การควบคุมเหมาะที่สุดของระบบที่กำกับด้วย
สมการเชิงอนุพันธ์ปริพันธ์อันดับเศษส่วนที่มีอิมพัลส์

นายวิชัย วิทยาเกียรติเลิศ

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรดุษฎีบัณฑิต
สาขาวิชาคณิตศาสตร์ ภาควิชาคณิตศาสตร์
คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย
ปีการศึกษา 2553

OPTIMAL CONTROL OF SYSTEMS GOVERNED BY IMPULSIVE FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS

Mr. Wichai Witayakiatilerd

A Dissertation Submitted in Partial Fulfillment of the Requirements
for the Degree of Doctor of Philosophy Program in Mathematics
Department of Mathematics
Faculty of Science
Chulalongkorn University
Academic Year 2010

Thesis Title	OPTIMAL CONTROL OF SYSTEMS GOVERNED BY IMPULSIVE FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS
By	Mr.Wichai Witthayakiattilerd
Field of Study	Mathematics
Thesis Advisor	Assistant Professor Anusorn Chonwerayuth, Ph.D.

Accepted by the Faculty of Science, Chulalongkorn University in Partial Fulfillment of the Requirements for the Doctoral Degree

.... Dean of the Faculty of Science

(Professor Supot Hannongbua, Dr.rer.nat.)

THESIS COMMITTEE

.... Chairman

(Associate Professor Pornchai Satravaha, Ph.D.)

.... Thesis Advisor

(Assistant Professor Anusorn Chonwerayuth, Ph.D.)

.... Examiner

(Khamron Mekchay, Ph.D.)

.... Examiner

(Sujin Khomrutai, Ph.D.)

.... External Examiner

(Professor Pairote Sattayatham, Ph.D.)

วิชัย วิทยาเกียรติเลิศ : การควบคุมเหมาะที่สุดของระบบที่กำกับด้วยสมการเชิงอนุพันธ์ปริพันธ์อันดับเศษส่วนที่มีอิมพัลส์ (OPTIMAL CONTROL OF SYSTEMS GOVERNED BY IMPULSIVE FRACTIONAL INTEGRO-DIFFERRENTAIL EQUATIONS) อ.ที่ปรึกษาวิทยานิพนธ์หลัก : ผศ. ดร.อนุสรณ์ ชนวีระยุทธ, 106 หน้า

ในวิทยานิพนธ์นี้ เราพิสูจน์การมีผลเฉลยของการควบคุมเหมาะสุดของระบบที่กำกับด้วยสมการเชิงอนุพันธ์ปริพันธ์อันดับเศษส่วนที่มีการหน่วงเชิงเวลา ซึ่งอธิบายระบบควบคุมนี้โดยใช้กึ่งกลุ่มของตัวดำเนินการปิดบนปริภูมิบานาค และใช้ทฤษฎีจุดตรึงบนปริภูมิบานาคเป็นเครื่องมือในการแก้ปัญหาการมีอยู่จริงของผลเฉลย และอธิบายระบบควบคุมนี้โดยใช้ตัวดำเนินการผลเฉลย ซึ่งไม่มีสมบัติกึ่งกลุ่มของตัวดำเนินการปิดบนปริภูมิบานาค เราใช้ผลดังกล่าวพิสูจน์การมีผลเฉลยของการควบคุมเหมาะที่สุดของระบบที่กำกับด้วยสมการเชิงอนุพันธ์ปริพันธ์อันดับเศษส่วนที่มีอิมพัลส์และมีการหน่วงด้วยเวลาซึ่งเป็นผลลัพธ์หลักของวิทยานิพนธ์นี้ นอกจากนี้ได้ประยุกต์ผลลัพธ์ที่ได้กับปัญหาต่างๆที่สอดคล้องอีกด้วย

ภาควิชา ...คณิตศาสตร์....
สาขาวิชา ...คณิตศาสตร์... ลายมือชื่อนิสิต......................................
ปีการศึกษา.....2553...... ลายมือชื่อ อ.ที่ปรึกษาวิทยานิพนธ์หลัก......................

##5073876923: MAJOR MATHEMATICS
KEYWORDS: BANACH FIXED POINT THEOREM/ SEMIGROUP OF OPERATOR/ SOLUTION OPERATOR/ GRONWALL LEMMA/ FRACTIONAL CALCULUS/ IMPULSIVE DIFFERENTIAL EQUATION/ OPTIMAL CONTROL.

WICHAI WITAYAKIATILERD: OPTIMAL CONTROL OF SYSTEMS GOVERNED BY IMPULSIVE FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS. ADVISOR: ASST. PROF. ANUSORN CHONWERAYUTH, Ph.D., 106 pp.

In this thesis, the system governed by the fractional integro-differential equations with time delays are considered, the local and global existences and uniqueness of mild solution with respect to the control in the admissible control set describing by the solution operator and also describing by the C_0−semigroup on Banach space are proved. The results are used extending to the fractional integro-differential equations with impulses. Also, the piecewise continuous mild solutions for the impulsive system with respect to the control in the admissible control set are proved. Furthermore, the optimal control problems of all systems are solved and the results are clarified by some examples.

Department:	....Mathematics.....	Student's Signature	Wichai
Field of Study:	....Mathematics.....	Advisor's Signature	A. Chonwerayuth
Academic Year:	.........2010............		

ACKNOWLEDGEMENTS

Beside the efforts I put in writing up this thesis, many people have involved in this project and contributed to its success. I appreciate their help and I am happy that I had the opportunity to learn from and work with them.

First of all, I would like to thank Assistant Professor Dr.Anusorn for being my adviser. He gave me advices and especially helped me finding a topic and getting started. But he also encouraged me all of the time. He guided me through the all the necessary paper work. He gave me comments, suggestion and every things. I would like to say from my heart "thank you very much".

Furthermore, we thanks for the supporting by the CEM scholarships of Thailand. Hopefully, I do not forget to mention anyone. If this show to be the case, I do apologize. It does not happen by intention.

Contents

Page

CHAPTER I
INTRODUCTION

The mathematical models of many real world problems can be described by impulsive differential equations. They have been studied quite extensively [5, 8, 10, 16, 17, 18] because they have advantage over the traditional initial value problems. They can be used to model other phenomena that can not be modeled by the traditional initial value, such as the dynamics of the systemic arterial pressure [2], the dynamics of populations subjected to abrupt changes (harvesting, diseases, etc.).

Some phenomena in physics or in the other fields, some time they can not be described or characterized by the differential equations of integer order but they can efficiently be described by the fractional order.

Among the previous research, little is concerned with integro-differential equations with fractional order and impulse. In 2006, Chonwerayuth [3] proved the existence and uniqueness of a classical solution of an integro-differential equation;

$$\begin{cases} x'(t) + Ax(t) = f(t, x(t), x_t) + \int_{-r}^{t} h(t-s)g(s, Kx(s))ds + Bu(t), \ t \in [0, T] \\ x(t) = \varphi(t), \ t \in [-r, 0], \end{cases} \tag{1.1}$$

where $x_t(\theta) = x(t+\theta)$, $-r - t \leq \theta \leq 0$, and Wei.W [16] has done a portion of work on the nonlinear impulsive integro-differential equation;

$$\begin{cases} x'(t) + Ax(t) = F(t, x(t), Gx(t), Sx(t)), \quad t \neq t_i, \quad t \in [0, T] \\ \Delta x(t_i) = J_i(x(t_i)), \quad t = t_i, \quad i = 1, 2, ..., n \\ x(0) = x_0, \end{cases} \tag{1.2}$$

where G and S are nonlinear integral operators given by

$$Gx(t) = \int_0^t k(t,\tau)g(\tau, x(\tau))d\tau, \quad Sx(t) = \int_0^T h(t,\tau)s(\tau, x(\tau))d\tau. \tag{1.3}$$

In 2008, Gastaö S.F.Frederico [4] has studied on the fractional optimal control in the sense of Caputo and the fractional Neether's theorem. In 2009, Gisele M.Mophou [5] proved existence and uniqueness of mild solution to impulsive fractional differential equations;

$$\begin{cases} D_t^\alpha x(t) = Ax(t) + f(t, x(t)), \quad t \neq t_i, \ t \in [0,T] \\ \Delta x(t_i) = J_i(x(t_i)), \quad t = t_i, \ i = 1,2,...,n; \\ x(0) = x_0, \end{cases} \tag{1.4}$$

where $0 < \alpha < 1$ and D_t^α denote the Caputo fractional derivative. These researches motivate our work. In this thesis, we consider the main objective, a class of nonlinear impulsive fractional integro-differential equation;

$$\begin{cases} D_t^\alpha x(t) = Ax(t) + f(t, x(t), Kx(t)) + Gx(t) + B(t)u(t), \quad t \neq t_k, \quad t \in [0,T] \\ \Delta x(t_k) = J_k(x(t_k)), \quad t = t_k, \quad k = 1,2,...,n; \\ x(t) = \varphi(t), \quad t \in [-r, 0], \end{cases} \tag{1.5}$$

where $\varphi \in C([-r,0], X)$ is fixed, the integral operator $G : X \to X$ is defined by

$$Gx(t) = \int_{-r}^t h(t,s)g(s, Hx(s))ds \tag{1.6}$$

and for $0 < \alpha < 1$, D_t^α denote the Riemann-Liouville fractional derivative. A is a densely closed operator on a Banach space X, $f : [0,T] \times X \times X \to X$ is given continuous function, $J_k : X \to X$, $k = 1,2,..,n$ is a given bounded map, $\Delta x(t_k) = x(t_k^+) - x(t_k^-) = x(t_k^+) - x(t_k)$ presents the jump in the state x at $t = t_k$ with J_k, $(k = 1,2,...)$ determining the size of the jump at $t = t_k$. Traditional initial value problems are replaced by the impulsive conditions. Then, we study a optimal control problem of system (1.5) via the Bolza problem (P). Find $u_0 \in U_{ad}$ such that

$$J(u_0) \leq J(u), \quad \text{for all} \quad u \in U_{ad}, \tag{1.7}$$

where $J(u)=\int_0^T l(t,x^u(t),x_t^u,u(t))dt+\Phi(x^u(T))$, U_{ad} denote the set of all admissible controls, x^u denote the $PC-$mild solution of system (1.5) corresponding to the control $u\in U_{ad}$. We can see that our system different is from the previous works.

1.1 Scope

In this section, we talk about the scope of this research. Certainly, the main objective of this study is proving the existence of the system (1.5) (the nonlinear impulsive fractional integro-differential equation). However, this system is very complex, it is difficult to seek the form of the solution. So, we first consider the system that is less complex than the system (1.5) : the nonlinear fractional integro-differential equations without impulse;

$$\begin{cases} D_t^\alpha x(t)=Ax(t)+f(t,x(t),Kx(t))+Gx(t)+B(t)u(t), \quad t\in[0,T]\\ x(t)=\varphi(t), \quad t\in[-r,0], \end{cases} \tag{1.8}$$

under the same assumptions without impulse. The objective of this part is getting a solution called a mild solution with respect to the control $u\in U_{ad}$. Then we apply this result to construct a solution for the main system (1.5);

$$\begin{cases} D_t^\alpha x(t)=Ax(t)+f(t,x(t),Kx(t))+Gx(t)+B(t)u(t), \quad t\neq t_k, \quad t\in[0,T]\\ \Delta x(t_k)=J_k(x(t_k)), \quad t=t_k, \quad k=1,2,...,n;\\ x(t)=\varphi(t), \quad t\in[-r,0]. \end{cases}$$

This solution is called a piecewise continuous mild solution or (for short) a $PC-$mild solution with respect to the control $u\in U_{ad}$. Moreover, we show the other scheme to prove the existence of a $PC-$mild solution. This scheme use the compactness property of semigroup and the Leray-Shauder fixed point theorem to imply that the system (1.5) has at least one $PC-$mild solution. Then the control problem of system (1.5) via the Bolza problem will be considered. We exemplify some construction examples which satisfy our results and try to link our results to the real world problems. Furthermore, we consider a fractional integro-differential equations of mixed type with the solution operator;

$$\begin{cases} D_t^\alpha x(t) = Ax(t) + f(t, x(t), Gx(t), Sx(t)), & t \in [0, T] \\ x(t) = \varphi(t) \quad t \in [-r, 0]. \end{cases} \tag{1.9}$$

We prove the existence of the Lagrange problem for (1.9). The last, we conclude all results in this thesis and open idea to new interesting problems that close to our problems.

1.2 Research Objectives

The following statements are our objectives in the thesis,

1) To obtain existence and uniqueness of mild solution to the fractional integro-differential equations without impulses by using C_0-semigroup on a Banach space.

2) To obtain existence and uniqueness of piecewise continuous mild solution to the impulsive fractional integro-differential equations with C_0-semigroup on a Banach space.

3) To obtain the existence and uniqueness of mild solution to the fractional integro-differential equations of mixed type with solution operator on a Banach space.

4) To obtain an existence result of optimal controls for the Bolza problem and the Lagrange problem.

CHAPTER II
MATHEMATIC BACKGROUND

Functional analysis plays a central role in modern control theory. For convenience, we summarize, in this chapter, some of definitions and theorems which are required in subsequent chapters, with appropriate references given wherever necessary. Moreover, we will introduce the idea of control via the basic problem in the last section of this chapter.

2.1 Elements of Functional Analysis

Let X be a Banach space with norm $||\cdot||$.

Definition 2.1.1. *A sequence x_n in X is said to be strongly convergent to an element x in X if $||x_n - x|| \to 0$ as $n \to \infty$. We denote by $x_n \xrightarrow{s} x$.*

Let Y be another Banach space with norm $||\cdot||_Y$. A linear transformation from X into Y is bounded on a domain of T, $D(T)$, if there exists a constant c such that $||Tx||_Y \leq c||x||$ for all $x \in D(T)$. The linear space of all bounded linear operators from X into Y , is denoted by $L(X,Y)$ and denote $L(X,X)$ by $L(X)$.

Theorem 2.1.2. *(Uniform Boundedness Principle). Let $\{T_\alpha \mid \alpha \in \Lambda\}$ be a family of operators from $L(X,Y)$. If for each $x \in X$ there is a constant c_x such that $\sup_{\alpha\in\Lambda}||T_\alpha|| \leq c_x$, then the operator $\{T_\alpha\}$ are uniformly bounded.*

Let X be a Banach space and X^* be its dual space. Element of X^* can be used to generate a new topology for X called the *weak topology*. Note that the norm topology on X was called the *strong topology*. So the new topology is weaker than the strong (norm) topology. Particularly, the linear functionals on X that are continuous in the weak topology are precisely the functionals in X^*. The concept of open (closed) sets, compactness, convergence, etc., are topological, hence they must be qualified by referring to the topology involved. In the case of normed linear spaces, when one speaks of open (closed) sets, compactness, convergence, etc., one refer to strong (norm) topology, while, with reference to its weak topology, they are called weakly open (weakly closed) sets, weak compactness, weak convergence, etc. Thus a sequence $\{x_n\}$ in X is said to converge weakly to an element x in X if, for every $x^* \in X^*$, $x^*(x_n) \to x^*(x)$, written by $x_n \overset{w}{\to} x$. Every weakly convergent sequence is bounded. Every strongly convergent sequence is weakly convergent, but the converse is not true.

2.2 Bochner Integral

A Banach space setting of evolution equations requires taking the derivative in the Banach space. Hence, integration of Banach space valued function is an important tool of this setting. We define the Bochner integral of such functions and derive its basic properties. In the following, a subset of $\Re^n$ is said to be measurable if and only if it is Lebesgue measurable. The functions will be defined on the nonempty measurable set $S \subseteq \Re^n$, with range in a Banach space X.

The map $x : S \to X$ is called weakly measurable if $s \mapsto \ell(x(s))$ is a Lebesgue measurable function for each $\ell \in X^*$.

The map $x : S \to X$ is called almost separably-valued if there exists $\{y_1, y_2, ...\}$, $\subseteq X$ such that $\inf_i ||x(s) - y_i|| = 0$ for almost all $s \in S$.

The map $x : S \to X$ is called strongly measurable if it is weakly measurable and almost separably valued.

The map $x : S \to X$ is said to be Bochner integrable if x is strongly measurable and the functions $s \mapsto ||x(s)||$ is Lebesgue integrable.

The set of all such functions x is a vector space and will be denoted by $L_1(S, X)$, is a Banach space. Similarly, the equivalence class of strongly measurable $X-$valued functions on S such that

$$\int_S ||x(s)||^p ds < \infty, \quad \text{for} \quad 1 \leq p < \infty$$

and

$$esssup\{||x(s)|| \ \mid \ s \in S\} < \infty \quad \text{for} \quad p = \infty$$

form a Banach space with respect to the norms

$$||x||_p \equiv [\int_S ||x(s)||^p ds]^{1/p} \quad \text{for} \quad 1 \leq p < \infty$$

and

$$||x||_p \equiv esssup\{||x(s)|| \ \mid \ s \in S\} < \infty \quad \text{for} \quad p = \infty.$$

They are denoted by $L_p(S, X)$, $1 \leq p \leq \infty$.

The following Theorem 2.2.1 enables us to define the Bochner integral $\int_S x$ of $x \in L_1(S, X)$ to be $y \in X$ which satisfies (2.1).

Theorem 2.2.1. *If $x \in L(S, X)$, then there exists a unique $y \in X$ such that*

$$\ell(y) = \int_S \ell(x(s))ds, \quad \textit{for all} \quad \ell \in X^*. \tag{2.1}$$

Moreover, $||y|| \leq \int_S ||x(s)||ds$.

2.3 Fixed point Theorems

Fixed point theorem on Banach spaces or contraction mapping is an advantage tool that is for proving the existence and the uniqueness of solution. Consider a function $\varphi : \Re \to \Re$ and suppose that we require to solve the equation $\varphi(x) = 0$. This is equivalent to solving the equation

$$\psi(x) = x \tag{2.2}$$

where $\psi(x) = \varphi(x) + x$ for all $x \in \Re$. Thus x is a zero of φ if and only if x is a fixed point of ψ, i.e., a point which is left unaltered after the application of ψ. More generally, many problems are equivalent to solving

$$Af = f \tag{2.3}$$

where $A : D(A) \to R(A)$ is an operator (not necessarily linear), acting in some normed vector spaces, $D(A)$ and $R(A)$ are domain and range of A in X respectively, i.e., we seek a fixed point $f \in D(A)$ of the operator A (for simplicity, we write Af rather than $A(f)$). There are many fixed point theorems which guarantee existence and/or uniqueness of fixed points. We state here what is used in this thesis.

Definition 2.3.1. *Let X be a normed vector space and let $A : D(A) \to R(A)$ be an operator (not necessarily linear). Then*

(1) A is a contraction if there exists a constant c with $0 \leq c \leq 1$ such that

$$||Af_1 - Af_2|| \leq c||f_1 - f_2|| \quad \text{for all} \quad f_1, f_2 \in D(A) \tag{2.4}$$

(2) A is strictly contraction if there exists a constant c with $0 \leq c < 1$ such that (2.4) holds.

Theorem 2.3.2. *(The contraction mapping theorem; Banach fixed point theorem) Let X be a Banach space and let $A : X \to X$ be a strictly contraction. Then the equation $Af = f$ has a unique solution in X, .i.e., A has a unique fixed point f.*

The result of this theorem can be easily generalized as follows:

Corollary 2.3.3. *Let X_0 be a closed subset of the Banach space X and assume that the operator A maps X_0 into itself and is a strictly contraction on X_0. Then the equation $Af = f$ has a unique solution $f \in X_0$.*

Corollary 2.3.4. *(Leray-Schauder theorem) Let T be a continuous compact mapping of a Banach space X into itself such that the set*

$$\{x \in X \mid \; x = \lambda Tx \quad \text{for some} \quad 0 \leq \lambda \leq 1\} \tag{2.5}$$

is bounded. Then T has a fixed point.

2.4 Semigroup of Bounded Linear Operators

Consider a dynamical system, the state of which is evolving with time according to some law. For example, we may be interested in the temperature distribution along a rod which is being heated at one end. Suppose the initial state of the system is x_0; in this case $x_0(z)$ would measure the initial temperature at the point z of the rod. At a subsequent time $t > 0$, the state of the system will be given by $x(z,t)$; this state would measure the temperature at the point x at time t. Since, for each $t > 0$, the state $x(z,t)$ is an element of a Banach space X. We shall use the symbol $x(t)$ to indicate such a state, i.e., $x(t)(z) = x(z,t)$.

The state $x(t)$ will be related to the original state x_0 by some transition operator $T(t)$ so that

$$x(t) = T(t)x_0, \quad t \geq 0. \tag{2.6}$$

We shall thus obtain a family $\{T(t)\}_{t\geq 0}$ of such operators. It is natural to ask what properties this family should have.

Firstly, each operator $T(t)$ acts in a set of state x_0, where the states can typically represented by functions. Hence the domain of $T(t)$ will be a subspace of function.

Next, it is clear that $T(0)$ must be I, the identity operator on X since at $t = 0$ there is no transition. Further, for any $s, t \geq 0$ we should require that $T(s+t)x_0 = T(s)T(t)x_0$. Indeed, the left hand side describes the evolution over a time interval of length $s + t$. The right hand side effectively say that the system evolves from x_0 to $T(t)x_0$ in t units of time and then continues to evolve from $T(t)x_0$ to $T(s)[T(t)x_0]$ in a subsequent time interval of length s, from t to $s + t$. The net effect should be the same as going nonstop from 0 to $s+t$, without taking a snapshot at time t. Thus we are led to the two conditions

$$T(0) = I, \quad T(s)T(t) = T(s+t) \quad \text{for} \;\; s, \; t \geq 0. \tag{2.7}$$

Finally it is natural to expect that if s is closed to t, then $T(s)x_0$ should be close to $T(t)x_0$ in some sense. This is concept to define a family of transition operator say semigroup of operators. We are now ready to make the following formal definition. Throughout this section X will be a Banach space.

Definition 2.4.1. *A one-parameter family $\{T(t)\}_{t\geq 0}$ of bounded linear operators from X into X is a semigroup of bounded linear operators on X if*

1. *$T(0) = I$, (I is the identity operator on X),*

2. $T(t+s) = T(t)T(s)$ *for every* $t, s \geq 0$ *(the semigroup property).*

A semigroup of bounded linear operators $\{T(t)\}_{t_0}$ *is uniformly continuous if*

$$\lim_{t \to 0^+} ||T(t) - I||_{L(X)} = 0. \tag{2.8}$$

The linear operator A *defined by*

$$Ax = \lim_{t \to 0^+} \frac{T(t)x - x}{t} = \frac{d^+T(t)x}{dt}|_{t=0} \quad \textit{for all} \;\; x \in D(A) \tag{2.9}$$

with

$$D(A) = \{x \in X | \lim_{t \to 0^+} \frac{T(t)x - x}{t} \quad \textit{exists in} \;\; X\} \tag{2.10}$$

is called the infinitesimal generator of the semigroup $\{T(t)\}_{t \geq 0}$, $D(A)$ *is the domain of* A.

From Definition 2.4.1, we have a semigroup $\{T(t)\}_{t \geq 0}$ with a unique infinitesimal generator. If $T(t)$ is uniformly continuous, its infinitesimal generator is a bounded operator. On the other hand, every bounded linear operator A is the infinitesimal generator of a uniformly continuous semigroup $\{T(t)\}_{t \geq 0}$ and this semigroup is unique.

Definition 2.4.2. *A semigroup* $\{T(t)\}_{t \geq 0}$ *of a bounded linear operator on* X *is a strongly continuous semigroup of a bounded linear operators if*

$$\lim_{t \to 0^+} T(t)x = x, \quad \textit{for every} \;\; x \in X. \tag{2.11}$$

A strongly continuous semigroup of bounded linear operators on X *will be called a semigroup of a* C_0- *semigroup.*

Example 2.4.3. *Let* $X = L^p(\Re)$ *with* $1 \leq p < \infty$. *Define* $T(0) = I$ *and for* $t > 0$ *define* $T(t)$ *on* X *by*

$$(T(t)f)(x) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{\frac{-(x-y)^2}{4t}} f(y)dy \tag{2.12}$$

for all $f \in X$ *and* $x \in \Re$. *Then* $\{T(t)\}_{t\geq 0}$ *is* C_0-*semigroup called the Gauss-Weierstrass semigroup. The right hand side of (2.12) represents the Fourier convolution of the function* $f \in X$ *with the function* k *defined by*

$$k(x,t) = \frac{1}{\sqrt{4\pi t}} e^{\frac{-x^2}{4t}}, \quad \text{for all } \ x \in \Re, \quad t > 0. \tag{2.13}$$

This function k *is the fundamental solution of the heat equation*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad \text{for all } \ x \in \Re. \tag{2.14}$$

Example 2.4.4. *Another important partial differential equation is the wave equation. For one dimension space, this equation takes the form*

$$\frac{\partial^2 u}{\partial t^2} = \nu^2 \frac{\partial^2 u}{\partial x^2}, \quad \text{for all } \ x \in \Re, t > 0. \tag{2.15}$$

we now take for simplicity $\nu = 1$. *The analogue of the Gauss-Weierstrass semigroup is the Poisson semigroup. Let* $X = L^p(\Re)$ *with* $1 \leq p < \infty$. *For* $t > 0$ *define* $T(t)$ *on* X *by*

$$(T(t)f)(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{t}{t^2 + (x-y)^2} f(y) dy \tag{2.16}$$

for all $f \in X$ *and* $x \in \Re$ *and define* $T(0) = I$. *Then* $\{T(t)\}_{t_0}$ *is the* C_0- *semigroup on* X. *We can see that (2.16) represents the Fourier convolution of the function* $f \in X$ *with the function* k *defined by*

$$k(x,t) = \frac{1}{\pi} \frac{t}{t^2 + x^2}, \quad \text{for all } \ x \in \Re, \quad t > 0. \tag{2.17}$$

Conditions (1) and (2) in Definition 2.4.1 are reminiscent of the basic properties of the exponential function. More precisely, we have the following result.

Theorem 2.4.5. *(A functional equation of Cauchy)Let* $\phi : [0, \infty) \to \Re$ *be such that*

1. $\phi(0) = I$,

2. $\phi(s+t) = \phi(s)\phi(t)$ *for all* $s, t \geq 0$,

3. ϕ *is continuous on* $[0, \infty)$.

Then ϕ has the form

$$\phi(t) = e^{at} \quad \textit{for some constant} \;\; a \in \Re. \tag{2.18}$$

From this theorem and the definition of semigroup so now we may conjecture that the operators $\{T(t)\}_{t\geq 0}$ forming a C_0-semigroup have the form

$$T(t) = e^{At} \quad \text{for some operator } A. \tag{2.19}$$

This conjecture leads to many important properties of the C_0-semigroup such $\frac{d(T(t)x)}{dt} = \frac{d(e^{At}x)}{dt} = Ae^{At}x = AT(t)x$.

Theorem 2.4.6. *[15] Let A be an infinitesimal generator of the $C_0 -$ semigroup $\{T(t)\}_{t\geq 0}$. Then*

a) for all $x \in X$

$$\lim_{h\to 0^+} \frac{1}{h}\int_t^{t+h} T(s)xds = T(t)x; \tag{2.20}$$

b) for all $x \in X$, $\int_0^t T(s)xds \in D(A)$ and

$$A\int_0^t T(s)xds = T(t)x - x; \tag{2.21}$$

c) for all $x \in D(A)$, $T(t)x \in D(A)$ and

$$\frac{d}{dt}T(t)x = AT(t)x = T(t)Ax; \tag{2.22}$$

d) for all $x \in D(A)$

$$T(s)x - T(t)x = \int_s^t T(\tau)Axd\tau = \int_s^t AT(\tau)xd\tau. \tag{2.23}$$

Theorem 2.4.6 have some simple consequences which we now state.

Corollary 2.4.7. *If A is the infinitesimal generator of a C_0-semigroup $\{T(t)\}_{t\geq 0}$ then $D(A)$ is dense in X and A is a closed linear operator.*

2.5 Extremal Set and Extremal Points

A subset C of a real or complex vector space X is said to be **a convex** if, for $\alpha \in [0,1]$, $\alpha x_1 + (1-\alpha)x_2 \in C$ for every $x_1, x_2 \in X$.

Definition 2.5.1. *Let K be a subset of a real or complex vector space X. A nonempty subset E of K is said to be* ***an extremal subset*** *of K if a proper convex combination $\alpha x_1 + (1-\alpha)x_2$, $0 < \alpha < 1$, of two point x_1, x_2 of K lies in E only if both x_1 and x_2 lie in E. An extremal subset of K consisting of just one point is called* an extremal point *of K.*

Definition 2.5.2. *A real value function f defined on a topological vector space X is said to be* convex *(strictly) if for every $x_1, x_2 \in X$;*

$$f(\alpha x_1 + (1-\alpha)x_2,) \leq (<) \alpha f(x_1) + (1-\alpha) f(x_2), \text{ for all } 0 \leq \alpha \leq 1 \qquad (2.24)$$

This is a classical result from the theory of extremal.

Theorem 2.5.3. *[1] Let C be a weakly compact subset of a Banach space X and f a weakly lower semi-continuous function on C, that is, $f(x_0) \leq \underline{\lim}_{n\to\infty} f(x_n)$ whenever $x_n \to x_0$. Then f attains its minimum on C. Furthermore, if C is also convex and f strictly convex, then it has a unique minimum in C.*

2.6 Basic Concept of Control

In this section we will introduce a basic concept of control via the basic problem. Let X be a Banach space. We open our discussion by considering an $X-$value system in the form;

$$\begin{cases} x'(t) = Ax(t) + f(t, x(t)), \quad t > 0 \\ x(0) = x_0. \end{cases} \qquad (2.25)$$

We are given here the initial point x_0, the function $f \in L^2([0,T], X)$ and A is an infinitesimal generator of a C_0-semigroup $\{T(t)\}_{t\geq 0}$. The unknown is $x : [0,\infty) \to X$, which x interpret as the state of system (2.25).

Let us denote by U the Hilbert space of controls. We are given an operator,

$$B \in L(U, L^2([0,T], X)). \tag{2.26}$$

We denote by x^u a solution with respect to a control $u \in U$ of

$$\begin{cases} x'(t) = Ax(t) + f(t, x(t)) + Bu(t), \quad t > 0, \\ x(0) = x^0. \end{cases} \tag{2.27}$$

More generally, we call a function $u : [0, \infty) \to U$ a control.
We also introduce

$$U_{ad} = \{u : [0, \infty) \to U \ \mid \ u \text{ measurable}\}$$

to denote the collection of all *admissible controls*. Note very carefully that our solution $x(\cdot)$ of system (2.27) depends upon the control $u(\cdot)$ and the initial condition. We write for short,

$$x^u(\cdot) = x(\cdot, u(\cdot), x_0). \tag{2.28}$$

For this we need to specify a cost functional (or payoff) criterion. Let us define *the cost functional*

$$P(u(\cdot)) \equiv \int_0^T r(x(t), u(t))dt + g(x(T)), \tag{2.29}$$

where $x(\cdot)$ is a solution corresponding the control $u(\cdot)$. Here $r : X \times U \to \Re$ and $g : X \to \Re$ are given, and we call r the *running cost* and g the *terminal cost*. The terminal time $T > 0$ is given as well.

Our overall task will be to determine what is the **best control** for our system. That is, we seek a control $u_0 \in U_{ad}$ such that

$$P(u_0) \leq P(u), \text{ for all } u \in U_{ad}, \text{ i.e., } P(u_0) = \inf_{u \in U_{ad}} P(u). \tag{2.30}$$

An example is given to illustrate the concept of control.

