$

$$w' = -\frac{3}{2}w\left[1 + \lambda(x)xz^2 - w^2 - \frac{4}{3}u^2 - v^2\right],$$

$$u' = -\frac{3}{2}u\left[\frac{4}{3} + \lambda(x)xz^2 - w^2 - \frac{4}{3}u^2 - v^2\right].$$

We summarize all the fixed points in the Table 5.4. We define new parameters as

$$A \equiv \left(9\overline{\gamma} + 3\sqrt{-3 + 9\overline{\gamma}^2}\right)^{1/3},$$

$$B \equiv -\overline{\gamma}^2 - 3\sqrt{6}\frac{\overline{\gamma}}{\lambda}.$$

Point	y_*	v_*^2	w_*	u_*	w_{tot}	Existence	Stability
A1	$\frac{1}{A} + \frac{A}{3}$	$1 - y_*^2$	0	0	$-\frac{1}{\sqrt{3}} \leqslant \overline{\gamma}$	$-\frac{1}{\sqrt{3}} \leqslant \overline{\gamma} \leqslant 0$	$\overline{\gamma} > 0$
A2	$-\frac{1+i\sqrt{3}}{2A} - \frac{(1-i\sqrt{3})A}{6}$	$1 - y_*^2$	0	0	$\overline{\gamma} \leqslant \frac{1}{\sqrt{3}}$	$0 \leqslant \overline{\gamma} \leqslant \frac{1}{\sqrt{3}}$	$\overline{\gamma} < 0$
A3	$-\frac{1-i\sqrt{3}}{2A} - \frac{(1+i\sqrt{3})A}{6}$	$1 - y_*^2$	0	0	No	$\lvert\overline{\gamma}\rvert \leqslant \frac{1}{\sqrt{3}}$	$\lvert\overline{\gamma}\rvert > \frac{1}{\sqrt{3}}$
B1	$\frac{\overline{\gamma}}{3} + \frac{1}{3}\sqrt{9-B}$	$-\frac{2}{3}\overline{\gamma}y_*$	0	0	-1	Figure 5.3(a)	Figure 5.3(a)
B2	$\frac{\overline{\gamma}}{3} - \frac{1}{3}\sqrt{9-B}$	$-\frac{2}{3}\overline{\gamma}y_*$	0	0	-	No	Figure 5.3(b)

Table 5.4: The properties of the fixed points for the coupling model (I). Note that, $x_* = \sqrt{\frac{2}{3}}y_*$ at the fixed points.

- **Fixed points A**

These fixed points are $(x, y, v, w, u) = (\sqrt{2/3}y_*, y_*, v_*, 0, 0)$, where

$$y_*^3 - y_* - \frac{2}{3}\overline{\gamma} = 0 \quad \text{and} \quad v_*^2 = 1 - y_*^2.$$

There are three solutions for y_*

$$\frac{1}{A} + \frac{A}{3}, -\frac{1+i\sqrt{3}}{2A} - \frac{(1-i\sqrt{3})A}{6} \quad \text{and} \quad -\frac{1-i\sqrt{3}}{2A} - \frac{(1+i\sqrt{3})A}{6}.$$

We will consider each solution to y_*.

- **Fixed points A1**

These fixed points correspond to

$$y_{1*} = \frac{1}{A} + \frac{A}{3}.$$

They exist when

$$-\frac{1}{\sqrt{3}} \leqslant \overline{\gamma} \leqslant 0,$$

while they are stable when

$$\overline{\gamma} > 0.$$

That is they are not stable.

- **Fixed points A2**

These fixed points correspond to

$$y_{2*} = -\frac{1+i\sqrt{3}}{2A} - \frac{(1-i\sqrt{3})A}{6}.$$

They exist when

$$0 \leqslant \overline{\gamma} \leqslant \frac{1}{\sqrt{3}},$$

while they are stable when

$$\overline{\gamma} < 0.$$

Thus they are not stable.

- **Fixed points A3**

These fixed points correspond to

$$y_{3*} = -\frac{1-i\sqrt{3}}{2A} - \frac{(1+i\sqrt{3})A}{6}.$$

They exist when

$$|\overline{\gamma}| \leqslant \frac{1}{\sqrt{3}},$$

while they are stable when

$$|\overline{\gamma}| > \frac{1}{\sqrt{3}}.$$

Again they are not stable.

- **Fixed points B**

These fixed points are $(x, y, v, w, u) = (\sqrt{2/3}y_*, y_*, v_*, 0, 0)$, where

$$y_*^2 - \frac{2}{3}\overline{\gamma}y_* - \left(1 + \sqrt{\frac{2}{3}}\frac{\overline{\gamma}}{\lambda}\right) = 0 \tag{5.32}$$

and

$$v_* = \pm\sqrt{-\frac{2}{3}\overline{\gamma}y_*}.$$

We see that the solutions to y_* depend on the potential. We will consider both of the potentials.

For the exponential potential, from (4.40) we have $\lambda = \eta$. Then y_* has two solutions

$$y_* = \frac{\overline{\gamma}\eta \pm \sqrt{\eta\left(3\sqrt{6}\overline{\gamma} + 9\eta + \overline{\gamma}^2\eta\right)}}{3\eta}. \tag{5.33}$$

- **Fixed points B1**

These fixed points correspond to

$$y_{1*} = \frac{\overline{\gamma}\eta + \sqrt{\eta\left(3\sqrt{6}\overline{\gamma} + 9\eta + \overline{\gamma}^2\eta\right)}}{3\eta} = \frac{\overline{\gamma}}{3} + \frac{1}{3}\sqrt{9 - B}.$$

The existence condition and the stability of these fixed points are shown in the Figure 5.3 (a).

From (5.19) and (5.32) the total equation of state at these fixed points is

$$w_{tot} = -1,$$

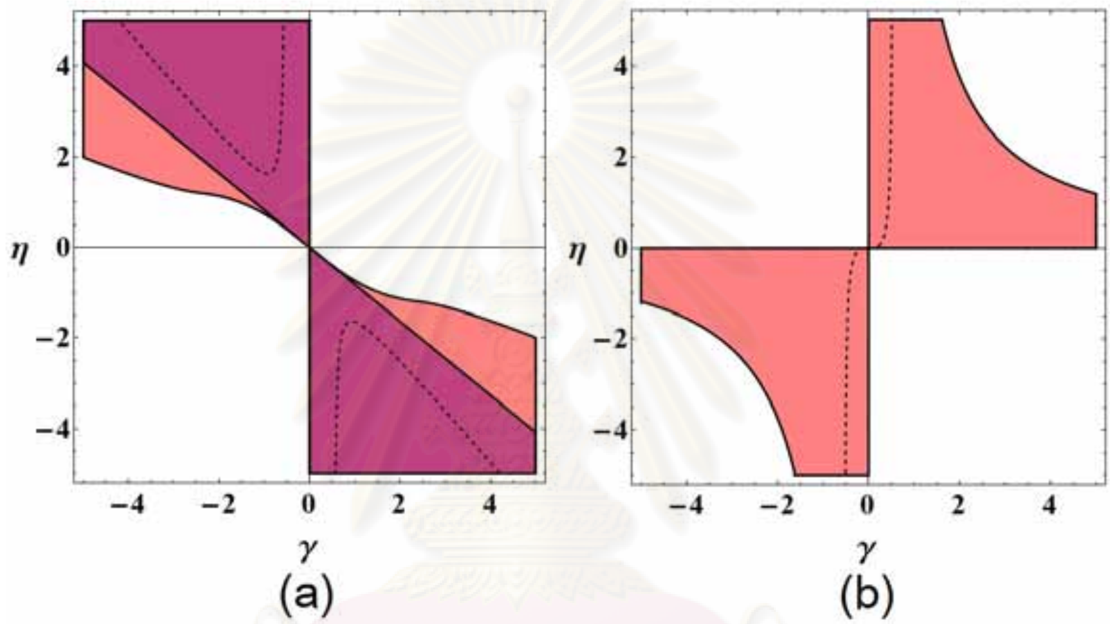

Figure 5.3: This figure shows the region of stability (red, shaded) and existence (blue, shaded) in the $(\overline{\gamma}, \eta)$ parameter space for the exponential potential $V = V_0 e^{-\eta x}$. The violet region indicates the compatible region of two conditions. Dashed line represents the point that the energy parameter ratio of dark energy to dark matter is 7:3. Figure (a) and (b) correspond to y_+ and y_- solution respectively.

therefore the three-form field at these fixed points acts as the cosmological constant like the previous models. From (5.17), we obtain

$$\Omega_X = \frac{9\eta + 2\overline{\gamma}^2\eta + 2\overline{\gamma}\sqrt{\eta\left(3\sqrt{6}\overline{\gamma} + 9\eta + \overline{\gamma}^2\eta\right)}}{9\eta}.$$

From (5.21), the density parameter for the cold dark matter at these fixed points is

$$\Omega_c = -\frac{2\overline{\gamma}^2\eta + 2\overline{\gamma}\sqrt{\eta\left(3\sqrt{6}\overline{\gamma} + 9\eta + \overline{\gamma}^2\eta\right)}}{9\eta}.$$

Therefore the ratio of the dark energy to the dark matter is

$$\frac{\Omega_X}{\Omega_c} = -\frac{9\eta + 2\overline{\gamma}^2\eta + 2\overline{\gamma}\sqrt{\eta\left(3\sqrt{6}\overline{\gamma} + 9\eta + \overline{\gamma}^2\eta\right)}}{2\overline{\gamma}^2\eta + 2\overline{\gamma}\sqrt{\eta\left(3\sqrt{6}\overline{\gamma} + 9\eta + \overline{\gamma}^2\eta\right)}}.$$

To solve the coincidence problem we require $\Omega_X/\Omega_c = 7/3$, that is

$$\eta = \frac{400\sqrt{6}\overline{\gamma}^3}{243 - 840\overline{\gamma}^2}.$$

This is represented by the dashed line in the Figure 5.3 (a). At these fixed points, there is the matter dominated period as shown in the Figure 5.4. Therefore at these fixed points the coincidence problem can be solved.

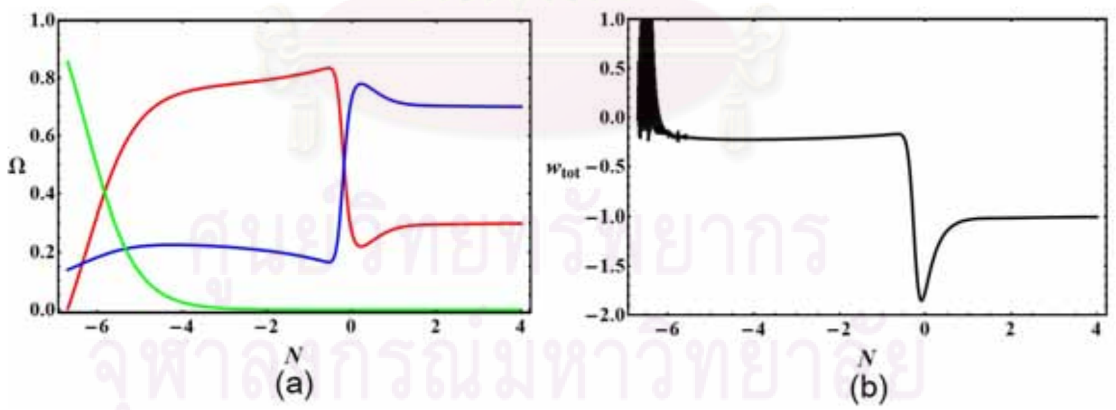

Figure 5.4: (a) shows the evolution of the dynamical variables for the fixed points B1 in extension the coupling model with exponential potential. We use $\gamma = -0.56$ and $\eta = 400\sqrt{6}\overline{\gamma}^3/\left(243 - 840\overline{\gamma}^2\right)$ for this simulation. The red line represents the energy parameter of dark matter (Ω_c), The blue line represents the energy parameter of dark energy (Ω_X) and The green line represents the energy parameter of radiation (Ω_r). (b) shows the evolution behavior of the total equation of state parameter by using the same parameter with (a).

- **Fixed points B2**

These fixed points correspond to

$$y_{2*} = \frac{\overline{\gamma}\eta - \sqrt{\eta\left(3\sqrt{6}\overline{\gamma} + 9\eta + \overline{\gamma}^2\eta\right)}}{3\eta} = \frac{\overline{\gamma}}{3} - \frac{1}{3}\sqrt{9-B};$$

however, they do not exist. Therefore, we are not interested in them.