Example 2.6.1. *We consider the following problem*

$$\begin{cases} \frac{\partial x(t,y)}{\partial t} = \frac{\partial^2 x(t,y)}{\partial y^2} + f_1(t, y, x(t,y)) + \int_\Omega K(y,\tau)u(\tau,t)d\tau, \quad y \in \Omega, \quad 0 < t \leq T, \\ x(t,y) = 0, \quad y \in \partial\Omega, \quad 0 \leq t \leq T, \end{cases} \tag{2.31}$$

where $\Omega \subset \Re^N$ *is a bounded open domain with sufficiently smooth boundary,* Δ *is the Laplacian operator,* $u \in L_p(\Omega \times [0,T])$, $(p > 1)$ *and* $K : \Omega \times \Omega \to \Re$ *is continuous.*

Suppose $f_1 : [0,T] \times \overline{\Omega} \times \Re \to \Re$ *is continuous and there exist constant* C_1, $C_2 \geq 0$ *such that*

$$|f_1(t,y,\xi)| \leq C_1(1+|\xi|); \tag{2.32}$$

$$|f_1(t,y,\xi_1) - f_1(s,y,\xi_2)| \leq C_2(|t-s| + |\xi_1 - \xi_2|). \tag{2.33}$$

Let $X = L_p(\Omega)$, *define* $Ax = \Delta x$ *for all* $x \in D(A)$ *where* $D(A) = W^{2p}(\Omega) \cap W_0^{1p}(\Omega)$. *It is well known from* L_p-*theory that* A *is the infinitesimal generator of a* C_0-*semigroup* $\{T(t)\}_{t\geq 0}$. *Define* $x(t)(y) = x(t,y)$, $\frac{dx(t)(y)}{dt} = \lim_{t\to 0}\frac{x(t+h)(y) - x(t)(y)}{h}$ *and* $f(t,x(t))(y) = f(t,y,x(t,y))$. *It is obvious that* f *satisfies,*

$$|f(t,x)| \leq \bar{C}_1(1+||x||); \tag{2.34}$$

$$|f(t,x_1) - f_1(s,x_2)| \leq \bar{C}_2(|t-s| + ||x_1 - x_2||), \tag{2.35}$$

for some constants $\bar{C}_1$, $\bar{C}_2 > 0$. Then the problem (2.31) can be written as

$$\begin{cases} x'(t) = Ax(t) + f(t,x(t)) + \int_\Omega K(\tau)u(\tau,t)d\tau, \quad 0 \leq t \leq T, \\ x(0) = x_0. \end{cases} \tag{2.36}$$

Use Theorem 2.1 in the paper of Wei W. and Xiang X.[16] to guarantee that the system (2.36) has a unique mild solution with respect to $u \in L_p(\Omega \times [0,T])$ when delay is zero.

Let $U_{ad} = \{u \in L_p(\Omega \times [0,T]) \mid \ ||u||_{L_p} \leq 1\}$. Then U_{ad} is closed and convex. We consider the following cost functional ;

$$J(u) = \int_0^T \int_\Omega |x(t,\tau)|d\tau dt + \int_0^T \int_0^1 |u(t,\tau)|^2 dt. \tag{2.37}$$

Aplying Theorem 2.2 in paper of Wei W. and Xiang X.[16], we also guarantee that there exists a $u^0 \in U_{ad}$ such that $J(u^0) = \inf_{u\in U_{ad}} J(u)$.

2.7 Impulsive Differential Equations

We begin this section by describing a set of relations which characterize an evolution process subject to impulsive effects. Let us consider an evolution process described by

i) a system of differential equation

$$x'(t) = Ax(t) + f(t, x) \tag{2.38}$$

where $f : \Re \times \Omega \to X$, is an open subset of a Banach space X, A is an operator.

ii) the set $M(t), N(t) \subseteq \Omega$ for each $t \in \Re$

iii) the operator $B(t) : M(t) \to N(t)$ for each $t \in \Re$.

Let $x(t) = x(t, t_0, x_0)$ be a solution of (2.38) starting at (t_0, x_0). The evolution process behaves as follows: the point $P_{t_0} = (t_0, x(t_0))$ begins its motion from the initial point $P_{t_0} = (t_0, x_0)$ and move along a curve $\{(t, x) \mid t \geq t_0,\ x = x(t)\}$ until the time $t_1 > t_0$ at which point P_t meets the set $M(t)$. At $t = t_1$ the operator $B(t)$ transfers the point $P_{t_1} = (t_1, x(t_1))$ into $P_{t_1^+} = (t_1, x_1^+) \in N(t_1)$ where $x_1^+ = B(t_1)x(t_1)$. Then the point P_t continues to move further along the curve with $x(t) = x(t_1, x_1^+)$ as a solution of (2.38) starting at $P_{t_1}^+ = (t_1, x_1^+)$ until it hit the set $M(t)$ at the moment $t_2 > t_1$. Then once again the point $P_t = (t_2, x_2)$ is transfered to the point $P_{t^+} = (t_2, x_2^+) \in N(t_2)$ where $x_2^+ = B(t_2)x(t_2)$.
As before, the point P_t continues to move forward with $x(t) = x(t, t_2, x_2^+)$ as the solution of (2.38) starting at (t_2, x_2^+). Thus the evolution process continues forward as long as the solution of (2.38) exists. The set of relations i), ii) and iii) is called the characterize the above mentioned evolution process an impulsive differential system, the curve which described by the point P_t the integral curve and the function that defines the integral curve a solution of the impulsive differential system. A solution of an impulsive differential system may be

a) a continuous function, if the integral curve does not intersect the set $M(t)$ or hit at the fixed point of operator $B(t)$;

b) a piecewise continuous function having finite number of discontinuous of the first kind if the integral curve meets $M(t)$ at a finite number of points which are not the fixed point of the operator $B(t)$;

c) a piecewise continuous function having a countable number of discontinuous of the first kind if the integral curve encounters the set $M(t)$ at a countable number of points that are the fixed point of the operator $B(t)$.

The moment t_i at which the point P_t hits the set $M(t)$ are called moments of impulsive effect. We will assume that the solution $x(t)$ of the impulsive differential system is left continuous at t_i, $i \in N$, that is

$$x(t_i^-) = \lim_{h \to 0^+} x(t_i - h) = x(t_i).$$

The meaning of the impulsive differential systems gives rise to several types of systems such as

1) systems with impulses at fixed times;

2) systems with impulses at variable times;

3) autonomous systems with impulses.

Now, we will give description for only one type say, type 1 that we use in this thesis. Let $M(t)$ be a set to represent a sequence of planes $t = t_i$ where $\{t_i\}$ is a sequence of time such that $t_i \to +\infty$ as $i \to +\infty$. Let us define the operator $B(t)$ for $t = t_i$ only so that the sequence of operator $B(i)$ is given by

$$B(i) : \Omega \to \Omega, \quad B(i)(x) = x + J_i(x),$$

where $J_i : \Omega \to \Omega$. As a result, the set $N(t)$ is also defined for $t = t_i$ and therefore $N(i) = B(i)M(i)$. With this choice of $M(i), N(i)$ and $B(i)$, the differential equation with impulses at fixed times may be described by

$$\begin{cases} x'(t) + Ax(t) = f(t, x(t)), \quad t \neq t_i \\ \Delta x(t_i) = J_i(x(t_i)), \quad t = t_i, i \in N \\ x(t_0) = x_0. \end{cases} \tag{2.39}$$

Example 2.7.1. *Consider the impulsive differential equation*

$$\begin{cases} x'(t) = 1 + [x(t)]^2, & t \neq t_i, \\ \Delta x(t_i) = -1, & t_i = \frac{i\pi}{4}, \quad i \in N. \end{cases} \tag{2.40}$$

The solution $x(t)$ *with* $x(0) = 0$ *is continual for all* $t \geq 0$. *In fact, we have* $x(t) = tan(t - \frac{i\pi}{4})$, $t \in (\frac{i\pi}{4}, \frac{(i+1)\pi}{4}]$ *which is periodic with period* $\frac{\pi}{4}$. *However, the corresponding differential equation has the solution* $x(t) = tant$ *whose interval of existence is* $[0, \frac{\pi}{2})$ *since* $\lim\limits_{t \to \frac{\pi}{2}^-} x(t) = +\infty$. *This means that we can control blow-up system to periodic bounded solution by using an impulsive control.*

CHAPTER III
FRACTIONAL CALCULUS BACKGROUND

Fractional calculus is a branch of mathematical analysis that studies the possibility of taking real number power or complex number power of the differential operator

$$Df(x) = \frac{df(x)}{dx}$$

and the integration operator

$$D^{-1}f(x) = \int_0^x f(t)dt.$$

In this context the term powers refers to iterative application or composition, in the same sense that $f^2 = f(f(x))$. For example, one may ask the equation of meaningfully interpreting

$$\sqrt{D} = D^{\frac{1}{2}}$$

as a square root of the differentiation operator (an operator half iterate), i.e., an expression for some operator that when applied twice to a function will have the some effect as differentiation. More generally, one can look at the equation of defining

$$D^{\alpha} \text{ and } D^0 = I \text{ identity operator}$$

for real number values of α in such a way that when α takes an integer value n, the usual power of $n-$fold differentiation is recover for $n > 0$ and the $n-th$ power of integration for $n < 0$.

3.1 Fractional Derivative

In this section, we give the generalization definition to the derivative of the fractional order (in fact real order and more generally complex order).

3.1.1 Binomial formula Derivative

We will give some definitions of fractional derivative by the binomial formula.

Definition 3.1.1. *Let $f : \Re \to \Re$ be a continuous(but not necessarily differentiable) function and let $h > 0$ denote a constant discretization span. The fractional difference of order α ($\alpha > 0$) of $f(x)$ is defined by the expression*

$$\Delta^\alpha f(x) \equiv \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f[x + (\alpha - k)h] \tag{3.1}$$

where $\binom{\alpha}{k} = \frac{\Gamma(\alpha+1)}{k!\Gamma(\alpha-k+1)}$, $\Gamma(\cdot)$ is gamma function and its fractional derivative of oder α is

$$D^\alpha f(x) = \lim_{h\to 0} \frac{\Delta^\alpha f(x)}{h^\alpha}. \tag{3.2}$$

This definition is similar to the standard definition of derivative and as a direct result the $n-th$ derivative of a constant is zero.

Example 3.1.2. *Consider the exponential function is specially simple and gives some clues about the generalization of derivative, following (3.2) in the definition 3.1.1,*

$$\begin{aligned} D^\alpha e^{ax} &= \lim_{h\to 0} \frac{1}{h^\alpha} \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} e^{a[x+(\alpha-k)h]} \\ &= e^{ax} \lim_{h\to 0} \frac{1}{h^\alpha} \sum_{k=0}^{\infty} \binom{\alpha}{k} (-1)^k (e^{ah})^{(\alpha-k)} \\ &= e^{ax} \lim_{h\to 0} \frac{(e^{ah}-1)^\alpha}{h^\alpha} \\ &= a^\alpha e^{ax}. \end{aligned} \tag{3.3}$$

The above limit exists for any real number α (in fact any complex number). However, in the expression (3.2) some functions allow the substitution of the binomial formula such as Example 3.1.2, but this is not for any given function. For applying this substitution, we require the other definitions.

3.1.2 Riemann-Liouville-Caputo Derivative

Riemann-Liouville derivative is the most used generalization of the derivative. It is based on Cauchy 's formula for calculation of iterated integrals. The idea is started at the first integral of function, is as follows,

$$D^{-1}f(x) = \int_0^x f(t)dt. \tag{3.4}$$

It is not difficult generalized to non-integer values, in what is the Riemann-Liouville integral,

$$D^{-\alpha}f(x) \equiv \frac{1}{\Gamma(\alpha)} \int_0^x f(t)(x-t)^{\alpha-1}dt \tag{3.5}$$

where $\alpha > 0$.

The problem with this generalization is, that if α is negative or zero the integral diverges. This problem was solved by Caputo and adopted by Caputo and Mainardi in the frame work of the theory of linear viscoelasticity. So the Caputo fractional derivative of order $\alpha > 0$ is defined by

$$D^{\alpha}f(x) = D^{\alpha-n}(D^n f(x)) \tag{3.6}$$

where $0 < n - \alpha < 1$ and D^n denote the ordinary derivative of order integer n.

However that in the above formulas the selection of 0 as the lower limit of integration can be arbitrary, and any other number could be chosen. Generally, the election of the integration limits in this and other generalizations of the derivative is indicated with subscripts. The Riemann-Liouville integral with the lower integration limit a would be

$$_aD_x^{-\alpha}f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x f(t)(x-t)^{\alpha-1}dt \tag{3.7}$$

and

$$_aD^{\alpha}f(x) =_a D^{\alpha-n}(D^n f(x)). \tag{3.8}$$

In particularly, the Riemann-Liouville derivative with the lower integration limit of $-\infty$ is known as the Weyl derivative.

Example 3.1.3. *consider the powers x^m, by using (3.5), (3.6), and integrating by part, for $n \in \mathbb{N}$ such $0 < n - \alpha < 1$*

$$\begin{aligned}
D^{\alpha}x^m &= D^{\alpha-n}(D^n x^m) = D^{\alpha-n}[\frac{m!}{(m-n)!}x^{m-n}] \\
&= \frac{m!}{(m-n)!}D^{\alpha-n}(x^{m-n}) = \frac{m!}{(m-n)!}\frac{1}{\Gamma(n-\alpha)}\int_0^x t^{m-n}(x-t)^{n-\alpha-1}dt \\
&= \frac{m!}{(m-n)!}\frac{1}{\Gamma(n-\alpha)}[\frac{-t^{m-n}(x-t)^{n-\alpha}}{n-\alpha}|_{t=0}^{t=x} + \frac{m-n}{\alpha-n}\int_0^x t^{m-n-1}(x-t)^{n-\alpha}dt] \\
&= \frac{m!}{(m-n)!}\frac{1}{\Gamma(n-\alpha)}\frac{(m-n)}{(n-\alpha)}\int_0^x t^{m-n-1}(x-t)^{n-\alpha}dt \\
&= \frac{m!}{(m-n)!}\frac{1}{\Gamma(n-\alpha)}\frac{(m-n)(m-n-1)}{(n-\alpha)(n-\alpha+1)}\int_0^x t^{m-n-2}(x-t)^{n-\alpha+1}dt \\
&= \frac{m!}{(m-n)!}\frac{1}{\Gamma(n-\alpha)}\frac{(m-n)(m-n-1)\dots(m-n-(m-n-1))}{(n-\alpha)(n-\alpha+1)\dots(n-\alpha+(m-n+1))} \\
&\quad \int_0^x t^{m-n-(m-n)}(x-t)^{n-\alpha+(m-n-1)}dt \\
&= \frac{\Gamma(m+1)}{\Gamma(m-\alpha+1)}x^{m-\alpha}.
\end{aligned}$$

Domain Transforms

The Laplace and Fourier transforms to the frequency domain can be used to get generalizations of the derivative validity for functions that allow such transformations. The Laplace transform is defined by

$$\mathcal{L}\{f(x)\} = \int_0^{\infty} e^{-tx}f(x)dx, \tag{3.9}$$

while its inverse transform is

$$\mathcal{L}^{-1}\{f(x)\} = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{tx}f(x)dx, \tag{3.10}$$

where a is chosen so that it is greater than the real part of any of the singularities of $f(x)$. An important of the Laplace transform is related to the $m-th$ derivative of a function,

$$\mathcal{L}\{D^m f(x)\} = t^m\mathcal{L}\{f(x)\} - \sum_{k=0}^{m-1} t^k(D^{m-k-1}f)(0). \tag{3.11}$$

In the cases that the terms in the summation are zero the relation is particularly simple, and for which the generalized derivative can defined as

$$D^{\alpha}f(x) = \mathcal{L}^{-1}\{t^{\alpha}\mathcal{L}\{f(x)\}\}. \tag{3.12}$$

On the other hand, the Fourier transform is defined by

$$\mathcal{F}\{f(x)\} = \int_{-\infty}^{\infty} e^{-itx} f(x)dx \tag{3.13}$$

while its inverse transform is

$$\mathcal{F}^{-1}\{f(x)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{itx} f(x)dx. \tag{3.14}$$

This transform also has an analogous property related to the transform of the $n-th$ derivative of a function,

$$\mathcal{F}\{D^{n}f(x)\} = (it)^{n}\mathcal{F}\{f(x)\} \tag{3.15}$$

and the derivative can be generalized so that this property holds true for non-integer values of α

$$\mathcal{F}\{D^{\alpha}f(x)\} = (it)^{\alpha}\mathcal{F}\{f(x)\} \tag{3.16}$$

yielding the following definition of the generalized derivative

$$D^{\alpha}f(x) = \mathcal{F}^{-1}\{(it)^{\alpha}\mathcal{F}\{f(x)\}\}. \tag{3.17}$$

In these two generalizations the implicit limits of differentiation should be determined. In the cases of Laplace transform, the generalized derivative is a Riemann-Liouville derivative with the lower limit of 0, whereas in the cases of Fourier transform it is a Weyl derivative.

Convolution

The generalize of the derivative as expressed in sense of Riemann-Liouville suggest that they can be formulated in terms of the convolution which would be important the convolution is a simple operation in the frequency space achieved by Laplace and Fourier transforms. The following development show how this is the case, and how after all derivative of a fraction is its convolution with certain function;

$$\phi_\alpha(x) \equiv \frac{x^{\alpha-1}}{\Gamma(\alpha)}. \tag{3.18}$$

The Laplace transform of ϕ_α is

$$\mathcal{L}\{\phi_\alpha(x)\} = \mathcal{L}\{\frac{x^{\alpha-1}}{\Gamma(\alpha)}\} = t^{-\alpha} \tag{3.19}$$

and the Laplace transform of convolution

$$\mathcal{L}\{\phi_\alpha(x) * \phi_\beta(x)\} = \mathcal{L}\{\phi_\alpha(x)\}\mathcal{L}\{\phi_\beta(x)\} = t^{-(\alpha+\beta)} = \mathcal{L}\{\phi_{\alpha+\beta}(x)\}. \tag{3.20}$$

Implying that these functions satisfy the semigroup property,

$$\phi_\alpha * \phi_\beta = \phi_{\alpha+\beta}. \tag{3.21}$$

So the Riemann-Liouville fractional derivative of order $-\alpha$, $\alpha > 0$ (or the fractional integration) can be defined as,

$$D^{-\alpha} f(x) \equiv (\phi_\alpha * f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x f(t)(x-t)^{\alpha-1} dt. \tag{3.22}$$

Moreover, we obtain the Laplace transform,

$$\mathcal{L}\{D^{-\alpha} f(x)\} = \mathcal{L}\{\phi_\alpha(x) * f(x)\} = \mathcal{L}\{\phi_\alpha(x)\}\mathcal{L}\{f(x)\} = t^{-\alpha}\mathcal{L}\{f(x)\}. \tag{3.23}$$

And by using relation (3.21) and associativity of the convolution we obtain

$$\begin{aligned} \mathcal{L}\{D^{-\alpha}D^{-\beta}f(x)\} &= t^{-\alpha}\mathcal{L}\{D^{-\beta}f(x)\} = t^{-\alpha}t^{-\beta}\mathcal{L}\{f(x)\} \\ &= t^{-(\alpha+\beta)}\mathcal{L}\{f(x)\} = \mathcal{L}\{D^{-\alpha-\beta}f(x)\}. \end{aligned} \tag{3.24}$$

These imply that the operator of fractional integration obey the semigroup property,

$$D^{-\alpha}D^{-\beta}f(x) = D^{-\alpha-\beta}f(x). \tag{3.25}$$

Example 3.1.4. *(Half derivative of a simple function.) Let us assume that $f(x)$ is a monomial of the form $f(x) = x^k$. By using Laplace transform*

$$\mathcal{L}\{D^\alpha x^k\} = s^\alpha \mathcal{L}\{x^k\} = \frac{\Gamma(k+1)}{s^{k+1-\alpha}}. \tag{3.26}$$

Hence

$$D^{\alpha}x^{k} = \mathcal{L}^{-1}\left\{\frac{\Gamma(k+1)}{s^{k+1-\alpha}}\right\} = \frac{\Gamma(k+1)x^{k-\alpha}}{\Gamma(k+1-\alpha)}. \tag{3.27}$$

Such as, the half derivative of x,

$$D^{\frac{1}{2}}x = \frac{\Gamma(1+1)x^{1-\frac{1}{2}}}{\Gamma(1+1-\frac{1}{2})} = \frac{\Gamma(2)}{\Gamma\frac{3}{2}} = \frac{2\sqrt{x}}{\sqrt{\pi}}. \tag{3.28}$$

Moreover,

$$D^{\frac{1}{2}}(D^{\frac{1}{2}}x) = D^{\frac{1}{2}}(\frac{2\sqrt{x}}{\sqrt{\pi}}) = \frac{2D^{\frac{1}{2}}x^{\frac{1}{2}}}{\sqrt{\pi}} = \frac{2}{\sqrt{\pi}}\frac{\Gamma(\frac{3}{2})}{\Gamma(1)} = \frac{2}{\sqrt{\pi}}\frac{\sqrt{\pi}}{2} = 1 \tag{3.29}$$

which is indeed expected result of

$$D^{1/2}(D^{1/2}x) = Dx = 1. \tag{3.30}$$

Proposition 3.1.5. *Assume that the function in the definition 3.1.1 has a Laplace 's transform. Then its fractional derivative of order* α *is defined by the following expression*

$$D^{-\alpha}f(x) \equiv \frac{1}{\Gamma(\alpha)}\int_0^x f(t)(x-t)^{\alpha-1}dt \tag{3.31}$$

where $\alpha > 0$.

For positive, one will set

$$D^{\alpha}f(x) = D^{n}(D^{\alpha-n}f(x)) \tag{3.32}$$

where $0 < n-\alpha < 1$ *and* D^n *denote the ordinary derivative of order integer* n.

Definition 3.1.6. *(Riemann-Liouville derivative.) Refer to the function of Proposition 3.1.5. Then its fractional derivative of order* α *is defined by the expression (3.31).*

With this definition, the Laplace transform of the fractional derivative is

$$\mathcal{L}\{D^{\alpha}f(x)\} = s^{\alpha}\mathcal{L}\{f(x)\} \text{ for all } 0 < \alpha < 1. \tag{3.33}$$

Proposition 3.1.7. *Let $f : \Re \to \Re$ be a continuous function and has fractional derivative of order $k\alpha$, $k \in \mathbb{N}$ and $0 < \alpha \leq 1$. Then the following fractional Taylor series holds, which is*

$$f(x+h) = \sum_{k=0}^{\infty} \frac{h^{\alpha k} f^{\alpha k}(x)}{\Gamma(1+\alpha k)}, \quad 0 < \alpha \leq 1 \tag{3.34}$$

this is equivalent to

$$f(x) = \sum_{k=0}^{\infty} \frac{(x-a)^{\alpha k} f^{\alpha k}(a)}{\Gamma(1+\alpha k)}, \quad 0 < \alpha \leq 1. \tag{3.35}$$

We say, the ***fractional Taylor expansion*** *about point $x = a$, if $a = 0$, this expansion is called the* ***fractional Mac-Loaurin expansion****.*

Corollary 3.1.8. *Assume that a function f in Proposition 3.1.7 is the $\alpha - th$ differentiable. Then the following equality holds, which are*

$$f^{\alpha}(x) = \lim_{h \to 0} \frac{\Delta^{\alpha} f(x)}{h^{\alpha}} = \Gamma(1+\alpha) \lim_{h \to 0} \frac{\Delta f(x)}{h^{\alpha}}, \quad 0 < \alpha \leq 1. \tag{3.36}$$

Moreover, the equation (3.36) provides the useful relation

$$\Delta^{\alpha} f(x) \cong \Gamma(1+\alpha) \Delta f(x) \quad or \quad d^{\alpha} f(x) \cong \Gamma(1+\alpha) df. \tag{3.37}$$

We obtain some properties for $0 < \alpha \leq 1$ (or more detail see [7]);

$$D^{\alpha}[u(x)v(x)] = u(x) D^{\alpha} v(x) + v(x) D^{\alpha} u(x), \tag{3.38}$$

$$D^{\alpha} f(u(x)) = \frac{df(u)}{du} D^{\alpha} u(x) = D_u^{\alpha} f(u) (\frac{du}{dx})^{\alpha}. \tag{3.39}$$

Note in the previous part that for $\alpha = 0$, D^0 denote the identity operator that is $D^0 f = f$, for $\alpha > 0$, D^{α} is denote the fractional derivative and $D^{-\alpha}$ denote the fractional integration, in particularly if α is an integer value it should be equal to ordinary derivative and ordinary integration respectively.

As in the case of differentiation and integration of integer order, D^n is a left inverse of D^{-n}, but in general it is not a right inverse. More precisely, we have the following theorem.

Theorem 3.1.9. *Let $\alpha > 0$ and integer n, $0 < \alpha - n < 1$. Assume that the function f has a Laplace's transform. Then*

$$D^{\alpha} D^{-\alpha} f(x) = f(x) \tag{3.40}$$

but in general not a right inverse;

$$D^{-\alpha} D^{\alpha} f(x) = f(x) - \sum_{k=0}^{n-1} f^{(k)}(0)\phi_{k+1}(t). \tag{3.41}$$

Note from Theorem 3.1.9 that in the sense of Riemann-Liouville derivative D_t^{α} is again a left inverse of $D_t^{-\alpha}$ but in general not a right inverse:

$$D^{\alpha} D^{-\alpha} f(x) = f(x)$$

$$D^{-\alpha} D^{\alpha} f(x) = f(x) - \sum_{k=0}^{n-1} (\phi_{n-\alpha} * f)^{(k)}(0)\phi_{\alpha+k+1-n}(t).$$

3.2 Mittag-Leffler functions

In this section, we summarize some properties of the general exponential function that called the **Mittag-Leffler function** which plays an important role in the study of fractional differential equations.

Definition 3.2.1. *For each α, $\beta > 0$ and $z \in \mathbb{C}$, the **Mittag-Leffler function** is defined as follows;*

$$E_{\alpha,\beta}(z) \equiv \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)} = \frac{1}{2\pi i} \int_C \frac{u^{\alpha-\beta} e^u}{u^{\alpha} - z} du, \tag{3.42}$$

where C is a contour which starts and ends at $-\infty$ and encircles the disc $|u| \leq |z|^{1/\alpha}$ counter-clockwise.

For short, $E_{\alpha}(z) \equiv E_{\alpha,1}(z)$. It is provided a simple generalization of the exponential function: $E_1(z) = e^z$ and the cosine function: $E_2(z^2) = cosh(z)$, $E_2(-z^2) = cos(z)$, and plays an important role in the theory of the fractional differential equations. Similarly to the differential equation $d/dt(e^{\omega t}) = \omega e^{\omega t}$ the Mittag-Leffler function $E_{\alpha(z)}$ satisfies a more general differential relation

$$D_t^{\alpha} E_{\alpha}(\omega t^{\alpha}) = \omega E_{\alpha}(\omega t^{\alpha}). \tag{3.43}$$

The most interesting properties of the Mittag-Leffler functions are associated with their Laplace integral

$$\int_0^\infty e^{-\lambda t} t^{\beta-1} E_{\alpha,\beta}(\omega t^\alpha)dt = \frac{\lambda^{\alpha-\beta}}{\lambda^\alpha - \omega}, \quad \mathrm{Re}\lambda > \omega^{1/\alpha},\ \omega > 0, \tag{3.44}$$

and with their asymptotic expansion as $z \to \infty$. If $0 < \alpha < 2$, $\beta = 1$ then

$$\begin{cases} E_\alpha(z) = \frac{1}{\alpha} e^{1/\alpha} + \epsilon_\alpha(z), \quad |arg(z)| \le \frac{1}{2}\alpha\pi, \\ E_\alpha(z) = \epsilon_\alpha(z), \quad |arg(-z)| \le (1 - \frac{1}{2}\alpha)\pi, \end{cases} \tag{3.45}$$

where

$$\epsilon_\alpha(z) = -\sum_{n=1}^{N-1} \frac{z^{-n}}{\Gamma(1-\alpha n)} + O(|z|^{-N}), \quad z \to \infty \ \text{ and for some } N \in \mathbb{N}.$$

Let us consider the ordinary fractional differential equation

$$D_t^\alpha u(t) = -\omega u(t), \ \ 0 < \alpha < 2,\ \omega > 0. \tag{3.46}$$

According to the cases $0 < \alpha < 1$ and $1 < \alpha < 2$ it can be referred to as the **fractional relaxation** and the **fractional oscillation** equation, respectively. In the former cases, it must be equipped with an initial, say $u(0) = u_0$, and in the later with two initial conditions, say $u(0) = u_0$ and $u'(0) = u_1$. The solution of (3.46) can be obtained by applying the Laplace transform technique which implies;

$$\begin{aligned} u(t) &= u_0 E_\alpha(-\omega t^\alpha), \quad \alpha \in (0,1), \\ u(t) &= u_0 E_\alpha(-\omega t^\alpha) + u_1 t E_{\alpha,2}(-\omega t^\alpha), \quad \alpha \in (1,2). \end{aligned}$$

CHAPTER IV
FRACTIONAL INTEGRO-DIFFERENTIAL EQUATION WITH C_0−SEMIGROUP

In this chapter, we introduce a mild solution for the fractional integro-differential equation with time delay by using semigroup approach. Some useful theorems (say Gronwall lemma) are listed in section 4.1. They can be used to estimate the integral inequalities. In section 4.2, we configure a mild solution to the nonlinear integro-differential controlled system with time delay;

$$\begin{cases} D_t^{\alpha}x(t) = Ax(t) + f(t, x(t), Kx(t)) + Gx(t) + B(t)u(t), \quad t \in [0, T] \\ x(t) = \varphi(t), \quad t \in [-r, 0] \end{cases} \tag{4.1}$$

where $Gx(t) = \int_{-r}^{t} h(t, s)g(s, Hx(s))ds$. Then the optimal control will be discussed in section 4.3 and an example is established to clarify our results in section 4.4.

4.1 Gronwall Lemma with Singularity

Throughout this thesis, we denote $[0, T]$ by I. Let X and Y be two Banach spaces, $L(X, Y)$ denote the space of bounded linear operators from X to Y. Particularly $L(X) = L(X, X)$ whose norm is denoted by $|| \cdot ||_{L(X)}$. Suppose that $r > 0$. Let $C([-r, a], X)$ be the Banach spaces of continuous functions from $[-r, a]$ to X with the usual supremum norm. If $a = 0$, we denote this space simply by C and its norm by $|| \cdot ||_C$. Now we state the Gronwall lemma and a generalized Gronwall lemma with singularity.

Lemma 4.1.1. *(Gronwall Lemma) For $t \geq 0$, let a function $x \in C([-r, T], X)$ such that*

$$||x(t)|| \leq a + \int_0^t b(s)||x(s)||ds, \quad t \in I \tag{4.2}$$

where $a > 0$, $b(s)$ is a nonnegative integrable function. Then

$$||x(t)|| \leq ae^{\int_0^t b(s)ds}, \quad \textit{for all} \;\; t \in I. \tag{4.3}$$

Proof. Let $g(t)$ be the right hand side of equation (4.2), we obtain

$$g'(t) = b(t)||x(t)|| \leq b(t)g(t), \quad g(0) = a, \tag{4.4}$$

which yield after integrating from 0 to t, hence

$$||x(t)|| \leq g(t) \leq ae^{\int_0^t b(s)ds}. \tag{4.5}$$

This completes the proof. □

Let φ be a given continuous function, we denote

$$B = \{x \in C([-r,T],X) \mid x(t) = \varphi(t) \;\; \text{for} \quad -r \leq t \leq 0\} \tag{4.6}$$

whose moving norm is defined by

$$||x_t||_B = \sup_{-r \leq s \leq t} ||x(s)||. \tag{4.7}$$

So a generalized Gronwall lemma with time delay is established.

Lemma 4.1.2. *Suppose that $x \in C([-r,T],X)$ satisfies the following inequality*

$$\begin{cases} ||x(t)|| \leq a + \int_0^t b(s)||x(s)||ds + \int_0^t c(s)||x_s||_B ds, \quad t \in I \\ x(t) = \varphi(t), \quad t \in [-r,0], \end{cases} \tag{4.8}$$

where $a > 0$, $b(s)$ and $c(s)$ are nonnegative continuous functions. Then

$$||x(t)|| \leq [a + (1 - e^{-2\bar{b}t})||\varphi||_C]e^{\bar{b}t} \;\; \textit{for all} \;\; t \in I, \tag{4.9}$$

where $\bar{b} = \sup_{s \in I}[b(s) + c(s)]$.

Proof. Let $x \in C([-r,T],X)$ which satisfies the inequality (4.8). Note that $||x(t)|| \leq \sup_{-r \leq s \leq t} ||x(s)|| = ||x_t||_B$ for all $t \in I$. Then for any $t \in I$, we have

$$||x(t)|| \leq a + 2\bar{b}\int_0^t ||x_s||_B ds$$

where $\bar{b} = \sup_{s\in I}[b(s) + c(s)]$. Setting

$$g(t) = a + 2\bar{b}\int_0^t ||x_s||_B ds \quad \text{for} \quad t \in I. \tag{4.10}$$

Then g is monotonously increasing and $||x(t)|| \le g(t)$ for all $t \in I$. Moreover, we obtain that

$$g'(t) = 2\bar{b}||x_t||_B = 2\bar{b}\sup_{-r\le s\le t}||x(s)|| \le 2\bar{b}||\varphi||_C + 2\bar{b}g(t), \quad g(0) = a, \tag{4.11}$$

That is,

$$de^{-2\bar{b}t}g(t) \le e^{-2\bar{b}t}2\bar{b}||\varphi||_C dt \tag{4.12}$$

which yield after integrating from 0 to t, hence

$$g(t) \le [a + (1 - e^{-2\bar{b}t})||\varphi||_C]e^{2\bar{b}t}. \tag{4.13}$$

Therefore,

$$||x(t)|| \le [a + (1 - e^{-2\bar{b}t})||\varphi||_C]e^{2\bar{b}t} \quad \text{for all} \quad t \in I,$$

where $\bar{b} = \sup_{s\in I}[b(s) + c(s)]$.
The proof is completed. □

We give a generalized Gronwall inequality with singularity.

Lemma 4.1.3. *Suppose $x \in C([-r, T], X)$ satisfies the following inequality*

$$\begin{cases} ||x(t)|| \le a + \int_0^t b(s)(t-s)^{\beta-1}||x(s)||ds + \int_0^t c(s)(t-s)^{\beta-1}||x_s||_B ds, \quad t \in I, \\ x(t) = \varphi(t); \quad t \in [-r, 0], \end{cases} \tag{4.14}$$

where $0 < \beta \le 1$ and $a > 0$, $b(s)$ and $c(s)$ are nonnegative continuous functions. Then

$$||x(t)|| \le [||\varphi||_C + a]e^{\frac{\bar{b}t^\beta}{\beta}}, \quad \textit{for all } t \in I, \tag{4.15}$$

where $\bar{b} = \sup_{s\in I}[b(s) + c(s)]$.

Proof. Note that $||x(t)|| \leq \sup\limits_{-r\leq s\leq t} ||x(s)|| = ||x_t||_B$ for all $t \in I$. So

$$\begin{aligned} ||x(t)|| &\leq a + \int_0^t [b(s) + c(s)](t-s)^{\beta-1}||x_s||_B ds \\ &\leq a + b\int_0^t (t-s)^{\beta-1}||x_s||_B ds \qquad (4.16) \end{aligned}$$

where $\bar{b} = \sup\limits_{s\in I}(b(s) + c(s))$.

Let $g(t) = \int_0^t (t-s)^{\beta-1}||x_s||_B ds$. Then g is monotonously increasing. Indeed, use the fact that $||x_t||_B$ is monotonously increasing, so for $0 \leq \tau < t$,

$$\begin{aligned} g(t) - g(\tau) &= \int_0^t (t-s)^{\beta-1}||x_s||_B ds - \int_0^\tau (\tau-s)^{\beta-1}||x_s||_B ds \\ &= \int_0^t u^{\beta-1}||x_{t-u}||_B du - \int_0^\tau u^{\beta-1}||x_{\tau-u}||_B du \\ &= \int_0^\tau u^{\beta-1}\left[||x_{t-u}||_B - ||x_{\tau-u}||_B\right] du + \int_\tau^t u^{\beta-1}||x_{t-u}||_B du > 0. \qquad (4.17) \end{aligned}$$

This implies that $g(s) \leq g(t)$ for all $0 \leq s \leq t$,

$$\begin{aligned} ||x_t||_B &\leq \sup_{-r\leq s\leq 0} ||\varphi(s)|| + \sup_{0\leq s\leq t} ||x(s)|| \\ &\leq ||\varphi||_C + \sup_{0\leq s\leq t} [a + bg(s)] \\ &\leq ||\varphi||_C + a + \bar{b}\int_0^t (t-s)^{\beta-1}||x_s||_B ds. \qquad (4.18) \end{aligned}$$

Applying the lemma 4.1.1, we obtain

$$||x(t)|| \leq ||x_t||_B \leq [||\varphi||_C + a]e^{\int_0^t \bar{b}(t-s)^{\beta-1}ds} \leq [||\varphi||_C + a]e^{\frac{\bar{b}t^\beta}{\beta}}.$$

This completes the proof. □

Using lemma 4.1.3, we devise the following new generalized Gronwall lemma which is very important for our work.

Lemma 4.1.4. *Suppose $x \in C([-r,T], X)$ satisfies the following inequality*

$$\begin{cases} ||x(t)|| \leq a + b\int_0^t (t-s)^{\beta-1}||x(s)||ds + c\int_0^t (t-s)^{\beta-1}||x_s||_B ds \\ \qquad\qquad + e\int_0^t (t-s)^{\beta-1}||x(s)||^\gamma ds, \quad t \in I \\ x(t) = \varphi(t), \quad t \in [-r, 0], \end{cases}$$

where $0 < \gamma,\ \beta \le 1,\ a,\ b,\ c,\ e$ *are nonnegative constants. Then*

$$||x(t)|| \le [||\varphi||_C + a + \frac{eT^\beta}{\beta}]e^{\frac{(b+c+e)t^\beta}{\beta}},\ t \in I. \qquad (4.19)$$

Proof. Note that $||x(s)|| \le \sup_{-r\le\tau\le s} ||x(\tau)|| = ||x_s||_B$, for $s \in I$ and $||x_t||_B$ is increasing function. We will prove this theorem by considering 4 cases;

Case 1)$||x_t||_B \le 1$ for all $t \in I$.

Case 2)$||x_t||_B \ge 1$ for all $t \in I$.

Case 3) There is a $t_0 \in [0,T]$ such that $||x_t||_B \le 1$ for all $t \in [0,t_0]$ and $||x_t||_B > 1$ for all $t \in [t_0,T]$.

Case 4) There is a $t_0 \in [0,T]$ such that $||x_t||_B > 1$ for all $t \in [0,t_0]$ and $||x_t||_B \le 1$ for all $t \in [t_0,T]$.

The proof in each cases are similar, we will show only Case 3). If there is a $t_0 \in [0,T]$ such that $||x_t||_B \le 1$ for all $t \in [0,t_0]$ and $||x_t||_B > 1$ for all $t \in [t_0,T]$. For $t \in [0,T]$, $||x_t||_B > 1$, we have

$$\begin{aligned}
||x(t)|| &\le a + b\int_0^t (t-s)^{\beta-1}||x(s)||ds + c\int_0^t (t-s)^{\beta-1}||x_s||_B ds + e\int_0^t (t-s)^{\beta-1}||x_s||^\gamma ds \\
&\le a + b\int_0^t (t-s)^{\beta-1}||x(s)||ds + c\int_0^t (t-s)^{\beta-1}||x_s||_B ds + e\int_0^{t_0} (t_0-s)^{\beta-1}||x_s||^\gamma ds \\
&+ e\int_{t_0}^t (t-s)^{\beta-1}||x_s||^\gamma ds \\
&\le a + \frac{et_0^\beta}{\beta} + b\int_0^t (t-s)^{\beta-1}||x(s)||ds + (c+e)\int_0^t (t-s)^{\beta-1}||x_s||_B ds.
\end{aligned}$$

Applying the lemma 4.1.3, we obtain that

$$||x(t)|| \le [||\varphi||_C + a + \frac{eT^\beta}{\beta}]e^{\frac{(b+c+e)t^\beta}{\beta}}, \quad \text{for all} \quad t \in [0,T].$$

□

4.2 Existence of Solution to Controlled System with Delay

In what follow, let X be a separable Banach space and Y be a reflexive Banach space. For $1 < q < \infty$, the Banach space $L_q(I,Y)$ consist of the usual strongly measurable $Y-$ value functions having $q-th$ power summable norms. Let A :

$D(A) \to X$ be an infinitesimal generator of a C_0-semigroup $\{T(t)\}_{t\geq 0}$ satisfying $||T(t)||_{L(X)} \leq Me^{\omega t}$ for some $M \geq 1$, $\omega > 0$, for all $t \geq 0$.

Definition 4.2.1. *Let $(X, ||\cdot||)$ be a Banach space and let $F : D(F) \to X$, $D(F)$ is a subset of X denoting the domain of F.*

(i) F satisfied a Lipschitz condition on $D(F)$ if there exists a positive constant κ such that

$$||F(f) - F(g)|| \leq \kappa ||f - g||, \quad \text{for all } \; f, g \in D(F). \tag{4.20}$$

(ii) F satisfies a local Lipschitz condition if, given $u_0 \in D(F)$, a closed ball $B(u_0, r) = \{f \in X \mid ||f - u_0|| \leq r\}$ exists such that

$$||F(f) - F(g)|| \leq k||f - g||, \quad \text{for all } \; f, g \in B(u_0, r) \cap D(F) \tag{4.21}$$

where k will in general depend on u_0 and r.

Consider the controlled system with delay;

$$\begin{cases} D_t^\alpha x(t) = Ax(t) + f(t, x(t), Kx(t)) + Gx(t) + B(t)u(t), \quad t \in I \\ x(t) = \varphi(t), \quad t \in [-r, 0] \end{cases} \tag{4.22}$$

where $\varphi \in C([-r, 0], X)$(or $\varphi \in PC([-r, 0], X)$) is fixed. The integral operator $G : X \to X$ is defined by

$$Gx(t) = \int_{-r}^{t} h(t, s)g(s, Hx(s))ds, \quad \text{for all } \; x \in X \tag{4.23}$$

where h is kernel function of G and g is an input function. For $0 < \alpha < 1$, D_t^α denote the Riemann-Liouville fractional derivative, $f : I \times X \times X \to X$ is a given continuous function. Suppose:

(HK) $K : X \to X$ is bounded linear operator.

(HF1) $f : I \times X \times X \to X$ is uniformly continuous in t and locally Lipschitz in x and y that is, for some $\rho > 0$, there is constant $a_f = a_f(\rho, \tau)$ such that

$$||f(t, x_1, y_1) - f(t, x_2, y_2)|| \leq a_f[||x_1 - x_2|| + ||y_1 - y_2||] \tag{4.24}$$

provided $||x_1||$, $||x_2||$, $||y_1||$, $||y_2|| \leq \rho$ and for all $t \in [0, \tau]$.

(HF2) There exists $c \geq 0$ such that $||f(t,x,y)|| \leq c(1+||x||+||y||)$, for all x, $y \in X$, for all $t \in I$.

(HB) Y is another separable reflexive Banach space from which the controls u take the value $B(s) \in L(L_q(I,Y), L_p(I,X))$, $1 < p, q < \infty$ for all $s \in [0,T]$.

Before proving the existence of system (4.22), we will prove some properties of the integral operator $G : X \to X$ in the delay system such that it is defined by

$$Gx(t) = \int_{-r}^{t} h(t,s)g(s, Hx(s))ds, \quad \text{for all} \quad t \in I, \quad x \in X,$$

under the following assumptions, say condition (HG)

(HG1) $g : [-r,T] \times X \to X$ is measurable function in t on $[-r,T]$ and locally Lipschitz continuous in x, i.e., for $\rho > 0$, for all x_1, $x_2 \in X$ satisfying $||x_1||$, $||x_2|| \leq \rho$ there exists a constant $L_g = L_g(\rho) > 0$ such that

$$||g(t,x_1) - g(t,x_2)|| \leq L_g||x_1 - x_2||, \quad \text{for all} \quad t \in I. \tag{4.25}$$

(HG2) There exist a constant $a_g > 0$ such that

$$||g(t,x)|| \leq a_g(1 + ||x||) \quad \text{for all} \quad t \in I, \quad x \in X. \tag{4.26}$$

(HG3) $h \in C([-r,T]^2, \Re)$ and H is a bounded linear operator.

By using the moving norm $|| \cdot ||_B$, we obtain the following lemmas;

Lemma 4.2.2. *Under the assumption (HG), the operator G has the following properties;*

1) $G : C([-r,T],X) \to C([-r,T],X)$.

2) For each $x_1, x_2 \in C([-r,T],X)$ such that $||x_1||$, $||x_2|| \leq \rho$, we have

$$||Gx_1(t) - Gx_2(t)|| \leq L_g||h||(T+r)||(x_1)_t - (x_2)_t||_B, \quad \textit{for all} \quad t \in I. \tag{4.27}$$

3) For each $x \in C([-r,T],X)$, we have

$$||Gx(t)|| \leq a_g(T+r)||h||(1+||x_t||_B), \quad \text{for all} \;\; t \in I. \tag{4.28}$$

Proof. (1) Let $x \in C([-r,T],X)$. Since h is continuous on the compact set $[-r,T]^2$, h is uniformly continuous. So, for each $\epsilon > 0$ there exists $\delta_1 > 0$ such that if $s \in I$ and $|t-a| < \delta_1$, then $|h(t,s) - h(a,s)| < \epsilon$ for all a, $t \in I$. Given $\epsilon > 0$. Choose $\delta = \min\{\delta_1, \frac{\epsilon}{||h||(1+||H||||x||)}\}$ and $0 < \tau < \delta$. Then for each $t \in I$.

$$\begin{aligned}||Gx(t+\tau) - Gx(t)|| =& || \int_{-r}^{t+\tau} h(t+\tau,s)g(s,Hx(s))ds - \int_{-r}^{t} h(t,s)g(s,Hx(s))ds|| \\ \leq& \int_{-r}^{t} ||h(t+\tau,s) - h(t,s)|| ||g(s,Hx(s))||ds \\ &+ \int_{t}^{t+\tau} ||h(t+\tau,s)|| ||g(s,Hx(s))||ds \\ \leq& (T+r)\epsilon a_g(1+||H||||x||) + \delta||h||(1+||H||||x||) \qquad (4.29) \\ \leq& [(T+r)a_g(1+||H||||x||)+1]\epsilon. \qquad (4.30)\end{aligned}$$

Since ϵ is arbitrary, $Gx \in C([-r,T],X)$.

(2) Let $x_1, x_2 \in C([-r,T],X)$ such that $||x_1||, ||x_2|| \leq \rho$. Then, for any $t \in I$, we have

$$\begin{aligned}||Gx_1(t) - Gx_2(t)|| =& || \int_{-r}^{t} h(t,s)g(s,Hx_1(s))ds - \int_{-r}^{t} h(t,s)g(s,Hx_2(s))ds|| \\ \leq& \int_{-r}^{t} ||h(t,s)|| ||g(s,Hx_1(s)) - g(s,Hx_2(s))||ds \\ \leq& ||h||(T+r)L_g||H|| ||(x_1)_t - (x_2)_t||_B. \qquad (4.31)\end{aligned}$$

(3) Let $x \in C([-r,T],X)$. Then, for any $t \in I$, we have

$$\begin{aligned}||Gx(t)|| \leq& \int_{-r}^{t} ||h(t,s)|| ||g(s,Hx(s))||ds \\ \leq& ||h||(T+r)(1+||H||||x_t||_B). \qquad (4.32)\end{aligned}$$

□

Now, we will prove the existence and uniqueness of a mild solution for the system (4.22) which state as following definition. Recall the system (4.22);

$$\begin{cases} D_t^\alpha x(t) = Ax(t) + f(t, x(t), Kx(t)) + Gx(t) + B(t)u(t), \quad t \in I \\ x(t) = \varphi(t), \quad t \in [-r, 0]. \end{cases}$$

Let A be an infinitesimal generator of a C_0-semigroup $\{T(t)\}_{t\geq 0}$ and $0 < \alpha < 1$. Define a function ϕ by $\phi_\alpha(t) = \frac{t^\alpha}{\Gamma(\alpha+1)}$ for all $t \geq 0$, for short, we denote $\phi_\alpha(t)$ by $\phi(t)$. If x is a solution of (4.22), then the $X-$valued function $w(s) = T(\phi(t) - \phi(s))x(s)$ is $\alpha-$differentiable for $0 < s < t$ and we use the properties (3.38) and (3.39) to obtain that

$$\begin{aligned} D_s^\alpha w(s) &= T(\phi(t) - \phi(s))D_s^\alpha x(s) - AT_\alpha(\phi(t) - \phi(s))x(s) \\ &= T(\phi(t) - \phi(s))[Ax(s) + f(s, x(s), Kx(s)) + Gx(s) + B(s)u(s)] \\ &\quad - AT(\phi(t) - \phi(s))x(s) \\ &= T(\phi(t) - \phi(s))[f(s, x(s), Kx(s)) + Gx(s) + B(s)u(s)]. \end{aligned} \tag{4.33}$$

If f is integrable, then the right hand side of (4.33) is integrable in the sense of Bochner and integrating (4.33) of order α from 0 to t and apply the initial value $w(0) = T(\phi(t))\varphi(0)$, yields

$$\begin{aligned} x(t) =& T(\phi(t))\varphi(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} T(\phi(t) - \phi(s))[f(s, x(s), Kx(s)) + Gx(s) \\ &+ B(s)u(s)]ds, \quad \text{for all } \ t \in I. \end{aligned}$$

So we give the definition of mild solution for the system (4.22)

Definition 4.2.3. *For every $u \in L_q(I, Y)$, $1 < q < \infty$, if there exists a $t_0 = t_0(u) > 0$ and $x \in C([-r, t_0], X)$ such that*

$$\begin{cases} x(t) = T(\phi(t))\varphi(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} T(\phi(t) - \phi(s))[f(s, x(s), Kx(s)) + Gx(s) \\ \qquad + B(s)u(s)]ds, \quad t \in [0, t_0], \\ x(t) = \varphi(t), \quad t \in [-r, 0] \end{cases} \tag{4.34}$$

then the system (4.22) is called mildly solvable with respect to (w.r.t) u on $[-r, t_0]$ *and this x is said to be a mild solution w.r.t. u on* $[-r, t_0]$.

Now, for each $\tau > 0$, C^τ denote the space $C([-r,\tau], X)$ with the usual supremum norm and for $\lambda > 0$, we set

$$S(\lambda,\tau) = \{y \in C^\tau \mid \max_{0\le t\le\tau} ||y(t) - \varphi(0)|| \le \lambda, y(t) = \varphi(t) \text{ for } -r \le t \le 0\}. \quad (4.35)$$

Then $S(\lambda,\tau)$ is nonempty closed convex subset of C^τ. Define $P : S(\lambda,\tau) \to C^\tau$ by

$$\begin{cases} Py(t) = T(\phi(t))\varphi(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} T(\phi(t)-\phi(s))[f(s,y(s),Ky(s)) + Gy(s) \\ \qquad + B(s)u(s)]ds, \quad t \in [0,\tau], \\ Py(t) = \varphi(t), \quad t \in [-r, 0] \end{cases} \quad (4.36)$$

for all $y \in S(\lambda,\tau)$.

To prove the existence of mild solution, we construct the map P as in (4.36) and show that it contains a fixed point by following these lemmas.

Lemma 4.2.4. *Assume the hypotheses* (HB), (HF), (HK) *and* (HG). *For* $\tau > 0$, *the map P defined as (4.36) is bounded, i.e., there exists* $M > 0$ *such that* $||Py(t)|| \le M$ *for all* $y \in S(\lambda,\tau)$.

Proof. Let $y \in S(\lambda,\tau)$. By using $(HF2)$ and lemma 4.2.2, there exist L_1, $L_2 > 0$ such that for all $s \in [0,\tau]$

$$\begin{aligned} ||f(s,y(s),Ky(s)) + Gy(s)|| &\le ||f(s,y(s),Ky(s))|| + ||Gy(s)|| \\ &\le L_1(1 + ||y(s)||) + L_2(1 + ||y_s||_B)) \le N \end{aligned} \quad (4.37)$$

for some $N > 0$ since $||y||$ and $||y_s||_B$ are continuous on $[0,\tau]$. Then apply the condition (HB) and for each $t \in [0, \tau]$, we obtain that

$$||Py(t)|| \leq ||T(\phi(t))||_{L(X)}||\varphi(0)||$$
$$+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}||T(\phi(t)-\phi(s))||_{L(X)}||f(s, y(s), Ky(s)) + Gy(s)||ds$$
$$+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}||T(\phi(t)-\phi(s))||_{L(X)})||B(s)u(s)||ds$$
$$\leq Me^{\omega\phi(T)}||\varphi||_C + \frac{NMe^{\omega\phi(T)}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}ds + \frac{Me^{\omega\phi(T)}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}||B(s)u(s)||ds.$$
$$\leq Me^{\omega\phi(T)}||\varphi||_C + \frac{NMe^{\omega\phi(T)}T^\alpha}{\alpha\Gamma(\alpha)}$$
$$+ \frac{Me^{\omega\phi(T)}}{\Gamma(\alpha)}[\int_0^T (t-s)^{\frac{p(\alpha-1)}{p-1}}ds]^{\frac{p-1}{p}}[\int_0^T ||B(s)u(s)||^p ds]^{\frac{1}{p}}$$
$$\leq Me^{\omega\phi(T)}||\varphi||_C + \frac{NMe^{\omega\phi(T)}T^\alpha}{\alpha\Gamma(\alpha)} + \frac{Me^{\omega\phi(T)}(p-1)T^{\frac{p(\alpha-1)}{p-1}}||B(\cdot)u||_{L_p(I,X)}}{(p\alpha-1)\Gamma(\alpha)} < \infty.$$

Then the map P is bounded. □

Lemma 4.2.5. *For $\tau > 0$, the operator P is well-defined on $S(\lambda, \tau)$. Moreover, there exists $\tau_0 > 0$ such that P maps $S(\lambda, \tau_0)$ into itself, i.e., $P(S(\lambda, \tau_0)) \subseteq S(\lambda, \tau_0)$.*

Proof. For $\tau > 0$, let $\{y_n\}$ be a sequence in $S(\lambda, \tau)$ and $y \in S(\lambda, \tau)$ such $y_n \to y$. Then by using $(HK), (HF2)$ and lemma 4.2.2, there exist N_1, $N_2 > 0$ such that for all $s \in [0, \tau]$,

$$||f(s, y_n(s), Ky_n(s)) - f(s, y(s), Ky(s))|| \leq N_1||y_n - y||_{S(\lambda,\tau)} \quad (4.38)$$
$$||Gy_n(s) - Gy(s)|| \leq N_2||(y_n)_\tau - y_\tau||_B. \quad (4.39)$$

Note that $||(y_n)_\tau - y_\tau||_B = \sup_{0\leq s\leq\tau} ||y_n(s) - y(s)|| \leq ||y_n - y||_{S(\lambda,\tau)}$, so, we have

$$||Py_n(t) - Py(t)||$$
$$\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}||T(\phi(t)-\phi(s))||||f(s, y_n(s), Ky_n(s)) - f(s, y(s), Ky(s))||ds$$
$$+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}||T(\phi(t)-\phi(s))||||Gy_n(s) - Gy(s)||ds$$
$$\leq \frac{Me^{\omega\phi(T)}N_1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}ds||y_n - y||_{S(\lambda,\tau)} + \frac{Me^{\omega\phi(T)}N_2}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}ds||(y_n)_\tau - y_\tau||_B$$
$$\leq \frac{Me^{\omega\phi(T)}(N_1+N_2)T^\alpha}{\alpha\Gamma(\alpha)}||y_n - y||_{S(\lambda,\tau)}.$$

Since $||y_n - y||_{S(\lambda,\tau)} \to 0$ as $n \to +\infty$, $||Py_n - Py|| \to 0$ as $n \to +\infty$.
This implies that the map P is well-defined.
We next will show that there exists τ_0 such P map $S(\lambda, \tau_0)$ into itself.
Given $\rho > 0$ and $y \in S(\lambda, \tau)$. By using assumptions (HF) and lemma 4.2.2, there exists $\kappa > 0$ such that

$$\begin{aligned} ||f(0, y(0), Ky(0)) + Gy(0)|| &\le ||f(0, y(0), Ky(0))|| + ||Gy(0)|| \\ &\le \kappa(2 + ||\varphi||_C + ||y_0||_B) \le 2\kappa(1 + ||\varphi||_C), \end{aligned} \tag{4.40}$$

and for all $s \in [0, \tau]$, there exists $a(\rho, \tau) > 0$

$$\begin{aligned} &||f(s, y(s), Ky(s)) + Gy(s) - f(0, y(0), Ky(0)) - Gy(0)|| \\ &\le ||f(s, y(s), Ky(s)) + Gy(s) - f(0, y(0), Ky(0))|| + ||Gy(s) - Gy(0)|| \\ &\le a(\rho, \tau)[||y(s) - y(0)|| + ||K||||y(s) - y(0)|| + ||y_\tau - y_0||_B] \\ &\le a(\rho, \tau)(||K|| + 2)\lambda. \end{aligned} \tag{4.41}$$

So, we obtain

$$\begin{aligned} &||Py(t) - \varphi(0)|| \\ \le &||T(\phi(t))\varphi(0) - \varphi(0)|| + \frac{Me^{\omega\phi(\tau)}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ||f(0, y(0), Ky(0)) + Gy(0)|| ds \\ &+ \frac{Me^{\omega\phi(\tau)}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ||f(s, y(s), Ky(s)) + Gy(s) - f(0, y(0), Ky(0)) - Gy(0)|| ds \\ &+ \frac{Me^{\omega\phi(\tau)}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ||B(s)u(s)|| ds \end{aligned}$$

$$\leq \max_{0\leq t\leq\tau} ||T(\phi(t))\varphi(0) - \varphi(0)|| + \frac{Me^{\omega\phi(\tau)}2\kappa(1+||\varphi||_C)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds$$
$$+ \frac{Me^{\omega\phi(\tau)}a(\rho,\tau)(||K||+2)\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds$$
$$+ \frac{Me^{\omega\phi(\tau)}}{\Gamma(\alpha)}[\int_0^\tau (t-s)^{\frac{p(\alpha-1)}{p-1}} ds]^{\frac{p-1}{p}} [\int_0^\tau ||B(s)u(s)||^p ds]^{\frac{1}{p}}$$
$$\leq \max_{0\leq t\leq\tau} ||T(\phi(t))\varphi(0) - \varphi(0)|| + \frac{Me^{\omega\phi(\tau)}(p-1)\tau^{\frac{p(\alpha-1)}{p-1}}||B(\cdot)u||_{L_p([0,\tau],X)}}{(p\alpha-1)\Gamma(\alpha)}$$
$$+ \frac{Me^{\omega\phi(\tau)}[2\kappa(1+||\varphi||_C) + a(\rho,\tau)(||K||+2)\lambda]\tau^\alpha}{\alpha\Gamma(\alpha)}$$
$$\leq \lambda q(u,\tau)$$

where

$$q(u,\tau) = \frac{1}{\lambda}\left[\max_{0\leq t\leq\tau} ||T(\phi(t))\varphi(0) - \varphi(0)|| + \frac{Me^{\omega\phi(\tau)}(p-1)\tau^{\frac{p(\alpha-1)}{p-1}}||B(\cdot)u||_{L_p([0,\tau],X)}}{(p\alpha-1)\Gamma(\alpha)}\right.$$
$$\left. + \frac{Me^{\omega\phi(\tau)}[2\kappa(1+||\varphi||_C) + a(\rho,\tau)(||K||+2)\lambda]\tau^\alpha}{\alpha\Gamma(\alpha)}\right].$$

Since $q(u,\tau) \to 0$ as $\tau \to 0^+$, a suitable τ_0 can be found such that $0 < q(u,\tau_0) < 1$. We conclude that P maps $S(\lambda,\tau_0)$ into itself, i.e., $P(S(\lambda,\tau_0)) \subseteq S(\lambda,\tau_0)$. □

Theorem 4.2.6. *Suppose (HK), (HF), (HB) and (HG) hold. Then for each $u \in L_q(I,Y)$ and $1 < q < \infty$, there exist a $t_0 > 0$ such that the system (4.22) is mildly solvable on $[-r,t_0)$ w.r.t. u and the mild solution is unique.*

Proof. For $\tau > 0$, setting

$$S(1,\tau) = \{y \in C^\tau \mid \max_{0\leq t\leq\tau} ||y(t) - \varphi(0)|| \leq 1, y(t) = \varphi(t) \text{ for all } -r \leq t \leq 0\}.$$

Then $S(1,\tau)$ is the nonempty close convex set.
Define the operator $P : S(1,\tau) \to C^\tau$ by

$$\begin{cases} Py(t) = T(\phi(t))\varphi(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} T(\phi(t)-\phi(s))[f(s,y(s),Ky(s)) + Gy(s) \\ \qquad + B(s)u(s)]ds, \quad t \in [0,\tau] \\ Py(t) = \varphi(t), \quad t \in [-r,0]. \end{cases}$$

By lemma 4.2.4, the operator P is well-defined on $S(1,\tau)$ and there exist τ_0 such P maps $S(1,\tau_0)$ into itself . We now only show that P is strictly contraction on $S(1,\tau_0)$. Given $\rho > 0$, let $y_1,\ y_2 \in S(1,\tau_0)$ such that $||y_1||,\ ||y_2|| \le \rho$. By lemma 4.2.4 and condition (HF) , there exists $b(\rho) > 0$ such that for all $s \in [0,\tau]$

$$\begin{aligned} &||f(s,y_1(s),Ky_1(s)) - f(s,y_2(s),Ky_2(s))|| + ||Gy_1(s) - Gy_2(s)|| \\ &\le b(\rho)(||y_1(s) - y_2(s)|| + ||(y_1)_s - (y_2)_s||_B) \le 2b(\rho)||y_1 - y_2||_{S(1,\tau)}. \end{aligned} \tag{4.42}$$

So, we obtain

$$\begin{aligned} &||Py_1(t) - Py_2(t)|| \\ &\le \frac{Me^{\omega\phi(\tau)}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ||f(s,y_1(s),Ky_1(s)) - f(s,y_2(s),Ky_2(s))|| + ||Gy_1(s) - Gy_2(s)||ds \\ &\le \frac{Me^{\omega\phi(\tau)}2b(\rho)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds ||y_1 - y_2||_{S(1,\tau)} \\ &\le \frac{Me^{\omega\phi(\tau)}2b(\rho)\tau^\alpha}{\alpha\Gamma(\alpha)} ||y_1 - y_2||_{S(1,\tau)} = q(u,\tau)||y_1 - y_2||_{S(1,\tau)}. \end{aligned}$$

where $q(u,\tau) = \frac{Me^{\omega\phi(\tau)}2b(\rho)\tau^\alpha}{\alpha\Gamma(\alpha)}$. Since $q(u,\tau) \to 0^+$ as $\tau \to 0^+$, a suitable $\bar{\tau}_0$ can be found such $0 < q(u,\bar{\tau}_0) < 1$, so the map P is strictly contraction. By contraction mapping on Banach space, P has a unique fixed point $x \in S(1,\tau_0)$ such that $Px(t) = x(t)$, i.e.,

$$\begin{cases} x(t) = T(\phi(t))\varphi(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} T(\phi(t) - \phi(s))[f(s,x(s),Kx(s)) + Gx(s) \\ \qquad + B(s)u(s)]ds, \quad t \in [0,\tau_0) \\ x(t) = \varphi(t), \quad t \in [-r,0]. \end{cases} \tag{4.43}$$

In other word, we say that $x(t)$ is the unique mild solution of system (4.34) w.r.t. u on $[-r,\tau_0)$. □

The problem now is to investigate what happens if $t \ge t_0$, i.e., $t = t_1 + \tau$ with $\tau \ge 0$ which is showed in the following corollary.