For the Gaussian potential, from (4.40) we have $\lambda = 2\eta x = \sqrt{8/3}\eta y$ and (5.32) becomes

$$y_*^2 - \left(\frac{2}{3}\right)\overline{\gamma}y_* - \left(1 + \frac{\overline{\gamma}}{2\eta y_*}\right) = 0. \tag{5.34}$$

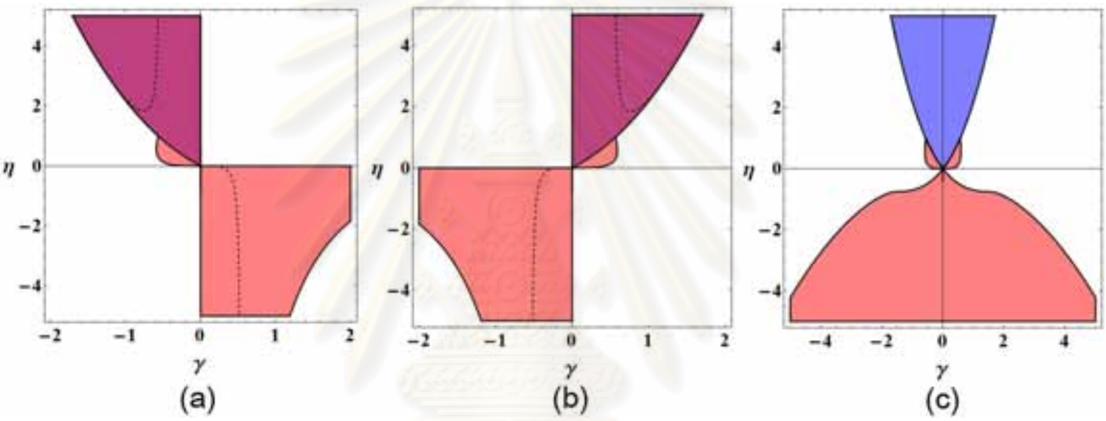

Figure 5.5: This figure shows the region of stability (red, shaded) and existence (blue, shaded) in the (γ, η) parameter space for the Gaussian potential $V = V_0 e^{-\eta x^2}$. The violet region indicates the compatible region of two conditions. Dashed line represents the point that the energy parameter ratio of dark energy and dark matter is 7:3. Figure (a), (b) and (c) correspond to three solutions of equation (5.34).

Then y_* has three solutions. The properties of all the fixed points are shown in the Figure 5.5. That is the fixed points for the Gaussian potential can also solve the coincidence problem.

Chapter VI

Conclusions and discussions

In this thesis, we investigate the possibility of the three-form field to drive inflation and to be dark energy. We find that the three-form field can drive inflation without the slow-roll conditions. This is different from the scalar field for which the slow-roll conditions play an important role in inflation. In the context of dark energy, the three-form field can act as dark energy. However, only the three-form field alone cannot solve the coincidence problem. In order to solve the coincidence problem we need to couple the three-form field to the dark matter. In this thesis, we study four types of the couplings.

In the coupling model (I), there are three types of the fixed points. The fixed points A are not stable. Although the fixed points B are stable, they cannot solve the coincidence problem since they give the dark energy dominated universe. For the fixed points C they can solve the coincidence problem as shown by the dashed line in the Figure 5.1 (a). However, there is no matter dominated period as shown in the Figure 5.1 (b).

For the coupling model (II), there are two types of the fixed points. The fixed points A are not stable as in the model (I). The fixed points B are stable, however they cannot solve the coincidence problem since they give the dark energy dominated universe as in the model (I).

In the coupling model (III), there are three types of the fixed points. The fixed points A are not stable. The properties of the fixed points B are the same as in the previous models. That is they cannot solve the coincidence problem since they give the dark energy dominated universe although they are stable. For the fixed points C they give $\dot{\rho}_c = 0$.

For the coupling model (IV), there are two main types of the fixed points. All of the fixed points A are not stable. For the exponential potential, the fixed points B1 are stable. They can solve the coincidence problem, represented by the dashed line in the Figure 5.3 (a). Moreover, they give the matter dominated period in agreement with the observations. The fixed points B2 do not exist. For the Gaussian potential, the fixed points can also solve the coincidence problem as shown by the dashed line in the Figure 5.5 (a) and (b).

REFERENCES

[1] Riess, A.G. *et al.* Observational Evidence from Supernovae for an Accelerating Universe and a Cosmological Constant. Astron. J. 116 (1998): 1009 - 1038.

[2] Perlmutter, S. *et al.* Measurements of Omega and Lambda from 42 High-Redshift Supernovae. Astrophys. J. 517 (1999): 565 - 586.

[3] Wetterich, C. Cosmology and the fate of dilatation symmetry. Nucl. Phys. B 302 (1988): 668.

[4] Caldwell, R.R., Dave, R., and Steinhardt, P.J. Cosmological Imprint of an Energy Component with General Equation-of-State. Phys. Rev. Lett. 80 (1998): 1582.

[5] Kiselev, V.V. Vector field as a quintessence partner. Class. Quant. Grav. 21 (2004): 3323-3336.

[6] Armendariz-Picon, C. Could dark energy be vector-like? JCAP 0407 (2004): 007.

[7] Boehmer, C.G., and Harko, T. Dark energy as a massive vector field. Eur. Phys. J. C 50 (2007): 423-429.

[8] Novello, M., Perez Bergliaffa S.E., and Salim J. Nonlinear electrodynamics and the acceleration of the Universe. Phys. Rev. D 69 (2004): 127301.

[9] Himmetoglu, B., Contaldi, C.R., and Peloso, M. Instability of the ACW model, and problems with massive vectors during inflation. Phys. Rev. D 79 (2009): 063517.

[10] Germani, C., and Kehagias, A. P-nflation: generating cosmic Inflation with p-forms. JCAP 0903 (2009): 028.

[11] Koivisto, T.S., Mota, D.F., and Pitrou, C. Inflation from N-forms and its stability. JHEP 09 (2009): 092.

[12] Koivisto, T.S., and Nunes, N.J. Inflation and dark energy from three-forms. Phys. Rev. D 80 (2009): 103509.

[13] Ford, L.H. Inflation driven by a vector field. Phys. Rev. D 40 (1989): 967.

[14] Golovnev, A., Mukhanov, V., and Vanchurin, V. Vector Inflation. JCAP 0806 (2008): 009.

[15] Edmund, J.C., Sami, M., and Tsujikawa, S. Dynamics of dark energy. Int. J. Mod. Phys. D 15 (2006): 1753-1936.

[16] Boehmer, C.G., Caldera-Cabral, G., Lazkoz, R., and Maartens, R. Dynamics of dark energy with a coupling to dark matter. Phys. Rev. D 78 (2008): 023505.

[17] Luca, A. Scaling solutions in general non-minimal coupling theories. Phys. Rev. D 60 (1999): 043501.

[18] Luca, A., Quatin, M., and Tsujikawa, S. Challenges for scaling cosmologies. Phys. Rev. D 74 (2006): 023525.

[19] Zimdahl, W., Pavon, D., and Chimento, L.P. Interacting Quintessence. Phys. Lett. B 521 (2001): 133-138.

[20] Olivares, G., Atrio-Barandela, F., and Pavon, D. Dynamics of Interacting Quintessence Models: Observational Constraints. Phys. Rev. D 77 (2008): 063513.

APPENDICES

Appendix A

Calculation of Einstein tensors

The Ricci tensor can be expressed as

$$R_{\mu\nu} = \partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\nu \Gamma^\alpha_{\mu\alpha} + \Gamma^\alpha_{\beta\alpha}\Gamma^\beta_{\mu\nu} - \Gamma^\alpha_{\beta\nu}\Gamma^\beta_{\mu\alpha}, \tag{A.1}$$

where $\Gamma^\alpha_{\mu\nu}$ is the Christoffel symbol given by

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\beta}(\partial_\mu g_{\nu\beta} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu}). \tag{A.2}$$

The metric tensor and its inverse in the FRW metric (2.1) take the form

$$g_{\mu\nu} = diag(-1, a^2(t), a^2(t), a^2(t)), \tag{A.3}$$

$$g^{\mu\nu} = diag(-1, 1/a^2(t), 1/a^2(t), 1/a^2(t)). \tag{A.4}$$

Using the FRW metric (2.1), we can obtain all components of the Christoffel symbol,

$$\begin{aligned}
\Gamma^0_{00} &= \frac{1}{2} g^{0\beta}(\partial_0 g_{0\beta} + \partial_0 g_{\beta 0} - \partial_\beta g_{00}) \\
&= \frac{1}{2} g^{00}(\partial_0 g_{00} + \partial_0 g_{00} - \partial_0 g_{00}) \\
&= \frac{1}{2} g^{00}\partial_0 g_{00} \\
&= 0,
\end{aligned} \tag{A.5}$$

$$\begin{aligned}
\Gamma^0_{0i} &= \frac{1}{2} g^{0\beta}(\partial_0 g_{i\beta} + \partial_i g_{\beta 0} - \partial_\beta g_{0i}) \\
&= \frac{1}{2} g^{00}(\partial_0 g_{i0} + \partial_i g_{00}) \\
&= 0,
\end{aligned} \tag{A.6}$$

$$\begin{aligned}
\Gamma^0_{ij} &= \frac{1}{2}g^{0\beta}(\partial_i g_{j\beta} + \partial_j g_{\beta 0} - \partial_\beta g_{ij}) \\
&= \frac{1}{2}g^{00}(\partial_i g_{j0} + \partial_j g_{00} - \partial_0 g_{ij}) \\
&= -\frac{1}{2}(-\partial_0 a^2 \delta_{ij}) \\
&= a\dot{a}\delta_{ij}, \qquad \text{(A.7)} \\
\Gamma^i_{00} &= \frac{1}{2}g^{i\beta}(\partial_0 g_{0\beta} + \partial_0 g_{\beta 0} - \partial_\beta g_{00}) \\
&= \frac{1}{2}g^{ij}(\partial_0 g_{0j} + \partial_0 g_{j0} - \partial_j g_{00}) \\
&= 0, \qquad \text{(A.8)} \\
\Gamma^i_{j0} &= \frac{1}{2}g^{i\beta}(\partial_j g_{0\beta} + \partial_0 g_{\beta j} - \partial_\beta g_{j0}) \\
&= \frac{1}{2}g^{ik}(\partial_j g_{0k} + \partial_0 g_{kj} - \partial_k g_{j0}) \\
&= \frac{1}{2a^2}\delta^{ik}\partial_0 a^2 \delta_{kj} \\
&= \frac{\dot{a}}{a}\delta^i_j, \qquad \text{(A.9)} \\
\Gamma^i_{jk} &= \frac{1}{2}g^{i\beta}(\partial_j g_{k\beta} + \partial_k g_{\beta j} - \partial_\beta g_{jk}) \\
&= \frac{1}{2}g^{il}(\partial_j g_{kl} + \partial_k g_{lj} - \partial_l g_{jk}) \\
&= 0. \qquad \text{(A.10)}
\end{aligned}$$

Then we can derive the Ricci tensors

$$\begin{aligned}
R_{00} &= \partial_\alpha \Gamma^\alpha_{00} - \partial_0 \Gamma^\alpha_{0\alpha} + \Gamma^\alpha_{\beta\alpha}\Gamma^\beta_{00} - \Gamma^\alpha_{\beta 0}\Gamma^\beta_{0\alpha} \\
&= -\partial_0 \Gamma^i_{0i} - \Gamma^i_{\beta 0}\Gamma^\beta_{0i} \\
&= -3\partial_0(\frac{\dot{a}}{a}) - \Gamma^i_{j0}\Gamma^j_{0i} \\
&= -3(\frac{a\ddot{a} - \dot{a}^2}{a^2}) - 3(\frac{\dot{a}}{a})^2 \\
&= -3(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} + \frac{\dot{a}^2}{a^2}) \\
&= -3\frac{\ddot{a}}{a}, \qquad \text{(A.11)} \\
R_{0i} &= \partial_\alpha \Gamma^\alpha_{0i} - \partial_i \Gamma^\alpha_{0\alpha} + \Gamma^\alpha_{\beta\alpha}\Gamma^\beta_{0i} - \Gamma^\alpha_{\beta i}\Gamma^\beta_{0\alpha} \\
&= \partial_j \Gamma^j_{0i} - \partial_i \Gamma^j_{0j} + \Gamma^\alpha_{j\alpha}\Gamma^j_{0i} - \Gamma^j_{\beta i}\Gamma^\beta_{0j} \\
&= 0, \qquad \text{(A.12)}
\end{aligned}$$

$$
\begin{aligned}
R_{ij} &= \partial_\alpha \Gamma^\alpha_{ij} - \partial_j \Gamma^\alpha_{i\alpha} + \Gamma^\alpha_{\beta\alpha}\Gamma^\beta_{ij} - \Gamma^\alpha_{\beta j}\Gamma^\beta_{i\alpha} \\
&= \partial_0 \Gamma^0_{ij} + \Gamma^\alpha_{0\alpha}\Gamma^0_{ij} - \Gamma^0_{\beta j}\Gamma^\beta_{i0} - \Gamma^k_{\beta j}\Gamma^\beta_{ik} \\
&= \partial_0(a\dot{a})\delta_{ij} + \Gamma^k_{0k}\Gamma^0_{ij} - \Gamma^0_{kj}\Gamma^k_{i0} - \Gamma^k_{0j}\Gamma^0_{ik} \\
&= (a\ddot{a} + \dot{a}^2)\delta_{ij} + 3\frac{\dot{a}}{a}a\dot{a}\delta_{ij} - a\dot{a}\delta_{kj}\frac{\dot{a}}{a}\delta^k_i - \frac{\dot{a}}{a}\delta^k_j a\dot{a}\delta_{ik} \\
&= (a\ddot{a} + \dot{a}^2)\delta_{ij} + 3\dot{a}^2\delta_{ij} - \dot{a}^2\delta_{ij} - \dot{a}^2\delta_{ij} \\
&= (a\ddot{a} + 2\dot{a}^2)\delta_{ij}. \qquad \text{(A.13)}
\end{aligned}
$$