Corollary 4.2.7. *Under the assumptions of theorem 4.2.6, the system (4.1) has a unique mild solution on any given interval* $[-r,\tau_0)$*. (Such a solution is called global in time.)*

Proof. We start by showing that for every $\tau_0 \geq 0$, $x_0 \in X$, there exists a $\delta = \delta(\tau, ||x_0||)$ such that the system (4.1) has a unique mild solution x on an interval $[\tau_0, \tau_0 + \delta]$ whose length δ is defined by,

$$\delta(\tau_0, ||x_0||) = \min\{1, [\frac{||x_0||\alpha\Gamma(\alpha)}{\rho(\tau_0)L(\rho(\tau_0), \tau_0 + 1) + N(\tau_0)}]^{1/\alpha}\} \tag{4.44}$$

where $L(c, t)$ is the local Lipschitz constant of f and G following from $(HF1)$ and lemma 4.2.2, $M(\tau_0) = \sup\{||T(\phi(t))|| \mid 0 \leq t \leq \tau_0 + 1\}$, $\rho(\tau_0) = 2||x_0||M(\tau_0)$ and $N(\tau_0) = \max\{||f(t, 0, 0)|| + ||G0(t)|| + ||B(t)||||u||_{L_p([0,\tau_0+1],Y)} \mid 0 \leq t \leq \tau_0 + 1\}$.
Indeed, Let $\tau_1 = \tau_0 + \delta$ where δ is given by (4.44).
Define a map $P : C([\tau_0, \tau_1], X) \to C^{\tau_1}$ by

$$\begin{aligned} Px(t) =& T(\phi(t) - \phi(\tau_0))x_0 + \int_{t_0}^{t} (t-s)^{\alpha-1} T(\phi(t) - \phi(s))[f(s, x(s), Kx(s)) \\ &+ Gx(s) + B(s)u(s)]ds. \end{aligned} \tag{4.45}$$

As in the proof of theorem 4.2.6, one can show that the map P is well-defined and maps the ball of radius $\rho(\tau_0)$ centered at 0 of $C([\tau_0, \tau_1], X)$ into itself. This follows from the estimate,

$$\begin{aligned} &||Px(t)|| \\ &\leq M(\tau_0)||x_0|| + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} ||T(\phi(t) - \phi(s))||(||f(s, x(s), Kx(s)) - f(s, 0, 0)|| \\ &+ ||Gx(s) - G0(x)||) + ||f(s, 0, 0)|| + ||G0(x)|| + ||B(s)||||u||_{L_p([0,\tau_0+1],Y)})ds \\ &\leq M(\tau_0)||x_0|| + \frac{M(\tau_0)\rho(\tau_0)L(\rho(\tau_0), \tau_0 + 1)}{\alpha\Gamma(\alpha)}(t - \tau_0)^\alpha + \frac{M(\tau_0)N(\tau_0)}{\alpha\Gamma(\alpha)}(t - \tau_0)^\alpha \\ &\leq 2M(\tau_0)||x_0|| = \rho(\tau_0) \end{aligned}$$

where the last inequality follows from the definition of τ_1. In this ball, P satisfies a uniform Lipschitz condition with constant $L = L(\rho(\tau_0), \tau_0 + 1)$ and thus in the proof of theorem 4.2.6, it possesses a unique fixed point x in the ball. This fixed point is the desired solution of (4.1) on the interval $[-r, \tau_1]$, i.e.,

$$\begin{cases} x(t) = T(\phi(t)))x_0 + \int_0^t (t-s)^{\alpha-1} T(\phi(t) - \phi(s))[f(s, x(s), Kx(s)) \\ +Gx(s) + B(s)u(s)]ds, \quad \text{for } \ t \in [0, t_1] \\ x(t) = \varphi(t), \quad \text{for } \ t \in [-r, 0]. \end{cases} \tag{4.46}$$

From what we have just proved, it follows that if x is a mild solution of (4.1) on the interval $[-r,\tau]$, it can be extended to the interval $[-r,\tau+\delta]$ with $\delta>0$ by defining on $[\tau,\tau+\delta]$, $x(t)=x_1(t)$ where $x_1(t)$ is the solution of the integral equation, for $t\in[\tau,\tau+\delta]$,

$$x_1(t) = T(\phi(t)-\phi(\tau))x(\tau)+\frac{1}{\Gamma(\alpha)}\int_\tau^t (t-s)^{\alpha-1}T(\phi(t)-\phi(s))[f(s,x_1(s),Kx_1(s)) + Gx_1(s)+B(s)u(s)]ds.$$

Moreover, δ depends only on $||x(\tau)||$, $\rho(\tau)$ and $N(\tau)$. Corresponding, equation (4.1) has a unique mild solution on $[-r,2\tau_1]$. Since the above procedure can be iterated any finite number of times (always using the same δ), we conclude that (4.1) has a unique mild solution on any given interval $[-r,\tau_0)$ and hence the unique mild solution that is global in time. □

4.3 Existence of optimal control

In this section we consider the optimal control of the fractional controlled system (4.34). Suppose Y is a separable reflexive Banach space and system (4.34) is mildly solvable on $[-r,T]$ for every $u\in L_q(I,Y)$, $1<q<\infty$. Let U_{ad} be the admissible control set. We consider the Bolza problem :

(P) Find $(x^0,u^0)\in X\times U_{ad}$ such that

$$J(x^0,u^0)\le J(x,u) \quad ,u\in U_{ad}$$

where

$$J(x^u,u)=\int_0^T l(t,x^u(t),x^u_t,u(t))dt+\Phi(x^u(T)),$$

x^u denote the mild solution of system (4.34) corresponding to the control $u\in U_{ad}$ and $\Phi:X\to\Re$ is continuous function. We call (x^u,u) an admissible state-control pair. Since solution x is corresponding to the control u, so for short, we denote $J(x^u,u)$ by $J(u)$. We will minimize the fractional controlled system (4.34) under the following assumptions:

(HU) $U_{ad}=L_q(I,Y)$, $B(s)\in L(L_q(I,Y),L_p(I,X))$ for all $s\in I$, $1<p,q<\infty$ and $B(\cdot)$ is strongly continuous.

(HL) $l : I \times X \times X \times Y \to [0, \infty]$ is Borel measurable satisfying these conditions:

1) $l(t, \cdot, \cdot, \cdot)$ is sequentially lower semicontinuous on $X \times X \times Y$ for a.e. $t \in I$.

2) $l(t, \xi, \nu, \cdot)$ is convex on Y for each $\xi \in X$, $\nu \in X$ and for a.e. $t \in I$.

3) There exist constants a, $b \geq 0$, $c > 0$ and $\eta \in L_1(I, \Re^+)$ such that

$$l(t, \xi, \nu_t, u) \geq \eta(t) + a||\xi|| + b||\nu_t||_B + c||u||_Y^q.$$

Theorem 4.3.1. *Under the assumption (HK), (HF), (HU) and (HL) the optimal control problem (P) has a solution that is, there exists an admissible state-control pair (x^0, u^0) such that*

$$J(x^0, u^0) = \int_0^T l(t, x^0(t), x_t^0, u^0(t))dt + \Phi(x^0(T)) \leq J(x, u), \quad \textit{for all} \;\; u \in U_{ad}.$$

Proof. If $inf\{J(u) \mid u \in U_{ad}\} = +\infty$ there is nothing to prove. So we assume that $inf\{J(u) \mid u \in U_{ad}\} = m < +\infty$. By $(HL3)$, there are constants a, $b \geq 0, c > 0$ and $\eta \in L_1(I, \Re^+)$ such that

$$l(t, x, x_t, u) \geq \eta(t) + a||x^u|| + b||x_t^u||_B + c||u||_Y^q.$$

Since η is nonnegative, we have

$$\begin{aligned} J(u) &= \int_0^T l(t, x^u(t), x_t^u, u(t))dt + \Phi(x^u(T)) \\ &\geq \int_0^T \eta(t)dt + a\int_0^T ||x^u(t)||dt + b\int_0^T ||x_t^u||_B dt + c\int_0^T ||u(t)||_Y^q dt + \Phi(x^u(T)) \\ &\geq -\sigma > -\infty, \end{aligned}$$

for some $\sigma > 0$, for all $u \in U_{ad}$. Hence $m \geq -\sigma > -\infty$. By definition of minimum, there exists a minimizing sequence $\{u_n\}$ of J , that is $\lim\limits_{n\to\infty} J(u_n) = m$ and

$$J(u_n) \geq \int_0^T \eta(t)dt + a\int_0^T ||x^{u_n}(t)||dt + b\int_0^T ||x_t^{u_n}||_B dt + c\int_0^T ||u_n(t)||_Y^p dt + \Phi(x^{u_n}(T)).$$

So there exist $N_0 > 0$ such that for all $n \geq N_0$,

$$m + \tilde{m} \geq J(u_n) \geq c\int_0^T ||u(t)||_Y^q dt$$

for some $\tilde{m} > 0$ and hence $||u_n||^q_{L_q(I,Y)} \leq \frac{\tilde{m}+m}{c}$.
This show that u_n is contained in a bounded subset of the reflexive Banach space $L_q(I,Y)$. So u_n has a convergence subsequence relabeled as u_n and $u_n \to u^0$ for some $u^0 \in U_{ad} = L_q(I,Y)$. Let $x_n \subseteq C([-r,T],X)$ be the corresponding sequence of solutions for the integral equation;

$$\begin{cases} x_n(t) = T(\phi(t))\varphi(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}T(\phi(t)-\phi(s))[f(s,x_n(s),Kx_n(s)) + Gx_n(s) \\ \qquad + B(s)u_n(s)]ds, \quad t \in I \\ x_n(t) = \varphi(t), \quad t \in [-r,0]. \end{cases} \tag{4.47}$$

From the a priori estimate, there exists a constant $\rho > 0$ such that

$$||x_n||_{C([-r,T],X)} \leq \rho, \quad \text{for all} \quad n = 0,1,2,...$$

where x^0 denote the solution corresponding to u^0, that is

$$\begin{cases} x^0(t) = T(\phi(t))\varphi(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}T(\phi(t)-\phi(s))[f(s,x^0(s),Kx^0(s)) + Gx^0(s) \\ \qquad + B(s)u^0(s)]ds, \quad t \in I \\ x^0(t) = \varphi(t), \quad t \in [-r,0]. \end{cases} \tag{4.48}$$

By (HF), (HK), (HG) and lemma 4.2.2 there are constants $a(\rho)$, $b(\rho)$ such that

$$||f(s,x_n(s),Kx_n(s)) - f(s,x^0(s),Kx^0(s))|| \leq a(\rho)||x_n(s) - x^0(s)||$$
$$||Gx_n(t) - Gx_0(t)|| \leq b(\rho)||(x_n)_t - (x_0)_t||_B$$

for each $s \in I$ and $t \in [-r, T]$.

$$\begin{aligned}
||x_n(t) - x^0(t)|| &\leq \frac{Me^{\omega\phi(T)}a(\rho)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ||x_n(s) - x^0(s)|| ds \\
&+ \frac{Me^{\omega\phi(T)}b(\rho)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ||(x_n)_s - (x^0)_s||_B ds \\
&+ \frac{Me^{\omega\phi(T)}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ||B(s)u_n(s) - B(s)u^0(s)|| ds \\
&\leq \frac{Me^{\omega\phi(T)}a(\rho)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ||x_n(s) - x^0(s)|| ds \\
&+ \frac{Me^{\omega\phi(T)}b(\rho)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ||(x_n)_s - (x^0)_s||_B ds \\
&+ \frac{Me^{\omega\phi(T)}(p-1)T^{\frac{\alpha p-1}{p-1}}}{(\alpha p-1)\Gamma(\alpha)} ||B(\cdot)u_n - B(\cdot)u^0||_{L_p(I,X)}.
\end{aligned}$$

Note that $x_n(s) - x^0(s) = 0$ for $s \in [-r, 0]$ and use lemma 4.1.2, then

$$||x_n(t) - x^0(t)|| \leq \tilde{M} ||B(\cdot)u_n - B(\cdot)u^0||_{L_p(I,X)}$$

where $\tilde{M}$ is a constant, independent of u, n and t. Since B is strongly continuous, we have $||B(\cdot)u_n - B(\cdot)u^0||_{L_p(I,X)} \to 0$. This implies that $||x_n - x^0|| \to 0$ in $C([-r, T], X)$. Let us set $l_n(t) = l(t, x_n(t), (x_n)_t, u_n(t))$ for all $t \in [0, T]$. Then by $(HL3)$, $\{l_n(t)\}$ is a sequence of non-negative measurable functions. So, by using Fatou 's Lemma,

$$\varliminf_{n\to\infty} \int_0^T l_n(t) dt \geq \int_0^T \varliminf_{n\to\infty} l_n(t) dt. \tag{4.49}$$

By $(HL1)$ and (4.49),

$$\begin{aligned}
m = \lim_{n\to\infty} J(u_n) &\geq \varliminf_{n\to\infty} [\int_0^T l_n(t) dt + \Phi(x_n(T))] \\
&\geq \int_0^T \varliminf_{n\to\infty} l_n(t) dt + \Phi(\varliminf_{n\to\infty} x_n(T)) \\
&= \int_0^T \varliminf_{n\to\infty} l(t, x_n(t), (x_n)_t, u_n(t)) dt + \Phi(x^0(T)) \\
&\geq \int_0^T l(t, x^0(t), x_t^0, u^0(t)) dt + \Phi(x^0(T)) = J(u^0).
\end{aligned}$$

This show that $J(u^0) = m$, i.e., $J(u^0) \leq J(u)$ for all $u \in U_{ad}$. □

4.4 Application to Nonlinear Schrödinger Equation

In this section, we consider a simple application of the results of section 4.2 and section 4.3 to the control problem for the following generalization nonlinear time dependent Schrödinger Equation with delay in $\Re^N$,

$$\begin{aligned}\frac{1}{i}\frac{\partial^\alpha \Psi(x,t)}{\partial t^\alpha} =& \Delta\Psi(x,t) + f(t,x,\Psi(x,t),\nabla\Psi(x,t))\\ &+ \int_{-r}^{t} h(t-s)g(s,x,\Psi(x,s),\nabla\Psi(x,t))ds\\ &+ \int_\Omega B(x,\xi)u(\xi,t)d\xi, \quad (x,t)\in\Omega\times I, \end{aligned} \tag{4.50}$$

$$\Psi(x,t) = \varphi(x,t), \quad (x,t)\in\bar{\Omega}\times[-r,0], \tag{4.51}$$

$$\Psi(x,t) = 0, \quad (x,t)\in\partial\Omega\times I, \tag{4.52}$$

where Ω is boundary domain of $\Re^N$, $\varphi\in C([-r,0]\times\bar{\Omega})$, $u\in L_q(\Omega\times I)$, $1<q<\infty$, $h\in L_1([-r,T],\Re)$ and $B:\bar{\Omega}\times\bar{\Omega}\to\Re$ is continuous. The space in which this problem will be considered is $L_2(\Re^N)$.

(AAf) Suppose that $f: I\times\bar{\Omega}\times\mathbb{C}\times\Re^N\to\Re$ and $g:[-r,T]\times\bar{\Omega}\times\mathbb{C}\times\Re^N\to\Re$ are satisfied the following conditions, there are L_1, $L_2\geq 0$ such that

$$|f(t,x,\xi,\eta)| + |g(t,x,\xi,\eta)| \leq L_1(1+|\xi|+|\eta|) \tag{4.53}$$

$$|f(t,x,\xi,\eta)-f(s,x,\tilde{\xi},\tilde{\eta})|+|g(t,x,\xi,\eta)-g(s,x,\tilde{\xi},\tilde{\eta})| \leq L_2(|t-s|+|\xi-\tilde{\xi}|+|\eta-\tilde{\eta}|) \tag{4.54}$$

for all $s,t\in[-r,T]$, $x\in\bar{\Omega}$, $\xi,\bar{\xi}\in\mathbb{C}$ and $\eta,\bar{\eta}\in\Re^N$.

Let $U_{ad} = L_q(I\times\Omega)$ be the admissible control set. We consider the Bolza problem :

(P_0) Find $u^0\in U_{ad}$ such that

$$J(u^0)\leq J(u) \quad , u\in U_{ad}$$

where

$$\begin{aligned}J(u) =& \int_0^T\int_\Omega |\Psi(\xi,t)|^2 d\xi dt + \int_0^T\int_\Omega\int_{-r}^0 |\Psi(\xi,t+s)|^2 ds d\xi dt\\ &+ \int_0^T\int_\Omega |u(\xi,t)|^q d\xi dt + z(\Psi(x,T)),\end{aligned}$$

and here $z \in C(\mathbb{C}, \Re^+)$.

We known that the Schrödinger equation can apply extensively in quantum mechanics. A complex value function $\Psi(x,t)$ is called wave function that depend on both position variable (x) and time variable(t). We introduce the integral $\int_{-r}^{t} h(t-s)g(x,s,\Psi(x,s),\nabla\Psi(x,s))ds$ denoting in sense of delay term that is impacted from the initial delay function $\varphi(x,t)$ for $t \in [-r,0]$ in the condition (4.51). Moreover, the system is controlled by the control u via the mapping $\int_\Omega B(x,\xi)u(\xi,t)d\xi$. In doing we will use the following notations; $x = (x_1, x_2, ..., x_N)$ is a variable point in the $N-$dimensional Euclidean space $\Re^N$. For any two such point $x = (x_1, x_2, ..., x_N)$, $y = (y_1, y_2, ..., y_N)$ we set $x \cdot y = \sum_{i=1}^{N} x_i y_i$ and $||x||^2 = x \cdot x$. An $N-$tuple of nonnegative integer $\beta = (\beta_1, \beta_2, ..., \beta_N)$ is called a multi-index and we define $|\beta| = \sum_{i=1}^{N} \beta_i$ and $x^\beta = x_1^{\beta_1} x_2^{\beta_2} \cdots x_N^{\beta_N}$ for $x = (x_1, x_2, ..., x_N)$. Denoting $D_k = \partial/\partial x_k$ and $D = (D_1, D_2, ..., D_N)$ we have $D^\beta = D_1^{\beta_1} D_2^{\beta_2} \cdots D_N^{\beta_N} = \frac{\partial^{\beta_1}}{\partial x_1^{\beta_1}} \frac{\partial^{\beta_2}}{\partial x_2^{\beta_2}} \cdots \frac{\partial^{\beta_N}}{\partial x_N^{\beta_N}}$. Let Ω be a fixed domain in $\Re^N$ with boundary $\partial\Omega$ and closure $\bar{\Omega}$. We will usually assume that $\partial\Omega$ is smooth, i.e., $\partial\Omega \in C^k$ for some suitable $k \geq 1$. By $C^m(\Omega)$ we denote the set of all $m-$times continuously differentiable real-valued or complex-valued functions in Ω. $C_0^m(\Omega)$ will denote the subspace of $C^m(\Omega)$ consisting of those functions which have compact support in Ω. For $x \in C^m(\Omega)$ and $1 \leq p < \infty$ we define

$$||u||_{m,p} = (\int_\Omega \sum_{|\beta| \leq m} |D^\beta u|^p dx)^{1/p}. \tag{4.55}$$

If $p = 2$ and $u, v \in C^m(\Omega)$ we also define

$$(u,v)_m = \int_\Omega \sum_{|\beta| \leq m} D^\beta u \bar{D^\beta} v dx. \tag{4.56}$$

Denoting by $\tilde{C}_p(\Omega)$ the subset of $C^m(\Omega)$ consisting of those functions u which $||u||_{m,p} < \infty$, we define $W^{m,p}(\Omega)$ and $W_0^{m,p}(\Omega)$ to be the completions in the norm $||\cdot||_{m,p}$ of $\tilde{C}_p(\Omega)$ and $C^m(\Omega)$ respectively. It is well known that $W^{m,p}(\Omega)$ and $W_0^{m,p}(\Omega)$ are Banach spaces and obviously $W_0^{m,p}(\Omega) \subset W^{m,p}(\Omega)$. For $p = 2$ we denote $W^{m,2}(\Omega) = H^m(\Omega)$ and $W_0^{m,p}(\Omega) = H_0^m(\Omega)$. The spaces $H^m(\Omega)$ and $H_0^m(\Omega)$ are Hilbert spaces with the scalar product $(\cdot,\cdot)_m$ given by (4.56).

We will transform the system (4.50) to the abstract form. Let $X = L_2(\Omega)$ and for $t \in (-r, T]$ define $\Psi(t) : \Omega \to X$ by

$$\Psi(t)(x) = \Psi(x,t) \quad \text{for all} \;\; x \in \Omega,$$

and define

$$D_t^\alpha \Psi(t)(x) = \frac{\partial^\alpha \Psi(x,t)}{\partial t^\alpha}, \;\; \text{for all} \;\; \Psi \in X, \;\; x \in \Omega.$$

Define $f : I \times X \times X \to X$ by

$$f(t, \Psi(t), H\Psi(t))(x) = if(x, t, \Psi(x,t), \nabla\Psi(x,t)), \tag{4.57}$$

$$G\Psi(t)(x) = i\int_{-r}^{t} h(t-s)g(x, s, y(x,s), \nabla y(x,t))ds, \tag{4.58}$$

$$B(t)u(t)(x) = i\int_{\Omega} B(x,\xi)u(\xi,t)d\xi. \tag{4.59}$$

We define an operator A_0 associated with the differential operator $i\Delta\Psi$;

$$A_0\Psi = i\Delta\Psi \quad \text{for all} \;\; \Psi \in D(A_0)$$

where $D(A_0) = H^2(\Omega)$. Then the system (4.50) is transformed to the abstract problem;

$$\begin{cases} D_t^\alpha \Psi(t) = A\Psi(t) + f(t, \Psi(t), Ky(t)) + G\Psi(t) + B(t)u(t), \quad t \in I \\ \Psi(t) = \varphi(t), \quad t \in [-r, 0]. \end{cases} \tag{4.60}$$

Lemma 4.4.1. *[15] The operator iA_0 is self adjoint in $L^2(\Re^N)$.*

Corollary 4.4.2. *[15] A_0 is the infinitesimal generator of a group of unitary operators on $L^2(\Re^N)$.*

From corollary 4.4.2, it follows that the operators A_0 is the infinitesimal generator of a group of unitary operators $\{T(t)\}_{t\in\Re}$ on $L^2(\Re^N)$. A simple application of the Fourier transform gives the following explicit formula for $T(t)$;

$$(T(t)v)(x) = \frac{1}{4\pi i t}\int_{\Re^2} e^{i|x-y|/4t}v(y)dy. \tag{4.61}$$

Lemma 4.4.3. *[15] Let $\{T(t)\}_{t\geq 0}$ be the semigroup given by (4.61). If $\leq p \leq \infty$ and $1/p+1/q=1$ then $T(t)$ can be extended in a unique way to a bounded operator from $L^q(\Re^2)$ into $L^p(\Re^2)$ and*

$$||T(t)v||_{0,p} \leq (4\pi t)^{1-2/q}||v||_{0,q}. \tag{4.62}$$

Lemma 4.4.3 guarantee that $A = i\Delta$ be the infinitesimal generator of the C_0-semigroup of bounded linear operator $\{T(t)\}_{t\geq 0}$.

Theorem 4.4.4. *Suppose assumption (AAf) holds. Then the control problem (P_0) for the generalization nonlinear time dependent Schrödinger equation with delay in $\Re^N$ (system(4.50)) has a solution, that is, there exists an admissible state-control pair (Ψ^0, u^0) such that*

$$J(\Psi^0, u^0) \leq J(\Psi, u) \quad \textit{for all} \quad u \in U_{ad}.$$

Proof. We solve the control problem (P_0) for system(4.50) via the Cauchy abstract form (4.60). By using the assumptions (AAf) and definitions of f, g and the cost functional J, it satisfies all the assumptions given in theorem 4.2.6 and theorem 4.3.1. Then the control problem (P_0) for system(4.50) has a solution, that is, there exists an admissible state-control pair (Ψ^0, u^0) such

$$J(\Psi^0, u^0) \leq J(\Psi, u) \quad \text{for all} \quad u \in U_{ad}.$$

□

CHAPTER V

IMPULSIVE FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS WITH C_0−SEMIGROUP

The main objective of this chapter is discussing to impulsive fractional integro-differential equations;

$$\begin{cases} D_t^\alpha x(t) = Ax(t) + f(t, x(t), Kx(t)) + G(t) + B(t)u(t), \quad t \in I \backslash D \\ \Delta x(t_k) = J_k(x(t_k)), \quad t_k \in D \\ x(t) = \varphi(t), \quad t \in [-r, 0] \end{cases} \tag{5.1}$$

where $I = [0, T]$, $D = \{t_1, t_2, ..., t_n\}$, the integral operator G is defined by

$$Gx(t) = \int_{-r}^{t} h(t, s)g(s, Hx(s))ds,$$

A is an infinitesimal generator of a compact semigroup $\{T(t)\}_{t \geq 0}$ satisfying $\|T(t)\| \leq Me^{\omega t}$, $M \geq 1$, $\omega > 0$, $t \geq 0$ for $t_k \in D$, $\Delta x(t_k) = x(t_k^+) - x(t_k^-) = x(t_k^+) - x(t_k)$ denote the jump of state X at t_k with the size of jump J_k, $k = 1, 2, ..., n$. We prove the existence of solution for system (5.1) under the assumptions (HG), (HK), (HF), (HB) as same as the assumptions in chapter 4;

(HK) $K : X \to X$ is bounded linear operator.

(HF1) $f : I \times X \times X \to X$ is uniformly continuous in t and locally Lipschitz in x and y, that is for any $\tau > 0$ and $\rho > 0$, there exists $a_f(\rho, \tau) > 0$ such that

$$||f(t, x_1, y_1) - f(t, x_2, y_2)|| \leq a_f(\rho, \tau)[||x_1 - x_2|| + ||y_1 - y_2||]$$

provided $||x_1||$, $||x_2||$, $||y_1||$, $||y_2|| \leq \rho$, for all $t \in [0, \tau]$.

(HF2) There exists $c \geq 0$ such that $||f(t, x, y)|| \leq c(1 + ||x|| + ||y||)$ for all $x, y \in X$ and $t \in [0, T]$.

(HB) Y is another separable reflexive Banach space from which the controls u take the value $B(s) \in L(L_q(I,Y), L_p(I,X))$ for all $s \in [-r,T]$.

and with the another assumption, say (HJ);

(HJ1) $J_k : X \to X$ is a map such that $J_k(X)$ is a bounded subset of X,

(HJ2) there exist $e_k > 0$, $k = 1, 2, ..., n$ such that

$$\|J_k(x_1(t)) - J_k(x_2)(t)\| \leq e_k \|x_1(t) - x_2(t)\|, \tag{5.2}$$

for all x_1, $x_2 \in X$ and $t \in [0,T]$.

5.1 Useful Definitions and Theorems

In this section, we will state some definitions and theorems that play important for proving the main results. Let $PC([-r,T],X) \equiv \{x : [-r,T] \to X|\ x$ continues at $t \in [-r,T]\backslash D$ and x is continuous from left and right hand limit at $t \in D\}$ where D denote $\{t_1, t_2, ..., t_n\}$. Then we will extend the integral operator G to $PC([-r,T],X)$.

Lemma 5.1.1. *Assume (HG) holds. Then the operator G has the following properties*

1) $G : PC([-r,T],X) \to PC([-r,T],X)$.

2) For each x_1, $x_2 \in PC([-r,T],X)$ *such that* $||x_1||$, $||x_2|| \leq \rho$,

$$||Gx_1(t) - Gx_2(t)|| \leq L_g ||h|| (T+r) ||(x_1)_t - (x_2)_t||_B, \quad \text{for all} \quad t \in I. \tag{5.3}$$

3) For each $x \in PC([-r,T],X)$, *we have*

$$||Gx(t)|| \leq a_g T ||h|| (1 + ||x_t||_B), \quad \text{for all} \quad t \in I. \tag{5.4}$$

Proof. The proof is similar to the proof of lemma4.2.2. □

Since the proving existence of solutions for system (5.1) is complexity, we will use some technique about constructing the fixed point operator, it meant that we must show this operator is contraction and map any compact subsets of X to compact subsets of X. The Ascoli-Arzela Theorem is an advantage choice that we choose to solve this problem. But we cannot directly apply the Ascoli-Arzela to our problem on $PC([-r,T],X)$. This is a reason why we need the generalized of the Ascoli-Arzela Theorem for $PC([-r,T],X)$.

Definition 5.1.2. *A set S of a norm vector space $(X, \|.\|)$ is (sequentially) compact if every sequence of S contain a convergence subsequence, i.e., a sequence converging to an element in S.*

Definition 5.1.3. *Let $\{T_\alpha | \alpha \in \Lambda\}$ be a family of operators from $L(X,Y)$. If for each $x \in X$, there exist c_x such that*

$$\sup\{||T_\alpha x|| \mid \alpha \in \Lambda\} \leq c_x \tag{5.5}$$

then the operators $\{T_\alpha | \alpha \in \Lambda\}$ are uniformly bounded, i.e, there exist $M > 0$ such that

$$||T_\alpha x|| \leq M \text{ for all } \alpha \in \Lambda \text{ and for all } x \in X. \tag{5.6}$$

Theorem 5.1.4. *(Ascoli-Arzela Theorem) A subset X_0 of $C([a,b],X)$ is compact if and only if it is bounded and equicontinuous , i.e., if and only if ;*

1) there exists $M > 0$ that $||f||_{C([a,b],X)} \leq M$ for all $f \in X_0$;

2) for all $\epsilon > 0$ there exists $\delta > 0$ that $|f(x) - f(y)| < \epsilon$ for all $f \in X_0$ and for all $x, y \in [a,b]$ such that $|x - y| < \delta$.

Theorem 5.1.5. *(Generalized Ascoli-Arzela Theorem) Suppose $W \subseteq \{x \in PC([-r,T],X) \mid x(t) = \varphi(t)$ for $t \in [-r,0]\}$. If the following conditions are satisfied;*

1. W is a uniformly bounded subset of $PC([-r,T],X)$

2. W is equicontinuous in $I \setminus D$

3. Its $t-$sections

$$W(t) \equiv \{x(t)|\ x \in W,\ t \in I \setminus D\};$$
$$W(t^+) \equiv \{x(t^+)|\ x \in W\};$$
$$W(t^-) \equiv \{x(t^-)|\ x \in W\}$$

are relatively compact subsets of X. Then W is a relatively compact subset $PC([-r,T],X)$.

Proof. Let $\{x_m\}$ be any sequence of W. Then $\{x_m|_{[0,t_1)}\} \subset C([0,t_1),X)$. Using the Ascoli-Arzela theorem in $[0,t_1)$, there exists a subsequence of $\{x_m\}$, again labeled by $\{x_m\}$, such that

$$x_m|_{[0,t_1)} \to x^1 \qquad \text{in} \quad C([0,t_1),X) \quad \text{as} \ \ m \to \infty.$$

Consider $\{x_m|_{[t_1,t_2]}\} \subset C([t_1,t_2],X)$ and set $x_m(t_1) = x_m(t_1^+)$. Due to the Ascoli-Arzela theorem in $[t_1,t_2)$, $\{x_m|_{[t_1,t_2)}\}$ is a relatively compact subset of $C([t_1,t_2),X)$. Then there exists a subsequence, again labeled by $\{x_m\}$, such that

$$x_m|_{[t_1,t_2)} \to x^2 \qquad \text{in} \quad C([t_1,t_2),X) \quad \text{as} \ \ m \to \infty.$$

Repeat the procedures until interval $[t_m,T]$. We know that there is a subsequence $\{x_m\}$, such that

$$x_m|_{[t_n,T]} \to x^{n+1} \qquad \text{in} \quad C([t_n,T],X) \quad \text{as} \ \ m \to \infty.$$

Define $x(t) = x^i(t)$, $t \in [t_{i-1},t_i)$ for $i = 1,...,n+1$. Then $x \in PC([-r,T],X)$ and

$$x_m|_{[t_n,T]} \to x \qquad \text{in} \quad PC([-r,T],X) \quad \text{as} \ \ m \to \infty.$$

Therefore W is a relatively compact set. This complete the proof. □

5.2 Impulsive Integral Inequalities

The following theorems are most useful integral inequalities which is of Gronwall type. Let $PC^\alpha(\Re^+,\Re)$ denotes the set of all functions map from $\Re^+$ to $\Re$ such that their derivatives of order α exist on $\Re^+ - \{t_k\}$, $k = 1,2,...$ and left continuous at t_k, $k = 1,2,...$ for $0 < t_k < t_{k+1}$.

Theorem 5.2.1. *Let $m \in PC^{\alpha}(\Re^+, \Re)$ and for $t \geq 0$,*

$$m^{\alpha}(t) \leq m(t)p(t) + q(t),\ t \neq t_k \tag{5.7}$$

$$m(t_k^+) \leq a_k m(t_k),\ m(0) = a_0 \tag{5.8}$$

where $0 < \alpha \leq 1$, $p, q \in C(\Re^+, \Re)$ and $a_k \geq 0$ are constants. Then

$$m(t) \leq \sum_{0 \leq t_k \leq t} \Big(\prod_{t_k \leq t_j \leq t} a_j e^{\int_{t_j}^{t_j+1} \phi_{\alpha-1}(t_{k+1}-s)p(s)ds} \Big) \cdot \Big(\int_{t_k}^{t_k+1} \phi_{\alpha-1}(t_{k+1}-s)q(s)e^{-\int_{t_k}^{s} \phi_{\alpha-1}(s-r)p(r)dr} ds \Big) \tag{5.9}$$

where $\phi_{\alpha}(t) = \frac{t^{\alpha}}{\Gamma(\alpha+1)}$.