Now, we can find the Ricci scalar by contracting the Ricci tensors,

$$
\begin{aligned}
R &\equiv g^{\mu\nu}R_{\mu\nu} \\
&= 3\frac{\ddot{a}}{a} + \frac{3}{a^2}(a\ddot{a} + 2\dot{a}^2) \\
&= 6(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}). \qquad \text{(A.14)}
\end{aligned}
$$

The Einstein tensors can be obtained.

$$
\begin{aligned}
G_{00} &= R_{00} - \frac{1}{2}Rg_{00} \\
&= -3\frac{\ddot{a}}{a} + 3(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}) \\
&= 3\frac{\dot{a}^2}{a^2}, \qquad \text{(A.15)} \\
G_{0i} &= R_{0i} - \frac{1}{2}Rg_{0i} \\
&= 0, \qquad \text{(A.16)} \\
G_{ij} &= R_{ij} - \frac{1}{2}Rg_{ij} \\
&= (a\ddot{a} + 2\dot{a}^2)\delta_{ij} - 3a^2(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2})\delta_{ij} \\
&= (a\ddot{a} + 2\dot{a}^2)\delta_{ij} - 3a\ddot{a}\delta_{ij} - 3\dot{a}^2\delta_{ij} \\
&= (-2a\ddot{a} - \dot{a}^2)\delta_{ij}. \qquad \text{(A.17)}
\end{aligned}
$$

We can raise a lower index of the Einstein tensors by using the inverse metric tensor

$$
\begin{aligned}
G^0_0 &= g^{0\mu}G_{\mu 0} \\
&= g^{00}G_{00} \\
&= -3\frac{\dot{a}^2}{a^2}, \qquad \text{(A.18)} \\
G^i_j &= g^{i\mu}G_{\mu j} \\
&= g^{ik}G_{kj} \\
&= \frac{1}{a^2}\delta^{ik}(-2a\ddot{a} - \dot{a}^2)\delta_{kj} \\
&= (-\frac{2\ddot{a}}{a} - \frac{\dot{a}^2}{a^2})\delta^i_j. \qquad \text{(A.19)}
\end{aligned}
$$

Appendix B

Equations of motion of an n-form field from the variational principle

B.1 n-form field

We consider an n-form field A in 4-dimensional spacetime with the action

$$S_A = \int d^4x\sqrt{-g}\left[\frac{R}{16\pi G} - \frac{1}{2(n+1)!}F^2 - V(A^2) - \frac{1}{2n!}\xi A^2 R\right], \tag{B.1}$$

where $F_{\mu_1 \dots \mu_{n+1}} \equiv (n+1)\nabla_{[\mu_1}A_{\mu_2 \dots \mu_{n+1}]}$ and $A^2 \equiv A^{\mu_1 \cdots \mu_n}A_{\mu_1 \dots \mu_n}$. The equation of motion of the n-form field is obtained by using the Euler-Lagrange equations

$$\frac{\partial L}{\partial A^{\mu_n}} - \nabla^\mu\left(\frac{\partial L}{\partial(\nabla^\mu A^{\mu_n})}\right) = 0, \tag{B.2}$$

where the Lagrangian of the n-form field L is given by

$$L = -\frac{1}{2(n+1)!}F^2 - V(A^2) - \frac{1}{2n!}\xi A^2 R. \tag{B.3}$$

Each term in (B.2) can be computed as follows.

$$\begin{aligned}
\frac{\partial L}{\partial A^{\mu_1 \cdots \mu_n}} &= -\frac{dV}{dA^2}\frac{\partial(A^{\alpha_1 \dots \alpha_n}A^{\beta_1 \dots \beta_n}g_{\beta_1\alpha_1}\dots g_{\beta_n\alpha_n})}{\partial A^{\mu_1 \cdots \mu_n}} - \frac{\xi R}{2n!}\frac{\partial(A^{\alpha_1 \dots \alpha_n}A^{\beta_1 \dots \beta_n}g_{\beta_1\alpha_1}\dots g_{\beta_n\alpha_n})}{\partial A^{\mu_1 \cdots \mu_n}} \\
&= \left(-V' - \frac{\xi R}{2n!}\right)(g_{\beta_1\alpha_1}\dots g_{\beta_n\alpha_n})\frac{\partial(A^{\alpha_1 \dots \alpha_n}A^{\beta_1 \dots \beta_n})}{\partial A^{\mu_1 \cdots \mu_n}} \\
&= -\left(V' + \frac{\xi R}{2n!}\right)(g_{\beta_1\alpha_1}\dots g_{\beta_n\alpha_n})(A^{\alpha_1 \dots \alpha_n}\delta^{\beta_1}_{\mu_1}\dots\delta^{\beta_n}_{\mu_n} + \delta^{\alpha_1}_{\mu_1}\dots\delta^{\alpha_n}_{\mu_n}A^{\beta 1 \dots \beta_n}) \\
&= -\left(V' + \frac{\xi R}{2n!}\right)(A_{\mu_1 \dots \mu_n} + A_{\mu_1 \dots \mu_n}) \\
&= -\left(2V' + \frac{\xi R}{n!}\right)A_{\mu_1 \dots \mu_n}.
\end{aligned}$$

Since

$$\begin{aligned}
\frac{\partial L}{\partial(\nabla^\mu A^{\mu_1\cdots\mu_n})} &= -\frac{1}{2(n+1)!}(g_{\beta_1\alpha_1}...g_{\beta_{n+1}\alpha_{n+1}})\frac{\partial}{\partial(\nabla^\mu A^{\mu_1\cdots\mu_n})}(F^{\alpha 1\ldots\alpha_{n+1}}F^{\beta_1\ldots\beta_{n+1}}) \\
&= -\frac{1}{2(n+1)!}(g_{\beta_1\alpha_1}...g_{\beta_{n+1}\alpha_{n+1}})\left[F^{\alpha_1\ldots\alpha_{n+1}}\frac{\partial}{\partial(\nabla^\mu A^{\mu_1\cdots\mu_n})}\right. \\
&\quad \left(\frac{n+1}{(n+1)!}(\nabla^{\beta_1}A^{\beta_2\ldots\beta_{n+1}}+...)\right)+F^{\beta_1\ldots\beta_{n+1}} \\
&\quad \left.\frac{\partial}{\partial(\nabla^\mu A^{\mu_1\cdots\mu_n})}\left(\frac{n+1}{(n+1)!}(\nabla^{\alpha_1}A^{\alpha_2\ldots\alpha_{n+1}}+...)\right)\right] \\
&= -\frac{1}{2(n+1)!n!}(g_{\beta_1\alpha_1}...g_{\beta_{n+1}\alpha_{n+1}})\left[F^{\alpha_1\ldots\alpha_{n+1}}\left(\delta^{\beta_1}_{\mu}\delta^{\beta_2}_{\mu_1}...\delta^{\beta_{n+1}}_{\mu_n}\right)\right. \\
&\quad \left.+F^{\beta_1\ldots\beta_{n+1}}\left(\delta^{\alpha_1}_{\mu}\delta^{\alpha_2}_{\mu_1}...\delta^{\alpha_{n+1}}_{\mu_n}\right)\right] \\
&= -\frac{1}{2(n+1)!n!}\left[F^{\alpha_1\ldots\alpha_{n+1}}(g_{\mu\alpha_1}...g_{\mu_n\alpha_{n+1}})+F^{\beta_1\ldots\beta_{n+1}}(g_{\beta_1\mu}...g_{\beta_{n+1}\mu_n})\right] \\
&= -\frac{1}{2(n+1)!n!}\left(F_{\mu\mu_1\ldots\mu_n}+...+F_{\mu\mu_1\ldots\mu_n}\right) \\
&= -\frac{1}{n!}F_{\mu\mu 1\ldots\mu_n},
\end{aligned}$$

we have

$$\nabla^\mu\left(\frac{\partial L}{\partial(\nabla^\mu A^{\mu_1\cdots\mu_n})}\right) = -\frac{1}{n!}\nabla^\mu F_{\mu\mu 1\ldots\mu_n}.$$

Then the equation of motion (B.2) can be written as

$$\nabla^\mu F_{\mu\mu_1\ldots\mu_n} = (2n!V' + \xi R)\, A_{\mu_1\ldots\mu_n}. \tag{B.4}$$

Variation of action (B.1) yields

$$\begin{aligned}
\delta S_A = \int d^4x &\left[\sqrt{-g}\,\delta\left(-\frac{1}{2(n+1)!}F^2 - V(A^2) - \frac{1}{2n!}\xi A^2 R\right)\right. \\
&\left.+\delta\sqrt{-g}\left(-\frac{1}{2(n+1)!}F^2 - V(A^2) - \frac{1}{2n!}\xi A^2 R\right)\right].
\end{aligned} \tag{B.5}$$

In order to calculate the right-handed side of (B.5), we consider

$$\begin{aligned}
\delta F^2 &= \delta(F_{\mu_1...\mu_{n+1}} F_{\nu_1...\nu_{n+1}} g^{\mu_1\nu_1}...g^{\mu_{n+1}\nu_{n+1}}) \\
&= F_{\mu_1...\mu_{n+1}} F_{\nu_1...\nu_{n+1}} \delta(g^{\mu_1\nu_1}...g^{\mu_{n+1}\nu_{n+1}}) \\
&= F_{\mu_1...\mu_{n+1}} F_{\nu_1...\nu_{n+1}} \left[(g^{\mu_2\nu_2}...g^{\mu_{n+1}\nu_{n+1}})\delta g^{\mu_1\nu_1}\right. \\
&\quad \left. +(g^{\mu_1\nu_1} g^{\mu_3\nu_3}...g^{\mu_{n+1}\nu_{n+1}})\delta g^{\mu_2\nu_2} + ... + (g^{\mu_1\nu_1}...g^{\mu_n\nu_n})\delta g^{\mu_{n+1}\nu_{n+1}}\right] \\
&= F_{\mu_1\mu_2...\mu_{n+1}} F_{\nu_1\nu_2...\nu_{n+1}} (g^{\mu_2\nu_2}...g^{\mu_{n+1}\nu_{n+1}})\delta g^{\mu_1\nu_1} \\
&\quad +F_{\mu_1\mu_2...\mu_{n+1}} F_{\nu_1\nu_2...\nu_{n+1}} (g^{\mu_1\nu_1} g^{\mu_3\nu_3}...g^{\mu_{n+1}\nu_{n+1}})\delta g^{\mu_2\nu_2} \\
&\quad +... + F_{\mu_1...\mu_{n+1}} F_{\nu_1...\nu_{n+1}} (g^{\mu_1\nu_1}...g^{\mu_n\nu_n})\delta g^{\mu_{n+1}\nu_{n+1}} \\
&= F_{\mu_1\mu_2...\mu_{n+1}} F_{\nu_1\nu_2...\nu_{n+1}} (g^{\mu_2\nu_2}...g^{\mu_{n+1}\nu_{n+1}})\delta g^{\mu_1\nu_1} \\
&\quad +F_{\mu_2\mu_1\mu_3...\mu_{n+1}} F_{\nu_2\nu_1\nu_3...\nu_{n+1}} (g^{\mu_2\nu_2} g^{\mu_3\nu_3}...g^{\mu_{n+1}\nu_{n+1}})\delta g^{\mu_1\nu_1} \\
&\quad +... + F_{\mu_{n+1}\mu_2...\mu_n\mu_1} F_{\nu_{n+1}\nu_2...\nu_n\nu_1} (g^{\mu_{n+1}\nu_{n+1}} g^{\mu_2\nu_2}...g^{\mu_n\nu_n})\delta g^{\mu_1\nu_1} \\
&= F_{\mu_1\mu_2...\mu_{n+1}} F_{\nu_1\nu_2...\nu_{n+1}} (g^{\mu_2\nu_2}...g^{\mu_{n+1}\nu_{n+1}})\delta g^{\mu_1\nu_1} \\
&\quad +F_{\mu_1\mu_2\mu_3...\mu_{n+1}} F_{\nu_1\nu_2\nu_3...\nu_{n+1}} (g^{\mu_2\nu_2} g^{\mu_3\nu_3}...g^{\mu_{n+1}\nu_{n+1}})\delta g^{\mu_1\nu_1} \\
&\quad +... + F_{\mu_1\mu_2...\mu_n\mu_{n+1}} F_{\nu_1\nu_2...\nu_n\nu_{n+1}} (g^{\mu_2\nu_2}...g^{\mu_n\nu_n} g^{\mu_{n+1}\nu_{n+1}})\delta g^{\mu_1\nu_1} \\
&= (n+1) F_{\mu_1\mu_2...\mu_{n+1}} F_{\nu_1\nu_2...\nu_{n+1}} (g^{\mu_2\nu_2}...g^{\mu_{n+1}\nu_{n+1}})\delta g^{\mu_1\nu_1} \\
&= (n+1) F_{\mu_1\mu_2...\mu_{n+1}} F_{\nu_1}^{\mu_2...\mu_{n+1}} \delta g^{\mu_1\nu_1}.
\end{aligned} \tag{B.6}$$