Proof. Let $t \in [0, t_1]$. Then, we get from (5.7),

$$D_t^{\alpha}[m(t)e^{-\int_0^t \phi_{\alpha-1}(t-s)p(s)ds}] \leq q(t)e^{-\int_0^t \phi_{\alpha-1}(t-s)p(s)ds} \tag{5.10}$$

which yields after integrating order α from 0 to t,

$$m(t) \leq e^{\int_0^t \phi_{\alpha-1}(t-s)p(s)ds}[a_0 + \int_0^t \phi_{\alpha-1}(t-s)q(s)e^{-\int_0^s \phi_{\alpha-1}(s-r)p(r)dr} ds]$$
$$= a_0 e^{\int_0^t \phi_{\alpha-1}(t-s)p(s)ds} + e^{\int_0^t \phi_{\alpha-1}(t-s)p(s)ds} \int_0^t \phi_{\alpha-1}(t-s)q(s)e^{-\int_0^s \phi_{\alpha-1}(s-r)p(r)dr} ds.$$

For $t \in (t_1, t_2]$, by (5.7) we have,

$$D^{\alpha}[m(t)e^{-\int_{t_1}^t \phi_{\alpha-1}(t-s)p(s)ds}] \leq q(t)e^{-\int_{t_1}^t \phi_{\alpha-1}(t-s)p(s)ds} \tag{5.11}$$

which yields after integrating order α from t_1 to t,

$$m(t) \leq m(t_1^+)e^{\int_{t_1}^t \phi_{\alpha-1}(t-s)p(s)ds} + e^{\int_{t_1}^t \phi_{\alpha-1}(t-s)p(s)ds} \int_{t_1}^t \phi_{\alpha-1}(t-s)q(s)e^{-\int_{t_1}^s \phi_{\alpha-1}(s-r)p(r)dr} ds \tag{5.12}$$

and from (5.8), we get

$$m(t_1^+) \leq a_1 m(t_1)$$
$$\leq a_0 a_1 e^{\int_0^{t_1} \phi_{\alpha-1}(t_1-s)p(s)ds} + a_1 e^{\int_0^{t_1} \phi_{\alpha-1}(t_1-s)p(s)ds} \int_0^{t_1} \phi_{\alpha-1}(t_1-s)q(s)e^{-\int_0^s \phi_{\alpha-1}(s-r)p(r)dr} ds. \tag{5.13}$$

Hence, we obtain for $t \in (t_1, t_2]$,

$$\begin{aligned} m(t) \leq & a_0 a_1 e^{\int_0^{t_1} \phi_{\alpha-1}(t_1-s)p(s)ds} e^{\int_{t_1}^{t} \phi_{\alpha-1}(t-s)p(s)ds} \\ & + a_1 e^{\int_0^{t_1} \phi_{\alpha-1}(t_1-s)p(s)ds} e^{\int_{t_1}^{t} \phi_{\alpha-1}(t-s)p(s)ds} \int_0^{t_1} \phi_{\alpha-1}(t_1-s)q(s)e^{-\int_0^s \phi_{\alpha-1}(s-r)p(r)dr} ds \\ & + e^{\int_{t_1}^{t} \phi_{\alpha-1}(t-s)p(s)ds} \int_{t_1}^{t} \phi_{\alpha-1}(t-s)q(s)e^{-\int_{t_1}^{s} \phi_{\alpha-1}(s-r)p(r)dr} ds. \end{aligned} \tag{5.14}$$

Assume that (5.13) holds for $t \in [0, t_k]$ some integer $k > 1$. Then for $t \in (t_k, t_{k+1}]$, it follows from (5.7) that

$$D^{\alpha}[m(t)e^{-\int_{t_k}^{t} \phi_{\alpha-1}(t-s)p(s)ds}] \leq q(t)e^{-\int_{t_k}^{t} \phi_{\alpha-1}(t-s)p(s)ds}. \tag{5.15}$$

So

$$m(t) \leq m(t_k^+)e^{\int_{t_k}^{t} \phi_{\alpha-1}(t-s)p(s)ds} + e^{\int_{t_k}^{t} \phi_{\alpha-1}(t-s)p(s)ds} \int_{t_k}^{t} \phi_{\alpha-1}(t-s)q(s)e^{-\int_{t_k}^{s} \phi_{\alpha-1}(s-r)p(r)dr} ds. \tag{5.16}$$

Using (5.8) we obtain for $t \in (t_k, t_{k+1}]$,

$$m(t) \leq a_k m(t_k)e^{\int_{t_k}^{t} \phi_{\alpha-1}(t-s)p(s)ds} + e^{\int_{t_k}^{t} \phi_{\alpha-1}(t-s)p(s)ds} \int_{t_k}^{t} \phi_{\alpha-1}(t-s)q(s)e^{-\int_{t_k}^{s} \phi_{\alpha-1}(s-r)p(r)dr} ds. \tag{5.17}$$

By the induction hypothesis, (5.16) can reduced to

$$m(t) \leq \sum_{0 \leq t_k \leq t} \Big(\prod_{t_k \leq t_j \leq t} a_j e^{\int_{t_j}^{t_j+1} \phi_{\alpha-1}(t_{k+1}-s)p(s)ds} \Big) \Big(\int_{t_k}^{t_k+1} \phi_{\alpha-1}(t_{k+1}-s)q(s)e^{-\int_{t_k}^{s} \phi_{\alpha-1}(s-r)p(r)dr} ds \Big) \tag{5.18}$$

which on simplification give the estimate (5.8) for $t \in [0, t_{k+1}]$. The proof is completed. □

Theorem 5.2.2. *Let $m \in PC^{\alpha}(\Re^+, \Re)$, $0 < \alpha < 1$ which satisfies,*

$$m(t) \leq a + \sum_{0 \leq t_k \leq t} \int_{t_k}^{t_{k+1}} \phi_{\alpha-1}(t_{k+1}-s)p(s)m(s)ds + \sum_{0 < t_k < t} c_k m(t_k) \quad \text{for } t \geq 0 \tag{5.19}$$

where $c_k \geq 0$ and a are constant and $\phi_{\alpha}(t) = \frac{t^{\alpha}}{\Gamma(\alpha+1)}$. Then,

$$m(t) \leq \prod_{0 \leq t_k \leq t} (1 + c_k)e^{\int_{t_k}^{t_{k+1}} \phi_{\alpha-1}(t_{k+1}-s)p(s)m(s)ds}, \quad t \geq 0 \tag{5.20}$$

where $c_0 = a - 1$.

Proof. Setting the right hand side equal to $v(t)$ we have,

$$\begin{cases} v^{\alpha}(t) = p(t)m(t); \quad t \neq t_k \\ v(t_k^+) = v(t_k) + c_k m(t_k), \quad v(0) = a. \end{cases} \tag{5.21}$$

Since $m(t) \leq v(t)$, we then have

$$\begin{cases} v^{\alpha}(t) = p(t)v(t); \quad t \neq t_k \\ v(t_k^+) = (1 + c_k)v(t_k), \quad v(0) = a = c_0 + 1. \end{cases} \tag{5.22}$$

Applying theorem 5.2.1, we obtain

$$m(t) \leq \prod_{0 \leq t_k \leq t} (1 + c_k) e^{\int_{t_k}^{t_{k+1}} \phi_{\alpha-1}(t_{k+1}-s)p(s)ds}, \quad t \geq 0. \tag{5.23}$$

□

Theorem 5.2.3. *If* $x \in PC^{\alpha}([-r, T], X)$*,* $0 < \alpha < 1$*, such that*

$$\begin{cases} \|x(t)\| \leq a + \sum_{0 \leq t_k \leq t} \int_{t_k}^{t_{k+1}} \phi_{\alpha-1}(t_{k+1} - s) b(s) \|x(s)\| ds \\ + \sum_{0 \leq t_k \leq t} \int_{t_k}^{t_{k+1}} \phi_{\alpha-1}(t_{k+1} - s) c(s) \|x_s\|_B ds + \sum_{0 < t_k < t} d_k \|x(x_k)\|, \quad t \in I \\ x(t) = \varphi(t), \quad t \in [-r, 0]. \end{cases} \tag{5.24}$$

Then

$$\|x(t)\| \leq (a + \|\varphi\|_C) \prod_{0 < t_k \leq t} (1 + d_k) e^{\frac{b(t_{k+1}-t_k)^{\alpha}}{\alpha\Gamma(\alpha)}}, \quad \text{for all} \;\; t \in I \tag{5.25}$$

where $\phi_{\alpha}(t) = \frac{t^{\alpha}}{\Gamma(\alpha+1)}$ *and* $b = \sup_{s \in I}[b(s) + c(s)]$*.*

Proof. Note that $\|x(t)\| \leq \|x_t\|_B$ for all $t \in I$. So

$$\|x(t)\| \leq a + b \sum_{0 \leq t_k \leq t} \int_{t_k}^{t_{k+1}} \phi_{\alpha-1}(t_{k+1} - s) \|x_s\|_B ds. \tag{5.26}$$

Setting

$$g(t) = \sum_{0 \leq t_k \leq t} \int_{t_k}^{t_{k+1}} \phi_{\alpha-1}(t_{k+1} - s) \|x_s\|_B ds, \quad \text{for all} \;\; t \in I. \tag{5.27}$$

Then $g(t)$ is monotonous increasing. Indeed, for $t_k < \tau < t \leq t_{k+1}$, $k = 1, 2, \ldots$, we have

$$\begin{aligned} g(t) - g(\tau) &= \int_{t_k}^{t} \phi_{\alpha-1}(t-s)\|x_s\|_B ds - \int_{t_k}^{\tau} \phi_{\alpha-1}(\tau-s)\|x_s\|_B ds \\ &= \int_0^{t-t_k} u^{\alpha-1}\|x_{t-u}\|_B du - \int_0^{\tau-t_k} u^{\alpha-1}\|x_{\tau-u}\|_B du \\ &= \int_0^{\tau-t_k} u^{\alpha-1}[\|x_{t-u}\|_B - \|x_{\tau-u}\|_B] du + \int_0^{t-t_k} u^{\alpha-1}\|x_{t-u}\|_B du. \quad (5.28) \end{aligned}$$

Since $\|x_t\|_B$ is monotonous increasing, $\|x_{t-u}\|_B - \|x_{\tau-u}\|_B > 0$. Hence $g(t) > g(\tau)$. We know that

$$\|x_t\|_B \leq \|\varphi\|_C + \sup_{t\in[0,T]} \|x(t)\| \leq (a + \|\varphi\|_C) + b \sum_{0\leq t_k \leq t} \int_{t_k}^{t_{k+1}} \phi_\alpha(t_{k+1}-s)\|x_s\|_B ds. \quad (5.29)$$

Therefore by using theorem 5.2.2,

$$\|x(t)\| \leq \|x_t\|_B \leq (a + \|\varphi\|_C) \prod_{0<t_k\leq t} (1+d_k) e^{\frac{b(t_{k+1}-t_k)^\alpha}{\alpha\Gamma(\alpha)}}, \quad \text{for all} \quad t \in I. \quad (5.30)$$

□

5.3 Existence of Solution of Impulsive Fractional Differential system

In the following, we consider the impulsive fractional differential equations with time delay;

$$\begin{cases} D_t^\alpha x(t) = Ax(t) + f(t, x(t), Kx(t)) + G(t) + B(t)u(t), & t \in I\backslash D \\ \Delta x(t_k) = J_k(x(t_k)), & t_k \in D \\ x(t) = \varphi(t), & t \in [-r, 0] \end{cases} \quad (5.31)$$

where $I = [0, T]$, $D = \{t_1, t_2, ..., t_n\}$, the integral operator G is defined by

$$Gx(t) = \int_{-r}^{t} h(t,s)g(s, Hx(s))ds,$$

A is a infinitesimal generator of a compact semigroup $\{T(t)\}_{t\geq 0}$ satisfying $\|T(t)\| \leq Me^{\omega t}$, $M \geq 1$, $\omega > 0$, $t \geq 0$ for $t_k \in D$,

$$\Delta x(t_k) = x(t_k^+) - x(t_k^-) = x(t_k^+) - x(t_k)$$

denote the jump of state X at t_k with the size of jump J_k, $k = 1, 2, ..., n$. Assume that the assumptions (HG), (HK), (HF), (HB) and (HJ) hold. We will prove the existence of a solution for system (5.31) by starting at this delay system,

$$\begin{cases} D_t^\alpha x(t) = Ax(t) + f(t, x(t), Kx(t)) + G(t) + B(t)u(t), \quad t \in [0, t_1] \\ x(t) = \varphi(t), \quad t \in [-r, 0]. \end{cases} \tag{5.32}$$

Then, by corollary 4.2.7 we obtain,

$$\begin{cases} x_1(t) = T(\phi(t))\varphi(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} T(\phi(t) - \phi(s))[f(s, x_1(s), Kx_1(s)) \\ \qquad\qquad + Gx_1(s) + B(s)u(s)]ds, \quad t \in [0, t_1] \\ x_1(t) = \varphi(t), \quad t \in [-r, 0] \end{cases} \tag{5.33}$$

be a solution of system (5.32) and

$$\begin{aligned} x_1(t_1) =& T(\phi(t_1))\varphi(0) + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - s)^{\alpha-1} T(\phi(t_1) - \phi(s))[f(s, x_1(s), Kx_1(s)) \\ &+ Gx_1(s) + B(s)u(s)]ds \end{aligned} \tag{5.34}$$

where $\phi(t) \equiv \phi_\alpha(t) = \frac{t^\alpha}{\Gamma(\alpha+1)}$. Next, we consider the system;

$$\begin{cases} D_t^\alpha x(t) = Ax(t) + f(t, x(t), Kx(t)) + G(t) + B(t)u(t), \quad t \in (t_1, t_2] \\ x(t_1) = x_1(t_1) + J_1(x_1(t_1)), \quad t = t_1 \\ x(t) = x(t_1), \quad t \in [-r, t_1). \end{cases} \tag{5.35}$$

Again using corollary 4.2.7, we get,

$$\begin{cases} x_2(t) = T(\phi(t) - \phi(t_1))[x_1(t_1) + J_1(x_1(t_1))] + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t-s)^{\alpha-1} T(\phi(t) - \phi(s)) \\ \qquad\qquad [f(s, x_2(s), Kx_2(s)) + Gx_2(s) + B(s)u(s)]ds, \quad t \in [t_1, t_2] \\ x_2(t) = x_1(t), \quad t \in [-r, t_1). \end{cases} \tag{5.36}$$

We can reform (5.36) to;

$$\begin{cases} x_2(t) = T(\phi(t))\varphi(0) + \frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}T(\phi(t)-\phi(s))[f(s,x_2(s),Kx_2(s)) \\ +Gx_2(s)+B(s)u(s)]ds, +\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t}(t-s)^{\alpha-1}T(\phi(t)-\phi(s))[f(s,x_2(s),Kx_2(s)) \\ +Gx_2(s)+B(s)u(s)]ds + T(\phi(t)-\phi(t_1))J_1(x_1(t_1)), \quad t\in[t_1,t_2] \\ x_2(t) = x_1(t), \quad t\in[-r,t_1) \end{cases} \tag{5.37}$$

and

$$\begin{aligned} x_2(t_2) = T(\phi(t_2))\varphi(0) &+ \frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}T(\phi(t_1)-\phi(s))[f(s,x_2(s),Kx_2(s)) \\ &+ Gx_2(s)+B(s)u(s)]ds + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}T(\phi(t)-\phi(s))[f(s,x_2(s),Kx_2(s)) \\ &+ Gx_2(s)+B(s)u(s)]ds + T(\phi(t_2)-\phi(t_1))J_1(x_1(t_1)). \end{aligned} \tag{5.38}$$

Continues this process and consider the delay system;

$$\begin{cases} D_t^\alpha x(t) = Ax(t) + f(t,x(t),Kx(t)) + Gx(t) + B(t)u(t), \quad t\in(t_k,t_{k+1}] \\ x(t_k) = x_k(t_k) + J_k(x_{k+1}(t_k)), \quad t = t_k \\ x(t) = x_k(t), \quad t\in[-r,t_k). \end{cases} \tag{5.39}$$

Then, by corollary 4.2.7, we obtain,

$$\begin{cases} x_{k+1}(t) = T(\phi(t)-\phi(t_k))[x_k(t_k)+J_k(x_k(t_k))] + \frac{1}{\Gamma(\alpha)}\int_{t_k}^{t}(t-s)^{\alpha-1}T(\phi(t)-\phi(s)) \\ \qquad [f(s,x_{k+1}(s),Kx_{k+1}(s)) + Gx_{k+1}(s) + B(s)u(s)]ds, \quad t\in[t_k,t_{k+1}] \\ x_{k+1}(t) = x_k(t), \quad t\in[-r,t_k). \end{cases} \tag{5.40}$$

Since $x_{k+1}(t) = x_k(t)$ for all $t \in [-r, t_k]$ and for $t \in (t_k, t_{k+1}]$

$$x_{k+1}(t) = T(\phi(t) - \phi(t_k))[T(\phi(t_k))\varphi(0) + \frac{1}{\Gamma(\alpha)} \sum_{0 \le t_i \le t_k} \int_{t_i}^{t_{i+1}} (t_{i+1} - s)^{\alpha - 1}$$

$$\cdot T(\phi(t_k) - \phi(s))[f(s, x_k(s), Kx_k(s)) + Gx_k(s) + B(s)u(s)]ds + J_k(x_k(t_k))]$$

$$+ \sum_{0<t_i<t_k} T(\phi(t_k) - \phi(t_i))J_i x_k(t_i)] + \frac{1}{\Gamma(\alpha)} \int_{t_k}^{t} (t-s)^{\alpha-1} T(\phi(t) - \phi(s))$$

$$\cdot [f(s, x_{k+1}(s), Kx_{k+1}(s)) + Gx_{k+1}(s) + B(s)u(s)]ds$$

$$= T(\phi(t))\varphi(0) + \frac{1}{\Gamma(\alpha)} \sum_{0 \le t_i \le t} \int_{t_i}^{t_{i+1}} (t_{i+1} - s)^{\alpha-1} T(\phi(t) - \phi(s))[f(s, x_k(s), Kx_{k+1}(s))$$

$$+ Gx_{k+1}(s) + B(s)u(s)]ds + \sum_{0<t_i<t} T(\phi(t) - \phi(t_i))J_i x_{k+1}(t_i). \tag{5.41}$$

So for $k = 0, 1, ..., n$ where $t_0 = 0, t_{n+1} = T$ we obtain,

$$\begin{cases} x_{k+1}(t) & = T(\phi(t))\varphi(0) + \frac{1}{\Gamma(\alpha)} \sum_{0 \le t_i \le t} \int_{t_i}^{t_i+1} (t_{i+1} - s)^{\alpha-1} T(\phi(t) - \phi(s)) \\ & \cdot [f(s, x_{k+1}(s), Kx_{k+1}(s)) + Gx_{k+1}(s) + B(s)u(s)]ds \\ & + \sum_{0<t_k<t} T(\phi(t) - \phi(t_k))J_k x_{k+1}(t_k), \quad t \in (t_k, t_{k+1}] \\ x_{k+1}(t) & = x_k(t), \quad t \in [-r, t_k) \end{cases} \tag{5.42}$$

be a solution for system (5.40). Moreover, from these process we obtain a solution of system (5.31) is

$$\begin{cases} x(t) = T(\phi(t))\varphi(0) + \frac{1}{\Gamma(\alpha)} \sum_{0 \le t_k \le t} \int_{t_k}^{t_k+1} (t_{k+1} - s)^{\alpha-1} T(\phi(t) - \phi(s))[f(s, x(s), Kx(s)) \\ \quad + Gx(s) + B(s)u(s)]ds + \sum_{0<t_k<t} T(\phi(t) - \phi(t_k))J_k x(t_k), \quad t \in I \\ x(t) = \varphi(t), \quad t \in [-r, 0]. \end{cases} \tag{5.43}$$

The solution in this form is called **a piecewise continuous mild solution of system (5.31)** with respect to a control u in admissible control set U_{ad} on $[-r, T]$ and for short, we call a $PC-$mild solution.

Definition 5.3.1. *For any $u \in U_{ad}$ and $x \in PC([-r,T],X)$ such that*

$$\begin{cases} x(t) = T(\phi(t))\varphi(0) + \frac{1}{\Gamma(\alpha)} \sum\limits_{0\le t_k \le t} \int_{t_k}^{t_k+1} (t_{k+1}-s)^{\alpha-1} T(\phi(t)-\phi(s))[f(s,x(s),Kx(s)) \\ \qquad + Gx(s) + B(s)u(s)]ds + \sum\limits_{0<t_k<t} T(\phi(t)-\phi(t_k))J_k x(t_k), \quad t \in I \\ x(t) = \varphi(t), \quad t \in [-r,0] \end{cases} \tag{5.44}$$

then the system (5.44) is call a mildly solvable with respect to u on $[-r,T]$ and this x is called a $PC-$ mild solution with respect to u on $[-r,T]$.

Theorem 5.3.2. *Suppose the assumptions (HF), (HG), (HK), (HU), (HJ) holds and the operator A is the infinitesimal generator of a C_0- semigroup $\{T(t)\}_{t\ge0}$ with $\|T(t)\| \le Me^{\omega t}$, $M \ge 1$, $\omega > 0$, $t \ge 0$, then the system (5.1) has a unique $PC-$ mild solution with respect to $u \in U_{ad}$ on $[-r,T]$.*

Proof. Apply the result of corollary 4.2.7 directly to each interval $[t_k, t_{k+1}]$, $k = 0,1,...,n$ where $t_0 = 0$, $t_n = T$. □

After this, we will show the other method to prove the existence of a $PC-$ mild solution of the system (5.1) by using the Leray-Schauder fixed point theorem and the compactness of semigroup $\{T(t)\}_{t\ge0}$. From the definition of a $PC-$ mild solution we define the operator F by

$$\begin{cases} Fx(t) = T(\phi(t))\varphi(0) + \frac{1}{\Gamma(\alpha)} \sum\limits_{0\le t_k \le t} \int_{t_k}^{t_k+1} (t_{k+1}-s)^{\alpha-1} T(\phi(t)-\phi(s))[f(s,x(s),Kx(s)) \\ \qquad + Gx(s) + B(s)u(s)]ds + \sum\limits_{0<t_k<t} T(\phi(t)-\phi(t_k))J_k x(t_k), \quad t \in I \\ Fx(t) = \varphi(t), \quad t \in [-r,0] \end{cases} \tag{5.45}$$

for all $x \in PC([-r,T],X)$. Then F is well-defined. Let $x \in PC([-r,T],X)$. By $(HF2)$, (HK), (HB), (HJ) and lemma 5.1.1, there are constants $a_f > 0$ such that

$$\|f(s,x(s),Kx(s))\| + \|Gx(s)\| \le a_f(2 + \|x(s)\| + \|x_s\|_B) \le N \tag{5.46}$$

for some $N > 0$, for all $s \in [0,T]$ by continuity of $\|x(t)\|$ and $\|x_t\|_B$. So, we obtain that

$$|Fx(t)| \leq Me^{\omega\phi(t)}\|\varphi\|_C + \frac{Me^{\omega\phi(t)}}{\Gamma(\alpha)} \sum_{0\leq t_k \leq t} \int_{t_k}^{t_k+1} (t_{k+1}-s)^{\alpha-1}[\|f(s,x(s),Kx(s))\|$$

$$+ \|Gx(s)\| + \|B(s)u(s)\|]ds + Me^{\omega\phi(t)} \sum_{0<t_k<t} \|J_k x(t_k)\|$$

$$\leq Me^{\omega\phi(t)}\|\varphi\|_C + \frac{MNe^{\omega\phi(t)}}{\Gamma(\alpha)} \sum_{0\leq t_k \leq t} \int_{t_k}^{t_k+1} (t_{k+1}-s)^{\alpha-1}ds$$

$$+ \frac{Me^{\omega\phi(t)}}{\Gamma(\alpha)} \sum_{0\leq t_k \leq t} \int_{t_k}^{t_{k+1}} (t_{k+1}-s)^{\alpha-1}\|B(s)u(s)\|ds + Me^{\omega\phi(t)} \sum_{0<t_k<t} e_k\|x(t_k)\|$$

$$\leq Me^{\omega\phi(t)}\|\varphi\|_C + \frac{MN(n+1)e^{\omega\phi(t)}T^\alpha}{\alpha\Gamma(\alpha)}$$

$$+ \frac{Me^{\omega\phi(t)}\tilde{K}}{\Gamma(\alpha)} \sum_{k=0}^{n} [\int_{t_k}^{t_k+1} (t_{k+1}-s)^{\frac{p(\alpha-1)}{p-1}}ds]^{\frac{p-1}{p}} [\int_{t_k}^{t_k+1} \|B(s)u(s)\|^p ds]^{\frac{1}{p}}$$

$$+ MNe^{\omega\phi(t)} \sum_{0<t_k<t} e_k$$

$$\leq Me^{\omega\phi(t)}\|\varphi\|_C + \frac{MNe^{\omega\phi(t)}T^\alpha}{\alpha\Gamma(\alpha)} + \frac{Me^{\omega\phi(t)}(p-1)(n+1)T^{\frac{p\alpha-1}{p-1}}\|B(\cdot)u\|_{L_p(I,X)}}{(p\alpha-1)\Gamma(\alpha)}$$

$$+ MNe^{\omega\phi(t)} \sum_{k=1}^{n} e_k < \infty.$$

Therefore the operator F is bounded.

Lemma 5.3.3. *Assume that assumption (HF), (HK), (HB), (HJ) holds. Then the operator F is continuous and bounded.*

Proof. Let x_n be a sequence in $PC([-r,T],X)$ that converging to x in $PC([-r,T],X)$. Then there exists $N_0 > 0$ and for all $n > N_0$, $\|x_n - x\|_{PC} \leq 1$. Then $\|x_n\| \leq 1 + \|x\| \equiv \rho$. By using $(HF2), (HK), (HJ)$, lemma 5.1.1, for $s \in (0,T)$ there exist $b(\rho)$, $\tilde{L_g} > 0$ such that

$$\|f(s,x_n(s),Kx_n(s)) - f(s,x(s),Kx(s))\| \leq b(\rho)\|x_n - x\|_{PC}$$

$$||Gx_n(s) - Gx(s)|| \leq \tilde{L_g}||(x_n)_t - x_t||_B.$$

So we have

$$|Fx_n(t) - Fx(t)| \leq \frac{1}{\Gamma(\alpha)} \sum_{0\leq t_k \leq t} \int_{t_k}^{t_k+1} (t_{k+1}-s)^{\alpha-1} \|T(\phi(t)-\phi(s))\|$$

$$\cdot [\|f(s,x_n(s),Kx_n(s)) - f(s,x(s),Kx(s))\| + \|Gx_n(s) - Gx(s)\|]ds$$

$$+ \sum_{0<t_k<t} \|T(\phi(t)-\phi(t_k))\| \|J_k x_n(t_k) - J_k x(t_k)\|$$

$$\leq \frac{Me^{\omega T}}{\Gamma(\alpha)} [b(\rho)||x_n - x||_{PC} + \tilde{L_g}||(x_n)_t - (x)_t||_B] \sum_{0\leq t_k \leq t} \int_{t_k}^{t_k+1} (t_{k+1}-s)^{\alpha-1} ds$$

$$+ Me^{\omega T} \sum_{0<t_k<t} e_k ||x_n(t_k) - x(t_k)||$$

$$\leq \frac{Me^{\omega T}}{\alpha\Gamma(\alpha)} [b(\rho)||x_n - x||_{PC} + \tilde{L_g}||(x_n)_t - (x)_t||_B](n+1)T^\alpha + \sum_{k=0}^{k=n} e_k ||x_n - x||_{PC}.$$

Since $||(x_n)_t - x_t||_B = \sup_{0\leq s\leq t} ||(x_n)_s - x_s||_B = \sup_{0\leq s\leq t} ||x_n(s) - x(s)|| \leq ||x_n - x||_{PC} \to 0$, as $n \to +\infty$, so $||Fx_n - Fx|| \to 0$, as $n \to +\infty$. This implies that the map F is continuous on $PC([-r,T],X)$. □

Corollary 5.3.4. *The operator F maps bounded sets into bounded sets.*

Proof. Let us prove that for any $r > 0$ the exists a $\gamma > 0$ such that for each $x \in B_r \equiv \{x \in PC([-r,T],X) \mid ||x||_{PC} \leq r\}$, we have $||Fx||_{PC} \leq \gamma$. The result is followed from the proof of lemma 5.3.3. □

Lemma 5.3.5. *Suppose conditions (HF), (HK), (HB), (HJ) holds and A is the infinitesimal generator of a compact semigroup $\{T(t)\}_{t\geq 0}$. Then F is a compact operator.*

Proof. Let B be a bounded subset of $PC([-r,T],X)$. By corollary 5.3.4, $F(B)$ is bounded. Define

$$Q = F(B) \text{ and } Q(t) = \{Fx(t) \mid x \in B\}. \tag{5.47}$$

Clearly, for $t \in [-r,0]$, $Q(t) = \{\varphi(t)\}$ is compact.
We only necessary consider for $t > 0$. Given $\epsilon > 0$. For $0 < \epsilon \leq t \leq T$, for short

we denote $f(s,x(s),Kx(s)+Gx(s)+B(s)u(s))$ by $\tilde{f}_u(s,x(s))$. Define

$$\begin{aligned}
Q_\epsilon(t) &\equiv F_\epsilon(B)(t) = T(\phi(\epsilon))T(\phi(t)-\phi(\epsilon))\varphi_0 \\
&+ \frac{T(\phi(\epsilon))}{\Gamma(\alpha)} \sum_{0\le t_k<t} \int_{t_k}^{t_{k+1}} (t_{k+1}-s)^{\alpha-1} T(\phi(t_{k+1})-\phi(\epsilon)-\phi(s))\tilde{f}_u(s,x(s))ds \\
&+ \frac{T(\phi(\epsilon))}{\Gamma(\alpha)} \int_{t_k}^{t-\epsilon} (t-s)^{\alpha-1} T(\phi(t)-\phi(\epsilon)-\phi(s))\tilde{f}_u(s,x(s))ds \\
&+ T(\phi(\epsilon)) \sum_{0<t_k<t} T(\phi(t_k)-\phi(\epsilon)-\phi(s))J_k(x(t_k)). \qquad (5.48)
\end{aligned}$$

Since $\phi(t)$ is continuous and $T(t)$, for $t\ge 0$ is compact in X, the set $\{Q_\epsilon(t) \mid \ x\in B\}$ is relatively compact in X for every ϵ sufficiently small, $t\in(\epsilon,T]$. For $t\in(0,t_1]$ the equation (5.48) reduce to

$$\begin{aligned}
Q_\epsilon(t) &= F_\epsilon(B)(t) = T(\phi(\epsilon))T(\phi(t)-\phi(\epsilon))\varphi_0 \\
&\quad + \frac{T(\phi(\epsilon))}{\Gamma(\alpha)} \int_0^{t-\epsilon} (t-s)^{\alpha-1} T(\phi(t)-\phi(\epsilon)-\phi(s))\tilde{f}_u(s,x(s))ds. \qquad (5.49)
\end{aligned}$$

Furthermore, since $||x(t)||$ and $||x_t||$ are continuous on $(0,t_1]$, there exist $N>0$ such that $||x(t)||$, $||x_t||_B \le N$. By assumptions $(HF2)$, (HB) and lemma 5.1.1, there exist L_g, $L_k>0$ such that

$$\begin{aligned}
||\tilde{f}_u(s,x(s))|| &\le ||f(s,x(s)),Kx(s)|| + ||Gx(s)|| + ||B(s)u(s)|| \\
&\le L_k(1+||x||) + L_g(1+||x_t||_B) + ||B(\cdot)u||_{L_p(I,X)} \\
&\le (L_k+L_g)(1+N) + ||B(\cdot)u||_{L_p(I,X)}|| \equiv L_u. \qquad (5.50)
\end{aligned}$$

Then for $t\in(\epsilon,t_1]$

$$\begin{aligned}
&\sup_{x\in B} ||Fx(t)-F_\epsilon x(t)|| = \frac{1}{\Gamma(\alpha)} \sup_{x\in B} || \int_0^t (t-s)^{\alpha-1} T(\phi(t)-\phi(s))\tilde{f}_u(s,x(s))ds \\
&- T(\phi(\epsilon)) \int_0^{t-\epsilon} (t-s)^{\alpha-1} T(\phi(t)-\phi(\epsilon)-\phi(s))\tilde{f}_u(s,x(s))ds|| \\
&= \frac{1}{\Gamma(\alpha)} \sup_{x\in B} || \int_{t-\epsilon}^t (t-s)^{\alpha-1} T(\phi(t)-\phi(s))\tilde{f}_u(s,x(s))ds|| \\
&\le \frac{ML_u}{\Gamma(\alpha)} \int_{t-\epsilon}^t (t-s)^{\alpha-1} ds = \frac{ML_u\epsilon^\alpha}{\Gamma(\alpha+1)}.
\end{aligned}$$