Similarly, we get

$$\delta A^2 = n A_{\mu_1\mu_2...\mu_n} A_{\nu_1}^{\mu_2...\mu_n} \delta g^{\mu_1\nu_1}. \tag{B.7}$$

Next we compute the variation of the metric $\delta g_{\mu\nu}$. Since

$$g^{\mu\lambda} g_{\lambda\nu} = \delta^\mu_\nu,$$

then

$$\begin{aligned}
g^{\mu\lambda}\delta g_{\lambda\nu} + g_{\lambda\nu}\delta g^{\mu\lambda} &= 0 \\
g^{\mu\lambda}\delta g_{\lambda\nu} &= -g_{\lambda\nu}\delta g^{\mu\lambda} \\
g_{\alpha\mu} g^{\mu\lambda}\delta g_{\lambda\nu} &= -g_{\alpha\mu} g_{\lambda\nu}\delta g^{\mu\lambda} \\
\delta^\lambda_\alpha \delta g_{\lambda\nu} &= -g_{\alpha\mu} g_{\lambda\nu}\delta g^{\mu\lambda} \\
\delta g_{\alpha\nu} &= -g_{\alpha\mu} g_{\nu\lambda}\delta g^{\mu\lambda}.
\end{aligned} \tag{B.8}$$

Next we will find $\delta\sqrt{-g}$. To do this we use the fact that for any square matrix M with nonvanishing determinant

$$\ln(\det M) = \mathrm{Tr}(\ln M), \tag{B.9}$$

where we use the fact that $\exp(\ln M) = M$. The variation of (B.9) gives

$$\frac{1}{\det M}\delta(\det M) = \mathrm{Tr}(M^{-1}\delta M). \tag{B.10}$$

For the metric $g_{\mu\nu}$, we get

$$\delta g = g(g^{\mu\nu}\delta g_{\mu\nu}).$$

Using (B.8), we get

$$\begin{aligned} \delta g &= -g(g^{\mu\nu}g_{\mu\alpha}g_{\nu\beta}\delta g^{\alpha\beta}) \\ &= -g(\delta^{\nu}_{\alpha}g_{\nu\beta}\delta g^{\alpha\beta}) \\ &= -g(g_{\alpha\beta}\delta g^{\alpha\beta}). \end{aligned} \tag{B.11}$$

Finally, we obtain

$$\begin{aligned} \delta\sqrt{-g} &= -\frac{1}{2\sqrt{-g}}\delta g \\ &= \frac{g}{2\sqrt{-g}}g_{\mu\nu}\delta g^{\mu\nu} \\ &= -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}. \end{aligned} \tag{B.12}$$

Now we compute the variation of the Christoffel symbols $\delta\Gamma^{\alpha}_{\mu\nu}$:

$$\begin{aligned} \delta\Gamma^{\alpha}_{\mu\nu} &= \frac{1}{2}\delta\left[g^{\alpha\beta}(\partial_\mu g_{\nu\beta} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu})\right] \\ &= \frac{1}{2}\left[\delta g^{\alpha\beta}(\partial_\mu g_{\nu\beta} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu})\right. \\ &\quad \left. + g^{\alpha\beta}(\partial_\mu \delta g_{\nu\beta} + \partial_\nu \delta g_{\beta\mu} - \partial_\beta \delta g_{\mu\nu})\right], \end{aligned}$$

and $\partial_\mu\delta g_{\nu\beta}$,

$$\begin{aligned} \partial_\mu\delta g_{\nu\beta} &= -\partial_\mu\left(g_{\nu\rho}g_{\beta\gamma}\delta g^{\rho\gamma}\right) \\ &= -(\partial_\mu g_{\nu\rho})g_{\beta\gamma}\delta g^{\rho\gamma} - g_{\nu\rho}(\partial_\mu g_{\beta\gamma})\delta g^{\rho\gamma} - g_{\nu\rho}g_{\beta\gamma}\partial_\mu\delta g^{\rho\gamma}. \end{aligned}$$

Then, we get

$$
\begin{aligned}
\delta\Gamma^{\alpha}_{\mu\nu} &= \frac{1}{2}\left[(\partial_\mu g_{\nu\beta})\delta g^{\alpha\beta} + (\partial_\nu g_{\beta\mu})\delta g^{\alpha\beta} - (\partial_\beta g_{\mu\nu})\delta g^{\alpha\beta} + g^{\alpha\beta}(-(\partial_\mu g_{\nu\rho})g_{\beta\gamma}\delta g^{\rho\gamma}\right.\\
&\quad -g_{\nu\rho}(\partial_\mu g_{\beta\gamma})\delta g^{\rho\gamma} - g_{\nu\rho}g_{\beta\gamma}\partial_\mu\delta g^{\rho\gamma} - (\partial_\nu g_{\beta\rho})g_{\mu\gamma}\delta g^{\rho\gamma} - g_{\beta\rho}(\partial_\nu g_{\mu\gamma})\delta g^{\rho\gamma}\\
&\quad \left.-g_{\beta\rho}g_{\mu\gamma}\partial_\nu\delta g^{\rho\gamma} + (\partial_\beta g_{\mu\rho})g_{\nu\gamma}\delta g^{\rho\gamma} + g_{\mu\rho}(\partial_\beta g_{\nu\gamma})\delta g^{\rho\gamma} + g_{\mu\rho}g_{\nu\gamma}\partial_\beta\delta g^{\rho\gamma})\right]\\
&= \frac{1}{2}\left[(\partial_\mu g_{\nu\beta})\delta g^{\alpha\beta} + (\partial_\nu g_{\beta\mu})\delta g^{\alpha\beta} - (\partial_\beta g_{\mu\nu})\delta g^{\alpha\beta} - (\partial_\mu g_{\nu\rho})\delta^\alpha_\gamma\delta g^{\rho\gamma}\right.\\
&\quad -g^{\alpha\beta}g_{\nu\rho}(\partial_\mu g_{\beta\gamma})\delta g^{\rho\gamma} - g_{\nu\rho}\delta^\alpha_\gamma\partial_\mu\delta g^{\rho\gamma} - (\partial_\nu g_{\beta\rho})g^{\alpha\beta}g_{\mu\gamma}\delta g^{\rho\gamma} - \delta^\alpha_\rho(\partial_\nu g_{\mu\gamma})\delta g^{\rho\gamma}\\
&\quad \left.-\delta^\alpha_\rho g_{\mu\gamma}\partial_\nu\delta g^{\rho\gamma} + (\partial_\beta g_{\mu\rho})g^{\alpha\beta}g_{\nu\gamma}\delta g^{\rho\gamma} + g^{\alpha\beta}g_{\mu\rho}(\partial_\beta g_{\nu\gamma})\delta g^{\rho\gamma} + g^{\alpha\beta}g_{\mu\rho}g_{\nu\gamma}\partial_\beta\delta g^{\rho\gamma}\right]\\
&= \frac{1}{2}\left[(\partial_\mu g_{\nu\beta})\delta g^{\alpha\beta} + (\partial_\nu g_{\beta\mu})\delta g^{\alpha\beta} - (\partial_\beta g_{\mu\nu})\delta g^{\alpha\beta} - (\partial_\mu g_{\nu\rho})\delta g^{\rho\alpha}\right.\\
&\quad -g^{\alpha\beta}g_{\nu\rho}(\partial_\mu g_{\beta\gamma})\delta g^{\rho\gamma} - g_{\nu\rho}\partial_\mu\delta g^{\rho\alpha} - (\partial_\nu g_{\beta\rho})g^{\alpha\beta}g_{\mu\gamma}\delta g^{\rho\gamma} - (\partial_\nu g_{\mu\gamma})\delta g^{\alpha\gamma}\\
&\quad \left.-g_{\mu\gamma}\partial_\nu\delta g^{\alpha\gamma} + (\partial_\beta g_{\mu\rho})g^{\alpha\beta}g_{\nu\gamma}\delta g^{\rho\gamma} + g^{\alpha\beta}g_{\mu\rho}(\partial_\beta g_{\nu\gamma})\delta g^{\rho\gamma} + g^{\alpha\beta}g_{\mu\rho}g_{\nu\gamma}\partial_\beta\delta g^{\rho\gamma}\right]\\
&= \frac{1}{2}\left[(\partial_\mu g_{\nu\beta})\delta g^{\alpha\beta} + (\partial_\nu g_{\beta\mu})\delta g^{\alpha\beta} - (\partial_\beta g_{\mu\nu})\delta g^{\alpha\beta} - (\partial_\mu g_{\nu\beta})\delta g^{\beta\alpha}\right.\\
&\quad -g^{\alpha\beta}g_{\nu\rho}(\partial_\mu g_{\beta\gamma})\delta g^{\rho\gamma} - g_{\nu\beta}\partial_\mu\delta g^{\beta\alpha} - (\partial_\nu g_{\beta\rho})g^{\alpha\beta}g_{\mu\gamma}\delta g^{\rho\gamma} - (\partial_\nu g_{\mu\beta})\delta g^{\alpha\beta}\\
&\quad \left.-g_{\mu\beta}\partial_\nu\delta g^{\alpha\beta} + (\partial_\beta g_{\mu\rho})g^{\alpha\beta}g_{\nu\gamma}\delta g^{\rho\gamma} + g^{\alpha\beta}g_{\mu\rho}(\partial_\beta g_{\nu\gamma})\delta g^{\rho\gamma} + g^{\alpha\beta}g_{\mu\rho}g_{\nu\gamma}\partial_\beta\delta g^{\rho\gamma}\right]\\
&= \frac{1}{2}\left[-(\partial_\beta g_{\mu\nu})\delta g^{\alpha\beta} - g^{\alpha\beta}g_{\nu\rho}(\partial_\mu g_{\beta\gamma})\delta g^{\rho\gamma} - g_{\nu\beta}\partial_\mu\delta g^{\alpha\beta}\right.\\
&\quad -(\partial_\nu g_{\beta\rho})g^{\alpha\beta}g_{\mu\gamma}\delta g^{\rho\gamma} - g_{\mu\beta}\partial_\nu\delta g^{\alpha\beta} + (\partial_\beta g_{\mu\rho})g^{\alpha\beta}g_{\nu\gamma}\delta g^{\rho\gamma}\\
&\quad \left.+g^{\alpha\beta}g_{\mu\rho}(\partial_\beta g_{\nu\gamma})\delta g^{\rho\gamma} + g^{\alpha\beta}g_{\mu\rho}g_{\nu\gamma}\partial_\beta\delta g^{\rho\gamma}\right].
\end{aligned}
$$

In the above equation, we make use of the metric compatibility $\nabla_\rho g_{\mu\nu} = 0$, to show that

$$\partial_\rho g_{\mu\nu} = \Gamma^\lambda_{\rho\mu}g_{\lambda\nu} + \Gamma^\lambda_{\rho\nu}g_{\mu\lambda}. \tag{B.13}$$

Therefore,

$$
\begin{aligned}
\delta\Gamma^{\alpha}_{\mu\nu} &= \frac{1}{2}\left[-\Gamma^\lambda_{\beta\mu}g_{\lambda\nu}\delta g^{\alpha\beta} - \Gamma^\lambda_{\beta\nu}g_{\mu\lambda}\delta g^{\alpha\beta} - g^{\alpha\beta}g_{\nu\rho}\Gamma^\lambda_{\mu\beta}g_{\lambda\gamma}\delta g^{\rho\gamma} - g^{\alpha\beta}g_{\nu\rho}\Gamma^\lambda_{\mu\gamma}g_{\beta\lambda}\delta g^{\rho\gamma}\right.\\
&\quad -g_{\nu\beta}\partial_\mu\delta g^{\alpha\beta} - \Gamma^\lambda_{\nu\beta}g_{\lambda\rho}g^{\alpha\beta}g_{\mu\gamma}\delta g^{\rho\gamma} - \Gamma^\lambda_{\nu\rho}g_{\beta\lambda}g^{\alpha\beta}g_{\mu\gamma}\delta g^{\rho\gamma}\\
&\quad -g_{\mu\beta}\partial_\nu\delta g^{\alpha\beta} + \Gamma^\lambda_{\beta\mu}g_{\lambda\rho}g^{\alpha\beta}g_{\nu\gamma}\delta g^{\rho\gamma} + \Gamma^\lambda_{\beta\rho}g_{\mu\lambda}g^{\alpha\beta}g_{\nu\gamma}\delta g^{\rho\gamma}\\
&\quad \left.+g^{\alpha\beta}g_{\mu\rho}\Gamma^\lambda_{\beta\nu}g_{\lambda\gamma}\delta g^{\rho\gamma} + g^{\alpha\beta}g_{\mu\rho}\Gamma^\lambda_{\beta\gamma}g_{\nu\lambda}\delta g^{\rho\gamma} + g^{\alpha\beta}g_{\mu\rho}g_{\nu\gamma}\partial_\beta\delta g^{\rho\gamma}\right]\\
&= \frac{1}{2}\left[-\Gamma^\lambda_{\beta\mu}g_{\lambda\nu}\delta g^{\alpha\beta} - \Gamma^\lambda_{\beta\nu}g_{\mu\lambda}\delta g^{\alpha\beta} - \delta^\alpha_\lambda g_{\nu\rho}\Gamma^\lambda_{\mu\gamma}\delta g^{\rho\gamma} - g_{\nu\beta}\partial_\mu\delta g^{\alpha\beta} - \Gamma^\lambda_{\nu\rho}\delta^\alpha_\lambda g_{\mu\gamma}\delta g^{\rho\gamma}\right.\\
&\quad \left.-g_{\mu\beta}\partial_\nu\delta g^{\alpha\beta} + \Gamma^\lambda_{\beta\rho}g_{\mu\lambda}g^{\alpha\beta}g_{\nu\gamma}\delta g^{\rho\gamma} + g^{\alpha\beta}g_{\mu\rho}\Gamma^\lambda_{\beta\gamma}g_{\nu\lambda}\delta g^{\rho\gamma} + g^{\alpha\beta}g_{\mu\rho}g_{\nu\gamma}\partial_\beta\delta g^{\rho\gamma}\right]\\
&= \frac{1}{2}\left[-\Gamma^\lambda_{\beta\mu}g_{\lambda\nu}\delta g^{\alpha\beta} - \Gamma^\lambda_{\beta\nu}g_{\mu\lambda}\delta g^{\alpha\beta} - g_{\nu\rho}\Gamma^\alpha_{\mu\gamma}\delta g^{\rho\gamma} - g_{\nu\beta}\partial_\mu\delta g^{\alpha\beta} - \Gamma^\alpha_{\nu\rho}g_{\mu\gamma}\delta g^{\rho\gamma}\right.\\
&\quad \left.-g_{\mu\beta}\partial_\nu\delta g^{\alpha\beta} + \Gamma^\lambda_{\beta\rho}g_{\mu\lambda}g^{\alpha\beta}g_{\nu\gamma}\delta g^{\rho\gamma} + g^{\alpha\beta}g_{\mu\rho}\Gamma^\lambda_{\beta\gamma}g_{\nu\lambda}\delta g^{\rho\gamma} + g^{\alpha\beta}g_{\mu\rho}g_{\nu\gamma}\partial_\beta\delta g^{\rho\gamma}\right]\\
&= \frac{1}{2}\left[-\Gamma^\beta_{\mu\lambda}g_{\beta\nu}\delta g^{\alpha\lambda} - \Gamma^\beta_{\nu\lambda}g_{\mu\beta}\delta g^{\alpha\lambda} - g_{\nu\beta}\Gamma^\alpha_{\mu\lambda}\delta g^{\lambda\beta} - g_{\nu\beta}\partial_\mu\delta g^{\alpha\beta} - \Gamma^\alpha_{\nu\lambda}g_{\mu\beta}\delta g^{\lambda\beta}\right.\\
&\quad \left.-g_{\mu\beta}\partial_\nu\delta g^{\alpha\beta} + \Gamma^\rho_{\beta\lambda}g_{\mu\rho}g^{\alpha\beta}g_{\nu\gamma}\delta g^{\lambda\gamma} + g^{\alpha\beta}g_{\mu\rho}\Gamma^\gamma_{\beta\lambda}g_{\nu\gamma}\delta g^{\rho\lambda} + g^{\alpha\beta}g_{\mu\rho}g_{\nu\gamma}\partial_\beta\delta g^{\rho\gamma}\right]\\
&= -\frac{1}{2}\left[g_{\nu\beta}\nabla_\mu\delta g^{\alpha\beta} + g_{\mu\beta}\nabla_\nu\delta g^{\alpha\beta} - g^{\alpha\beta}g_{\mu\rho}g_{\nu\gamma}\nabla_\beta\delta g^{\rho\gamma}\right].
\end{aligned}
\tag{B.14}
$$

From (2.3), the variation of the Ricci tensor is

$$
\begin{aligned}
\delta R_{\mu\nu} &= \partial_\alpha \delta\Gamma^\alpha_{\mu\nu} - \partial_\nu \delta\Gamma^\alpha_{\mu\alpha} + \Gamma^\alpha_{\beta\alpha}\delta\Gamma^\beta_{\mu\nu} + \delta\Gamma^\alpha_{\beta\alpha}\Gamma^\beta_{\mu\nu} - \Gamma^\alpha_{\beta\nu}\delta\Gamma^\beta_{\mu\alpha} - \delta\Gamma^\alpha_{\beta\nu}\Gamma^\beta_{\mu\alpha} \\
&= \partial_\alpha \delta\Gamma^\alpha_{\mu\nu} + \Gamma^\alpha_{\alpha\beta}\delta\Gamma^\beta_{\mu\nu} - \Gamma^\beta_{\alpha\mu}\delta\Gamma^\alpha_{\beta\nu} - \Gamma^\beta_{\alpha\nu}\delta\Gamma^\alpha_{\mu\beta} \\
&\quad -\partial_\nu \delta\Gamma^\alpha_{\mu\alpha} - \Gamma^\alpha_{\nu\beta}\delta\Gamma^\beta_{\mu\alpha} + \Gamma^\beta_{\nu\mu}\delta\Gamma^\alpha_{\beta\alpha} + \Gamma^\beta_{\nu\alpha}\delta\Gamma^\alpha_{\mu\beta} \\
&= \nabla_\alpha \delta\Gamma^\alpha_{\mu\nu} - \nabla_\nu \delta\Gamma^\alpha_{\mu\alpha} \\
&= -\frac{1}{2}\left[g_{\nu\beta}\nabla_\alpha\nabla_\mu\delta g^{\alpha\beta} + g_{\mu\beta}\nabla_\alpha\nabla_\nu\delta g^{\alpha\beta} - g^{\alpha\beta}g_{\mu\rho}g_{\nu\gamma}\nabla_\alpha\nabla_\beta\delta g^{\rho\gamma}\right] \\
&\quad +\frac{1}{2}\left[g_{\alpha\beta}\nabla_\nu\nabla_\mu\delta g^{\alpha\beta} + g_{\mu\beta}\nabla_\nu\nabla_\alpha\delta g^{\alpha\beta} - g^{\alpha\beta}g_{\mu\rho}g_{\alpha\gamma}\nabla_\nu\nabla_\beta\delta g^{\rho\gamma}\right] \\
&= -\frac{1}{2}\left[g_{\nu\beta}\nabla_\alpha\nabla_\mu\delta g^{\alpha\beta} - g_{\mu\rho}g_{\nu\gamma}\Box\delta g^{\rho\gamma}\right] + \frac{1}{2}\left[g_{\alpha\beta}\nabla_\nu\nabla_\mu\delta g^{\alpha\beta} - \delta^\beta_\gamma g_{\mu\rho}\nabla_\nu\nabla_\beta\delta g^{\rho\gamma}\right] \\
&= \frac{1}{2}\left[g_{\alpha\beta}\nabla_\nu\nabla_\mu\delta g^{\alpha\beta} - g_{\mu\rho}\nabla_\nu\nabla_\gamma\delta g^{\rho\gamma} - g_{\nu\beta}\nabla_\alpha\nabla_\mu\delta g^{\alpha\beta} + g_{\mu\rho}g_{\nu\gamma}\Box\delta g^{\rho\gamma}\right]. \qquad \text{(B.15)}
\end{aligned}
$$

The variation of the Ricci scalar is

$$
\begin{aligned}
\delta R &= \delta\left(g^{\mu\nu}R_{\mu\nu}\right) \\
&= g^{\mu\nu}\delta R_{\mu\nu} + R_{\mu\nu}\delta g^{\mu\nu} \\
&= \frac{1}{2}\left[g^{\mu\nu}g_{\alpha\beta}\nabla_\nu\nabla_\mu\delta g^{\alpha\beta} - g^{\mu\nu}g_{\mu\rho}\nabla_\nu\nabla_\gamma\delta g^{\rho\gamma} - g^{\mu\nu}g_{\nu\beta}\nabla_\alpha\nabla_\mu\delta g^{\alpha\beta}\right. \\
&\quad \left. + g^{\mu\nu}g_{\mu\rho}g_{\nu\gamma}\Box\delta g^{\rho\gamma}\right] + R_{\mu\nu}\delta g^{\mu\nu} \\
&= \frac{1}{2}\left[g_{\alpha\beta}\Box\delta g^{\alpha\beta} - \delta^\nu_\rho\nabla_\nu\nabla_\gamma\delta g^{\rho\gamma} - \delta^\mu_\beta\nabla_\alpha\nabla_\mu\delta g^{\alpha\beta} + \delta^\nu_\rho g_{\nu\gamma}\Box\delta g^{\rho\gamma}\right] + R_{\mu\nu}\delta g^{\mu\nu} \\
&= \frac{1}{2}\left[g_{\alpha\beta}\Box\delta g^{\alpha\beta} - \nabla_\rho\nabla_\gamma\delta g^{\rho\gamma} - \nabla_\alpha\nabla_\beta\delta g^{\alpha\beta} + g_{\rho\gamma}\Box\delta g^{\rho\gamma}\right] + R_{\mu\nu}\delta g^{\mu\nu} \\
&= g_{\mu\nu}\Box\delta g^{\mu\nu} - \nabla_\mu\nabla_\nu\delta g^{\mu\nu} + R_{\mu\nu}\delta g^{\mu\nu}. \qquad \text{(B.16)}
\end{aligned}
$$

From (B.5) we obtain

$$
\begin{aligned}
\delta S_A &= \int d^4x\left[\sqrt{-g}\left(-\frac{1}{2(n+1)!}\delta F^2 - V'\delta A^2 - \frac{\xi R}{2n!}\delta A^2 - \frac{\xi A^2}{2n!}\delta R\right)\right. \\
&\quad \left. -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\left(-\frac{1}{2(n+1)!}F^2 - V(A^2) - \frac{1}{2n!}\xi A^2 R\right)\delta g^{\mu\nu}\right] \\
&= \int d^4x\sqrt{-g}\left[\left(-\frac{1}{2(n+1)!}(n+1)F_{\mu\mu_1\ldots\mu_n}F_\nu^{\mu_1\ldots\mu_n}\delta g^{\mu\nu} - \left(V' + \frac{\xi R}{2n!}\right)\right.\right. \\
&\quad nA_{\mu\mu_1\ldots\mu_{n-1}}A_\nu^{\mu_1\ldots\mu_{n-1}}\delta g^{\mu\nu} - \frac{\xi A^2}{2n!}g_{\mu\nu}\Box\delta g^{\mu\nu} + \frac{\xi A^2}{2n!}\nabla_\mu\nabla_\nu\delta g^{\mu\nu} - \left.\frac{\xi A^2}{2n!}R_{\mu\nu}\delta g^{\mu\nu}\right) \\
&\quad \left. -\frac{1}{2}g_{\mu\nu}\left(-\frac{1}{2(n+1)!}F^2 - V(A^2) - \frac{1}{2n!}\xi A^2 R\right)\delta g^{\mu\nu}\right] \\
&= \int d^4x\sqrt{-g}\left[\left(-\frac{1}{2n!}F_{\mu\mu_1\ldots\mu_n}F_\nu^{\mu_1\ldots\mu_n} - \left(nV' + \frac{\xi R}{2(n-1)!}\right)A_{\mu\mu_1\ldots\mu_{n-1}}A_\nu^{\mu_1\ldots\mu_{n-1}}\right.\right. \\
&\quad \left.\left. -\frac{\xi A^2}{2n!}R_{\mu\nu}\right) - \frac{1}{2}g_{\mu\nu}\left(-\frac{1}{2(n+1)!}F^2 - V(A^2) - \frac{1}{2n!}\xi A^2 R\right)\right]\delta g^{\mu\nu} \\
&\quad -\int d^4x\sqrt{-g}\frac{\xi A^2}{2n!}\left[g_{\mu\nu}\Box\delta g^{\mu\nu} - \nabla_\mu\nabla_\nu\delta g^{\mu\nu}\right].
\end{aligned}
$$

By integrating by parts

$$\int d^n x\sqrt{-g}A^{\mu}\left(\nabla_{\mu}B\right) = -\int d^n x\sqrt{-g}\left(\nabla_{\mu}A^{\mu}\right)B + \text{boundary terms}, \tag{B.17}$$