Therefore there are relatively compact sets arbitrary close to the set $Q(t)$ for $t \in (0, t_1]$. Hence $Q(t)$ itself is relatively compact in X for $(0, t_1]$.
Consider for $t \in (t_1, t_2)$, we define

$$Q(t_1^+) \equiv Q(t_1^-) + J_1(Q(t_1^-)) = Q(t_1) + J_1(Q(t_1)).$$

By the condition (HJ), we get $J_1(Q(t_1))$ is relatively compact and this implies $Q(t_1^+)$ is also relatively compact. Let $x(t_1^+) = x_1$. Then for $t \in (t_1, t_2]$, the equation 5.48 reduce to

$$\begin{aligned} Q_\epsilon(t) = F_\epsilon(B)(t) &= T(\phi(\epsilon))T(\phi(t) - \phi(t_1) - \phi(\epsilon))x_1 \\ &+ \frac{T(\phi(\epsilon))}{\Gamma(\alpha)} \int_0^{t_{k+1}} (t_1 - s)^{\alpha-1} T(\phi(t_1) - \phi(\epsilon) - \phi(s)) \tilde{f}_u(s, x(s)) ds \\ &+ \frac{T(\phi(\epsilon))}{\Gamma(\alpha)} \int_{t_1}^{t-\epsilon} (t - s)^{\alpha-1} T(\phi(t) - \phi(\epsilon) - \phi(s)) \tilde{f}_u(s, x(s)) ds \\ &+ T(\phi(\epsilon))T(\phi(t_1) - \phi(\epsilon) - \phi(s)) J_1(x(t_1)). \end{aligned} \tag{5.51}$$

Furthermore, for $t \in (t_1 + \epsilon, t_2]$

$$\sup_{x \in B} \{||Fx(t) - Fx(t)||\} \leq \frac{ML_u \epsilon^\alpha}{\Gamma(\alpha + 1)}.$$

Therefore $Q(t)$ is relatively compact on $(t_1, t_2]$. In general , given any $t_k \in D = \{t_0 = 0, t_1, t_2, ..., t_n, t_{n+1} = T\}$, define $x(t_k^+) = x_k$ and

$$Q(t_k^+) \equiv Q(t_k^+) + J_k(Q(t_k^-)) = Q(t_k) + J_k(Q(t_k)) \quad \text{for} \quad t_k \in D$$

Similarly,for $t \in (t_k, t_{k+1}]$ the equation (5.48) reduce to

$$\begin{aligned} Q_\epsilon(t) = F_\epsilon(B)(t) &= T(\phi(\epsilon))T(\phi(t) - \phi(t_k) - \phi(\epsilon))x_k \\ &+ \frac{T(\phi(\epsilon))}{\Gamma(\alpha)} \sum_{0 \leq t_k < t} \int_{t_k}^{t_{k+1}} (t_{k+1} - s)^{\alpha-1} T(\phi(t_{k+1}) - \phi(\epsilon) - \phi(s)) \tilde{f}_u(s, x(s)) ds \\ &+ \frac{T(\phi(\epsilon))}{\Gamma(\alpha)} \int_{t_k}^{t-\epsilon} (t - s)^{\alpha-1} T(\phi(t) - \phi(\epsilon) - \phi(s)) \tilde{f}_u(s, x(s)) ds \\ &+ T(\phi(\epsilon)) \sum_{0 < t_k < t} T(\phi(t_k) - \phi(\epsilon) - \phi(s)) J_k(x(t_k)). \end{aligned} \tag{5.52}$$

Furthermore, for $t \in (t_k, t_{k+1}]$

$$\sup_{x\in B}\{||Fx(t) - Fx(t)||\} \leq \frac{ML_u\epsilon^\alpha}{\Gamma(\alpha+1)}. \tag{5.53}$$

By repeating these process till the time interval which expanded, $Q(t)$ is relatively compact for $t \in I\backslash D$ and $Q(t_k^+)$ is relatively compact for $t_k \in D$. Next, we will show that the map Q is equicontinuous on (t_k, t_{k+1}), $k = 0, 1, ..., n$. Since B is bounded and follow from the inequality(5.3.5), there exists a $L_u > 0$ such that

$$||\tilde{f}_u(s, x(s))|| \leq L_u. \tag{5.54}$$

Let $h > 0$ and for $0 < t < t+h < t_1$ and for $x \in B$, we obtain

$$\begin{aligned}
&||Fx(t+h) - Fx(t)|| \leq ||T(\phi(t+h))\varphi(0) - T(\phi(t))\varphi(0)|| \\
&+ ||\frac{1}{\Gamma(\alpha)}\int_0^{t+h}(t+h-s)^{\alpha-1}T(\phi(t+h)-\phi(s))\tilde{f}_u(s,x(s))ds \\
&- \frac{1}{\Gamma(\alpha)}\int_0^{t}(t-s)^{\alpha-1}T(\phi(t)-\phi(s))\tilde{f}_u(s,x(s))ds|| \\
\leq& ||T(\phi(t))||||T(\phi(h)) - I||||\varphi||_C \\
&+ \frac{1}{\Gamma(\alpha)}\int_t^{t+h}(t+h-s)^{\alpha-1}||T(\phi(t+h)-\phi(s))||||\tilde{f}_u(s,x(s))||ds \\
&+ \frac{1}{\Gamma(\alpha)}\int_0^{t}||T(\phi(t)-\phi(s))||||(t-s+h)^{\alpha-1}T(\phi(h)) - (t-s)^{\alpha-1}I||||\tilde{f}_u(s,x(s))||ds \\
\leq& Me^{\omega\phi(T)}||\varphi||_C||T(\phi(h)) - I|| + \frac{Me^{\omega\phi(T)}}{\alpha\Gamma(\alpha)}L_uh^\alpha \\
&+ \frac{Me^{\omega\phi(T)}}{\Gamma(\alpha)}L_u\int_0^t||(t-s+h)^{\alpha-1}T(h) - (t-s)^{\alpha-1}I||ds.
\end{aligned}$$

Since $\lim_{h\to 0}||(t-s+h)^{\alpha-1}T(\phi(h)) - (t-s)^{\alpha-1}I|| = 0$ for all $t > 0$ and $\lim_{h\to 0}||T(\phi(h)) - I|| = 0$, so the right hand side of this equation can be made as desired by choosing h sufficiently small. Hence F is equicontinuous on $(0, t_1)$. In general, for (t_k, t_{k+1}), $k = 0, 1, 2, ..., n$, for $t_k < t < t+h < t_{k+1}$

$$\begin{aligned}
&||Fx(t+h) - Fx(t)|| \leq ||T(\phi(t))||||T(\phi(h)) - I||||x_k|| \\
&+ \frac{1}{\Gamma(\alpha)}\int_t^{t+h}(t+h-s)^{\alpha-1}||T(\phi(t+h)-\phi(s))||||\tilde{f}_u(s,x(s))||ds \\
&+ \frac{1}{\Gamma(\alpha)}\int_0^{t}||T(\phi(t)-\phi(s))||||(t-s+h)^{\alpha-1}T(\phi(h)) - (t-s)^{\alpha-1}I||||\tilde{f}_u(s,x(s))||ds
\end{aligned}$$

$$\leq Me^{\omega\phi(T)}||x_k||||T(\phi(h)) - I|| + \frac{Me^{\omega\phi(T)}}{\alpha\Gamma(\alpha)}L_u h^\alpha$$
$$+ \frac{Me^{\omega\phi(T)}}{\Gamma(\alpha)}L_u \int_0^t ||(t-s+h)^{\alpha-1}T(h) - (t-s)^{\alpha-1}I||ds.$$

Using the same idea, one can show that Q is equicontinuous on (t_k, t_{k+1}), $k = 0, 1, 2, ..., n$. So, the generalized Ascoli-Arzela theorem implies that FB is a relatively compact subset of $PC([-r,T],X)$. Further, F is a compact operator. □

Lemma 5.3.6. *The set* $\Omega \equiv \{x \in PC([-r,T],X) \mid x = \sigma Fx,\ \sigma \in [0,1]\}$ *is bounded on* $PC([-r,T],X)$

Proof. Let $x \in \Omega$. Since $\varphi \in C$, Then, for $t \in [-r, 0]$

$$||x(t)|| = ||\sigma Fx(t)|| \leq ||Fx(t)||||\varphi(t)|| \leq M \quad \text{for some} \quad M > 0.$$

By using assumptions $(HF2)$, (HB) and lemma 5.1.1, there exist a_g, $a_f > 0$ such that for $t \in (0,T]$, we have

$$||x(t)|| = ||\sigma Fx(t)|| \leq ||Fx(t)|| \leq ||T(\phi(t))||||\varphi||_C$$
$$+ \frac{1}{\Gamma(\alpha)} \sum_{0\leq t_k \leq t} \int_{t_k}^{t_{k+1}} (t_{k+1}-s)^{\alpha-1}||T(\phi(t)-\phi(s))||[||f(s,x(s),Kx(s))|| + ||Gx(s)||$$
$$+ ||B(s)u(s)||]ds + \sum_{0<t_k<t} ||T(\phi(t)-\phi(t_k))||||J_k(x(t_k))||$$
$$\leq Me^{\omega\phi(T)}||\varphi||_C$$
$$+ \frac{Me^{\omega\phi(T)}}{\Gamma(\alpha)} \sum_{0\leq t_k \leq t} \int_{t_k}^{t_{k+1}} (t_{k+1}-s)^{\alpha-1}[a_f(1+||x(s)||) + a_g(1+||x_s||_B)]ds$$
$$+ \frac{Me^{\omega\phi(T)}}{\Gamma(\alpha)} \sum_{0\leq t_k \leq t} \int_{t_k}^{t_{k+1}} (t_{k+1}-s)^{\alpha-1}||B(s)u(s)||ds + Me^{\omega\phi(T)} \sum_{0<t_k<t} e_k||x(t_k)||$$

$$\le Me^{\omega\phi(T)}||\varphi||_C + \frac{Me^{\omega\phi(T)}}{\alpha\Gamma(\alpha)}(a_f + a_g)T^{\alpha}(n+1)$$

$$+ \frac{Me^{\omega\phi(T)}}{\Gamma(\alpha)} \sum_{0\le t_k \le t} \int_{t_k}^{t_{k+1}} (t_{k+1}-s)^{\alpha-1}||x(s)||ds$$

$$+ \frac{Me^{\omega\phi(T)}}{\Gamma(\alpha)} \sum_{0\le t_k \le t} \int_{t_k}^{t_{k+1}} (t_{k+1}-s)^{\alpha-1}||x_s||_B ds$$

$$+ \frac{Me^{\omega\phi(T)}}{\Gamma(\alpha)} \sum_{k=0}^{n} [\int_{t_k}^{t_{k+1}} (t_{k+1}-s)^{\frac{p(\alpha-1)}{p-1}} ds]^{\frac{p-1}{p}} [\int_{t_k}^{t_{k+1}} ||B(s)u(s)||^p ds]^{\frac{1}{p}}$$

$$+ Me^{\omega\phi(T)} \sum_{0<t_k<t} e_k ||x(t_k)||$$

$$\le \underbrace{Me^{\omega\phi(T)}[||\varphi||_C + \frac{(a_f+a_g)T^{\alpha}(n+1)}{\alpha\Gamma(\alpha)} + \frac{(a_f+a_g)T^{\frac{p\alpha-1}{p-1}}(n+1)(p-1)||B(\cdot)u||_{L_p(I,X)}}{(p\alpha-1)\Gamma(\alpha)}]}_{a^*}$$

$$+ \underbrace{\frac{Me^{\omega\phi(T)}}{\Gamma(\alpha)}}_{b^*} \sum_{0\le t_k \le t} \int_{t_k}^{t_{k+1}} (t_{k+1}-s)^{\alpha-1}||x(s)||ds$$

$$+ \underbrace{\frac{Me^{\omega\phi(T)}}{\Gamma(\alpha)}}_{c^*} \sum_{0\le t_k \le t} \int_{t_k}^{t_{k+1}} (t_{k+1}-s)^{\alpha-1}||x_s||_B ds + \underbrace{Me^{\omega\phi(T)}}_{d^*} \sum_{0<t_k<t} e_k ||x(t_k)||$$

$$\le a^* + b^* \sum_{0\le t_k \le t} \int_{t_k}^{t_{k+1}} (t_{k+1}-s)^{\alpha-1}||x(s)||ds + c^* \sum_{0\le t_k \le t} \int_{t_k}^{t_{k+1}} (t_{k+1}-s)^{\alpha-1}||x_s||_B ds$$

$$+ d^* \sum_{0<t_k<t} e_k ||x(t_k)||.$$

By theorem 5.2.3, there exists $M > 0$ such that $||x(t)|| \le M$ for all $t \in (0, T]$ for all $x \in \Omega$. Hence Ω is a bounded subset of $PC([-r, T], X)$. □

Theorem 5.3.7, the main result in this section guarantees the existence of $PC-$mild solution with respect to a control $u \in U_{ad}$ on $[-r, T]$ for system(5.31).

Theorem 5.3.7. *Suppose that assumptions (HF), (HG), (HK), (HJ), (HU) holds and the operator A is the infinitesimal generator of a compact semigroup $\{T(t)\}_{t\ge 0}$, then the system (5.31) has at least $PC-$mild solution with respect to a control $u \in U_{ad}$ on $[-r, T]$.*

Proof. Define the operator F by

$$\begin{cases} Fx(t) = T(\phi(t))\varphi(0) + \frac{1}{\Gamma(\alpha)} \displaystyle\sum_{0 \le t_k \le t} \int_{t_k}^{t_k+1} (t_{k+1} - s)^{\alpha-1} T(\phi(t) - \phi(s))[f(s, x(s), Kx(s)) \\ \qquad + Gx(s) + B(s)u(s)]ds + \displaystyle\sum_{0 < t_k < t} T(\phi(t) - \phi(t_k)) J_k x(t_k), \quad t \in I \\ Fx(t) = \varphi(t), \quad t \in [-r, 0] \end{cases}$$

Then by lemma 5.3.3 and lemma 5.3.5, we have F is continuous on $PC([-r,T],X)$ and compact. Set $\Omega \equiv \{x \in PC([-r,T],X) \mid x = \sigma Fx, \ \sigma \in [0,1]\}$. The lemma 5.3.6 implies Ω is a bounded subset of $PC([-r,T],X)$. Thus, by Leray-Schauder fixed point theorem we obtain F has a fixed point in $PC([-r,T],X)$. This implies that the system (5.16) has at least $PC-$mild solution with respect to the control $u \in U_{ad}$ on $PC([-r,T],X)$. □

5.4 Existence of Optimal Controls

In the previous section we already prove the existence of the $PC-$mild solution for the impulsive system. For this section we solve the optimize control problem to the impulsive system. Let U_{ad} be the admissible control set, we consider the Bolza problem say problem (P);
Find $u \in U_{ad}$ corresponding to x^0 such that

$$J(x^0, u^0) \le J(x, u) \quad \text{for all} \ \ u \in U_{ad} \tag{P}$$

where

$$J(x^u, u) = \int_0^T l(t, x^u(t), x_t^u, u(t))dt + \Phi(x^u(T)).$$

x^u denote the mild solution of system (5.16) corresponding to the control $u \in U_{ad}$ and $\Phi : X \to \Re$ is nonnegative continuous function. For short, we denote $J(x,u)$ by $J(u)$.
We solve the optimizing control problem under the following assumption (HL).
Let $l : I \times X \times Y \to (-\infty, \infty]$ be Borel measurable satisfying these conditions:

(HL1) $l(t,\cdot,\cdot,\cdot)$ is sequentially lower semicontinuous on $X \times X \times Y$ for a.e. on I.

(HL2) $l(t,\xi,\nu,\cdot)$ is convex on Y for each $\xi \in X$, $\nu \in X$ and for a.e. $t \in I$.

(HL3) There exist constants a, $b \geq 0, c > 0$ and $\eta \in L_1(I,\Re)$ such that

$$l(t,\xi,\nu_t,u) \geq \eta(t) + a||\xi|| + b||\nu_t||_B + c||u||_Y^q.$$

Theorem 5.4.1. *Under the assumption* (HF), (HU), (HB) *and* (HL) *the optimal control problem* (P) *has a solution, that is, there exists an admissible state-control pair* (x^0,u^0) *such that*

$$J(x^0,u^0) \leq J(x,u) \quad \textit{for all} \;\; u \in U_{ad}.$$

Proof. If $\inf\{J(u) \mid u \in U_{ad}\} = +\infty$ then there is nothing to prove. So we assume that $\inf\{J(u) \mid u \in U_{ad}\} = m < +\infty$. By (HL3), there are constants a, $b \geq 0, c > 0$ and $\eta \in L_1(I,\Re)$ such that

$$l(t,x,x_t,u) \geq \eta(t) + a||x|| + b||x_t||_B + c||u||_Y^q.$$

Since η is nonnegative we have

$$\begin{aligned} J(u) &= \int_0^T l(t,x^u(t),x_t^u,u(t))dt + \Phi(x^u(T)) \\ &\geq \int_0^T \eta(t)dt + a\int_0^T ||x^u(t)||dt + b\int_0^T ||x_t^u||_B dt + c\int_0^T ||u(t)||_Y^q dt \\ &\geq -\sigma > -\infty \quad \text{for some} \;\; \xi > 0, \;\text{for all} \;\; u \in U_{ad}. \end{aligned}$$

Hence $m \geq -\sigma > -\infty$. By definition of minimum, there exists a minimizing sequence $\{u_n\}$ of J , that is $\lim_{n\to\infty} J(u_n) = m$ and

$$J(u_n) \geq \int_0^T \eta(t)dt + a\int_0^T ||x^{u_n}(t)||dt + b\int_0^T ||x_t^{u_n}||_B dt + c\int_0^T ||u_n(t)||_Y^q dt.$$

So there exists $N_0 > 0$ such that for all $n \geq N_0$,

$$m + \widetilde{m} \geq J(u_n) \geq c\int_0^T ||u_n(t)||_Y^q dt$$

for some $\widetilde{m} > 0$ and hence $||u_n||^q_{L_q(I,Y)} \leq \frac{\widetilde{m}+m}{c}$. This show that u_n is contained in a bounded subset of the reflexive Banach space $L_q(I,Y)$. So u_n has a convergence subsequence relabeled as u_n and $u_n \to u_0$ for some $u_0 \in U_{ad} = L_q(I,Y)$. Let $x_n \in PC([-r,T],X)$ be the corresponding sequence of solutions for the integral equation;

$$\begin{cases} x_n(t) = T(\phi(t))\varphi(0) + \frac{1}{\Gamma(\alpha)} \sum_{0 \leq t_k \leq t} \int_{t_k}^{t_k+1} (t_{k+1} - s)^{\alpha-1} T(\phi(t) - \phi(s))[f(s, x_n(s), Kx_n(s)) \\ \qquad + Gx_n(s) + B(s)u_n(s)]ds + \sum_{0 < t_k < t} T(\phi(t) - \phi(t_k))J_k x_n(t_k), t \in [0,T] \\ x_n(t) = \varphi(t), t \in [-r, 0]. \end{cases}$$

From the a priori estimate, there exists a constant $\rho > 0$ such that

$$||x_n||_{PC([-r,T],X)} \leq \rho \quad \text{for all} \quad n = 0, 1, 2, ...$$

where x^0 denote the solution corresponding to u^0, that is

$$\begin{cases} x^0(t) = T(\phi(t))\varphi(0) + \frac{1}{\Gamma(\alpha)} \sum_{0 \leq t_k \leq t} \int_{t_k}^{t_k+1} (t_{k+1} - s)^{\alpha-1} T(\phi(t) - \phi(s))[f(s, x^0(s), Kx^0(s)) \\ \qquad + Gx^0(s) + B(s)u(s)]ds + \sum_{0 < t_k < t} T(\phi(t) - \phi(t_k))J_k x^0(t_k), \quad t \in I \\ x^0(t) = \varphi(t), \quad t \in [-r, 0]. \end{cases}$$

By (HF), (HK), (HG) and lemma 5.1.1 there are constants $a(\rho), b(\rho)$ such that

$$||f(s, x_n(s), Kx_n(s)) - f(s, x^0(s), Kx^0(s))|| \leq a(\rho)||x_n(s) - x^0(s)|| \quad \text{and}$$
$$||Gx_n(t) - Gx^0(t)|| \leq b(\rho)||(x_n)_t - (x^0)_t||_B$$

for each $s \in [0,T]$ and $t \in [-r,T]$. We use the fact that $x_n(s) - x^0(s) = 0$ for

$s \in [-r, 0]$, so we have

$$||x_n(s) - x^0(s)|| \leq \frac{1}{\Gamma(\alpha)} \sum_{0 \leq t_k \leq t} \int_{t_k}^{t_{k+1}} (t_{k+1} - s)^{\alpha-1} T(\phi(t) - \phi(s))$$

$$\cdot [||f(s, x_n(s), Kx_n(s)) - f(s, x^0(s), Kx^0(s))|| + ||Gx_n(t) - Gx^0(t)||$$

$$+ ||B(s)u_n(s) - B(s)u^0(s)||]ds + \sum_{0<t_k<t} T(\phi(t) - \phi(t_k))||J_k x_n(t_k) - J_k x^0(t_k)||$$

$$\leq \frac{Me^{\omega\phi(T)}}{\Gamma(\alpha)} \sum_{0 \leq t_k \leq t} \int_{t_k}^{t_{k+1}} (t_{k+1} - s)^{\alpha-1} [a(\rho)||x_n(s) - x^0(s)|| + b(\rho)||(x_n)_t - (x^0)_t||_B] ds$$

$$+ \sum_{0 \leq t_k \leq t} \int_{t_k}^{t_{k+1}} (t_{k+1} - s)^{\alpha-1} ||B(s)u_n(s) - B(s)u^0(s)||ds + Me^{\omega\phi(T)} \sum_{0<t_k<t} e_k ||x_n(t_k) - x^0(t_k)||$$

$$\leq \frac{[a(\rho) + b(\rho)]Me^{\omega\phi(T)}}{\Gamma(\alpha)} \sum_{0 \leq t_k \leq t} \int_{t_k}^{t_{k+1}} (t_{k+1} - s)^{\alpha-1} ||x_n(s) - x^0(s)||ds$$

$$+ \frac{Me^{\omega\phi(T)}}{\Gamma(\alpha)} \sum_{k=1}^{k=n} [\int_{t_k}^{t_{k+1}} (t_{k+1} - s)^{\frac{p\alpha-1}{p-1}} ds]^{\frac{p-1}{p}} [\int_{t_k}^{t_{k+1}} ||B(s)u_n(s) - B(s)u^0(s)||^p ds]^{\frac{1}{p}}$$

$$+ Me^{\omega\phi(T)} \sum_{0<t_k<t} e_k ||x_n(t_k) - x^0(t_k)||$$

$$\leq \frac{Me^{\omega\phi(T)}(p-1)(n+1)T^{\frac{p\alpha-1}{p-1}} ||B(\cdot)u_n - B(\cdot)u^0||_p}{(p\alpha - 1)\Gamma(\alpha)}$$

$$+ \frac{[a(\rho) + b(\rho)]Me^{\omega\phi(T)}}{\Gamma(\alpha)} \sum_{0 \leq t_k \leq t} \int_{t_k}^{t_{k+1}} (t_{k+1} - s)^{\alpha-1} ||x_n(s) - x^0(s)||ds$$

$$+ Me^{\omega\phi(T)} \sum_{0<t_k<t} e_k ||x_n(t_k) - x^0(t_k)||.$$

By using theorem 5.2.2, we obtain there exist $\widetilde{M} > 0$ independent on u, n and t such that

$$||x_n(t) - x^0(t)|| \leq \widetilde{M} ||B(\cdot)u_n - B(\cdot)u^0||_{L_q(I,Y)}.$$

Since $B(\cdot)$ is strongly continuous, we have $||B(\cdot)u_n - B(\cdot)u^0||_{L_q(I,Y)} \xrightarrow{s} 0$. This implies that $||x_n - x^0|| \xrightarrow{s} 0$ in $C([-r, T], X)$. Let us set $l_n(t) = l(t, x_n(t), (x_n)_t, u_n(t))$ for all $t \in [0, T]$. Then by $(HL3)$, $\{l_n(t)\}$ is a sequence of non-negative measurable functions. So, by using Fatou 's Lemma,

$$\underline{\lim}_{n\to\infty} \int_0^T l_n(t)dt \geq \int_0^T \underline{\lim}_{n\to\infty} l_n(t)dt. \quad (5.55)$$

By $(HL1)$ and (5.55),

$$
\begin{aligned}
m = \lim_{n\to\infty} J(u_n) &\geq \varliminf_{n\to\infty} [\int_0^T l_n(t)dt + \Phi(x_n(T))] \\
&\geq \int_0^T \varliminf_{n\to\infty} l_n(t)dt + \Phi(\varliminf_{n\to\infty} x_n(T)) \\
&= \int_0^T \varliminf_{n\to\infty} l(t, x_n(t), (x_n)_t, u_n(t))dt + \Phi(x^0(T)) \\
&\geq \int_0^T l(t, x^0(t), x_t^0, u^0(t))dt + \Phi(x^0(T)) = J(u^0).
\end{aligned}
$$

This show that $J(u^0) = m$, i.e., $J(u^0) \leq J(u)$ for all $u \in U_{ad}$. □

5.5 Application to Nonlinear Heat Equation

Consider the boundary value problem with delay and control;

$$
\begin{aligned}
\frac{\partial^\alpha y(x,t)}{\partial t^\alpha} =& \Delta y(x,t) + f(x,t,y(x,t),\nabla y(x,t)) \\
&+ \int_{-r}^t h(t-s) g(x,s,y(x,s),\nabla y(x,t))ds \\
&+ \int_\Omega B(x,\xi)u(\xi,t)d\xi, \quad (x,t) \in \Omega \times I \backslash D
\end{aligned} \tag{5.56}
$$

$$\Delta y(x,t_k) = J_k(y(x,t_k)), \quad t_k \in D \tag{5.57}$$

$$y(x,t) = \varphi(x,t), \quad (x,t) \in \bar{\Omega} \times [-r,0] \tag{5.58}$$

$$y(x,t) = 0, \quad (x,t) \in \partial\Omega \times I \tag{5.59}$$

where $I = [0,T]$, $D = \{t_1, t_2, ..., t_n\}$, Ω is boundary domain of $\Re^N$, $\varphi \in C([-r,0] \times \bar{\Omega})$, $u \in L_q(\Omega \times I)$, $h \in C([-r,T]^2, \Re)$ and $B : \bar{\Omega} \times \bar{\Omega} \to \Re$ is continuous.
(HHf) Suppose that $f : \bar{\Omega} \times I \times \Re \times \Re^N \to \Re$, $g : \bar{\Omega} \times I \times \Re \times \Re^N \to \Re$ and there are L_1, $L_2 \geq 0$ such that

$$|f(x,t,\xi,\eta)| + |g(x,t,\xi,\eta)| \leq L_1(1 + |\xi| + |\eta|), \tag{5.60}$$

and

$$|f(x,t,\xi,\eta) - f(x,s,\tilde{\xi},\tilde{\eta})| + |g(x,t,\xi,\eta) - g(x,s,\tilde{\xi},\tilde{\eta})| \leq L_2(|t-s| + |\xi - \tilde{\xi}| + |\eta - \tilde{\eta}|). \tag{5.61}$$

(HHJ) Suppose that $J_k : \Re \to \Re$, $k = 1, 2, ..., n$ satisfies the following conditions, there are $e_k > 0$, $k = 1, 2, ..., n$ such that

$$|J_k(\xi) - J_k(\tilde{\xi})| \leq e_k|\xi - \tilde{\xi}|. \tag{5.62}$$

If we interpret $y(x,t)$ as temperature at the point $x \in \Omega$ at time t, then condition (5.56) means that the temperature at the initial time $t = 0$ is prescribed. Condition (5.59) means that the temperature on the boundary $\partial\Omega$ is equal to zero. The function f describes an external heat sources,for this system f and u are given. We introduce the integral $\int_{-r}^{t} h(t-s)g(x,s,y(x,s),\nabla y(x,s))ds$ denoting in sense of delay term that is impacted from the initial delay function $\varphi(x,t)$ for $t \in [-r, 0]$ in the condition (5.58). Moreover, the system is controlled by the control u via the sensor mapping $\int_{\Omega} B(x,\xi)u(\xi,t)d\xi$. Let $U_{ad} = L_q(\Omega \times I)$ be the admissible control set. We will solve the optimal problem (P_0) via the cost functional;

$$\begin{aligned} J(u) = &\int_0^T \int_{\Omega} |y(\xi,t)|^2 d\xi dt + \int_0^T \int_{\Omega} \int_{-r}^{0} |y(\xi, t+s)|^2 ds d\xi dt \\ &+ \int_0^T \int_{\Omega} |u(\xi,t)|^2 d\xi dt + \Phi(y(x,T)), \end{aligned}$$

where $\Phi \in C(\Re, \Re^+)$.

That is, find $u_0 \in U_{ad}$ Let $X = L_p(\Omega)$. For $t \in [-r, T]$, define $y(t) : \Omega \to X$ by

$$y(t)(x) = y(x,t) \quad \text{for all} \quad x \in \Omega,$$

and define

$$D_t^{\alpha} y(t)x = \frac{\partial^{\alpha} y(x,t)}{\partial t^{\alpha}}, \quad \text{for all} \quad y \in X, \quad x \in \Omega.$$

We define

$$f(t, y(t), Hy(t))(x) = f(x,t,y(x,t),\nabla y(x,t)), \tag{5.63}$$

$$Gy(t)(x) = \int_{-r}^{t} h(t-s)g(x,s,y(x,s),\nabla y(x,t))ds, \tag{5.64}$$

$$B(t)u(t)(x) = \int_{\Omega} B(x,\xi)u(\xi,t)d\xi \tag{5.65}$$

$$J_k(y(t))(x) = J_k(y(x,t)). \tag{5.66}$$

Define an operator $A : X \to X$ as

$$Ay = \Delta y \quad \text{for all} \quad y \in D(A)$$

where $D(A)$ consists of all $C^2(\bar{\Omega})$ function vanishing on $\partial\Omega$. Now we introduce the eigenvalue problem for the negative Laplacian;

$$Ay = \lambda y \quad \text{for all} \quad y \in D(A).$$

Using the standard definition of the inner product, we define that for any y_1, $y_2 \in D(A)$;

$$< Ay_1, y_2 >= \int_\Omega \bar{y_2} \triangle y_1 dy = \int_\Omega \bar{y_1} \triangle y_2 dy =< y_1, Ay_2 > . \tag{5.67}$$

So that A is symmetric and its eigenvalues must be real. Furthermore, for any $y \in D(A)$, we have

$$< Ay, y >=< \triangle y, y >= \int_\Omega \bar{y} \triangle y dy = \int_\Omega |grady|^2 dy \geq 0. \tag{5.68}$$

The right hand side vanishes only if y is constant but the only constant in $D(A)$ is the zero constant. Thus, we obtain

$$\lambda ||y||^2 =< \lambda y, y >=< Ay, y >> 0, \text{ for all } y \neq 0 \text{ in } D(A). \tag{5.69}$$

This is precisely the definition of a positive operator, A is actually strongly positive. On account of equation (5.69), the eigenvalues of A must be positive and we obtain a following lemma.

Lemma 5.5.1. *[15] The operator A defined above is the infinitesimal generator of a compact C_0-semigroup $\{T(t)\}_{t \geq 0}$ on X.*

Then the system (5.56) can transform to the abstract problem as followed;

$$\begin{cases} D_t^\alpha y(t) = Ay(t) + f(t, y(t), Ky(t)) + Gy(t) + B(t)u(t), \quad t \in I \backslash D \\ \Delta y(t_k) = J_k(y(t_k)), \quad t_k \in D \\ y(t) = \varphi(t), \quad t \in [-r, 0]. \end{cases} \tag{5.70}$$

Theorem 5.5.2. *Suppose the assumptions* (HHf) *and* (HHJ) *hold. Then the control problem* (P_0) *for the generalization nonlinear heat equation with delay in* $\Re^N$ *(system (5.56)) has a solution, that is, there exists an admissible state-control pair* (y^0, u^0) *such*

$$J(y^0, u^0) \leq J(y, u) \quad \textit{for all} \quad u \in U_{ad}.$$

Proof. We solve the control problem (P_0) for system(5.56) via the Chauchy abstract form (5.70). By using the assumptions (HHf), (HHJ) and definitions of f, g, J_k $(k = 1, 2, ..., n)$ and the cost functional J, it satisfies all the assumptions given in theorem 5.3.7 and theorem 5.4.1. Then the control problem (P_0) for system(5.56) has a solution, that is, there exists an admissible state-control pair (y^0, u^0) such

$$J(y^0, u^0) \leq J(y^0, u) \quad \text{for all} \quad u \in U_{ad}.$$

□

CHAPTER VI
FRACTIONAL INTEGRO-DIFFERENTIAL EQUATION OF MIXED TYPE WITH SOLUTION OPERATOR

In this chapter, we consider a fractional integro-differential equations of mixed type;

$$\begin{cases} D_t^{\alpha} x(t) = Ax(t) + f(t, x(t), Gx(t), Sx(t)), & t \in I \\ x(t) = \varphi(t), & t \in [-r, 0], \end{cases} \tag{6.1}$$

on infinite dimensional Banach space X, where $I = [0, T]$, $0 < \alpha \leq 1$, D_t^{α} denote the fractional derivative in the sense of Riemann-Liouville, $f : I \times X \times X \times X \to X$, and $\varphi : [-r, 0] \to X$ are given continuous functions, A is an infinitesimal generator of a solution operator $\{T_{\alpha}(t)\}_{t \geq 0}$ in X and G, S are nonlinear integral operators given by

$$Gx(t) = \int_{-r}^{t} k(t, s) g(s, x(s)) ds, \qquad Sx(t) = \int_{0}^{T} h(t, s) q(s, x(s)) ds. \tag{6.2}$$

We will prove the existence and uniqueness of mild solution for system (6.1).

6.1 Background of Solution operator

In this section, the fractional evolution in the sense of Riemann-Liouville, which will be studied throughout this chapter, is formulated. The notion of solution operator plays a basic role in its study.

We study solution operator by starting at the Cauchy problem for the fractional evolution of order α, $0 < \alpha < 1$, ;

$$\begin{cases} D_t^{\alpha} x(t) - Ax(t) = f(t); & t > 0 \\ x(0) = x_0 \end{cases} \tag{6.3}$$

where a closed linear operator A densely defined in a Banach Space X. Certainly, if $\alpha = 1$,then from system (6.3), we get;

$$\begin{cases} D_t x(t) - Ax(t) = f(t); \quad t > 0 \\ x(0) = x_0. \end{cases} \tag{6.4}$$

We will seek a form of solution for (6.4). If f is integrable, then we have

$$\begin{aligned} D_s[e^{A(t-s)}x(s)] &= e^{A(t-s)}D_s x(s) - e^{A(t-s)}x(s) \\ &= e^{A(t-s)}[Ax(s) + f(s)] - e^{A(t-s)}x(s) \\ &= e^{A(t-s)}f(s). \end{aligned} \tag{6.5}$$

Integrating (6.5) from 0 to t and we have,

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)}f(s)ds. \tag{6.6}$$

It is well known that $\{T(t) = e^{At}\}_{t\geq 0}$ form a C_0-semigroup. So the equality (6.6) is equivalent to

$$x(t) = T(t)x_0 + \int_0^t T(t-s)f(s)ds. \tag{6.7}$$

This equation is called a **mild solution** of system (6.4) for $\alpha = 1$.

We can extend this concept to the fractional evolution of order α, $0 < \alpha < 1$ by using the generalized exponential(the Mittag Leffler function). Similarly, we seek the solution in the integral from by using the relation

$$D_t^\alpha f(u(t)) = D_u^\alpha f(u).(\frac{du}{dt})^\alpha$$

and

$$D_t^\alpha[u(t)v(t)] = u(t)D_t^\alpha v(t) + v(t)D_t^\alpha u(t),$$

so we obtain,

$$D_s^\alpha(E_\alpha(A(t-s)^\alpha)x(s)) = E_\alpha(A(t-s)^\alpha)D_s^\alpha x(s) - A[E_\alpha(A(t-s)^\alpha)]x(s). \tag{6.8}$$

Applying the equality (6.3), yields

$$D_s^\alpha(E_\alpha(A(t-s)^\alpha)x(s)) = E_\alpha(A(t-s)^\alpha)[Ax(s) + f(s)] - AE_\alpha(A(t-s)^\alpha)x(s) \tag{6.9}$$

So,

$$D_s^\alpha(E_\alpha(A(t-s)^\alpha)x(s)) = E_\alpha(A(t-s)^\alpha)f(s). \tag{6.10}$$

Integrating of order α from 0 to t and applying an initial $x(0) = x_0$,

$$x(t) = E_\alpha(At^\alpha)x_0 + \frac{1}{\Gamma(1+\alpha)}\int_0^t (t-s)^{\alpha-1}E_\alpha(A(t-s)^\alpha)f(s)ds \tag{6.11}$$

For short, we denote this family $\{E_\alpha(At^\alpha)\}_{t\geq 0}$ by $\{T_\alpha(t)\}_{t\geq 0}$. So the equality 6.11 can be written as

$$x(t) = T_\alpha(t)x_0 + \frac{1}{\Gamma(1+\alpha)}\int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)f(s)ds. \tag{6.12}$$

Also, (6.12) is called **mild solution** of system (6.4) for $0 < \alpha < 1$.

We conjecture that a family $\{E_\alpha(At^\alpha)\}_{t\geq 0}$ will be a C_0-semigroup. Unfortunately, it is not formed a C_0-semigroup. For example, the Mittag-Leffler function $E_\alpha(z)$ for $\alpha = 1/2$ is computed by

$$E_{1/2}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k/2+1)} = e^{z^2}\cdot erfc(-z),$$

where $erfc(z)$ is the complementary error function, which is defined by

$$erfc(z) = \frac{2}{\sqrt{\pi}}\int_z^\infty e^{-t^2}dt \tag{6.13}$$

Let $a = 1$ and $t = s = 1$. Then we have,

$$E_{1/2}(a(t+s)^{1/2}) = E_{1/2}(\sqrt{2}) = e^2\cdot erfc(-\sqrt{2}),$$
$$E_{1/2}(at^{1/2})E_{1/2}(as^{1/2}) = [E_{1/2}(1)]^{1/2} = e^2\cdot[erfc(-1)]^2.$$

Using the software computer to compute $erfc(z)$ with 0.1 percent precision, we get the result that $erfc(-1) \approx 1.8427$ and $erfc(-\sqrt{2}) \approx 1.9545$, which show that $E_{1/2}(a(t+s)^{1/2}) \neq E_{1/2}(at^{1/2})E_{1/2}(as^{1/2})$. This is an evidence guarantee that the family $\{E_\alpha(At^\alpha)\}_{t\geq 0}$ is not a C_0-semigroup.

However, the family $\{E_\alpha(At^\alpha)\}_{t\geq 0}$ is interesting and now we will discuss about its properties and study the equality (6.12) that is why we refer several times to this monograph for basic results on evolutionary equations. For shortness, we define the **solution operator** of (6.3) in terms of the corresponding the integral equation (6.12).

Definition 6.1.1. *Let A be a linear operator on Banach space X. For each $\alpha \in (0,1)$, a family $\{T_\alpha(t)\}_{t\geq 0}$ of bounded linear operators on X is called a solution operator generated by A if the following conditions are satisfied;*

1. *$T_\alpha(t)$ is strongly continuous for $t \geq 0$ and $T_\alpha(0) = I$;*
2. *$T_\alpha(t)x \in D(A)$ for all $x \in D(A)$ and $AT_\alpha(t)x = T_\alpha(t)Ax$;*
3. *$T_\alpha(t)x = x + \frac{1}{\Gamma(1+\alpha)}\int_0^t (t-\tau)^{\alpha-1} AT_\alpha(\tau)x d\tau$ for all $x \in D(A)$.*

Definition 6.1.2. *The solution operator $T_\alpha(t)$ is called **exponential bounded** if there is a constant $M \geq 1$ and $\omega > 0$ such that*

$$||T_\alpha(t)|| \leq Me^{\omega t}, \quad t \geq 0. \tag{6.14}$$

Example 6.1.3. *Let $X = L^p(\Re)$ with $1 \leq p < \infty$ and $A \in L(X)$ define the operator*

$$T_\alpha(t)x \equiv E_\alpha(At^\alpha)x = \sum_{n=0}^{\infty} \frac{A^n t^{\alpha n}}{x} \Gamma(\alpha n + 1). \tag{6.15}$$

Then the right hand side of (6.15) converge in norm for every $t \geq 0$ and defines a bounded linear operator $T_\alpha(t)$;

$$||T_\alpha(t)|| \leq \sum_{n=0}^{\infty} \frac{||A||^n t^{\alpha n}}{\Gamma(\alpha n + 1)} = E_\alpha(||A||t^\alpha). \tag{6.16}$$

If $\alpha \in (0,2)$, then the inequality (6.16) implies that $T_\alpha(t)$ is exponentially bounded. Indeed, the asymptotic expansion (3.45) and the continuity of the Mittag-Leffler function in $t \geq 0$ imply that if $\omega > 0$, there is a constant C such that

$$E_\alpha(\omega t^\alpha) \leq Ce^{\omega^{1/\alpha} t}, \quad \textit{for} \;\; t \geq 0, \;\alpha \in (0,2). \tag{6.17}$$

Therefore (6.16)and (6.17) imply,

$$||T_\alpha(t)|| \leq Ce^{||A||^{1/\alpha} t}, \quad \textit{for} \;\; t \geq 0, \;\alpha \in (0,2). \tag{6.18}$$

Then $T_\alpha(t)$ satisfies conditions of definition 6.1.1, hence $T_\alpha(t)$ define as (6.15) is the solution operator.

Moreover, estimating the power series yields

$$||T_\alpha(t) - I|| \leq \sum_{n=1}^{\infty} \frac{||A||^n t^{\alpha n}}{\Gamma(\alpha n + 1)} = t^\alpha ||A|| E_{\alpha,\alpha+1}(||A||t^\alpha),$$

therefore $\lim_{t\to 0^+} ||T_\alpha(t) - I|| = 0$, *i.e., the solution operator* $T_\alpha(t)$ *is uniformly continuous.*

From this definition we get some facts.

Proposition 6.1.4. *Let* A *a linear operator on* X. *If* $\{T_\alpha(t)\}_{t\geq 0}$ *is a solution operator generated by* A, *then* $T_\alpha(t)T_\alpha(s) = T_\alpha(s)T_\alpha(t)$ *for all* s, $t \geq 0$

Proof. For $x \in D(A)$, for each $t > 0$,

$$T_\alpha(t)x = x + \frac{1}{\Gamma(1+\alpha)} \int_0^t (t-\tau)^{\alpha-1} A T_\alpha(\tau) x d\tau \tag{6.19}$$

and

$$D_\tau^\alpha T_\alpha(\tau)x = AT_\alpha(\tau)x. \tag{6.20}$$

Similarly, for $x \in D(A)$ we have for all s, $\tau \geq 0$,

$$D_\tau^\alpha T_\alpha(\tau)T_\alpha(s)x = AT_\alpha(\tau)T_\alpha(s)x = T_\alpha(\tau)AT_\alpha(s)x = T_\alpha(\tau)T_\alpha(s)Ax. \tag{6.21}$$

Integrating of order α from 0 to t,

$$\begin{aligned} T_\alpha(t)T_\alpha(s)x &= T_\alpha(s)x + \frac{1}{\Gamma(1+\alpha)} \int_0^t (t-\tau)^{\alpha-1} T_\alpha(\tau)T_\alpha(s)Axd\tau \\ &= T_\alpha(s)[x + \frac{1}{\Gamma(1+\alpha)} \int_0^t (t-\tau)^{\alpha-1} T_\alpha(\tau)Axd\tau] \\ &= T_\alpha(s)T_\alpha(t)x \end{aligned}$$

□

Proposition 6.1.5. *If* $\{T_\alpha(t)\}_{t\geq 0}$ *is the solution operator generated by a linear operator* A *on* X *then*

$$Ax = \Gamma(\alpha+1) \lim_{t\to 0^+} \frac{T_\alpha(t)x - x}{t^\alpha}, \tag{6.22}$$

for any $x \in X$ *for which this limit exists.*

Proof. For any function $f \in C(\Re^+, X)$ we have

$$\Delta^\alpha f(t) \cong \Gamma(1+\alpha)\Delta f(t).$$

Hence, for any $x \in X$,

$$\Delta^\alpha T_\alpha(t)x \cong \Gamma(1+\alpha)\Delta T_\alpha(t)x. \tag{6.23}$$

Dividing by t^α and take a limit t near 0^+ on both sides of (6.23) we obtain,

$$\lim_{t\to 0^+} \frac{\Delta^\alpha T_\alpha(t)x}{t^\alpha} = \Gamma(1+\alpha) \lim_{t\to 0^+} \frac{\Delta T_\alpha(t)x}{t^\alpha}. \tag{6.24}$$

Using the condition (1) and (3) in definition 6.1.1 ,

$$AT_\alpha(0)x = \Gamma(1+\alpha) \lim_{t\to 0^+} \frac{T_\alpha(t)x - T_\alpha(0)x}{t^\alpha}. \tag{6.25}$$

We success and get the target equality (6.22). □

Proposition 6.1.6. *If $\{T_\alpha(t)\}_{t\geq 0}$ is the solution operator generated by a linear operator A on X then for every $x \in D(A)$, $\lim_{s\to 0^+} \frac{T_\alpha(t+s)x - T_\alpha(s)T_\alpha(t)x}{s^\alpha} = 0$.*

Proof. From the definition 6.1.1, for each $x \in D(A)$ we have $D_t^\alpha T_\alpha(t)x = AT_\alpha(t)x$ and

$$\begin{aligned}
D_t^\alpha T_\alpha(t)x &= \lim_{s\to 0} \frac{\Delta^\alpha T_\alpha(t)x}{s^\alpha} \\
&= \Gamma(\alpha+1) \lim_{s\to 0} \frac{T_\alpha(t+s)x - T_\alpha(t)x}{s^\alpha} \\
&= \Gamma(\alpha+1) \lim_{s\to 0} \frac{T_\alpha(t+s)x - T_\alpha(s)T_\alpha(t)x + T_\alpha(s)T_\alpha(t)x - T_\alpha(t)x}{s^\alpha} \\
&= \Gamma(\alpha+1)[\lim_{s\to 0} \frac{T_\alpha(t+s)x - T_\alpha(s)T_\alpha(t)x}{s^\alpha} + \lim_{s\to 0} \frac{T_\alpha(s)T_\alpha(t)x - T_\alpha(t)x}{s^\alpha}] \\
&= \Gamma(\alpha+1) \lim_{s\to 0} \frac{T_\alpha(t+s)x - T_\alpha(s)T_\alpha(t)x}{s^\alpha} + AT_\alpha(t)x.
\end{aligned}$$

This implies that $\lim_{s\to 0^+} \frac{T_\alpha(t+s)x - T_\alpha(s)T_\alpha(t)x}{s^\alpha} = 0$, for every $x \in D(A)$. □

Now we introduce the definition for the solution operator that equivalent to the definition 6.1.1, defined by purely algebraic conditions and give diverse properties of them as well.

Definition 6.1.7. *For each $\alpha \in (0,1)$, a family $\{T_\alpha(t)\}_{t\geq 0}$ of bounded linear operators on X is called a solution operator if the following condition are satisfied;*

1. *$T_\alpha(t)$ is strongly continuous for $t \geq 0$ and $T_\alpha(0) = I$;*
2. *for every s, $t \geq 0$, $T_\alpha(s)T_\alpha(t) = T_\alpha(t)T_\alpha(s)$;*
3. *for every $x \in X$, $\lim_{s\to 0^+} \frac{T_\alpha(t+s)x - T_\alpha(s)T_\alpha(t)x}{s^\alpha} = 0.$*

A solution operator $\{T_\alpha(t)\}_{t\geq 0}$ is uniformly continuous if

$$\lim_{t\to 0^+} ||T_\alpha(t) - I||_{L(X)} = 0. \tag{6.26}$$

The operator A defined by

$$Ax = \Gamma(\alpha+1) \lim_{t\to 0^+} \frac{T_\alpha(t)x - x}{t^\alpha} \quad \text{for all } \ x \in D(A) \tag{6.27}$$

is called the infinitesimal generator of solution operator $\{T_\alpha(t)\}_{t\geq 0}$ where

$$D(A) = \{x \in X \mid \lim_{t\to 0^+} \frac{T_\alpha(t)x - x}{t^\alpha} \ \text{ exists in } \ X\}$$

the domain of A.

From the definition (6.1.7), we have a $\{T_\alpha(t)\}_{t\geq 0}$ with a unique infinitesimal generator. If $T_\alpha(t)$ is uniformly continuous, its infinitesimal generator is a bounded operator. On the other hand, every bounded linear operator A is the infinitesimal generator of solution operator of a uniformly continuous solution operator $T_\alpha(t)$ and this solution operator is unique.

6.2 Existence of Solutions to Fractional Integro-differential equations of mixed type

Consider the nonlinear fractional system (6.1),

$$\begin{cases} D_t^\alpha x(t) = Ax(t) + f(t, x(t), Gx(t), Sx(t)), & t \in I \\ x(t) = \varphi(t), & t \in [-r, 0], \end{cases}$$

where $A : D(A) \to X$ be the infinitesimal generator of a solution operator $\{T_\alpha(t)\}_{t\geq 0}$ satisfying $||T_\alpha(t)||_{L(X)} \leq Me^{\omega t}$ for some $M \geq 1, \omega > 0$ for all $t \geq 0$, $f : I \times X \times X \times X \to X$ and $\varphi \in C([-r,T],X)$ are given functions satisfies following conditions (HF);

(HF1) $f : I \times X \times X \times X \to X$ is uniformly continuous in t and locally Lipschitz in x, ξ, η that is for any $\rho > 0$, there are constants $a_f = a_f(\rho,\tau)$ such that $||f(t,x_1,\xi_1,\eta_1) - f(t,x_2,\xi_2,\eta_2)|| \leq a_f[||x_1 - x_2|| + ||\xi_1 - \xi_2|| + ||\eta_1 - \eta_2||]$ provided $||x_1||$, $||x_2||$, $||\xi_1||$, $||\xi_2||$, $||\eta_1||$, $||\eta_2|| \leq \rho$ and for all $t \in [0,\tau]$.

(HF2) There exists $c_f \geq 0$ such that $||f(t,x,\xi,\eta)|| \leq c_f(1 + ||x|| + ||\xi|| + ||\eta||)$ for all x, ξ, $\eta \in X$ and $t \in I$.

First of all, we study the properties of integral operators;

$$Gx(t) = \int_{-r}^{t} k(t,s)g(s,x(s))ds, \quad Sx(t) = \int_{0}^{T} h(t,s)q(s,x(s))ds, \quad \text{for all} \;\; x \in X.$$

We introduce the following assumptions (HG) and (HS);

(HG1) $g : [-r,T] \times X \to X$ is measurable in t on I and locally Lipschitz in x, i.e., let $\rho > 0$, there exists a constant $L_g(\rho)$ such that

$$||g(t,x_1) - g(t,x_2)|| \leq L_g||x_1 - x_2||$$

provided $||x_1||$, $||x_2|| \leq \rho$, for all $t \in [-r,T]$.

(HG2) There exists a constant a_g such that

$$||g(t,x)|| \leq a_g(1 + ||x||), \quad \text{for all} \;\; t \in [-r,T], \; x \in X.$$

(HG3) $k \in C([-r,T]^2, \Re)$.

(HS1) $q : I \times X \to X$ is measurable in t on I and locally Lipschitz in x, i.e., let $\rho > 0$, there exists a constant $L_q(\rho)$ such that

$$||q(t,x_1) - q(t,x_2)|| \leq L_q||x_1 - x_2||$$

provided $||x_1||$, $||x_2|| \leq \rho$, for all $t \in I$.

(HS2) There exists a constant a_q such that for $0 < \gamma < 1$,

$$||q(t,x)|| \le a_q(1 + ||x||^\gamma), \quad \text{for all} \ \ t \in I,\ x \in X.$$

(HS3) $h \in C(I^2, \Re)$.

Using moving norm $||\cdot||_B$ one can verify that integral operator G and S have the following properties.

Lemma 6.2.1. *Under the assumption (HG), the operator G has the following properties;*

(1) $G : C([-r,T], X) \to C([-r,T], X)$.

(2) Let $x_1,\ x_2 \in C([-r,T], X)$ *and* $||x_1||,\ ||x_2|| \le \rho$, *then*

$$||Gx_1(t) - Gx_2(t)|| \le L_g(\rho)(T+r)||k||||(x_1)_t - (x_2)_t||_B, \quad \textit{for all} \ \ t \in [-r,T].$$

(3) For $x \in C([-r,T], X)$, *we have* $||Gx(t)|| \le a_g(T+r)||k||(1+||x_t||_B)$, *for all* $t \in [-r,T]$.

Proof. The proof is similar to the proof of lemma 4.2.2. □

We can similarly obtain the following lemma.

Lemma 6.2.2. *Under the assumption (HS), the operator S has the following properties;*

(1) $S : C(I, X) \to C(I, X)$.

(2) Let $x_1,\ x_2 \in C(I, X)$ *and* $||x_1||,\ ||x_2|| \le \rho$, *then*

$$||Sx_1(t) - Sx_2(t)|| \le L_q(\rho)||h||T||x_1 - x_2||_{C(I,X)}, \quad \textit{for all} \ \ t \in I.$$

(3) For $x \in C(I, X)$, *we have* $||Sx(t)|| \le a_q T||h||(1 + ||x||^\gamma_{C(I,X)})$, *for all* $t \in [-r,T]$.

Proof. The proof is similar to the proof of lemma 4.2.2. □

Recall the fractional integro-differential equations of mixed type system (6.1);

$$\begin{cases} D_t^\alpha x(t) = Ax(t) + f(t, x(t), Gx(t), Sx(t)), & t \in (0, T] \\ x(t) = \varphi(t), & t \in [-r, 0]. \end{cases}$$

Let $0 < \alpha < 1$. If x is a solution of (6.1), then the $X-$valued function $w(s) = T_\alpha(t-s)x(s)$ is $\alpha-$differentiable for $0 < s < t$ and

$$\begin{aligned} D_s^\alpha w(s) &= T_\alpha(t-s)D_s^\alpha x(s) - AT_\alpha(t-s)x(s) \\ &= T_\alpha(t-s)[Ax(s) + f(s, x(s), Gx(s), Sx(s))] - AT_\alpha(t-s)x(s) \\ &= T_\alpha(t-s)f(s, x(s), Gx(s), Sx(s)). \end{aligned} \tag{6.28}$$

Since f is integrable, the right hand side of (6.28) is integrable in the sense of Bochner and integrating (6.28) of order α from 0 to t and applying the initial $w(0) = T_\alpha(t)\varphi(0)$, yields

$$x(t) = T_\alpha(t)\varphi(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) f(s, x(s), Gx(s), Sx(s)) ds, \quad \text{for} \quad t \in I.$$

Therefore we will give a definition of mild solution for system (6.1) as follows.

Definition 6.2.3. *Let $x \in C([-r, t_0], X)$. If there exists a $t_0 > 0$ such that*

$$\begin{cases} x(t) = T_\alpha(t)\varphi(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) f(s, x(s), Gx(s), Sx(s)) ds, & t \in [0, t_0] \\ x(t) = \varphi(t), & t \in [-r, 0]. \end{cases} \tag{6.29}$$

*Then the system (6.1) is called **mildly solvable** on $[-r, t_0]$ and this x is said to be a **mild solution** on $[-r, t_0]$.*

Lemma 6.2.4. *(An a priori bound) If $x \in C([-r, T], X)$ is any solution of system (6.1) then x has an a priori bound, i.e., there is a $\rho > 0$, if x is solution of (6.1) on $[-r, T]$ then $||x(t)|| \leq \rho$, for all $t \in [-r, T]$.*

Proof. Let $x \in C([-r, T], X)$. For $t \in [0, T]$, we use $(HF2)$, lemma 6.2.1 and lemma 6.2.2, there exists a constant c_f such that for all $s \in [0, T]$

$$||f(s, x(s), Gx(s), Sx(s))|| \leq c_f(1 + ||x(s)|| + ||x_s||_B + ||x(s)||^\gamma) \tag{6.30}$$

and

$$||x(t)|| \leq Me^{\omega T}||\varphi||_C + \frac{Me^{\omega T}c_f}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(1+||x(s)||+||x_s||_B+||x(s)||^{\gamma})ds.$$

$$\leq Me^{\omega T}||\varphi||_C + \frac{Me^{\omega T}c_f T^{\alpha}}{\alpha\Gamma(\alpha)} + \frac{Me^{\omega T}c_f}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(||x(s)||+||x_s||_B)ds$$

$$+ \frac{Me^{\omega T}c_f}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}||x(s)||^{\gamma}ds.$$

By lemma 4.1.4, there exists a constant $\rho > 0$ such that $||x(t)|| \leq \rho$, for $t \in I$. □

We will prove the existence and uniqueness of mild solution for the system (6.1). We construct an operator F and prove that it is a strictly contraction by following next lemmas.

For $\tau > 0$, $C^{\tau} \equiv C([-r,\tau], X)$ with the usual supremum norm and for $\lambda > 0$, we set $S(\lambda,\tau) = \{y \in C^{\tau} | \max_{0\leq t\leq \tau} ||y(t) - y(0)|| \leq \lambda$ and $y(0) = \varphi(0),\ t \in [-r,0]\}$. Then $S(\lambda,\tau)$ is a nonempty closed convex subset of C^{τ}.

Define $F : S(\lambda,\tau) \to C^{\tau}$ by

$$\begin{cases} Fy(t) = T_{\alpha}(t)\varphi(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}T_{\alpha}(t-s)f(s,y(s),Gy(s),Sy(s))ds, & t \in [0,\tau], \\ Fy(t) = \varphi(t) \quad t \in [-r,0]. \end{cases} \tag{6.31}$$

Then the map F is bounded. Indeed, by using (6.30), we obtain

$$||Fy(t)|| \leq Me^{\omega T}||\varphi||_C + \frac{Me^{\omega T}c_f}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(1+||y(s)||+||y_s||_B+||y(s)||^{\gamma})ds.$$

Since $y \in C^{\tau}$, there is a constant $N > 0$ such that $1+||y(s)||+||y_s||_B+||y(s)||^{\gamma} \leq N$, so

$$||Fy(t)|| \leq Me^{\omega T}||\varphi||_C + \frac{Me^{\omega T}c_f N T^{\alpha}}{\alpha\Gamma(\alpha)} < \infty.$$

Moreover, some properties of the map F are listed as follows.

Lemma 6.2.5. *The operator F is well-defined on $S(\lambda,\tau)$ for $\tau > 0$. Moreover, there exists $\tau_0 > 0$ such that F maps $S(\lambda,\tau))$ into itself, i.e., $F(S(\lambda,\tau)) \subseteq S(\lambda,\tau))$.*

Proof. For $\lambda > 0$ and $\tau > 0$. Let $\{y_n\}$ be a sequence in $S(\lambda, \tau)$ and $y \in S(\lambda, \tau)$ such $y_n \to y$.

By condition $(HF1)$, lemma 6.2.1 and lemma 6.2.2, there exists a constant $\widetilde{L}(\lambda + ||\varphi||_C, \tau) > 0$ such that for all $s \in [0, \tau]$,

$$||f(s, y_n(s), Gy_n(s), Sy_n(s)) - f(s, y(s), Gy(s), Sy(s))|| \leq \widetilde{L}(\lambda + ||\varphi||_C, \tau)[||y_n(s) - y(s)|| + ||(y_n)_s - y_s||_B]$$

and for each $t \in [0, \tau]$

$$\begin{aligned} ||Fy_n(t) - Fy(t)|| &\leq \frac{Me^{\omega\tau}\widetilde{L}(\lambda + ||\varphi||_C, \tau)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}[||y_n(s) - y(s)|| + ||(y_n)_s - y_s||_B]ds \\ &\leq \frac{Me^{\omega\tau}\widetilde{L}(\lambda + ||\varphi||_C, \tau)\tau^\alpha}{\alpha\Gamma(\alpha)}[||y_n - y||_{C([0,\tau],X)} + ||(y_n)_t - y_t||_B]. \end{aligned}$$

Since $||(y_n)_t - y_t||_B = \sup_{0 \leq s \leq t} ||y_n(s) - y(s)|| \leq ||y_n - y||_{C([0,\tau],X)} \to 0$ as $n \to +\infty$, $||Fy_n - Fy|| \to 0$ as $n \to +\infty$. This implies that the map F is well-defined.

We next show that there is a τ_0 such that F map $S(\lambda, \tau_0)$ into itself.

For each $y \in S(\lambda, \tau)$ and $t \in [0, \tau]$, by assumptions (HF), lemma 6.2.1 and lemma 6.2.2, there exists a κ, $L(\lambda + ||\varphi||_C, \tau) > 0$ such that

$$||f(0, y(0), Gy(0), Sy(0))|| \leq \kappa(1 + ||\varphi||_C),$$

and for each $s \in [0, \tau]$,

$$\begin{aligned} ||f(s, y(s), Gy(s), Sy(s)) - f(0, y(0), Gy(0), Sy(0))|| &\leq L(\lambda, \tau)[||y(s) - \varphi(0)|| + ||y_s - y_0||_B] \\ &\leq 2\lambda L(\lambda + ||\varphi||_C, \tau). \end{aligned}$$

we obtain,

$$\begin{aligned} &||Fy(t) - \varphi(0)|| \\ &\leq ||T_\alpha(t)\varphi(0) - \varphi(0)|| + \frac{Me^{\omega\tau}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}||f(0, y(0), Gy(0), Sy(0))||ds \\ &+ \frac{Me^{\omega\tau}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}||f(s, y(s), Gy(s), Sy(s)) - f(0, y(0), Gy(0), Sy(0))||ds \\ &\leq \max_{0 \leq t \leq \tau} ||T_\alpha(t)\varphi(0) - \varphi(0)|| + \frac{Me^{\omega\tau}[\kappa(1 + ||\varphi||_C) + 2\lambda L(\lambda + ||\varphi||_C, \tau)]\tau^\alpha}{\alpha\Gamma(\alpha)} \leq \lambda q(\tau) \end{aligned}$$

where

$$q(\tau) = \frac{1}{\lambda}[\max_{0\le t\le\tau}||T_\alpha(t)\varphi(0) - \varphi(0)|| + \frac{Me^{\omega\tau}[\kappa(1+||\varphi||_C) + 2\lambda L(\lambda + ||\varphi||_C,\tau)]\tau^\alpha}{\alpha\Gamma(\alpha)}].$$

Since $q(\tau) \to 0^+$ as $\tau \to 0^+$, a suitable τ_0 can be found such that $0 < q(\tau_0) < 1$, so we conclude that the F maps $S(\lambda,\tau_0)$ into itself, i.e., $F(S(\lambda,\tau_0)) \subseteq S(\lambda,\tau_0)$. □

Theorem 6.2.6. *Suppose (HF), (HS), (HG) holds and A is an corresponding to a solution operator $\{T_\alpha(t)\}_{t\ge0}$ with exponentially bound. Then there exists a τ_0 such that the system (6.1) is mildly solvable on $[-r,\tau_0]$ and the mild solution is unique.*

Proof. For $\tau > 0$, set $S(1,\tau) = \{y \in C^\tau \mid \max_{0\le t\le\tau}||y(t) - \varphi(0)|| \le 1, y(0) = \varphi(t),\ t \in [-r,0]\}$. Then $S(1,\tau)$ is the nonempty close convex set. Define the operator $F : S(1,\tau)\to C^\tau$, by (6.31). Then, by lemma 6.2.5, the operator F is well-defined on $S(1,\tau)$ and there exists a τ_0 such that F maps $S(1,\tau_0)$ into itself. We now only show that F is strictly contraction on $S(1,\tau_0)$.
Given $\rho = 2$, let $y_1, y_2 \in S(1,\tau_0)$ such that $||y_1||, ||y_2|| \le 2$. By $(HF1)$, lemma 6.2.1, lemma 6.2.2 and lemma 6.2.5, for $0 \le s \le \tau \le \tau_0$, there exists $b(1+||\varphi||_C,\tau) > 0$ such that

$$||f(s,y_1(s),Gy_1(s),Sy_1(s)) - f(s,y_2(s),Gy_2(s),Sy_2(s))||$$
$$\le b(1+||\varphi||_C,\tau)[||y_1(s) - y_2(s)|| + ||(y_1)_s - (y_2)_s||_B] \le 2b(1+||\varphi||_C,\tau)||y_1 - y_2||_{C([0,\tau_0],X)}.$$

Then

$$||Fy_1(t) - Fy_2(t)|| \le \frac{2Me^{\omega\tau}b(1+||\varphi||_C,\tau)\tau^\alpha}{\alpha\Gamma(\alpha)}||y_1 - y_2||_{C([0,\tau_0],X)} = p(\tau)||y_1 - y_2||_{C([0,\tau_0],X)}$$

where $p(\tau) = \frac{2Me^{\omega\tau}b(1+||\varphi||_C,\rho)\tau^\alpha}{\alpha\Gamma(\alpha)}$. Since $p(\tau) \to 0$ as $\tau \to 0^+$, a suitable $\bar\tau_0 \le \tau_0$ can be found such $0 < p(\bar\tau_0) < 1$, so we conclude that the map F is strictly contraction. By the contraction mapping on Banach space, F has a unique fixed point $x \in S(1,\tau_0)$ such that $Fx(t) = x(t)$, i.e.,

$$\begin{cases} x(t) = T_\alpha(t)x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)f(s,x(s),Gx(s),Sx(s))ds,\ t\in[0,\tau_0] \\ x(t) = \varphi(t),\quad t\in[-r,0]. \end{cases}$$

In other word, we say that $x(t)$ is the unique mild solution of system (6.1) on $[0, \tau_0]$. $\square$

We break the main system (6.1) for a moment and consider the initial value problem,

$$\begin{cases} D_t^\alpha x(t) = Ax(t) + f(t, x(t), Gx(t), Sx(t)), \quad t \geq t_0 \\ x(t_0) = x_0 \end{cases} \tag{6.32}$$

where A is the infinitesimal generator of a solution operator $\{T_\alpha(t)\}_{t\geq 0}$ and $f : [t_0, T] \times X \times X \times X \to X$ is continuous in t on $[t_0, T]$ and uniformly Lipschitz continuous on X. We have the following results.