δS_A becomes

$$\begin{aligned}
\delta S_A = \int d^4x\sqrt{-g}\Bigg[\Bigg(&-\frac{1}{2n!}F_{\mu\mu_1\ldots\mu_n}F_{\nu}^{\mu_1\ldots\mu_n} \\
&-\left(nV' + \frac{\xi R}{2(n-1)!}\right)A_{\mu\mu_1\ldots\mu_{n-1}}A_{\nu}^{\mu_1\ldots\mu_{n-1}} \\
&-\frac{\xi A^2}{2n!}R_{\mu\nu} - \frac{\xi}{2n!}g_{\mu\nu}\Box A^2 + \frac{\xi}{2n!}\nabla_{\mu}\nabla_{\nu}A^2\Bigg) \\
&-\frac{1}{2}g_{\mu\nu}\left(-\frac{1}{2(n+1)!}F^2 - V(A^2) - \frac{1}{2n!}\xi A^2 R\right)\Bigg]\delta g^{\mu\nu} \\
= -\frac{1}{2}\int d^4x\sqrt{-g}\Bigg[\Bigg(&\frac{1}{n!}F_{\mu\mu_1\ldots\mu_n}F_{\nu}^{\mu_1\ldots\mu_n} + \left(2nV' + \frac{\xi R}{(n-1)!}\right)A_{\mu\mu_1\ldots\mu_{n-1}}A_{\nu}^{\mu_1\ldots\mu_{n-1}} \\
&+\frac{\xi A^2}{n!}R_{\mu\nu} + \frac{\xi}{n!}g_{\mu\nu}\Box A^2 - \frac{\xi}{n!}\nabla_{\mu}\nabla_{\nu}A^2\Bigg) \\
&+g_{\mu\nu}\left(-\frac{1}{2(n+1)!}F^2 - V(A^2) - \frac{1}{2n!}\xi A^2 R\right)\Bigg]\delta g^{\mu\nu} \\
= -\frac{1}{2}\int d^4x\sqrt{-g}\Bigg[&\frac{1}{n!}F_{\mu\mu_1\ldots\mu_n}F_{\nu}^{\mu_1\ldots\mu_n} + \left(2nV' + \frac{\xi R}{(n-1)!}\right)A_{\mu\mu_1\ldots\mu_{n-1}}A_{\nu}^{\mu_1\ldots\mu_{n-1}} \\
&+g_{\mu\nu}\left(-\frac{1}{2(n+1)!}F^2 - V(A^2)\right) \\
&+\frac{\xi}{n!}\left(A^2R_{\mu\nu} + g_{\mu\nu}\Box A^2 - \nabla_{\mu}\nabla_{\nu}A^2 - \frac{1}{2}Rg_{\mu\nu}A^2\right)\Bigg]\delta g^{\mu\nu} \\
= -\frac{1}{2}\int d^4x\sqrt{-g}\Bigg[&\frac{1}{n!}F_{\mu\mu_1\ldots\mu_n}F_{\nu}^{\mu_1\ldots\mu_n} + 2nV'A_{\mu\mu_1\ldots\mu_{n-1}}A_{\nu}^{\mu_1\ldots\mu_{n-1}} \\
&-g_{\mu\nu}\left(\frac{1}{2(n+1)!}F^2 + V(A^2)\right) \\
&+\frac{\xi}{n!}\left[nRA_{\mu\mu_1\ldots\mu_{n-1}}A_{\nu}^{\mu_1\ldots\mu_{n-1}} + \left(G_{\mu\nu} + g_{\mu\nu}\Box - \nabla_{\mu}\nabla_{\nu}\right)A^2\right]\Bigg]\delta g^{\mu\nu}.
\end{aligned} \tag{B.18}$$

The energy momentum tensor of the n-form field is derived by

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_A}{\delta g^{\mu\nu}}, \tag{B.19}$$

which can be written as

$$\begin{aligned}
T_{\mu\nu} = &\frac{1}{n!}F_{\mu\mu_1\ldots\mu_n}F_{\nu}^{\mu_1\ldots\mu_n} + 2nV'A_{\mu\mu_1\ldots\mu_{n-1}}A_{\nu}^{\mu_1\ldots\mu_{n-1}} - g_{\mu\nu}\left(\frac{1}{2(n+1)!}F^2 + V(A^2)\right) \\
&+\frac{\xi}{n!}\left[nRA_{\mu\mu_1\ldots\mu_{n-1}}A_{\nu}^{\mu_1\ldots\mu_{n-1}} + \left(G_{\mu\nu} + g_{\mu\nu}\Box - \nabla_{\mu}\nabla_{\nu}\right)A^2\right].
\end{aligned} \tag{B.20}$$

B.2 Scalar field

The action of the scalar field is given by

$$S = \int d^4x\sqrt{-g}\left[\frac{R}{16\pi G} - \frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - V(\phi)\right], \tag{B.21}$$

where $V(\phi)$ is the potential of the scalar field. The equation of motion of the scalar field is obtained by using

$$\frac{\partial L}{\partial\phi} - \nabla_\mu\left(\frac{\partial L}{\partial(\nabla_\mu\phi)}\right) = 0, \tag{B.22}$$

where L is the Lagrangian of the scalar field given by

$$L = -\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - V(\phi). \tag{B.23}$$

We will compute each term of (B.22) as follows.

$$\begin{aligned}
\frac{\partial L}{\partial\phi} &= -\frac{dV}{d\phi} \\
\frac{\partial L}{\partial(\nabla_\mu\phi)} &= -\frac{1}{2}g^{\alpha\beta}\left[\nabla_\alpha\phi\frac{\partial(\nabla_\beta\phi)}{\partial(\nabla_\mu\phi)} + \frac{\partial(\nabla_\alpha\phi)}{\partial(\nabla_\mu\phi)}\nabla_\beta\phi\right] \\
&= -\frac{1}{2}g^{\alpha\beta}\left[\nabla_\alpha\phi\delta^\mu_\beta + \delta^\mu_\alpha\nabla_\beta\phi\right] \\
&= -\frac{1}{2}g^{\alpha\beta}\nabla_\alpha\phi\delta^\mu_\beta - \frac{1}{2}g^{\alpha\beta}\delta^\mu_\alpha\nabla_\beta\phi \\
&= -\frac{1}{2}g^{\alpha\mu}\nabla_\alpha\phi - \frac{1}{2}g^{\mu\beta}\nabla_\beta\phi \\
&= -\frac{1}{2}\nabla^\mu\phi - \frac{1}{2}\nabla^\mu\phi. \\
&= -\nabla^\mu\phi \\
\nabla_\mu\left(\frac{\partial L}{\partial(\nabla_\mu\phi)}\right) &= -\Box\phi,
\end{aligned}$$

where we introduce the covariant d' Alembertian operator as

$$\Box \equiv \nabla_\mu\nabla^\mu. \tag{B.24}$$

Now we arrive at the equation of motion

$$\Box\phi - \frac{\mathrm{d}V}{\mathrm{d}\phi} = 0. \tag{B.25}$$

For the FRW metric (2.1) we have

$$\begin{aligned}
\Box\phi &= \nabla_\mu \nabla^\mu \phi \\
&= g^{\mu\nu}\nabla_\mu \nabla_\nu \phi \\
&= g^{\mu\nu}\nabla_\mu \partial_\nu \phi \\
&= g^{\mu\nu}\partial_\mu \partial_\nu \phi - g^{\mu\nu}\Gamma^\lambda_{\mu\nu}\partial_\lambda \phi \\
&= -\ddot{\phi} - g^{ij}\Gamma^0_{ij}\partial_0\phi \\
&= -\ddot{\phi} - \frac{1}{a^2}\delta^{ij} a\dot{a}\delta_{ij}\dot{\phi} \\
&= -\ddot{\phi} - 3H\dot{\phi}. \qquad \text{(B.26)}
\end{aligned}$$

Then the equation of motion of the scalar field in the spacetime (2.1) is

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\mathrm{d}V}{\mathrm{d}\phi} = 0. \qquad \text{(B.27)}$$

The energy momentum tensor of the scalar field is derived by

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}}. \qquad \text{(B.28)}$$

The variation of the action (B.21) is

$$\begin{aligned}
\delta S &= \int d^4x \left[\sqrt{-g}\delta\{-\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - V(\phi)\} + \delta\sqrt{-g}\{-\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - V(\phi)\}\right] \\
&= \int d^4x \left[\sqrt{-g}\{-\frac{1}{2}\delta g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi\} - \frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}\{-\frac{1}{2}g^{\alpha\beta}\nabla_\alpha\phi\nabla_\beta\phi - V(\phi)\}\right] \\
&= \int d^4x\sqrt{-g} \left[-\frac{1}{2}\delta g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\delta g^{\mu\nu}\{-\frac{1}{2}g^{\alpha\beta}\nabla_\alpha\phi\nabla_\beta\phi - V(\phi)\}\right] \\
&= \int d^4x\sqrt{-g} \left[-\frac{1}{2}\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\{-\frac{1}{2}g^{\alpha\beta}\nabla_\alpha\phi\nabla_\beta\phi - V(\phi)\}\right]\delta g^{\mu\nu}. \qquad \text{(B.29)}
\end{aligned}$$

Therefore, from (B.28) we obtain the energy momentum tensor

$$T_{\mu\nu} = \nabla_\mu\phi\nabla_\nu\phi - g_{\mu\nu}\left[\frac{1}{2}g^{\alpha\beta}\nabla_\alpha\phi\nabla_\beta\phi + V(\phi)\right]. \qquad \text{(B.30)}$$

Non-vanishing components of the energy momentum tensor are

$$\begin{aligned}
T_{00} &= \nabla_0\phi\nabla_0\phi - g_{00}\left[\frac{1}{2}g^{\alpha\beta}\nabla_\alpha\phi\nabla_\beta\phi + V(\phi)\right] \\
&= \dot{\phi}^2 + \frac{1}{2}g^{\alpha\beta}\nabla_\alpha\phi\nabla_\beta\phi + V(\phi) \\
&= \dot{\phi}^2 + \frac{1}{2}g^{00}\nabla_0\phi\nabla_0\phi + \frac{1}{2}g^{jj}\nabla_j\phi\nabla_j\phi + V(\phi) \\
&= \dot{\phi}^2 - \frac{1}{2}\dot{\phi}^2 + V(\phi) \\
&= \frac{1}{2}\dot{\phi}^2 + V(\phi), \qquad (B.31)\\
T_{ii} &= \nabla_i\phi\nabla_i\phi - g_{ii}\left[\frac{1}{2}g^{\alpha\beta}\nabla_\alpha\phi\nabla_\beta\phi + V(\phi)\right] \\
&= -a^2(t)\left[\frac{1}{2}g^{\alpha\beta}\nabla_\alpha\phi\nabla_\beta\phi + V(\phi)\right] \\
&= -a^2(t)\left[\frac{1}{2}g^{00}\nabla_0\phi\nabla_0\phi + V(\phi)\right] \\
&= -a^2(t)\left[-\frac{1}{2}\dot{\phi}^2 + V(\phi)\right] \\
&= a^2(t)\left[\frac{1}{2}\dot{\phi}^2 - V(\phi)\right]. \qquad (B.32)
\end{aligned}$$

The energy density and pressure of the scalar field are

$$\begin{aligned}
\rho_\phi &= -T^0_0 \\
&= -g^{0\mu}T_{\mu 0} \\
&= -g^{00}T_{00} \\
&= T_{00} \\
&= \frac{1}{2}\dot{\phi}^2 + V(\phi) \qquad (B.33)\\
p_\phi &= T^i_i \\
&= g^{ij}T_{ji} \\
&= g^{ii}T_{ii} \\
&= \frac{1}{2}\dot{\phi}^2 - V(\phi). \qquad (B.34)
\end{aligned}$$

B.3 Three-form field

For the FRW metric (2.1), the timelike component of three-form field is nondynamical. This can be shown by observing that