Definition 6.2.7. *A continuous solution x of the integral equation,*

$$x(t) = T_\alpha(t-t_0)x_0 + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}T_\alpha(t-s)f(s, x(s), Gx(s), Sx(s))ds, \ t \in [t_0, T] \tag{6.33}$$

will be called a mild solution of the system (6.32).

Theorem 6.2.8. *Under the assumption $(HF2)$, (HG) and (HS), if $f : [t_0, T] \times X \times X \times X \to X$ is continuous in t on $[t_0, T]$ and uniformly Lipschitz continuous (with constant L) on X then for every $x_0 \in X$ the initial value problem (6.32) has a unique mild solution $x \in C([t_0, T], X)$. Moreover, the map $x_0 \to x$ is Lipschitz continuous from X into $C([t_0, T], X)$.*

Proof. For a given $x_0 \in X$, we define a mapping $F : C([t_0, T], X) \to C([t_0, T], X)$ by

$$Fx(t) = T_\alpha(t-t_0)x_0 + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}T_\alpha(t-s)f(s, x(s), Gx(s), Sx(s))ds, \ t \in [t_0, T]. \tag{6.34}$$

Then F is well-defined and bounded, it follows readily from the definition of F, lemma 6.2.1 and lemma 6.2.2 that

$$||Fx(t) - Fy(t)|| \leq M_\alpha L(t-t_0)||x-y||_{C([t_0,T],X)} \tag{6.35}$$

where M_α is a bound of $\frac{1}{\alpha\Gamma(\alpha)}||T_\alpha(t)||$ on $[t_0, T]$. Using (6.34), (6.35) and induction on n it follows that

$$||F^n x(t) - F^n y(t)|| \le \frac{(M_\alpha L(t-t_0)^\alpha)^n}{n!}||x-y||_{C([t_0,T],X)} \tag{6.36}$$

whence
$$||F^n x - F^n y|| \le \frac{(M_\alpha L T^\alpha)^n}{n!}||x-y||_{C([t_0,T],X)}. \tag{6.37}$$

For n large enough $\frac{(M_\alpha LT^\alpha)^n}{n!} < 1$ and by a well-known extension of the contraction principle, F has a unique fixed point x in $C([t_0, T], X)$. This fixed point is desired mild solution of (6.32).

The uniqueness of x and the Lipschitz continuity of the map $x_0 \to x$ are consequences of the following argument. Let y be a mild solution of (6.32) on $[t_0, T]$ with the initial value y_0. Then,

$$\begin{aligned}
&||x(t) - y(t)|| \le ||T_\alpha(t-t_0)x_0 - T_\alpha(t-t_0)y_0|| \\
&+ \frac{1}{\Gamma(\alpha)}\int_{t_0}^t (t-s)^{\alpha-1}||T_\alpha(t-s)||||f(s, x(s), Gx(s), Sx(s)) - f(s, y(s), Gy(s), Sy(s))||ds \\
&\le \alpha\Gamma(\alpha)M_\alpha||x_0 - y_0|| + M_\alpha L\int_{t_0}^t (t-s)^{\alpha-1}[||x(s) - y(s)|| + ||x_s + y_s||_B]ds
\end{aligned}$$

which implies, by lemma 4.1.3, that

$$||x(t) - y(t)|| \le \alpha\Gamma(\alpha)M_\alpha e^{M_\alpha L(T-t_0)^\alpha}||x_0 - y_0||, \quad \text{for all} \quad t \in [0, T]$$

and therefore

$$||x-y||_{C([t_0,T],X)} \le \alpha\Gamma(\alpha)M_\alpha e^{M_\alpha L(T-t_0)^\alpha}||x_0 - y_0||$$

which yields both the uniqueness of x and the Lipschitz continuity of the map $x_0 \to x$. □

From the result of theorem 6.2.8, if f is uniform Lipschitz, we have the existence and uniqueness of a global mild solution for system (6.1). But if we assume that f satisfies only local Lipschitz in x, uniformly continuous in t on bounded intervals, then we have the following local version of theorem 6.2.8.

Theorem 6.2.9. *Assume the assumptions of theorem 6.2.6 are holding. Then for every* $x_0 \in X$*, there is a* $t_{max} \leq \infty$ *such that the initial value problem*

$$\begin{cases} D_t^\alpha x(t) = Ax(t) + f(t, x(t), Gx(t), Sx(t)), \quad t > 0 \\ x(0) = x_0 \end{cases} \tag{6.38}$$

has a unique mild solution x *on* $[-r, t_{max})$*. Moreover, if* $t_{max} < \infty$*, then* $\lim_{t \to t_{max}} ||x(t)|| = \infty$.

Proof. We start by showing that for every $\tau_0 \geq 0$, $x_0 \in X$, there exists a $\delta = \delta(\tau, ||x_0||)$ such that the system (6.32) has a unique mild solution x on an interval $[\tau_0, \tau_0 + \delta]$ whose length δ is defined by,

$$\delta(\tau_0, ||x_0||) = \min\{1, [\frac{||x_0||\alpha\Gamma(\alpha)}{\rho(\tau_0)L(\rho(\tau_0), \tau_0 + 1) + N(\tau_0)}]^{1/\alpha}\} \tag{6.39}$$

where $L(c, t)$ is the local Lipschitz constant of f following from $(HF1)$, lemma 6.2.1 and lemma 6.2.2, $M(\tau_0) = \sup\{||T_\alpha(t)|| \mid 0 \leq t \leq \tau_0 + 1\}$, $\rho(\tau_0) = 2||x_0||M(\tau_0)$ and $N(\tau_0) = \max\{||f(t, 0, G0(t), S0(t))|| \mid 0 \leq t \leq \tau_0 + 1\}$. Indeed, Let $\tau_1 = \tau_0 + \delta$ where δ is given by (6.39). Define a map F by (6.34) maps the ball of radius $\rho(\tau_0)$ centered at 0 of $C([\tau_0, \tau_1], X)$ into itself. This follows from the estimate,

$$\begin{aligned} ||Fx(t)|| &\leq M(\tau_0)||x_0|| + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} ||T_\alpha(t-s)||(||f(s, x(s), Gx(s), Sx(s)) \\ &\quad - f(s, 0, G0(s), S0(s))|| + ||f(s, 0, G0(s), S0(s))||)ds \\ &\leq M(\tau_0)||x_0|| + \frac{M(\tau_0)\rho(\tau_0)L(\rho(\tau_0), \tau_0 + 1)}{\alpha\Gamma(\alpha)}(t - \tau_0)^\alpha + \frac{M(\tau_0)N(\tau_0)}{\alpha\Gamma(\alpha)}(t - \tau_0)^\alpha \\ &\leq 2M(\tau_0)||x_0|| = \rho(\tau_0) \end{aligned}$$

where the last inequality follows from the definition of τ_1. In this ball, F satisfies a uniform Lipschitz condition with constant $L = L(\rho(\tau_0), \tau_0 + 1)$ and thus in the proof of theorem 6.2.8, it possesses a unique fixed point x in the ball. This fixed point is the desired solution of (6.32) on the interval $[\tau_0, \tau_1]$.

From what we have just proved, it follows that if x is a mild solution of (6.38) on the interval $[0, \tau]$, it can be extended to the interval $[0, \tau + \delta]$ with $\delta > 0$ by defining on $[\tau, \tau+\delta]$, $x(t) = w(t)$ where $w(t)$ is the solution of the integral equation, for $t \in [\tau, \tau+\delta]$,

$$w(t) = T_\alpha(t-\tau)x(\tau) + \frac{1}{\Gamma(\alpha)} \int_\tau^t (t-s)^{\alpha-1} T_\alpha(t-s) f(s, w(s), Gw(s), Sw(s)) ds.$$

Moreover, δ depends only on $||x(\tau)||$, $\rho(\tau)$ and $N(\tau)$.

Let $[-r, t_{max})$ be the maximum interval of existence of mild solution x for (6.38). If $t_{max} < \infty$, then $\lim_{t \to t_{max}} ||x(t)|| = +\infty$, indeed, if it is false, then there exists a sequence $\{t_n\}$ and $C > 0$ such that $t_n \to t_{max}$ and $||x(t_n)|| \leq C$ for all n. This implies that for each t_n near enough to t_{max}, x define on $[-r, t_n]$ can be extended to $[-r, t_n+\delta]$ where $\delta > 0$ is independent of t_n, hence x can be extend beyond t_{max}, this contradicts the definition of t_{max}. So if $t_{max} < \infty$, then $\lim_{t \to t_{max}} ||x(t)|| = +\infty$.

To prove the uniqueness of the local mild solution of (6.38) we note that if y is a mild solution of (6.38), then on every closed interval $[-r, \tau_0]$ on which both x and y exist they coincide by the uniqueness argument given in the end of the proof of theorem 6.2.8. Therefore, both x and y have the same t_{max} and on $[-r, t_{max})$, $x = y$. □

Theorem 6.2.10. *If the assumptions of theorem 6.2.6 are holding, then the system (6.1) has a unique mild solution on* $[-r, T]$.

Proof. Let $[-r, t_{max})$ be the maximum interval of existence of mild solution x for (6.1). If $t_{max} > T$, there is nothing to prove. If $t_{max} < T$, by theorem 6.2.9, then $\lim_{t \to t_{max}} ||x(t)|| = +\infty$, contradicts with an a priori bound of solution. So the system (6.1) has a unique mild solution on $[-r, T]$. □

6.3 Existence of Optimal Controls

In this section, we discuss the existence of optimal controls of systems governed by the fractional integro-differential equation (6.1).

We suppose that A is the infinitesimal generator of a solution operator $\{T_\alpha(t)\}_{t \geq 0}$ and Y is another separable reflexive Banach space from which the controls u take the values. Let $U_{ad} = L_q(I, Y)$, $1 < q < \infty$ denoting the admissible controls set. Consider the following controlled system;

$$\begin{cases} D_t^\alpha x(t) = Ax(t) + f(t, x(t), Gx(t), Sx(t)) + B(t)u(t), & t \in [0, T] \\ x(t) = \varphi(t), & t \in [-r, 0]. \end{cases} \tag{6.40}$$

Suppose (HB):$B(s) \in L(L_q(I, Y), L_p(I, X))$ for all $s \in I$ and $B(\cdot)$ is strongly continuous where $1 < q < \infty$ and $p > 1/\alpha$. Then $B(\cdot)u \in L_p(I, X)$ for all $u \in U_{ad}$ and we give the definition of mild solution with respect to a control in U_{ad}.

Definition 6.3.1. *Let $x \in C([-r, T], X)$ and $u \in U_{ad}$. If x is a solution of,*

$$\begin{cases} x(t) = T_\alpha(t)\varphi(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s)[f(s, x(s), Gx(s), Sx(s)) + B(s)u(s)]ds, & t \in I \\ x(t) = \varphi(t), & t \in [-r, 0] \end{cases}$$

then this x is said to be a mild solution with respect to (w.r.t.) u on $[-r, T]$.

Theorem 6.3.2. *Under assumptions (HF), (HG), (HS), (HB) and A is the infinitesimal generator of a solution operator $\{T_\alpha(t)\}_{t\geq 0}$, for every $u \in U_{ad}$, system (6.40) has a mild solution corresponding to u.*

Proof. Let $u \in U_{ad}$, define $\widetilde{f}(t, x(t)) = f(t, x(t), Gx(t), Sx(t)) + B(t)u(t)$, for all $x \in X$. Use the fact that $B(\cdot)u \in L_p(I, X)$ for all $u \in U_{ad}$ and use assumption (HF), lemma 6.2.1 and lemma 6.2.2, we obtain that $\widetilde{f}$ satisfies the assumption (HF). By theorem 6.2.6, so we have complete the proof. □

We consider the Lagrange problem (P_0): Find $(x^0, u^0) \in X \times U_{ad}$ such that

$$J(x^0, u^0) \leq J(x^u, u), \quad \text{for all } u \in U_{ad} \tag{6.41}$$

where

$$J(x^u, u) = \int_0^T l(t, x^u(t), x_t^u, u(t))dt, \tag{6.42}$$

for short, denoting $J(x^u, u)$ by $J(u)$ and x^u denotes the mild solution of system (6.40) corresponding to the control $u \in U_{ad}$.

We impose some assumption on l. Assumption (HL);

1) $l : I \times X \times X \times Y \to (-\infty, \infty]$ is Borel measurable.

2) $l(t, \cdot, \cdot, \cdot)$ is sequentially lower semicontinuous on $X \times Y$ for a.e. on I.

3) $l(t, x, y_t, \cdot)$ is convex on Y for each $(x, y_t) \in X \times X$ and for a.e. $t \in I$.

4) There exist constants a, $b \geq 0$, $c > 0$ and $\eta \in C(I, \Re)$ such that

$$l(t, x, y_t, u) \geq \eta(t) + a||x|| + b||y_t||_B + c||u||_Y^q$$

A pair (x^u, u) is said to be feasible if it satisfies equation (6.40).

Theorem 6.3.3. *Suppose the assumption (HL) and the assumptions of theorem 6.3.2 hold. Then problem (P_0) for system (6.40) admits at least one optimal pair.*

Proof. If $inf\{J(u)|u \in U_{ad}\} = +\infty$ there is nothing to prove. So we assume that $inf\{J(u)|u \in U_{ad}\} = m < +\infty$. By $(HL4)$, there are constants $b \geq 0, c > 0$ and $\eta \in L_1(I, \Re)$ such that $l(t, x^u, x_t^u, u) \geq \eta(t) + a||x^u|| + b||x_t^u||_B + c||u||_Y^q$. Since η is nonnegative, we have

$$\begin{aligned} J(u) &= \int_0^T l(t, x^u(t), x_t^u, u(t))dt \\ &\geq \int_0^T \eta(t)dt + a\int_0^T ||x^u(t)||dtb\int_0^T ||x_t^u||dt + c\int_u^T ||u(t)||_Y^q dt \\ &\geq -\xi > -\infty \end{aligned}$$

for some $\xi > 0$, for all $u \in U_{ad}$. Hence $m \geq -\xi > -\infty$. By definition of minimum, there exists a minimizing sequence $\{u_n\}$ of J , that is $\lim_{n\to\infty} J(u_n) = m$ and

$$J(u_n) \geq \int_0^T \eta(t)dt + a\int_0^T ||x^{u_n}(t)||dt + b\int_0^T ||x_t^{u_n}||dt + c\int_0^T ||u_n(t)||_Y^q dt.$$

So there exists $N_0 > 0$ such that for all $n \geq N_0$,

$$m + \tilde{m} \geq J(u_n) \geq c\int_0^T ||u(t)||_Y^q dt$$

for some $\tilde{m} > 0$ and hence $||u_n||_{L_q(I,Y)}^q \leq \frac{\tilde{m}+m}{c}$.
This show that u_n is contained in a bounded subset of the reflexive Banach space

$L_q(I,Y)$. So u_n has a convergence subsequence relabeled as u_n and $u_n \to u^0$ for some $u^0 \in U_{ad} = L_q(I,Y)$. Let $x_n \subseteq C([-r,T],X)$ be the corresponding sequence of solutions for the integral equation;

$$\begin{cases} x_n(t) = T_\alpha(t)\varphi(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)[f(s,x_n(s),Gx_n(s),Sx_n(s)) \\ \qquad\qquad + B(s)u_n(s)]ds, \quad t \in I \\ x_n(t) = \varphi(t), \quad t \in [-r,0]. \end{cases} \tag{6.43}$$

From the a priori estimate, there exists a constant $\rho > 0$ such that

$$||x_n||_{C(I,X)} \leq \rho, \quad \text{for all} \quad n = 0,1,2,...$$

where x^0 denote the solution corresponding to u^0, that is

$$\begin{cases} x^0(t) = T_\alpha(t)\varphi(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)[f(s,x^0(s),Gx^0(s),Sx^0(s)) \\ \qquad\qquad + B(s)u^0(s)]ds, \quad t \in I, \\ x^0(t) = \varphi(t), \quad t \in [-r,0]. \end{cases} \tag{6.44}$$

By (HF), (HG), (HS), (HL), lemma 6.2.1 and lemma 6.2.2, there is a constant $a(\rho)$ such that for each $t \in I$,

$$\begin{aligned} ||x_n(t) - x^0(t)|| &\leq \frac{Me^{\omega T}a(\rho)}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[||x_n(s) - x^0(s)|| + ||(x_n)_t - (x^0)_t||_B]ds \\ &\quad + \frac{Me^{\omega T}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}||B(s)u_n(s) - B(s)u^0(s)||ds \\ &\leq \frac{Me^{\omega T}a(\rho)}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[||x_n(s) - x^0(s)|| + ||(x_n)_t - (x^0)_t||_B]ds \\ &\quad + \frac{Me^{\omega T}}{\Gamma(\alpha)}[\frac{(p-1)T^{(\alpha p-1)/(p-1)}}{\alpha p - 1}]^{\frac{p-1}{p}}||B(\cdot)u_n - B(\cdot)u^0||_{L_p(I,X)}. \end{aligned}$$

By using lemma 4.1.3,

$$||x_n(t) - x^0(t)|| \leq \tilde{M}||B(\cdot)u_n - B(\cdot)u^0||_{L_p(I,X)}$$

where $\tilde{M}$ is a constant,independent of u, n and t. Since $B(\cdot)$ is strongly continuous, we have $||B(\cdot)u_n - B(\cdot)u^0||_{L_p(I,X)} \to 0$. This implies that $||x_n - x^0|| \to 0$ in

$C([-r,T],X)$. Let us set $l_n(t) = l(t, x_n(t), (x_n)_t, u_n(t))$ for all $t \in [0,T]$. Then by $(HL1)$ and $(HL3)$, $\{l_n(t)\}$ is a sequence of non-negative measurable functions. So, by using Fatou 's Lemma,

$$\varliminf_{n\to\infty} \int_0^T l_n(t)dt \geq \int_0^T \varliminf_{n\to\infty} l_n(t)dt. \tag{6.45}$$

By $(HL2)$ and (6.45),

$$\begin{aligned} m = \lim_{n\to\infty} J(u_n) &\geq \varliminf_{n\to\infty} \int_0^T l_n(t)dt \\ &\geq \int_0^T \varliminf_{n\to\infty} l_n(t)dt \\ &= \int_0^T \varliminf_{n\to\infty} l(t, x_n(t), (x_n)_t, u_n(t))dt \\ &\geq \int_0^T l(t, x^0(t), x^0_t, u^0(t))dt = J(u^0). \end{aligned}$$

This show that $J(u^0) = m$, i.e., $J(u^0) \leq J(u)$ for all $u \in U_{ad}$. □

6.4 Application to Nonlinear Heat Equation

Consider the nonlinear heat equation control;

$$\begin{cases} \frac{\partial^\alpha y(x,t)}{\partial t^\alpha} = \Delta y(x,t) + f_1(x,t,y(x,t)) + \int_{-r}^t h(t-s)g(x,s,y(x,s))ds \\ + \int_0^T k(t-s)q(x,s,y(x,s))ds + \int_\Omega B(x,\xi)u(\xi,t)d\xi, \quad (x,t) \in \bar{\Omega}\times I \\ y(x,t) = 0, \quad (x,t) \in \partial\Omega \times I \\ y(x,t) = \varphi(x,t), \quad (x,t) \in \bar{\Omega}\times[-r,0], \end{cases} \tag{6.46}$$

where Ω is boundary domain of $\Re^N$, $u \in L_q(\Omega\times I)$, $h, k \in C(I^2, \Re)$ and $B : \bar{\Omega}\times\bar{\Omega} \to \Re$ and $\varphi : \bar{\Omega}\times[-r,0] \to \Re$ are continuous. Suppose that $f : \bar{\Omega}\times I\times\Re \to \Re$, $g : \bar{\Omega}\times[-r,T]\times\Re \to \Re$, $q : \bar{\Omega}\times I\times\Re \to \Re$, and for each $\rho > 0$ there are L_1, L_2, $L_3 > 0$ such that

$$|f(x,t,\xi) - f(x,s,\tilde{\xi})| \leq L_1(|t-s| + |\xi - \tilde{\xi}|), \tag{F}$$

$$|g(x,t,\xi) - g(x,s,\tilde{\xi})| \leq L_2(|t-s| + |\xi - \tilde{\xi}|), \tag{G}$$

$$|q(x,t,\xi) - q(x,s,\tilde{\xi})| \leq L_3(|t-s| + |\xi - \tilde{\xi}|), \tag{S}$$

provided $||\xi||$, $||\tilde{\xi}|| \leq \rho$ and $s,\, t \in I$. If we interpret $y(x,t)$ as temperature at the point $x \in \Omega$ at time t, then condition (6.46) means that the temperature at the initial time $t = 0$ is prescribed. Condition $y(x,t) = 0, \quad (x,t) \in \partial\Omega \times I$ means that the temperature on the boundary $\partial\Omega$ is equal to zero. The function f describes an external heat sources, for this system f and u are given. We introduce the integral $Gy(x,t) = \int_{-r}^{t} h(t-s)g(x,s,y(x,s))ds$ and $Sy(x,t) = \int_0^T k(t-s)q(x,s,y(x,s))ds$, both terms directly impact to the system. Moreover, the system is controlled by the control u via the sensor mapping $\int_\Omega B(x,\xi)u(\xi,t)d\xi$. Let $U_{ad} = L_q(\Omega \times I)$ be the admissible control set. We will solve the optimal problem (P_0) via the cost functional;

$$J(u) = \int_0^T \int_\Omega |y(\xi,t)|^2 d\xi dt + \int_0^T \int_\Omega \int_{-r}^0 |y(\xi,t+s)|^2 ds d\xi dt + \int_0^T \int_\Omega |u(\xi,t)|^2 d\xi dt.$$

That is, find $u_0 \in U_{ad}$ such that $J(u_0) \leq J(u)$ for all $u \in U_{ad}$.
Let $X = L_p(\Omega)$. For $t \in [-r,T]$, define $y(t) : \Omega \to X$ by

$$y(t)(x) = y(x,t) \quad \text{ for all } \ x \in \Omega,$$

and define

$$D_t^\alpha y(t)(x) = \frac{\partial^\alpha y(x,t)}{\partial t^\alpha}, \ \text{ for all } \ y \in X, \ \ x \in \Omega.$$

We define

$$f(t,y(t),Gy(t),Sy(t))(x) = f(x,t,y(x,t)) + Gy(t)(x) + S(t)(x), \tag{6.47}$$

$$B(t)u(t)(x) = \int_\Omega B(x,\xi)u(\xi,t)d\xi, \tag{6.48}$$

where

$$Gy(t)(x) = \int_{-r}^{t} h(t-s)g(x,s,y(x,s))ds, \tag{6.49}$$

$$Sy(t)(x) = \int_0^T k(t-s)q(x,s,y(x,s))ds. \tag{6.50}$$

Define an operator $A : X \to X$ as

$$Ay = \Delta y \quad \text{for all } \ y \in D(A)$$

where $D(A)$ consists of all $C^2(\bar{\Omega})$ function vanishing on $\partial\Omega$.

Lemma 6.4.1. *The operator A defined above is the infinitesimal generator of a solution operator $\{T_\alpha(t)\}_{t\geq 0}$ on X.*

Proof. Consider the general heat equation of fractional order $0 < \alpha \leq 1$,

$$D_t^\alpha u = Au, \quad u(0,x) = f(x). \tag{6.51}$$

Applying the Fourier transformation, we obtain

$$D_t^\alpha \hat{u} = -|\xi|^2 \hat{u}, \quad \hat{u}(0,\xi) = \hat{f}(\xi). \tag{6.52}$$

By solving (6.52),

$$\hat{u}(\xi,t) = E_\alpha(-t^\alpha |\xi|^2)\hat{u}(\xi). \tag{6.53}$$

Take the inverse Fourier formula, the solution of (6.51) is,

$$u(t,x) = E_\alpha(t^\alpha A)f(x) = (2\pi)^{-n/2}\int_{\Re^n} E_\alpha(-t^\alpha|\xi|^2)\hat{f}(\xi)e^{ix\xi}d\xi. \tag{6.54}$$

Set $T_\alpha(t) = E_\alpha(t^\alpha A)$. Then $T_\alpha(t)$ satisfies the conditions of lemma 6.1.7. Therefore The operator $A = \Delta$ is the infinitesimal generator of a solution operator $\{T_\alpha(t)\}_{t\geq 0}$ on X. □

Then by lemma 6.4.1 and all above, the system (6.46) can transform to the abstract problem as followed;

$$\begin{cases} D_t^\alpha y(t) = Ay(t) + f(t, y(t), Ky(t)) + Gy(t) + B(t)u(t), \quad t \in I \\ y(t) = \varphi(t), \quad t \in [-r, 0]. \end{cases} \tag{6.55}$$

Theorem 6.4.2. *Suppose assumptions (F), (G) and (S) hold. Then the control problem (P_0) for system(6.46) has a solution, that is there exists an admissible state-control pair (y^0, u^0) such*

$$J(y^0,u^0) \leq J(y^0,u) \;\; \textit{for all} \;\; u \in U_{ad}.$$

Proof. We solve the control problem (P_0) for system(6.46) via the Cauchy abstract form (6.55). By using the assumptions (F), (G), (S) and the cost functional J, it satisfies all the assumptions given in theorem 6.3.3 and theorem 6.2.6. Then the control problem (P_0) for system(6.46) has a solution. □

CHAPTER VII
CONCLUSIONS AND OUTLOOK

In this work, we start considering the nonlinear fractional integro-differential system (4.1) in Chapter 4 when $-A$ is the infinitesimal generator of C_0-semigroup satisfying the exponential stability. We win to prove the existence and uniqueness of mild solution. We propose a method for proving existence whose main component is the use semigroup of bounded linear operators and Banach fixed point theorem. We successfully apply this method and use some assumptions to prove the existence and uniqueness of mild solution. We win to prove the existence of the optimal control problem via the Bolza condition. Beside the study of the solution, we give some examples (model of problem in the real world). Then we transform them to the abstract form and use our main result to conclude that these systems have a mild solution with respect to a control in admissible control set and the Bolza problem for this system has a solution.

In Chapter 6, we consider the fractional integro-differential equations of mixed type, this system resemble the system (6.1) in Chapter 4. The idea of semigroup of bounded linear operators is replaced by the idea of solution operators. We win to prove the existence of a mild solution with respect to a control in the admissible controls set and also the existence of a solution of the Lagrange problem for the fractional integro-differential equations of mixed type, system (6.1).

In Chapter 5, we consider the impulsive fractional introgro-differential equation (5.1). We successfully prove the existence of piecewise continuous mild solution w.r.t a control in the admissible controls set with compact semigroup of bounded linear operators. In this case, we use Leray-Schauder theorem and the new version of generalization Gronwall lemma for the fractional order. Also we successfully prove the existence the optimal control problem by using the Bolza condition. In the last section, some example was established to supporting the main result.

Last but not least we should be interested in developing this method and use weakly assumptions to prove the existence and uniqueness of classical solution a little further. Moreover, we should be interested in studying the solution behaviors for example;the stable property. Even though it seems likely that efforts in this direction can be successful, there no guarantee for that. Therefore, we can only hope for the best, but have to expect the worst.

REFERENCES

[1] Ahmed N. U. and Teo K. L., *Optimal Control of Distributed Parameter Systems*, North Holland New York Oxford, 1982.

[2] Akhmet M.U., Bekmukhambetova G.A. and Serinğaoğlu, *The dynamics of the systemic arterial pressure through impulsive differential equations*, 2000.

[3] Chonwerayuth A., *Optimal Control of System Governed by a Class of Integro-Differential Equations*, Southeast Asian Bulletin of Mathematics, Vol.30:229-248, 2006.

[4] Gastão S.F.Frederico and Delfim F.M.Torres. *Fractional optimal control in the sense of Caputo and the fractional Noether's theorem*, International Mathematical forum, 3,no. 10, pp.479-493, 2008.

[5] Gisèle M.Mophou. *Existence and uniqueness of mild solution to impulsive fractional differential equations*, Nonlinear Analysis, doi:10.1016/ j.na.2009.08.046., 2009.

[6] Igor podlubny. *Geometric and physical interpretation of fractional integration and fractional differentiation*, Fractional calculus and applied analysis, vol.5, Number 4, 2002.

[7] Jumarie G. *Fractional Partial Differential Equations and Modified Riemann -Liouville Derivative new Methods for Solution.* J. Appl. Math and Computing, Vol. 24, No. 1-2, pp. 31-48, 2007.

[8] Lakshmikantham V., Binov D.D. and Simenov P.S., *Theory of Impulsive Differential Equations*, World Scientific, Singapore, 1989.

[9] Lions J. L., *Optimal Control of Systems Governed by Partial Differential Equations*, Springer-Verlag New York Heidelberg Berlin, 1971.

[10] Liu H.J., *Nonlinear Impulsive Evolution Equations*, AMS (MOS).

[11] Mainardi F., *Fractional relaxation-oscillation and fractional diffusion-wave phenomena. Chaos, Solutions and Fractals*, No 9, PP.1461-1477, 1996.

[12] McBride A.C., *Semigroups of Linear Operators: An introduction*, Longman, Harlow, U.K., 1987.

[13] Milan M., *Applied Functional Analysis Partial Differential Equations*, Michigan State University.

[14] Miller K. S. and Ross B., *An Introduction to the Fractional Calculus and Fractional Differential Equations*. Willey, New York, 1993.

[15] Pazy A., *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer, New York, 1983.

[16] Wei. W, Xiang X. and Peng Y. *Nonlinear impulsive integro-differential equations of mixed type and optimal controls, Optimization*, Vol.55, Issue 12 February, pp.141 - 156, 2006.

[17] Xiang X., *Impulsive Control Theory*, Springer, New York, 2001.

[18] Yang T., *Impulsive Control Theory*, Springer, New York, 2001.

VITA

Mr. Wichai Witayakiatilerd was born on Febuary 5, 1981 in NakornPathom, Thailand. He graduated with a Bachelor Degree of Science in Mathematics from Silapakorn University in 2003. He got the Development and Promotion Science and Technology talents project(DPST) scholarship in 1996 to 2005 for his study in Mathematics. For his Master degree and Doctoral degree, he has studied Mathematics at the Faculty of Science, Chulalongkorn University with the DPST and the Centre of Excellence in Mathematics(CEM) scholarship.