$$
\begin{aligned}
F_{\alpha\mu\nu\rho} &= 4\nabla_{[\alpha}A_{\mu\nu\rho]} \\
&= \frac{4}{4!}\left(\nabla_\alpha A_{\mu\nu\rho} - \nabla_\alpha A_{\mu\rho\nu} + \nabla_\alpha A_{\rho\mu\nu} - \nabla_\alpha A_{\rho\nu\mu} + \nabla_\alpha A_{\nu\rho\mu} - \nabla_\alpha A_{\nu\mu\rho}\right. \\
&\quad + \nabla_\nu A_{\alpha\mu\rho} - \nabla_\nu A_{\alpha\rho\mu} + \nabla_\nu A_{\rho\alpha\mu} - \nabla_\nu A_{\rho\mu\alpha} + \nabla_\nu A_{\mu\rho\alpha} - \nabla_\nu A_{\mu\alpha\rho} \\
&\quad + \nabla_\mu A_{\nu\alpha\rho} - \nabla_\mu A_{\nu\rho\alpha} + \nabla_\mu A_{\rho\nu\alpha} - \nabla_\mu A_{\rho\alpha\nu} + \nabla_\mu A_{\alpha\rho\nu} - \nabla_\mu A_{\alpha\nu\rho} \\
&\quad \left. + \nabla_\rho A_{\alpha\nu\mu} - \nabla_\rho A_{\alpha\mu\nu} + \nabla_\rho A_{\mu\alpha\nu} - \nabla_\rho A_{\mu\nu\alpha} + \nabla_\rho A_{\nu\mu\alpha} - \nabla_\rho A_{\nu\alpha\mu}\right) \\
&= \frac{1}{3!}\left(6\nabla_\alpha A_{\mu\nu\rho} + 6\nabla_\nu A_{\alpha\mu\rho} + 6\nabla_\mu A_{\nu\alpha\rho} + 6\nabla_\rho A_{\alpha\nu\mu}\right) \\
&= \nabla_\alpha A_{\mu\nu\rho} + \nabla_\nu A_{\alpha\mu\rho} + \nabla_\mu A_{\nu\alpha\rho} + \nabla_\rho A_{\alpha\nu\mu} \\
&= \partial_\alpha A_{\mu\nu\rho} - \Gamma^\lambda_{\alpha\mu}A_{\lambda\nu\rho} - \Gamma^\lambda_{\alpha\nu}A_{\mu\lambda\rho} - \Gamma^\lambda_{\alpha\rho}A_{\mu\nu\lambda} + \partial_\nu A_{\alpha\mu\rho} - \Gamma^\lambda_{\nu\alpha}A_{\lambda\mu\rho} \\
&\quad - \Gamma^\lambda_{\nu\mu}A_{\alpha\lambda\rho} - \Gamma^\lambda_{\nu\rho}A_{\alpha\mu\lambda} + \partial_\mu A_{\nu\alpha\rho} - \Gamma^\lambda_{\mu\nu}A_{\lambda\alpha\rho} - \Gamma^\lambda_{\mu\alpha}A_{\nu\lambda\rho} - \Gamma^\lambda_{\mu\rho}A_{\nu\alpha\lambda} \\
&\quad + \partial_\rho A_{\alpha\nu\mu} - \Gamma^\lambda_{\rho\alpha}A_{\lambda\nu\mu} - \Gamma^\lambda_{\rho\nu}A_{\alpha\lambda\mu} - \Gamma^\lambda_{\rho\mu}A_{\alpha\nu\lambda} \\
&= \partial_\alpha A_{\mu\nu\rho} - \Gamma^\lambda_{\alpha\mu}A_{\lambda\nu\rho} - \Gamma^\lambda_{\alpha\nu}A_{\mu\lambda\rho} - \Gamma^\lambda_{\alpha\rho}A_{\mu\nu\lambda} + \partial_\nu A_{\alpha\mu\rho} + \Gamma^\lambda_{\alpha\nu}A_{\mu\lambda\rho} \\
&\quad - \Gamma^\lambda_{\nu\mu}A_{\alpha\lambda\rho} - \Gamma^\lambda_{\nu\rho}A_{\alpha\mu\lambda} + \partial_\mu A_{\nu\alpha\rho} + \Gamma^\lambda_{\nu\mu}A_{\alpha\lambda\rho} + \Gamma^\lambda_{\alpha\mu}A_{\lambda\nu\rho} - \Gamma^\lambda_{\mu\rho}A_{\nu\alpha\lambda} \\
&\quad + \partial_\rho A_{\alpha\nu\mu} + \Gamma^\lambda_{\alpha\rho}A_{\mu\nu\lambda} + \Gamma^\lambda_{\nu\rho}A_{\alpha\mu\lambda} + \Gamma^\lambda_{\mu\rho}A_{\nu\alpha\lambda} \\
&= \partial_\alpha A_{\mu\nu\rho} + \partial_\nu A_{\alpha\mu\rho} + \partial_\mu A_{\nu\alpha\rho} + \partial_\rho A_{\alpha\nu\mu}, \qquad \text{(B.35)}
\end{aligned}
$$

and

$$
\begin{aligned}
\nabla^\alpha F_{\alpha\mu\nu\rho} &= \nabla^\alpha\partial_\alpha A_{\mu\nu\rho} + \nabla^\alpha\partial_\nu A_{\alpha\mu\rho} + \nabla^\alpha\partial_\mu A_{\nu\alpha\rho} + \nabla^\alpha\partial_\rho A_{\alpha\nu\mu} \\
&= g^{\alpha\beta}\nabla_\beta\partial_\alpha A_{\mu\nu\rho} + g^{\alpha\beta}\nabla_\beta\partial_\nu A_{\alpha\mu\rho} + g^{\alpha\beta}\nabla_\beta\partial_\mu A_{\nu\alpha\rho} + g^{\alpha\beta}\nabla_\beta\partial_\rho A_{\alpha\nu\mu} \\
&= g^{\alpha\beta}\partial_\beta\partial_\alpha A_{\mu\nu\rho} - g^{\alpha\beta}\Gamma^\lambda_{\beta\alpha}\partial_\lambda A_{\mu\nu\rho} - g^{\alpha\beta}\Gamma^\lambda_{\beta\mu}\partial_\alpha A_{\lambda\nu\rho} - g^{\alpha\beta}\Gamma^\lambda_{\beta\nu}\partial_\alpha A_{\mu\lambda\rho} - g^{\alpha\beta}\Gamma^\lambda_{\beta\rho}\partial_\alpha A_{\mu\nu\lambda} \\
&\quad + g^{\alpha\beta}\partial_\beta\partial_\nu A_{\alpha\mu\rho} - g^{\alpha\beta}\Gamma^\lambda_{\beta\nu}\partial_\lambda A_{\alpha\mu\rho} - g^{\alpha\beta}\Gamma^\lambda_{\beta\alpha}\partial_\nu A_{\lambda\mu\rho} - g^{\alpha\beta}\Gamma^\lambda_{\beta\mu}\partial_\nu A_{\alpha\lambda\rho} - g^{\alpha\beta}\Gamma^\lambda_{\beta\rho}\partial_\nu A_{\alpha\mu\lambda} \\
&\quad + g^{\alpha\beta}\partial_\beta\partial_\mu A_{\nu\alpha\rho} - g^{\alpha\beta}\Gamma^\lambda_{\beta\mu}\partial_\lambda A_{\nu\alpha\rho} - g^{\alpha\beta}\Gamma^\lambda_{\beta\nu}\partial_\mu A_{\lambda\alpha\rho} - g^{\alpha\beta}\Gamma^\lambda_{\beta\alpha}\partial_\mu A_{\nu\lambda\rho} - g^{\alpha\beta}\Gamma^\lambda_{\beta\rho}\partial_\mu A_{\nu\alpha\lambda} \\
&\quad + g^{\alpha\beta}\partial_\beta\partial_\rho A_{\alpha\nu\mu} - g^{\alpha\beta}\Gamma^\lambda_{\beta\rho}\partial_\lambda A_{\alpha\nu\mu} - g^{\alpha\beta}\Gamma^\lambda_{\beta\alpha}\partial_\rho A_{\lambda\nu\mu} - g^{\alpha\beta}\Gamma^\lambda_{\beta\nu}\partial_\rho A_{\alpha\lambda\mu} - g^{\alpha\beta}\Gamma^\lambda_{\beta\mu}\partial_\rho A_{\alpha\nu\lambda} \\
&= -\partial_0\partial_0 A_{\mu\nu\rho} - g^{ij}\Gamma^0_{ij}\partial_0 A_{\mu\nu\rho} + \Gamma^i_{0\mu}\partial_0 A_{i\nu\rho} + \Gamma^i_{0\nu}\partial_0 A_{\mu i\rho} + \Gamma^i_{0\rho}\partial_0 A_{\mu\nu i} \\
&\quad - \partial_0\partial_\nu A_{0\mu\rho} - g^{ij}\Gamma^0_{j\nu}\partial_0 A_{i\mu\rho} - g^{ij}\Gamma^0_{ij}\partial_\nu A_{0\mu\rho} - g^{\alpha\beta}\Gamma^\lambda_{\beta\mu}\partial_\nu A_{\alpha\lambda\rho} - g^{\alpha\beta}\Gamma^\lambda_{\beta\rho}\partial_\nu A_{\alpha\mu\lambda} \\
&\quad - \partial_0\partial_\mu A_{\nu 0\rho} - g^{ij}\Gamma^0_{j\mu}\partial_0 A_{\nu i\rho} - g^{\alpha\beta}\Gamma^\lambda_{\beta\nu}\partial_\mu A_{\lambda\alpha\rho} - g^{ij}\Gamma^0_{ij}\partial_\mu A_{\nu 0\rho} - g^{\alpha\beta}\Gamma^\lambda_{\beta\rho}\partial_\mu A_{\nu\alpha\lambda} \\
&\quad - \partial_0\partial_\rho A_{0\nu\mu} - g^{ij}\Gamma^0_{j\rho}\partial_0 A_{i\nu\mu} - g^{ij}\Gamma^0_{ij}\partial_\rho A_{0\nu\mu} - g^{\alpha\beta}\Gamma^\lambda_{\beta\nu}\partial_\rho A_{\alpha\lambda\mu} - g^{\alpha\beta}\Gamma^\lambda_{\beta\mu}\partial_\rho A_{\alpha\nu\lambda}. \qquad \text{(B.36)}
\end{aligned}
$$

Note that we have made use of the homogeneity and isotropy of the universe so that $\partial_i A_{\mu\nu\rho} = 0$. Consider a timelike component $\nabla^\alpha F_{\alpha ij0}$,

$$\begin{aligned}
\nabla^\alpha F_{\alpha ij0} &= -\partial_0\partial_0 A_{ij0} - g^{kl}\Gamma^0_{kl}\partial_0 A_{ij0} + \Gamma^k_{0i}\partial_0 A_{kj0} + \Gamma^k_{0j}\partial_0 A_{ik0} + \Gamma^k_{00}\partial_0 A_{ijk} \\
&\quad -\partial_0\partial_j A_{0i0} - g^{kl}\Gamma^0_{lj}\partial_0 A_{ki0} - g^{kl}\Gamma^0_{kl}\partial_j A_{0i0} - g^{\alpha\beta}\Gamma^\lambda_{\beta i}\partial_j A_{\alpha\lambda 0} - g^{\alpha\beta}\Gamma^\lambda_{\beta 0}\partial_j A_{\alpha i\lambda} \\
&\quad -\partial_0\partial_i A_{j00} - g^{kl}\Gamma^0_{li}\partial_0 A_{jk0} - g^{\alpha\beta}\Gamma^\lambda_{\beta j}\partial_i A_{\lambda\alpha 0} - g^{kl}\Gamma^0_{kl}\partial_i A_{j00} - g^{\alpha\beta}\Gamma^\lambda_{\beta 0}\partial_i A_{j\alpha\lambda} \\
&\quad -\partial_0\partial_0 A_{0ji} - g^{kl}\Gamma^0_{l0}\partial_0 A_{kji} - g^{kl}\Gamma^0_{kl}\partial_0 A_{0ji} - g^{\alpha\beta}\Gamma^\lambda_{\beta j}\partial_0 A_{\alpha\lambda i} - g^{\alpha\beta}\Gamma^\lambda_{\beta i}\partial_0 A_{\alpha j\lambda} \\
&= -\partial_0\partial_0 A_{ij0} - g^{kl}\Gamma^0_{kl}\partial_0 A_{ij0} + \Gamma^k_{0i}\partial_0 A_{kj0} + \Gamma^k_{0j}\partial_0 A_{ik0} - g^{kl}\Gamma^0_{lj}\partial_0 A_{ki0} \\
&\quad -g^{kl}\Gamma^m_{l0}\partial_j A_{kim} - g^{kl}\Gamma^0_{li}\partial_0 A_{jk0} - \partial_0\partial_0 A_{0ji} - g^{kl}\Gamma^0_{kl}\partial_0 A_{0ji} \\
&\quad +\Gamma^k_{0j}\partial_0 A_{0ki} - g^{kl}\Gamma^0_{lj}\partial_0 A_{k0i} + \Gamma^k_{0i}\partial_0 A_{0jk} - g^{kl}\Gamma^0_{li}\partial_0 A_{kj0} \\
&= -\partial_0\partial_0 A_{ij0} - g^{kl}\Gamma^0_{kl}\partial_0 A_{ij0} + \Gamma^k_{0i}\partial_0 A_{kj0} + \Gamma^k_{0j}\partial_0 A_{ik0} \\
&\quad -g^{kl}\Gamma^0_{lj}\partial_0 A_{ki0} - g^{kl}\Gamma^0_{li}\partial_0 A_{jk0} + \partial_0\partial_0 A_{ij0} + g^{kl}\Gamma^0_{kl}\partial_0 A_{ij0} \\
&\quad -\Gamma^k_{0j}\partial_0 A_{ik0} + g^{kl}\Gamma^0_{lj}\partial_0 A_{ki0} - \Gamma^k_{0i}\partial_0 A_{kj0} + g^{kl}\Gamma^0_{li}\partial_0 A_{jk0} \\
&= 0. \qquad\qquad (B.37)
\end{aligned}$$

From (3.18) its equation of motion is

$$\nabla^\alpha F_{\alpha ij0} = 12V'(A^2)A_{ij0}, \qquad (B.38)$$

we obtain

$$12V'(A^2)A_{ij0} = 0, \qquad (B.39)$$

which is an algebraic constraint. This implies $A_{ij0} = 0$.

Consider a spacelike component $\nabla^\alpha F_{\alpha 123}$. From (B.36) we have

$$\begin{aligned}
\nabla^\alpha F_{\alpha 123} &= -\partial_0\partial_0 A_{123} - g^{ij}\Gamma^0_{ij}\partial_0 A_{123} + \Gamma^i_{01}\partial_0 A_{i23} + \Gamma^i_{02}\partial_0 A_{1i3} + \Gamma^i_{03}\partial_0 A_{12i} \\
&\quad -\partial_0\partial_2 A_{013} - g^{ij}\Gamma^0_{j2}\partial_0 A_{i13} - g^{ij}\Gamma^0_{ij}\partial_2 A_{013} - g^{\alpha\beta}\Gamma^\lambda_{\beta 1}\partial_2 A_{\alpha\lambda 3} - g^{\alpha\beta}\Gamma^\lambda_{\beta 3}\partial_2 A_{\alpha 1\lambda} \\
&\quad -\partial_0\partial_1 A_{203} - g^{ij}\Gamma^0_{j1}\partial_0 A_{2i3} - g^{\alpha\beta}\Gamma^\lambda_{\beta 2}\partial_1 A_{\lambda\alpha 3} - g^{ij}\Gamma^0_{ij}\partial_1 A_{203} - g^{\alpha\beta}\Gamma^\lambda_{\beta 3}\partial_1 A_{2\alpha\lambda} \\
&\quad -\partial_0\partial_3 A_{021} - g^{ij}\Gamma^0_{j3}\partial_0 A_{i21} - g^{ij}\Gamma^0_{ij}\partial_3 A_{021} - g^{\alpha\beta}\Gamma^\lambda_{\beta 2}\partial_3 A_{\alpha\lambda 1} - g^{\alpha\beta}\Gamma^\lambda_{\beta 1}\partial_3 A_{\alpha 2\lambda} \\
&= -\partial_0\partial_0 A_{123} - g^{ij}\Gamma^0_{ij}\partial_0 A_{123} + \Gamma^1_{01}\partial_0 A_{123} + \Gamma^2_{02}\partial_0 A_{123} \\
&\quad +\Gamma^3_{03}\partial_0 A_{123} - g^{22}\Gamma^0_{22}\partial_0 A_{213} - g^{11}\Gamma^0_{11}\partial_0 A_{213} - g^{33}\Gamma^0_{33}\partial_0 A_{321} \\
&= -\partial_0\partial_0 A_{123} - g^{ij}\Gamma^0_{ij}\partial_0 A_{123} + 3\Gamma^1_{01}\partial_0 A_{123} \\
&\quad +g^{22}\Gamma^0_{22}\partial_0 A_{123} + g^{11}\Gamma^0_{11}\partial_0 A_{123} + g^{33}\Gamma^0_{33}\partial_0 A_{123} \\
&= -\partial_0\partial_0 A_{123} + 3\Gamma^1_{01}\partial_0 A_{123}.
\end{aligned}$$

From (3.21), we get

$$\begin{aligned}
\partial_0 A_{123} &= \partial_0 \left(a^3 X\right) \\
&= a^3 \dot{X} + 3a^2 \dot{a} X \\
\partial_0 \partial_0 A_{123} &= a^3 \ddot{X} + 3a^2 \dot{a} \dot{X} + 3a^2 \dot{a} \dot{X} + 3a^2 \ddot{a} X + 6a \dot{a}^2 X \\
&= a^3 \ddot{X} + 6a^2 \dot{a} \dot{X} + 3a^2 \ddot{a} X + 6a \dot{a}^2 X.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\nabla^\alpha F_{\alpha 123} &= -a^3 \ddot{X} - 6a^2 \dot{a} \dot{X} - 3a^2 \ddot{a} X - 6a \dot{a}^2 X + 3\frac{\dot{a}}{a} \left(a^3 \dot{X} + 3a^2 \dot{a} X\right) \\
&= -a^3 \ddot{X} - 6a^2 \dot{a} \dot{X} - 3a^2 \ddot{a} X - 6a \dot{a}^2 X + 3a^2 \dot{a} \dot{X} + 9a \dot{a}^2 X \\
&= -a^3 \ddot{X} - 3a^2 \dot{a} \dot{X} - 3a^2 \ddot{a} X + 3a \dot{a}^2 X.
\end{aligned}$$

From (2.19) and (2.20) we have

$$\begin{aligned}
\dot{a} &= Ha \\
\frac{\ddot{a}}{a} &= \dot{H} + H^2.
\end{aligned}$$

Then

$$\begin{aligned}
\nabla^\alpha F_{\alpha 123} &= -a^3 \ddot{X} - 3a^3 H \dot{X} - 3a^3 (\dot{H} + H^2) X + 3a^3 H^2 X \\
&= -a^3 \ddot{X} - 3a^3 H \dot{X} - 3a^3 \dot{H} X - 3a^3 H^2 X + 3a^3 H^2 X \\
&= -a^3 (\ddot{X} + 3H \dot{X} + 3\dot{H} X). \qquad \text{(B.40)}
\end{aligned}$$

From (3.18) we obtain the equation of motion of the field X

$$\begin{aligned}
\nabla^\alpha F_{\alpha 123} &= 12 V'(A^2) A_{123} \\
-a^3 (\ddot{X} + 3H \dot{X} + 3\dot{H} X) &= 12 V'(A^2) a^3 X \\
\ddot{X} + 3H \dot{X} + 3\dot{H} X &= -12 V'(A^2) X \\
\ddot{X} &= -3H \dot{X} - 3\dot{H} X - 12 V'(A^2) X.
\end{aligned}$$

Consider $V'(A^2)$

$$\begin{aligned}
V'(A^2) &= \frac{dV}{dA^2} \\
&= \frac{1}{6} \frac{dV}{dX^2} \\
&= \frac{1}{6} \frac{dV}{dX} \frac{dX}{dX^2} \\
&= \frac{1}{12X} \frac{dV}{dX} \\
&= \frac{1}{12X} V_{,X}\,, \qquad \text{(B.41)}
\end{aligned}$$

where $V_{,X} \equiv dV/dX$. Therefore, we get the equation of motion of X as

$$\ddot{X} = -3H\dot{X} - 3\dot{H}X - V_{,X}\,. \tag{B.42}$$

From (B.20), each component of the energy momentum tensor of the three-form field is

$$T_{00} = \frac{1}{3!}F_{0\mu_1\mu_2\mu_3}F_0^{\mu_1\mu_2\mu_3} + 6V'A_{0\mu_1\mu_2}A_0^{\mu_1\mu_2} + \frac{1}{48}F^2 + V(A^2).$$

From (B.35)

$$\begin{aligned}
F_{0\mu_1\mu_2\mu_3}F_0^{\mu_1\mu_2\mu_3} &= F_{0ijk}F_0^{ijk} \\
&= g^{il}g^{jm}g^{kn}F_{0ijk}F_{0lmn} \\
&= g^{il}g^{jm}g^{kn}\left(\partial_0 A_{ijk} + \partial_j A_{0ik} + \partial_i A_{j0k} + \partial_k A_{0ji}\right)(\partial_0 A_{lmn} \\
&\quad + \partial_m A_{0ln} + \partial_l A_{m0n} + \partial_n A_{0ml}) \\
&= g^{il}g^{jm}g^{kn}\left(\partial_0 A_{ijk}\right)\left(\partial_0 A_{lmn}\right) \\
&= g^{il}g^{jm}g^{kn}\epsilon_{ijk}\epsilon_{lmn}\left(a^3\dot{X} + 3a^2\dot{a}X\right)\left(a^3\dot{X} + 3a^2\dot{a}X\right) \\
&= \frac{1}{a^6}\delta^{il}\delta^{jm}\delta^{kn}\epsilon_{ijk}\epsilon_{lmn}\left(a^6\dot{X}^2 + 6a^5\dot{a}\dot{X}X + 9a^4\dot{a}^2X^2\right) \\
&= \epsilon_{ijk}\epsilon_{ijk}\left(\dot{X}^2 + 6H\dot{X}X + 9H^2X^2\right) \\
&= 6\left(\dot{X} + 3HX\right)^2,
\end{aligned} \tag{B.43}$$

$$\begin{aligned}
F^2 &= F_{\mu\nu\rho\sigma}F^{\mu\nu\rho\sigma} \\
&= 4!F_{0123}F^{0123} \\
&= 4!g^{0\mu}g^{1i}g^{2j}g^{3k}F_{0123}F_{\mu ijk} \\
&= -4!g^{11}g^{22}g^{33}F_{0123}F_{0123} \\
&= -4!\frac{1}{a^6}F_{0123}F_{0123} \\
&= -4!\frac{1}{a^6}\left(\partial_0 A_{123}\right)\left(\partial_0 A_{123}\right) \\
&= -4!\frac{1}{a^6}\left(a^6\dot{X}^2 + 6a^5\dot{a}\dot{X}X + 9a^4\dot{a}^2X^2\right) \\
&= -24\left(\dot{X} + 3HX\right)^2.
\end{aligned} \tag{B.44}$$

Therefore, the $(0,0)$ component of $T_{\mu\nu}$ is

$$\begin{aligned}
T_{00} &= \left(\dot{X} + 3HX\right)^2 - \frac{1}{2}\left(\dot{X} + 3HX\right)^2 + V(A^2) \\
&= \frac{1}{2}\left(\dot{X} + 3HX\right)^2 + V(A^2).
\end{aligned} \tag{B.45}$$

The (i,j) component of energy momentum tensor is

$$T_{ij} = \frac{1}{3!}F_{i\mu\nu\rho}F_j^{\mu\nu\rho} + 6V'A_{i\mu\nu}A_j^{\mu\nu} - g_{ij}\left(\frac{1}{48}F^2 + V(A^2)\right).$$

Observing that

$$
\begin{aligned}
F_{i\mu\nu\rho}F_j^{\mu\nu\rho} &= 3F_{i0kl}F_j^{0kl} \\
&= 3g^{0\mu}g^{km}g^{ln}F_{i0kl}F_{j\mu mn} \\
&= -3g^{km}g^{ln}F_{i0kl}F_{j0mn} \\
&= -3g^{km}g^{ln}\left(\partial_0 A_{kil}\right)\left(\partial_0 A_{mjn}\right) \\
&= -\frac{3}{a^4}\delta^{km}\delta^{ln}\epsilon_{kil}\epsilon_{mjn}\left(a^3\dot{X}+3a^2\dot{a}X\right)\left(a^3\dot{X}+3a^2\dot{a}X\right) \\
&= -3a^2\epsilon_{kil}\epsilon_{kjl}\left(\dot{X}+3HX\right)^2 \\
&= -6a^2\left(\dot{X}+3HX\right)^2\delta_{ij}, \qquad \text{(B.46)}
\end{aligned}
$$

and

$$
\begin{aligned}
A_{i\mu\nu}A_j^{\mu\nu} &= A_{ikl}A_j^{kl} \\
&= g^{km}g^{ln}A_{ikl}A_{jmn} \\
&= \frac{1}{a^4}\delta^{km}\delta^{ln}\epsilon_{ikl}\epsilon_{jmn}a^6X^2 \\
&= a^2\epsilon_{ikl}\epsilon_{jkl}X^2 \\
&= 2a^2X^2\delta_{ij}, \qquad \text{(B.47)}
\end{aligned}
$$

we can rewrite

$$
\begin{aligned}
T_{ij} &= -a^2\left(\dot{X}+3HX\right)^2\delta_{ij}+12V'a^2X^2\delta_{ij}-a^2\delta_{ij}\left[-\frac{1}{2}\left(\dot{X}+3HX\right)^2+V(A^2)\right] \\
&= a^2\left[-\frac{1}{2}\left(\dot{X}+3HX\right)^2+12V'X^2-V(A^2)\right]\delta_{ij}.
\end{aligned}
$$

From (B.41) we get

$$
T_{ij} = a^2\left[-\frac{1}{2}\left(\dot{X}+3HX\right)^2+V_{,X}X-V(A^2)\right]\delta_{ij}. \qquad \text{(B.48)}
$$

The energy density and pressure of the three-form field are given by

$$
\begin{aligned}
\rho_X &= -T_0^0 \\
&= -g^{0\mu}T_{\mu 0} \\
&= -g^{00}T_{00} \\
&= T_{00} \\
&= \frac{1}{2}\left(\dot{X}+3HX\right)^2+V(A^2) \qquad \text{(B.49)} \\
p_X &= T_i^i \\
&= g^{ij}T_{ji} \\
&= \frac{1}{a^2}\delta^{ij}a^2\left[-\frac{1}{2}\left(\dot{X}+3HX\right)^2+V_{,X}X-V(A^2)\right]\delta_{ij} \\
&= -\frac{1}{2}\left(\dot{X}+3HX\right)^2+V_{,X}X-V(A^2). \qquad \text{(B.50)}
\end{aligned}
$$

VITAE

Mr. Tritos Ngampitipan was born on 29 October 1986 and received his Bachelor's degree in physics with the first class honors from Chulalongkorn University in 2008. He has studied cosmology for his Master's degree. His research interests are in the three-form field dark energy.

Presentations

1. Three-Form Field Dark Energy: The 19^{th} National Graduate Research Conference, Rajabhat Rajanagarindra University, Chachoengsao, Thailand (23 - 24) December 2010.

International Schools

1. The 4^{th} Siam Symposium on General Relativity, High Energy Physics and Cosmology, Narasuan University, Phitsanulok, Thailand (26 - 28) July 2009.